School-Mathematics all over the World – some Differences

Ludwig Paditz, University of Applied Sciences Dresden, Germany
paditz@informatik.htw-dresden.de

Abstract:
The lecture is devoted on some differences of definitions in school-mathematics. In other countries we have sometimes other definitions compared with Germany. Sometimes we can discover even in Germany some differences, because we have in Germany 16 smaller federal countries and every country has its own ministry of education.
On the other hand every professor in any university is free and independent concerning the research and education. He could use his own definition in special fields of mathematics. We would have no problems, if we respect different definitions of different teachers and in different software:

1. What is the set of natural numbers?
   {0, 1, 2, …} or {1, 2, 3, …} and what means the symbol **N**?
2. What means $3^1/_2$? Is it $3 + {}^1/_2 = 3.5$ or $3 * {}^1/_2 = 1.5$?
3. Where the real function $y = \mathrm{f}(x) = x$^(1/3) is defined? x must be non-negative or $x \in \mathbf{R}$?
4. What means the notation $y = \tan^{-1}(x)$? Is this the arctan-function or the cot-function?
5. What is $y = \log(x)$? We know $y = \log_a(x)$. What is the base a in $y = \log(x)$?
   Without any a we know $y = \lg(x)$ or $y = \ln(x)$ or $y = \mathrm{lb}(x) = \mathrm{ld}(x)$ (older notation).
   Sometimes we find the notation $y = {}_a\log(x)$. Why different notations?
6. What is the main argument φ of a complex number in the Gaussian plane?
   $\varphi = \arg(-2-2\mathbf{j}) = -3\pi/4$ or $\varphi = \arg(-2-2\mathbf{j}) = 5\pi/4$?
7. What is the 3rd main root of the complex number -2-2**j** ? What is (-2-2**j**)^(1/3) ?
   What is (-8)^(1/3) ? Draw the real function $y = \mathrm{f}(x) = x$^(1/3) ≈ x^0.33333333 !
8. What is the α-quantile x_α of a probability distribution of a random variable X?
   x_α is a number with $P(X< x_\alpha) \le \alpha \le P(X \le x_\alpha)$ or $P(X< x_\alpha) \le 1-\alpha \le P(X \le x_\alpha)$
9. Is the distribution function $y = \mathrm{F}(x)$ of a random variable X right continuous or left continuous? Is $\mathrm{F}(x) = P(X< x)$ or $\mathrm{F}(x) = P(X\le x)$? Consider the binomial distribution!

We have to solve in this context two basic problems:
A. If students change the school or university (this is the mobility of our young people) and go to another country, than they remark or not remark that some differences exist in mathematical definitions. In the examination sometimes we observe some mistakes of our students and the reasons are some differences in education.
B. We observe sometimes differences between teaching and used software in calculators or PC or used books of several authors.

In the lecture we will discuss about the stated problems and show some examples by the help of CASIO **ClassPad330** (operating system 3.06, published 2011).
The well known software package **Mathematica** (Version 8) by Wolfram could be the basic for all teachers in the world to work with standard definitions which are used in Mathematica.
In Germany the "**German Institute for Standardization**" (Deutsches Institut für Normung, **DIN**) offers stakeholders a platform for the development of standards as a service to industry, the state and society as a whole. DIN has been based in Berlin since 1917. DIN's primary task is to work closely with its stakeholders to develop consensus-based standards that meet market requirements. By agreement with the German Federal Government, DIN is the acknowledged national standards body that represents German interests in European and international standards organizations. Ninety percent of the standards work now carried out

by DIN are international. Standards play a major deregulatory role. DIN's goal is to develop standards that have validity worldwide.
ISO (the **International Organization for Standardization**) is the world's **largest developer** and publisher of **International Standards**. It has its headquarters in Geneva, Switzerland. ISO considers this trend of utmost importance and believes in the fundamental contribution that educational institutions can give on teaching what international standardization is and what can be achieved through it. Cp. chapter 10, **SANS (**South African national standard).

References:

http://en.wikipedia.org/wiki/International_Organization_for_Standardization
http://www.din.de/cmd?level=tpl-home&languageid=en
http://www.iso.org/iso/home.htm
http://edu.casio.com/products/classpad/
http://www.wolfram.com/

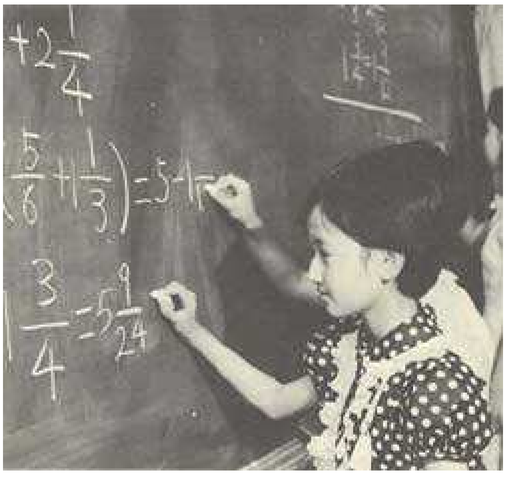
1. What is the set of natural numbers?

There are two conventions for the set of natural numbers: it is either the set of positive integers {1, 2, 3, ...} according to the traditional definition; or the set of non-negative integers {0, 1, 2, 3, ...} according to a definition first appearing in the 19th century.

Have a look in **DIN5374**(logic and set theory, symbols and concepts) or **DIN1302**(general mathematical symbols and concepts) or **ISO31-11**(quantities and units – part 11: Mathematical signs and symbols for the use in physical sciences and technology, revised by **ISO 80000-2**:2009): $\mathbf{N}=\{0, 1, 2, \ldots\}$ and $\mathbf{N}^*=\{1, 2, 3, \ldots\}$,
see: http://en.wikipedia.org/wiki/ISO_31-11

e.g. in the online-book
MATHEMATICAL PREPARATION COURSE BEFORE STUDYING PHYSICS
http://www.thphys.uni-heidelberg.de/~hefft/vk_download/vk1e.pdf (March 10, 2011)
you can read:
"We begin with the set of natural numbers {1, 2, 3, …}, given the name **N** by mathematicians and called "natural" because they have been used by mankind to count within living memory." Here $\mathbf{N}_0 =\{0, 1, 2, 3, ...\}$ (in Germany: ISBN 978-3-8274-1638-4)

2. What means $3^{1}/_{2}$?


http://lamar.colostate.edu/~hillger/faq-images/faq-frac.jpg

Using metric measurements, you don't need to do arithmetic with mixed numbers.

What means $3^1/_2$? Often our students are not sure is it $3 + {}^1/_2 = 3.5$ or $3 * {}^1/_2 = 1.5$?
For our students a problem of the mixed notation is that it can be misinterpreted as a product.
A **mixed number** is the sum of a whole number and a proper fraction.
In the expression $\mathbf{a}\ {}^{\mathbf{b}}/_{\mathbf{c}}$ is omitted the operator. Here it is a multiplication, because in terms with symbolic variables other arithmetic operators can not be omitted, cp. DIN1302.

The multiplication of quantity symbols (or numbers in parentheses or values of quantities in parentheses) may be indicated in one of the following ways: ab, a b, a·b, a×b.
When the dot is used as the decimal marker as in the United States, the preferred sign for the multiplication of numbers or values of quantities is a cross (that is, multiplication sign) (×), not a half-high (that is, centered) dot (·).
Example: Write 15×72 but not 15·72 , cp. http://physics.nist.gov/cuu/pdf/sp811.pdf

In some countries such as France, the mixed number is unusual.
See: http://en.wikipedia.org/wiki/Fractions and http://de.wikipedia.org/wiki/Bruchrechnung

You may, of course, ***say*** "three and a half" — 0.5 is often read aloud as "one half" — but you should always ***write*** it as a decimal fraction.
http://lamar.colostate.edu/~hillger/faq.html
http://lamar.colostate.edu/~hillger/decimal.htm

Why Decimal?

A room measures **15**ft. **3-3/4**in. by **21**ft. **7-1/2**in. (**4.667**m by **6.591**m).
Questions:

What is its floor area in *square yards*?
What is its floor area in *square meters*?

Answers:

36.79sq.yd., or **30.76**m^2

3. Where the real function *y* = f(*x*) = *x*^(1/3) is defined?

The 3rd root is a special case of exponentiation (real power with a fraction), cp.
http://en.wikipedia.org/wiki/Exponentiation

We know:
The exponentiation operation with integer exponents requires only elementary algebra.
By definition, raising a nonzero number to the −1 power produces its reciprocal.
Raising a positive real number *x* to a power that is not an integer, say 1/3, can be accomplished in two ways.

a) Rational number exponents can be defined in terms of n^{th} roots, and arbitrary nonzero exponents can then be defined by continuity: *x*^(1/3) ≈ *x*^0.333333333 = (*x*^333333333)^(1/1000000000) = (*x* ^(1/1000000000))^333333333 (the exponent is only a limit of a sequence with finite decimal numbers). Thus it seems, *x* can't be negative.
b) The natural logarithm can be used to define real exponents using the exponential function: *x*^(1/3) = exp(ln(*x*)/3). Thus it is clear, *x* can't be negative (and not zero)

In DIN1302: *y* = *x*^(1/n) is the n^{th} root **with a positive *y*** such that *y*^n=*x*. Here n∈**N*** and *x* is a nonnegative real number. In many **German school books** we find this definition of the n^{th} root in the domain of the real numbers.
e.g. Definition 1.6 in ISBN 978-3-427-21503-5(2007: Mathematik für Berufliche Gymnasien)

Another question is the real solution of the equation x^3 = -8.
The real solution is -8^(1/3)= -(8^(1/3)) = -2 but not (-8)^(1/3) or (-8)^0.333333333.
The last numbers are not real but complex (cp. principal value, complex main root in **C**).

In US-school books a negative base is allowed:
For any real numbers *a* and *b*, and any positive integer *n*, if $a^n = b$, then *a* is the n^{th} root of *b*.
Here (-8)^(1/3) = -2, where from the point of view of **C** the value -2 is not the principal root.

http://www.farmersville.k12.ca.us/aztecs/Department/Math/spradling/Algebra%20II/Alg%202%20unit%207/Review%20&%20tests/Ch%20review%20answers.pdf

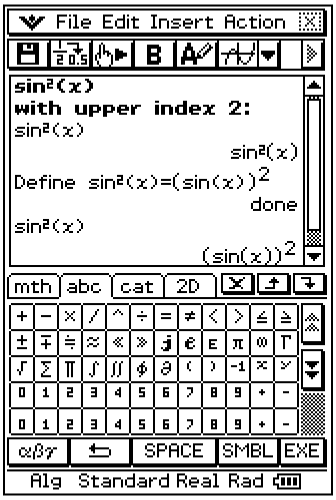
ISBN 0-618-39478-8 (2005: Precalculus with Limits – A graphical Approach)
ISBN 0-471-48273-0 (2005: Calculus)

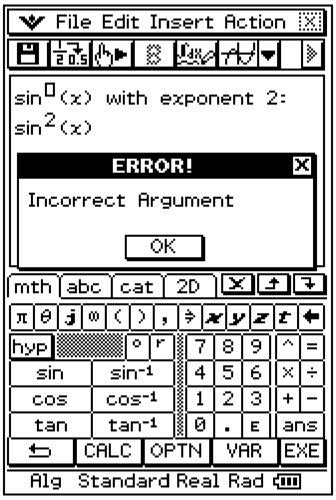
4. What means the notation $y = \tan^{-1}(x)$?

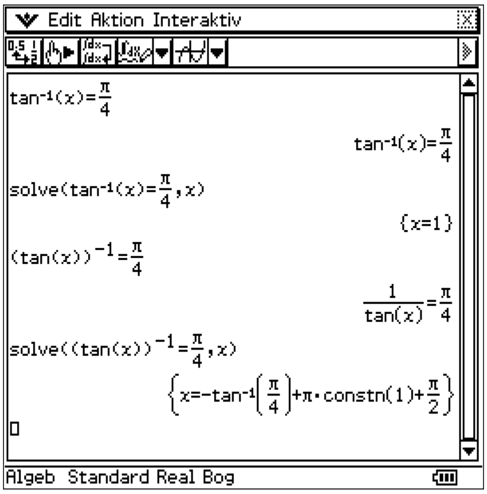
Is this the arctan-function or the cot-function?
For a given function y = f(*x*) we have the notation $y = f^{-1}(x)$ for the **inverse function**.
However the notation $(f(x))^{-1}$ means 1/f(*x*). Here we have the **exponentiation with -1**.

In textbooks we often read e.g. $\sin^2(x)+\cos^2(x) = 1$ (trigonometric Pythagoras)
http://www.qc.edu.hk/math/Certificate%20Level/Trigo%20Py%20Th.htm

If you work with a calculator: the input $\sin^2(x)+\cos^2(x) = 1$ is not defined. You have to write $(\sin(x))^2+(\cos(x))^2 = 1$ or you have to define the symbol $\sin^2$ and $\cos^2$ (a name with four characters!). On the other hand the symbol $\tan^{-1}$ is clear for the calculator.







Only in the case if the symbolic exponent equals -1 our students have problems e.g. with $y = \tan^{-1}(x)$. The calculators use the notation $\tan^{-1}(x)$, but in written form the students should use the notation arctan(*x*) (DIN1302) and not $\tan^{-1}(x)$ to avoid problems and misunderstandings.

5. What is $y = \log(x)$?

We know $y = \log_a(x)$. What is the base *a* in *y* = log(*x*)?
Without any *a* we know *y* = lg(*x*) or *y* = ln(*x*) or *y* = lb(*x*) = ld(*x*) (older notation). Sometimes we find the notation $y = {}_a\log(x)$. Why different notations? We should use modern notations given by DIN and ISO. We should avoid the old notations ld(*x*) and ${}_a\log(x)$. In calculators log(*x*) means lg(*x*). Why not a lg-key instead of log-key? In ISBN 978-3-8274-1638-4 (and English and Spanish translation 2011) again the old notation, e.g. $ld(x) = {}_2\log(x)$ – why ?

6. What is the main argument φ of a complex number in the Gaussian plane?

φ = arg(-2-2**j**) = -3π/4 or φ = arg(-2-2**j**) = 5π/4 ?

In every calculators and PC-software and DIN1302 the main argument is in the interval]-π, π] = (-π, π], but in many school books we can read: the main argument is in the interval [0, 2π[= [0, 2π).

Mathematica: In[1]:= Sqrt[-1] Out[1] = **i** In[2]:= Arg[-2-2**i**] Out[2] = -3π/4

ClassPad: √ (-1) = **j** arg(-2-2**j**) = -3π/4

We have two notations for the complex unit: **i** in mathematics and **j** in engineering.

In the engl. translation (2011) of the book ISBN 978-3-8274-1638-4 you can read:

"for the (only modulo 2π determined) argument of a complex number:
$0 < \varphi = \arg z := \arctan(y/x) \leq 2\pi$"

Here $z = x + y$**j**, i.e. (x, y) are the Cartesian coordinates of z. Why here not include $\varphi = 0$?

We know the real function f(t) = arctan(t) has its values in the interval]- π/2, π/2[= (- π/2, π/2), thus the definition **$0 < \varphi = \arg z := \arctan(y/x) \leq 2\pi$** is not correct!

In the German online script http://www.thphys.uni-heidelberg.de/~hefft/vk_download/vk1.pdf (March 11th, 2011) the author has added a remark:

„**Einschub: Alternative Phasenkonvention:** Natürlich kann man auch ein um den Ursprung symmetrisches Intervall der Länge 2π für die Argumente der komplexen Zahlen wählen: **$-\pi < \varphi = \arg z := \arctan(y/x) \leq \pi$**, was allerdings später die Herleitung der Wurzelfunktionen etwas komplizierter macht. Nach Prof. L. Paditz von der HTW Dresden empfiehlt die DIN diese Konvention."

Here the author is again not correct: He has to write **$-\pi < \varphi = \arg z \leq \pi$** whithout the arctan-function! Furthermore he remarks that with the DIN convention for the argument φ the definition of the root-function will be more difficult – this is not true! The definition of the root based on the main argument **$\varphi = \arg z$** is very easy:

The main root is **z_0 = z^(1/n) = |z|^(1/n) * exp(arg(z)j/n)**
and the other roots are **$z_k = z_0$ * exp(k*2πj/n), k = 1, 2, …, n-1.**

This definition of the main root and the other roots is implemented in all calculators and PC-software.

ClassPad: arg(x+yi)|x<0 yields the correct main argument **$\tan^{-1}$(y/x)+signum(y)*π**

Spanish version (March 11th, 2011) of the considered online book „MATHEMATICAL PREPARATION COURSE BEFORE STUDYING PHYSICS":

CURSO DE MATEMÁTICA PREPARATORIO - para el estudio de la Física
traducido por Prof. Dr. LAUTARO VERGARA, Universidad de Santiago de Chile

http://www.thphys.uni-heidelberg.de/~hefft/vk1/k0/000s.htm

"Argumento el número complejo: **$0 < \varphi = \arg z := \arctan(y/x) \leq 2\pi$** (sólo determinado módulo 2π)"

http://www.thphys.uni-heidelberg.de/~hefft/vk1/k8/814s.htm

Thus an obvious error goes around the world!

7. What is the 3[rd] main root of the complex number -2-2j ?

What is (-2-2**j**)^(1/3) ?

What is (-8)^(1/3) ? Draw the real function $y = f(x) = x$^(1/3) ≈ x^0.33333333 !

The main root is $\boldsymbol{z_0}$ **= z^(1/n) = |z|^(1/n) * exp(arg(z)j/n) = 1 – j** if z =-2-2**j**

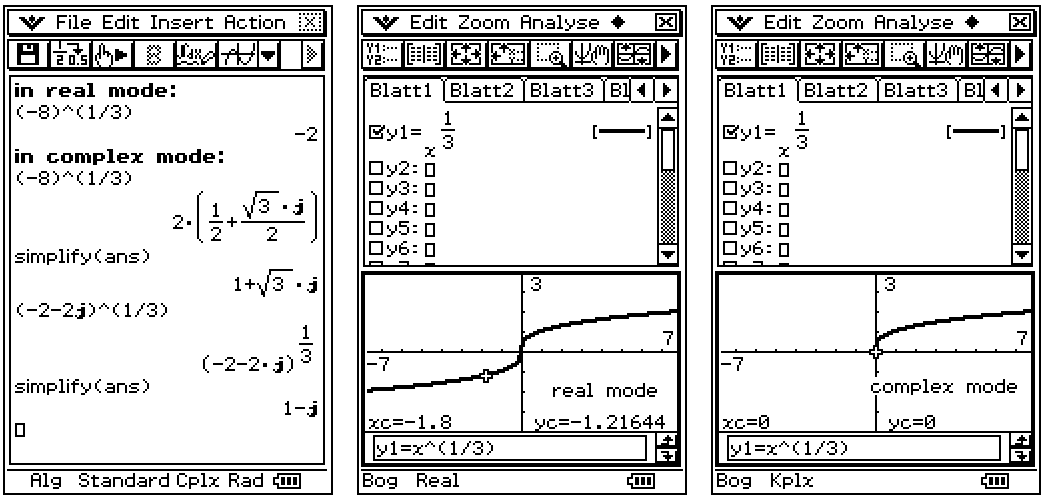
Here we use the main argument **arg(z)** = -3π/4, i.e. **arg(z)j/n** = -π/4.

If we follow the other point of view with **arg(z)** = 5π/4, i.e. **arg(z)j/n** = 5π/12, we would get the main root (-2-2**j**)^(1/3) = **√(3)/2 – 1/2 + (√(3)/2 + 1/2) j** and not **1-j.**

The same with (-8)^(1/3). The main root is 2*exp(πj/3) = **1 + √(3)j** and not **-2.**

ClassPad: in complex mode: (-8)^(1/3) = **1 + √(3)j, in real mode:** (-8)^(1/3) = -2.

In the real mode the ClassPad follows the US-definition of a root, thus we get here different graphics of $y = f(x) = x$^(1/3) in the x-y-plane, in dependence on real mode or complex mode:



However in real mode or complex mode we get the same graphics for $y = f(x) = x$^0.33333333!

8. What is the α-quantile x_α of a probability distribution of a random variable X?

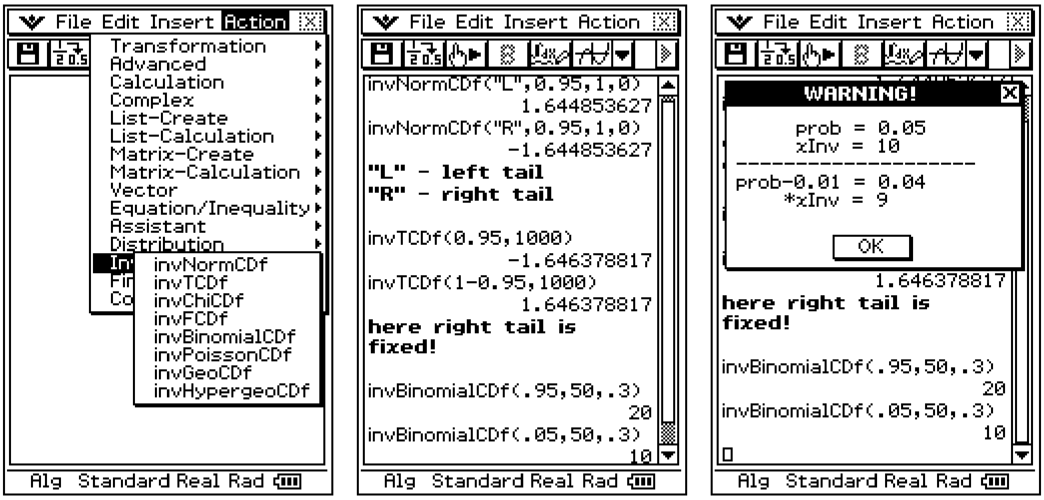
x_α is a number with $P(X < x_\alpha) \le \alpha \le P(X \le x_\alpha)$ or $P(X < x_\alpha) \le 1-\alpha \le P(X \le x_\alpha)$?

Follow **Mathematica** "**Quantile[dist, α]** is equivalent to **InverseCDF[dist, α]**" we get the definition: $P(X < x_\alpha) \le \alpha \le P(X \le x_\alpha)$, i.e. according to DIN ISO 3534-1 (2009) (Statistics – Vocabulary and symbols – Part 1: General statistical terms and terms used in probability) based on ISO 3534-1 (2006) (http://it.wikipedia.org/wiki/ISO_3534).

For a continuous distribution of X the inverse CDF at α is the value x_α such that **CDF[dist, x_α]= α.** For a discrete distribution of X the inverse CDF at α is the smallest integer x_α such that **CDF[dist, x_α] ≥ α.**

cp.
http://reference.wolfram.com/mathematica/ref/Quantile.html
http://reference.wolfram.com/mathematica/ref/InverseCDF.html
http://reference.wolfram.com/mathematica/ref/CDF.html
http://reference.wolfram.com/mathematica/ref/Probability.html

However ClassPad only follows this definition for discrete distributions but for continuous distributions ClassPad works with the other definition $\mathbf{P(X< x_\alpha) \leq 1-\alpha \leq P(X \leq x_\alpha)}$.



The last picture shows a good service during the computing the quantile of a discrete distribution. If the assumed probability (say $\alpha = 0.05$) is no value of the distribution function, than the computed quantile is the smallest integer x_α such that $\alpha \leq P(X \leq x_\alpha) = F(x_\alpha)$, i.e. for x_α-1 we have already $F(x_\alpha-1) = P(X \leq x_\alpha-1) = P(X< x_\alpha) < \alpha$. The “WARNING!” is connected with the last digit after the decimal point of the assumed probability prob = α and will not appear, if a change in the last digit by one step will not change the computed quantile.

9. Is the distribution function $y = F(x)$ of a random variable X right continuous or left continuous?

Is $F(x) = P(X< x)$ or $F(x) = P(X\leq x)$? Consider the binomial distribution!

For a continuous distribution both definitions of $F(x)$ are equal, but for discrete distributions we get here a left continuous function with $F(x) = P(X< x)$ and a right continuous function with $F(x) = P(X\leq x)$. The last definition is in context with **DIN ISO 3534-1 (2009)** based on **ISO 3534-1 (2006)** and all software, e.g. Mathematica or ClassPad.

http://en.wikipedia.org/wiki/Cumulative_distribution_function
http://en.wikipedia.org/wiki/Quantile_function

However in Russian school books we have the other definition $F(x) = P(X< x)$. This was the Russian standard during the time of the former Soviet Union, which is sometimes not compatible with DIN or ISO.
http://www.toehelp.ru/theory/ter_ver/3_2/

However in the Russian Wikipedia we find the new definition too
http://ru.wikipedia.org/wiki/Функция_распределения
http://www.exponenta.ru/educat/class/courses/tv/theme0/2.asp

In the Polish Wikipedia http://pl.wikipedia.org/wiki/Dystrybuanta e.g. you can read:

“Niech P będzie rozkładem prawdopodobieństwa na prostej. Funkcję F: R→R daną wzorem **F(t) = P((-∞,t])** nazywamy dystrybuantą rozkładu P. … Niekiedy w definicji dystrybuanty stosuje się przedział otwarty: **F(t) = P((-∞,t))** Dystrybuanta jest wówczas funkcją **lewostronnie ciągłą** (w przeciwieństwie do przypadku gdy w definicji stosuje się przedział prawostronnie domknięty i dystrybuanta jest funkcją **prawostronnie ciągłą**).”

Here the hint that both definitions exist – but what is the recommendation?

10. More references on ISO 3534-1 and ISO 31-11 around the world:

International standard, ISO 3534-1: Geneve, Switzerland : ISO, 2006.
Statistics - vocabulary and symbols. Part 1, Probability and general statistical terms = Statistique - vocabulaire et symboles. Partie 1, Probabilite et termes statistique generaux.

DIN ISO 3534-1, Berlin, Beuth-Verlag, 2009,
Titel (deutsch): Statistik - Begriffe und Formelzeichen - Teil 1: Wahrscheinlichkeit und allgemeine statistische Begriffe (ISO 3534-1:2006); Text Deutsch und Englisch

SANS 3534-1: Pretoria, South African national standard, 2007:
"This national standard is the identical implementation of **ISO 3534-1:2006** and is adopted with the permission of the International Organization for Standardization."
Cancels and replaces ed. 1 (SANS 3534-1/ISO 3534-1:1993), ISBN: 9780626189983

UNE-ISO 3534-1: Madrid, Asociación Española de Normalización y Certificación (AENOR) 2008, estadística : vocabulario y símbolos. Parte 1, Términos estadísticos generales y términos empleados en el cálculo de probabilidades

PN-ISO 3534-1: Warszawa : Polski Komitet Normalizacyjny 2002, Statystyka – Terminologia i symbole - Część 1: Ogólne terminy z zakresu rachunku prawdopodobieństwa i statystyki, ISBN: 9788323676690

SIST ISO 3534-1: Slovenski standard. Ljubljana: Slovenski inštitut za standardizacijo, 2008, Statistika - slovar in simboli. 1. del, Splošni statistični izrazi in izrazi v zvezi z verjetnostjo,

UNI ISO 3534-1 è la versione in lingua italiana, c.p.
http://it.wikipedia.org/wiki/UNI_ISO_3534-1

International standard, ISO 31-11, Geneve, Switzerland: ISO, 1992. Quantities and units = Grandeurs et unités. Part 11. = Partie 11, Mathematical signs and symbols for use in the physical sciences and technology = Signes et symboles mathématiques à employer dans les sciences physiques et dans la technique.

SIST ISO 31-11, Slovenski standard. Ljubljana: Urad Republike Slovenije za standardizacijo in meroslovje, cop. 2008.Veličine in enote. Del 11, Matematični znaki in simboli za uporabo v fizikalnih in tehniških vedah : (istoveten ISO 31-11:1992)

PN-ISO 31-11, Warszawa: Polski Komitet Normalizacyjny 2001, Wielkości fizyczne i jednostki miar - Znaki i symbole matematyczne do stosowania w naukach fizycznych i technice

New International standard, ISO 80000-2 (1st ed. 2009-12-01), Geneva, Switzerland: Quantities and units – Part 2: Mathematical signs and symbols to be used in the natural sciences and technology.
This edition cancels and replaces ISO 31-11 (1992), which has been technical revised.
Four clauses have been added, i.e. “Standard number sets and intervals”, “Elementary geometry”, “Combinatorics” and “Transforms”.

“Mathematics online and mathematics mobile – where is all this going?”

Douglas Butler debutler@argonet.co.uk
iCT Training Centre, Oundle (UK)
www.tsm-resources.com

Douglas Butler will explore the Aladdin's cave of online resources that busy teachers can now call on to add a sparkle to their lessons. But now students can explore these too on their mobile devices, so the learning process is no longer restricted to the classroom. Exciting and challenging times! But how to ensure secure understanding?

Since around 1995, both the quantity and quality of online resources for mathematics teachers have risen dramatically. Faced with this staggering choice, teachers naturally need a lot of guidance. The TSM Resources website is one solution, breaking the massive spectrum into digestible categories, and this presentation will give a whistle-stop tour of some of the possibilities, including:
- web broadcasting: straight into the classroom
- resources for the busy teacher
- calendar of training events
- useful excel files for mathematics
- integer lists
- resources for dynamic software
- mathematics links and data from around the world

But wait – there is another player in this equation: increasingly, the students themselves have their own access to all this on a device that sits in their pocket. And they are creating their own online resources and sharing their learning experiences online.

This has the potential to change the dynamic of teaching and learning forever! And does the subject of mathematics stand to be affected by all this more than other academic subjects? What other subject comes close to expecting a detailed and proactive knowledge of spread-sheets, dynamic software, computer algebra systems and internet resources? What are the implications on In-service training?

Finally there is the burning question: does all this innovation allow teaching and learning to be more efficient and more effective? How can we be sure that the learners are not cutting corners? Is their understanding secure? After all, a spell checker is generally not much use if you can't spell …

SETTING MATHEMATICS LABORATORY IN SCHOOLS

Adenegan, Kehinde Emmanuel
B.Sc.(Ed.), M.Sc., Dip.(Comp. Sc.), Cert.(ADAPT)
Lecturer of Numerical Analysis
Department of Mathematics
Adeyemi College of Education Ondo, Nigeria
ademmak07@yahoo.com, adenegankehinde@gmail.com

ABSTRACT

Mathematics as a subject is indispensable in the development of any nation with respect to science and technology since mathematics itself is the language of science. In this 21st century where virtually all attentions are shifted towards technological advancements and the mathematics education into the 21st century project is waxing stronger in objectively achieving all its goals in which mathematics is a veritable tool. Hence, this paper explicitly discuss the concept of mathematics and education, mathematics laboratory and its numerous advantages, mathematical equipment/materials in an ideal mathematics laboratory and how to set a mathematics laboratory. Pictorial representations of mathematics laboratory with some mathematical instructional materials were used to substantiate the paper.

INTRODUCTION

Mathematics involves thinking logically and reasonably so as to understand how formulae are derived and their applications. In order to enhance learners' mastery and meaningful learning of mathematics, it is necessary to reduce to the bearable minimum its level of abstraction with the use of instructional materials. Adenegan(2010) testified to this that instructional materials, when properly used in the teaching and learning situation, can supply concrete bases for conceptual thinking, high degree of interest for students in making learning more permanent.

According to Oyekan (2000), "instructional materials are those things that can facilitate effective teaching and pleasant learning that is teaching aids through which learning process may be encouraged and motivated under the classroom situation". These enhance the teaching learning process when adequately and appropriately used.

To this end, this paper focuses on setting mathematics laboratory which directly houses instructional materials. Specifically, this paper aims at defining a mathematics laboratory, listing and identifying mathematical equipment/materials that can be put in a mathematics laboratory and enumerating ways of setting mathematics laboratory in the school.

Concept of Mathematics and Education

Mathematics is the study of numbers, set of points and various abstract elements together with relation between them and operation performed on them. In the beginning, mathematics curriculum in school was arithmetic, since people were just able to calculate, but by the early 1950's the concept of mathematics in the school as subject had developed and was being taught in three different sessions as arithmetic, geometry and algebra.

One of the objectives of teaching mathematics in all strata of education, from primary school level upward is the attainment of an understanding of the nature of the subject within the umbrella of a science education in relation to everyday activities of one's life as asserted by Adenegan (2003). Mathematics leads people into discovering things. However, new discoveries cannot be made unless it is effectively taught through application of adequate and efficient human and physical facilities.

Mathematics cannot be pushed aside in our day to day activities, yet mere mentioning the name of the subject sends cold chill round majority of the students' spine. Nervousness and fear that followed are better seen than imagined! The mathematics students shiver and fear wrinkles up their youthful faces. Then one wonders why this repulsive and uncheerful attitude towards mathematics in our schools, (almost at all levels; primary schools not excluded), in a period when the government desires a technological breakthrough. Could this be attributed to ineffective and inefficient handling of mathematics or inadequacy and non availability of instructional facilities? It is, however, important to note that for pupils to develop interests and do well in mathematics, being the language of science, and a tool for their future career, care must be given to what is taught and how they are being taught in the various schools.

Education can be defined as the process of imparting and acquisition of knowledge through teaching and learning especially at a formal setting such as schools or similar institutions (Alao, 1997). Thus, education can be perceived as a process whereby a person learns how to learn. It actually begins at birth and ends at death. In fact, education is an age-long concept. Mathematics as a subject is part of the curriculum content taught and learnt at different educational strata. Education enriches man with information and when one is not informed, one is at the risk of being deformed.

Importance of Mathematics

Mathematics is a model for thinking, for developing scientific structure, for drawing conclusions and for solving problems. It is a subject that deals with facts. As a result, Olademo (1990) opined, "this subject-mathematics should be given much consideration and let no man think of it as abstract or as untrue". As posited by Balogun et al (2002), "Mathematics instruction is a training of logical thinking. It is a means of solving many problems. It is confronted with finding solutions to problems that have not been provided by a similar type. Its greatest virtue is its flexibility and the high esteem at which it is held as a tending discipline is partly due to its illustrious pedigree".

People who have become more and more skeptical towards mathematics saw it as discipline that pursues needless complications, inventing unrealistic problems and prescribing solving methods within the frame of elementary mathematics. To this end, Adenegan (2003) highlighted Mathematics importance under four broad functions-utilitarian, cultural, social and personal functions.

- Utilitarian functions: It is useful in everyday life that is it serves as a functional tool in studying individuals everyday problems; it is useful as a tool to other discipline, that is, serves as a hand maiden for explanation of quantitative situations in other subjects such as economics, physics, navigation, finance, biology and even the arts. This service of Mathematics is exceedingly important to future scientists, engineers, technologists, technicians and skilled mechanics; it is useful for national income and budgeting and useful for laying foundation for further education.

- Cultural functions: It is useful for calculation in local languages and useful for naming objects.

- Social functions: It is useful in voting, games and lotteries, birth and death rates and population census.

- Personal functions: It encourages correct or accurate thinking, allows for cooperation with others to achieve common goals, allows for character building (patience, persistent and perseverance) and remarkably, it makes one to be happy.

  In a nutshell, "Mathematics is now an enormously useful science which, in order to attain this status, has had to cross a desert of usefulness where Mathematics was nursed tenderly as a science of mind" (Balogun et al 2002)". Astronomy is a practical science of Mathematics. It is used to foretell the calendar, feast, eclipses, wars, pestilence, whirlwinds, storms and the future of nation and even of individuals. It is a useful application of Mathematics and would link on for at least the next two millennia.

The diverse applications of mathematics abundantly establish that mathematics, as a discipline, is fit for purpose, as mathematics continually drives the expansion of the frontiers of other disciplines through their progressive formalization and symbolization and the building of mathematical paradigms of real world systems.

In Nigeria, a credit in mathematics is required for admission to countless programmes of study at the tertiary level of education. Ekhaguere (2010) asserted that in view of this fate-determining place of mathematics in the nation's educational system, a policy must be formulated and implemented toward ensuring that no child is left behind in mathematics at the pre-tertiary level of education.

The Mathematics Laboratory

As defined by Adenegan (2003), the mathematics laboratory is a unique room or place, with relevant and up-to-date equipment known as instructional materials, designated for the teaching and learning of mathematics and other scientific or research work, whereby a trained and professionally qualified person (mathematics teacher) readily interact with learners (students) on specified set of instructions. The picture below is an example of a mathematics laboratory where the children are seen playing with educational toys under the supervision of their teacher.

figure 1: *A typical mathematics laboratory with educational toys for children*

In a related term, a current version (miniature) of mathematics laboratory is the "mathematics corner". This indeed is still a new concept. In a school where there is no mathematics laboratory, the teacher together with the students can readily improvise and create what we call the mathematics corner in the classroom as can be found in the picture below. The teacher can start by creating a corner in the class as mathematics corner where he can be depositing periodically mathematics equipment or ask the pupils to bring, with pride and boldness, local mathematics materials like different geometrical shapes so as to facilitate a successful take off and unhindered success of the establishment. The mathematics corner can contain some of the equipment found in the mathematics laboratory but will not be as full and well organized and assembled as what we found in the later.

Figure 2: *A typical mathematics corner at Mathematics Department, Adeyemi Coll. Of Edu. Ondo.*

The materials or equipment that can be found in the mathematics laboratory include, among others constructed (wooden/metal/plastic made) mathematical sets, charts and pictures, computer(s), computer software, audio-visual instructional materials such as projector, electronic starboard, radio, television set, tape recorder, video tape, etc, solid shapes (real or model), bulletin board, three-dimensional aids, filmstrips, tape photographs, portable board or whiteboard, abacus, cardboards, tape measure, graphics, workbooks, graphs, flannel boards, flash cards, etc.

Figure 3: *Teacher/Students-Made Instructional Materials:Solid Shapes and Geometrical Objects during a workshop for Junior Secondary School principals at Ebonyi State, Nigeria by the Author.*

Mathematics laboratory is relatively new in the teaching and learning of mathematics. It is a practical oriented classroom or place where materials useful for the effective teaching and learning of mathematics are kept. It is the latest design to make mathematics real. The term “laboratory method” is commonly used today to refer to an approach to teaching and learning of mathematics which provides opportunity to the learners to abstract mathematical ideas through their own experiences, that is to relate symbol to realities. It is uncommon in our schools today possibly as a result of lack of fund or the absence of any government policy on the provision of such laboratory facilities. In short, its non-existence in our schools is one of the major contributory factors to mass failure in mathematics. Thus, as highlighted by Adenegan (2003), the functions of mathematics laboratory include the followings:

- Permitting students to learn abstract concepts through concrete experiences and thus increase their understanding of those ideas.
- Enabling students to personally experience the joy of discovering principles and relationships.
- Arousing interest and motivating learning.
- Cultivating favourable attitudes towards mathematics.
- Enriching and varying instructions.
- Encouraging and developing creative problems solving ability.
- Allowing for individual differences in manner and speed at which students learn.
- Making students to see the origin of mathematical ideas and participating in “mathematics in the making”
- Allowing students to actually engage in the doing rather than being a passive observer or recipient of knowledge in the learning process.

SETTING MATHEMATICS LABORATORY

Having already discussed extensively the mathematics laboratory, we will proceed to itemize how to set a befitting and remarkable mathematics laboratory in the school.

1 Identify the necessary materials required in the laboratory by labeling them with name tags.

2 Put or assemble all related equipment or materials on the same side/place. e.g. geometric objects should not be placed where audio-visual materials are positioned.

3 Put the bulletin board close to the entrance door in case of any information display.

4 Arrange the benches and tables to allow for free movement in the laboratory.

5 Hang relevant pictures and charts on picture rails and boards.

6 The starboard or white board must be positioned where every student can readily see it.

7 Shelves can be constructed for keeping and demarcating materials.

8 Electronic materials such as projector, television, etc, should be properly displayed.

9 Electrification of the laboratory should be professionally done to allow for safety use.

10 Display materials on tables in an organized manner.

11 The laboratory should be set in such a way that it must be well ventilated.

12 Handy materials that can be easily destroyed or lost can be kept in a cabinet or separate shelve.

13 Arrange the materials in places (on tables, shelves, board, etc) in a way that they can be easily accessed when needed and returned appropriately after use.

CONCLUSION

It is expected that the 21st century mathematics educators/teachers should be readily acquainted with the modern day technique of teaching mathematics in our schools and possibly facilitate their teaching pedagogies with the aid of modern mathematics laboratories to be able to achieve the objectives of the mathematics education into the 21st century project. This paper hereby strongly *recommend* to all school teachers to liaise with their school principals/heads to facilitate the establishment of a mathematics laboratory or for a start, mathematics corner in their schools.

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Technology: The Bridge to Facilitate Learning of Adult Learners of Mathematics

LaVerne Alan, MA, MS
Instructor, Victory University
Memphis, Tenessee, USA
lchambers-alan@victory.edu

Abstract

With the advent of many adults returning back to college, math professors all over the country have been trying to find ways to facilitate adult student learning. Many students have been out of school for over ten or more years and are now required to take college algebra (in an accelerated format). However, many adult learners have forgotten facts of basic math. Although some adult students will need heavy remediation of the subject matter regardless of the venue, there must be a way to help adult learners recall basic math and algebra facts.

This paper seeks to explore whether or not blended courses using MyMathLab/MathXL are effective in expediting learning in developmental math courses such as basic math, elementary and intermediate algebra. Research concerning adult learners and technology is significant because learning new technology along with mathematics would seemingly pose an even bigger challenge than just recalling and learning basic math and algebra facts alone.

Why Technology for Adult Learners of Math?

Many adult math learners have not seen math in ten (10) or twenty 20 years. For their given degree program, college algebra is required, but they cannot remember basic math facts. With the notion of andragogy, some adult students are misplaced in accelerated (5 and 8 week) courses because previous life experiences, however this does not correlate in most math courses. Most adult learners need excessive remediation in order to fulfil the requirements of College Algebra due to no prior knowledge of the subject matter, lack of prior knowledge of the subject matter, inaccurate knowledge relating to the subject matter. There are misconceptions, preconceptions and illogical reasoning that adult learners must overcome. Many adult students have math phobias and math anxiety. Some adult students have little or no exposure to the computer and technology related to the math courses that they are taking (e.g, graphing calculators, pda's) In addition, many adult math learners lead busy lives and do not have extra time to study in order to fill in gaps in their learning.

Adult learners of mathematics face the same challenges that a typical math student may face. For example, they may know the material, but cannot remember it. Since there have been removed from an academic setting for so many years, many adult learners of mathematics may have poor study habits and note taking skills.

Other challenges that adult learners of mathematics face include time management and finding what Elayn Martin-Gay calls "teachable, math learning moments."[1] Moreover, working adult learners do not retain much of the material being presented in night classes, especially those with busy schedules in the day. Adult Students can follow the professor during class, but cannot retain the same level of understanding once they attempt the homework assignment outside of class.

As previously stated, adult student learners of mathematics face the same challenges as a typical math student. Victory University provides a robust college learning environment which admits over 600 students annually. The average age of a Victory University student is 35. Thus, specifically addressing the needs of the adult learner is essential. This paper will access the current status of adult learners of mathematics taking developmental courses at Victory University as well as consider the use of mathematical educational software as a means of enhancing the performance rates of adult learners of mathematics in developmental courses.

Methods

The analysis in this paper considers data taken from 433 students of Victory University (formerly known as Crichton College) who have taken math courses between the fall of 1991 until the fall of 2011. Student age and final grade has been taken into consideration. There were a few sections of the LE Basic Math Courses were given Satisfactory/Unsatisfactory assessments at the end of the course. These grades were tabulated as either 2.0= C for Satisfactory or 0=F for Unsatisfactory. Students who withdrew from the course or received an FA grades were translated as 0=F.

Hypothesis: Adult learners between the ages of 45-80 will have lower grade point averages than the traditional 18-26 year old student. In addition, the grades will be reviewed from two LE0114 Basic Math Courses taught by the author of this paper. The first course was given in the Fall of 2008 and the second course was given in the Spring of 2011. The course in the Spring of 2011 was a hybrid course that included MyMathLab mathematical learning software. The course given in the Fall of 2008 was given in a traditional format. The final component of this research paper examines individual adult student learners of mathematics. In particular, four students over 59 years of age were observed over the course of a semester in basic math, elementary and intermediate algebra. Two students were observed between the ages of 40-45, who openly shared their challenges in intermediate algebra.

Results

From the fall of 1991 until the fall of 2011, there were 260 students who had recorded grades for LE Basic Math at Victory University. The average age of the students taking LE Basic Math courses is 32 and unfortunately the average grade is 1.61 (D). The following histograms provide the age and grade demographics for each student.

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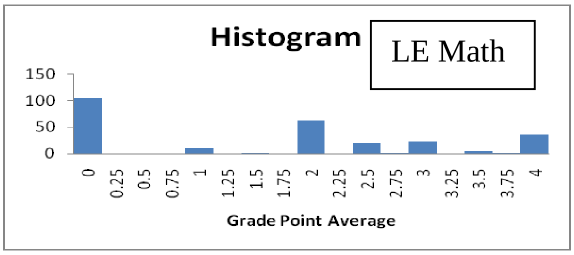
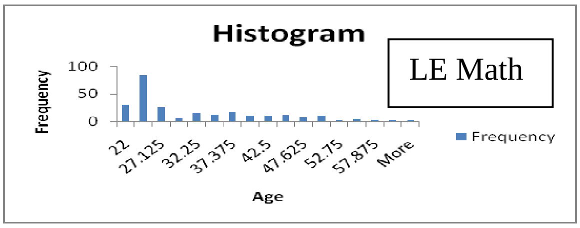
[1] Martin-Gay, E. (2009). *Beginning & Intermediate Algebra* (4th ed.). University of New Orleans: Prentice Hall.

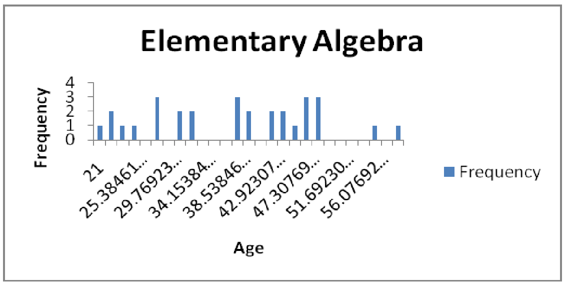


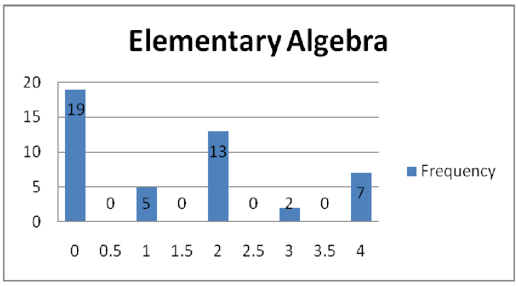
Because the regression coefficient r = .0412, the data also reveals that there is no

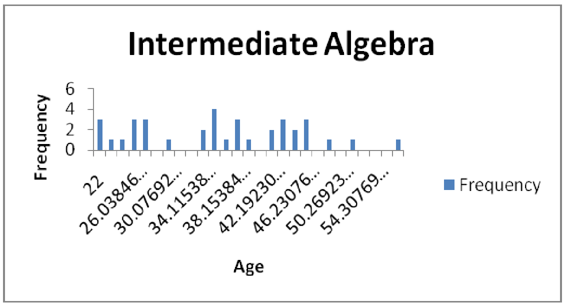


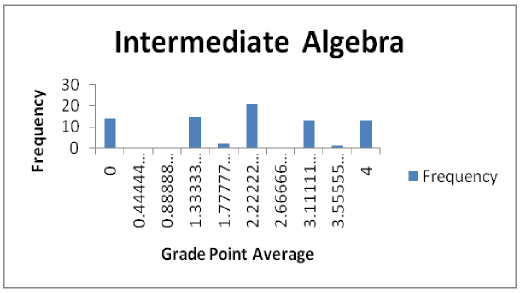
correlation between age and the grade in the course. Since there is no correlation, it is erroneous to assume that older students would perform better or worse in LE mathematics. Yet the grade point average for students 45 and older in LE mathematics, which is 1.72, is slightly higher than the group average. These results reveal that most students taking LE Basic Math have challenges passing the course.

The results for the Elementary and Intermediate Algebra courses are similar. Of the 30 recorded students who have taken elementary algebra from the Fall of 2008 until Fall 2009, the average age is 37 and the average grade is 1.43 (D). The correlation coefficient is r =.25—this implies that there is no relationship between the student's age and grade in Elementary Algebra. Yet again, the grade point average for students 43 and older is 2.0 (C), which is slightly higher than the average student. Of the 36 recorded students who have taken Intermediate Algebra from the Spring of 2006 until the Fall of 2011, the average age is 36 and the average grade is 1.95 (C). Moreover, the correlation coefficient for the student ages and grades of Intermediate Algebra students is r=.35—this also reveals that there is no relationship between student age and the grade that he/she will make in intermediate algebra. Note that .35 is closer to 1 than .042, so age is has a closer relationship to grades in intermediate algebra than in LE Basic Math. The graphs are given below:









Although this paper does not focus on non-developmental math courses, it is interesting to note that 70% of all students (433 students) taking math courses (developmental and regular courses) at Victory University earned an average grade of 2.0 (C).

The LE Basic Math Course in the Fall of 2008 had 11 students with an average age of 34 and an average grade of 1.09 (D). Similarly, the LE Basic Math Course in the spring of 2011 had 8 students with an average age of 48 and an average grade of 2.25 (C). For each course, the correlation coefficient r is under .8 (.34 and -.27 respectively) so one cannot make a strong case for a relationship between the student's age and academic performance. However, the spring of 2011 class has the higher grade point averages and the average age is also higher.

Three students over the age of 59 and two students over 40 were given extra reinforcement within MyMathLab/MathXL. Though these students are not strong in mathematics, they were able to pass their respective developmental math courses with the grade of a C. These students noted that the view an example feature and the help me solve this feature helped them to study concepts that they did not understand from the lecture. Unfortunately, students in the fall of 2008 LE Basic Math class were not afforded this option in the traditional format course.

Discussion

The sole purpose of observing the data above is to help older learners of mathematics excel at a faster rate. However, the data does not substantiate the hypothesis that older students will perform worse than the average/traditional age student. History reveals that older students have performed slightly better. Yet the average age of the students taking developmental math courses at Victory University is 36. What the data does reveal is that the average student taking a developmental math courses is an adult learner (not between the traditional ages of 17-26) who will struggle to pass the course with a grade of C or better. Adult learners in developmental math courses need more support in order to be successful in the course. Looking closely at the data suggests that there are other factors beside age that are instrumental in student performance in developmental math courses:

1. Time devoted to study
2. Aptitude
3. Study Skills
4. Comprehension
5. Utilization of teacher/tutor for outside support
6. Completion of homework and other classroom assignments

The hybrid course reinforces the aforementioned factors within the design of the course. Traditional courses place the burden of excelling these areas on the student with little or no help. With MyMathLab/MathXL students have access to an online tutor more hours of the day than their instructors. Students in the hybrid format can get help with several problems in a more time efficient manner.

Students who are returning back to school after being out of school ten (10) years or more usually are more challenged in recalling basic math facts. Though the data suggests that being out of school for many years does not automatically assume failure, it does not suggest that adult learners of developmental math could not use extra support.

Conclusions

All adult learners of mathematics could use help with mastering the material presented in developmental math courses at Victory University. As in the LE Basic Math courses observed in this study, students who are exposed to learning technologies outperform students within traditional class settings. Helping adult learners to use technology necessary for completing math assignments can be well worth the investment. However, there are some disadvantages to using the math learning technology.

The six students observed in this study needed a lot of help initially with understanding how to operate the computer application. Major disadvantages include:

- Helping students to understand MyMathLab/MathXL takes away from curriculum time
- Challenging internet connections and servers not being available is sometimes discouraging to the students
- Requiring an additional cost to the student sometimes is not favoured by colleagues
- Logging in with the access code and course id usually delay the course's start date by 1 or 2 weeks for most adult learners who have been out of school for several years
- Learning the new technology takes about 2 to 3 weeks in order to become proficient
- Some students will find learning both technology and mathematics overwhelming and drop the course

Although there are several disadvantages, they are not insurmountable. In fact, the benefits compensate for the disadvantages list above—the result is accelerated levels higher learning. The fact that the average student in the MyMathLab/MathXL based LE course has an average grade of 2.25 (C) versus the average grade of 1.09 (D) in a traditional LE math course reveals the potential usefulness math learning technology within the classroom.

Of the students observed, they all claimed that they were able to identify the areas that they are deficient in and spend the time on the computer "filling in the gaps" of their learning. For a couple of the students, MyMathLab/MathXL assignments were more effective than one on one tutoring, because they could play and re-play the explanation to the problems until the concept was understood. Of course, MyMathLab/MathXL is not a replacement for necessary tutoring, but should be utilized before the student access a tutor. MyMathLab/MathXL allows students to use homework time more effectively, which increases their likelihood of mastering the material. Moreover, 4 of the 5 students observed believed that using MyMathLab/MathXL to complete assignments helped them to develop confidence in solving math problems.

Implications

Assuredly, professors still need to enforce the learning and understanding of mathematical concepts of mathematics while using technology. However, more research is necessary to determine if technology can support true understanding of mathematical concepts vs. building basic skills. Another desire of many math professors is to give students feedback in real-time, while they are working on the math problem. Recommendation: Installation of real time software that assesses the student at each level of the problem and remediates the student accordingly. The “Help me solve this feature moves in this direction—precisely because it breaks the problems down into smaller chunks. However, more scaffolding is needed. At the level at which the student is unable to complete the problem, the student will be able to click a link to learn the missing concept via view an example, PowerPoint lesson, animation and/or video lecture. Currently, this technology is not available.

More research is necessary to ensure the viability of developmental math learning with the use of technology over learning within the traditional classroom lecture format. Yet there is a lot of research that supports this claim.[2] For example, if one were to track the amount of time spent in MyMathLab/MathXL by the average adult developmental math learner, would there be a correlation to the grade point average attained in the course? For what percentage of students does technology provide a hindrance for developmental math learning? These and other related questions provide implications for further research.

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USING A VALUES-BASED APPROACH TO PROMOTE SELF-EFFICACY IN MATHEMATICS EDUCATION

Pam Austin and Paul Webb

Faculty of Education, Nelson Mandela Metropolitan University, Port Elizabeth, South Africa

pamela.austin@nmmu.ac.za; paul.webb@nmmu.ac.za

Abstract

This study examines the effects of a values-based approach to teaching mathematics on promoting self-efficacy amongst university mathematics education students (n=130) registered for a BEd degree course at the Nelson Mandela Metropolitan University. Data were generated using students' written reflections (disposition statements), interviews, student journals, and observation of classroom practice video-clips. These data were analysed and triangulated to provide insights into aspects of the 'social reality' in terms of the moral values that participants brought to their mathematics education classes, how they perceived their experience of making explicit moral values in mathematics education classes, and whether these experiences influenced their mathematics self-efficacy. The data suggest that the strategy raised self-efficacy levels and that the participating students recognized the importance of their lecturer practicing core values in her classes, and that they also recognized the importance for themselves as future teachers.

Introduction

This study on a values-based approach to mathematics teaching and learning emanated from an interest in values and morals in education and broader concerns around low self-efficacy amongst many pre- and in-service teachers (Nieuwenhuis, 2007). The 'new scholarship' approach of Whitehead (1989) and McNiff and Whitehead (2006), which is premised on the view that teaching is about participatory learning and the living nature of educational enquiry, provided a framework for the intervention used and the analysis of the data generated.

The intervention used firstly introduced the 130 participating pre-service mathematics education students to a 'values wheel' (honesty, fairness, respect, accountability and compassion bounded within a ring (rim) of integrity, and turning on an axle of trust), after which whole-class discussions were held to provide opportunities to clarify their thinking about these values, as well as the possible influence that a values-based approach might have on their learning. The lecturer (first author) then informed them that she would attempt to live out the values of respect, fairness, accountability, honesty and compassion, and encouraged them to hold her accountable. She explained that she believed that by doing so a respectful and trustful relationship could be developed, which would provide opportunities for them to experience feelings of success and wellbeing, which are essential for the development of self-efficacy (Woolfolk, 2004). As such, the ultimate aim of the intervention was to assist students to adopt a self-efficacious and life-affirming view of themselves as individuals and as mathematics teachers (McNiff, 2002). Data on the effect of this approach was generated via interviews, written student reflections (affective-disposition statements), journal entries and video-recording of an aspect of the intervention.

Values and self-efficacy

Kidder (2006) refers to core values as the moral values which constitute humanity's common moral framework. Based on research conducted by the Institute for Global Ethics these core values are acknowledged as "the core moral and ethical values held in highest regard in diverse communities around the world, given the global diversity of culture, ethnicity, race, religion, gender, political persuasion, economic disparity and educational attainment" (p. 42-43). The core values are the five moral ideas of honesty, fairness, respect, responsibility and compassion (Kidder, 2006, p 64). Nieuwenhuis (2007) asserts that values are consciously and unconsciously at work in all our interpersonal interactions and are an integral part of all human actions, thus making education by its very nature a value-based and value-laden phenomenon. He further notes that there is general agreement that values are important concerns that education institutions will have to deal with if they are serious about issues of quality and effectiveness. He argues that we choose our behaviour based on our personal and socially constructed values, assumptions and beliefs, which in turn inform our understanding of what is morally right and morally wrong and of the type of conduct that would be just and ethical (Nieuwenhuis, 2007). Furthermore, values need to be clarified, discussed, refined and reinvented through a process of active deliberation, debate and the provision of concept clarifying within the socio-cultural milieu of the classroom and community.

Woolfolk (2004) points out that without students' trust, respect and cooperation even the best teaching materials and methods can fail, and she believes that enabling students to feel valued and respected plays a strong role in developing their sense of self-efficacy. The intervention used in this study attempted, via examining moral values in the classroom, to make these relationships explicit and tangible for students. It also attempted to provide opportunities for them to experience success and feelings of well-being and, as such, to create learning environments that foster positive self-efficacy.

The critical outcomes of the South African curriculum suggest that the government has accepted that inculcating values associated with being a 'good citizen' is a key objective in order to develop responsible citizens for the 21st century (Department of Education, 2002; 2007). It is this apparent acceptance that motivates and provides context for this study.

Research design and methodology

Seventy nine first-year and 51 second-year (n=130) mathematics education students, who came from diverse social, cultural, economic and political backgrounds, and who were registered for a B Ed degree at the NMMU in Port Elizabeth, South Africa, participated in the study. The participants were informed as to the aims of the project, that the data generated would be used for research purposes, and that they could withdraw from the project if they wished to do so.

Prior to the intervention, focus-group interviews were held with the students and they were required to provide written reflections (disposition statements) as to how they 'felt' about mathematics. The data generated by the post-intervention affective-disposition statements were used to probe the degree to which students had enjoyed participating in the values-based mathematics education strategy and whether their engagement had influenced their levels of mathematics self-efficacy. Pre-, mid- and post-intervention semi-structured interviews with focus groups of students generated similar data. The data generated by these interviews were classified into broad themes and analysed within the framework of the literature reviewed.

The students kept a personal journal in which they were asked (i) to respond to five formal prompts which were introduced sequentially throughout the semester, and (ii) to make *ad hoc* informal entries throughout the intervention. The prompts included questions as to whether they thought a values-based approach would benefit their teaching, what they enjoyed or did not enjoy about the approach, the influence of the strategy on their thinking and beliefs, their perceptions of what problem solving entailed, and their thoughts on the strategy for their own teaching using a values-based approach. The process of journaling allowed the students to record their perceptions, challenges and experiences, and their own 'transforming self-beliefs' (McNiff, 2002) in their ability to do mathematics. Video recordings of lessons also provided data as to the educational influence the strategy had with regard to students' self-efficacy and belief in their own ability to do and teach mathematics. Overall, the data generated, when triangulated, provided insights into aspects of the 'social reality' that participants brought to their mathematics education classes in terms of moral values, how they perceived these experiences, and whether they influenced their mathematics self-efficacy.

Results

Affective disposition statements

The results of the pre- and post- affective disposition statements suggest an improvement in self-efficacy over the course of the intervention. Twelve percent of the first year students and 20% of the second-year group indicated that they were more positive towards mathematics. Although no students from the second-year group initially expressed a desire to improve their mathematics skills, 15% responded in the post-intervention statement that they believed that they had in fact improved their mathematics skills during the course of the mathematics education lectures.

'Fear of failure' was expressed by 31% of second-year students and 7% of first-year students in the pre-intervention responses. However, in the post-intervention responses, only 17% of second-year students expressed reservations regarding their mathematics capabilities, i.e. that they were still fairly dependent on assistance in solving mathematics problems. No first-year student indicated that 'fear of failure' was a concern for them. These results suggest that both groups of students improved their self-efficacy levels during the intervention.

Interviews

Student responses from both groups indicated that they believed that their initial mathematics self-efficacy levels were directly linked to their success and achievement at mathematics, or lack thereof, during their school years. The results also indicate that a major motivating factor for enjoyment in mathematics was 'experiencing success' This finding concurs with Pajares' (2002) belief that students who perform well in mathematics are likely to develop a strong sense of confidence in their mathematical capabilities, whereas poor performance generally weakens students' confidence in their capability. Student responses clearly indicate that 'fear of failure' was a dominant cause of lack of enjoyment in mathematics. However, by the end of the semester, the data indicated that the intensity levels of students' 'fear of failure' had dissipated significantly, suggesting increased self-efficacy.

The majority of participating students (99%) regarded values as making a positive contribution to their engagement in mathematics education. Only two students felt that a focus of values in mathematics teaching was not important, and stated that they believed that teaching values at university level was too late to make a difference.

Reflective journals

Student responses indicated that the use of journals in their mathematics education classes was a major motivating factor in terms of reflecting and internalising values, and highlighted the role they had played in contributing to social responsibility and social cohesion, especially when shared in class. They appreciated the time the lecturer had spent responding to each of their journal entries and believed that the written communication with her had enabled them to feel that the she had a personal interest in them, despite the anonymity of the journal entries (students dropped off and collected their anonymous journals at agreed and predetermined times and places).

Observation

Observation of the video-clips suggest that the lecturer was able to make the core values explicit at appropriate times, that the values were evident in her attempts to guide the students towards experiencing success in the problem-solving activities, and that these activities elicited a positive response from most students who engaged in the process.

Discussion

Trustworthiness is defined by Kidder (2006) as the manner in which an individual acts in order to engender trust and merit the confidence of others. He describes the warm, solid 'gut feeling' you get from trust – from counting on yourself and in trusting and being trusted by others – as one of the great enablers of life. All student responses for both the mid- and post-intervention interviews suggested that they felt that they had developed a sense of trust in their lecturer. Journal entries and interview responses indicated that the development of a trust relationship was not only due to their perception of her personal commitment to both the students and her teaching, but predominantly to the fact that she had responded to every journal entry that they had submitted within a two-day period of time, as promised. Students' verbal and written responses and discussions suggest that her behaviour in promoting and 'living out' the core values also encouraged the development of a lecturer-student trust relationship.

As noted above, the data suggest that the students believed that their lecturer lived out the core values adopted, and that not only did they recognise the importance of her practising core values in their classes, but that they also recognised the importance of making values explicit for themselves as future teachers. They appreciated the opportunity to practise values in their mathematics education classes and believed that values enrich and "make better people". They also stated that they also believed that the implementation of a values-based approach to teaching and learning promotes more positive attitudes, improves performance and creates opportunities for the development of positive relationships. The importance of values for the wider community and society at large was also acknowledged.

It was the responses in which students suggested they trusted that their lecturer would not ridicule or ignore their contributions to the learning process that prompted her to believe that 'valuing students' opinions' had played an important role in the development of their student-lecturer trust relationship. This supports Llewellyn's theory (2005) of encouraging student inquiry as active participants in the learning process, as well as Brooks and Brooks'

(1993) model of building a classroom community by getting students to believe that their ideas count, promoting a trust relationship within a learning community, and thereby influencing their self-efficacy. We believe that without the continued purposeful recognition, acceptance and practice of the five core values which underpin this research study, the desired classroom climate of integrity (which depends on all of the core values being upheld) would have remained an unattainable goal. For this reason, we believe it was important to address issues which appeared to be in contradiction of the core values that the lecturer was trying so hard to uphold.

Although this research study is limited, and that more in-depth interrogation of students' conceptions of the role of values would be profitable in the search for greater understanding of the role of values in mathematics education, we believe it has a contribution to make in terms of informing teacher educators, teachers and policy-makers about perspectives and interventions which may contribute to pre-service teachers' mathematics self-efficacy levels, and provide pointers to strategies for developing teachers who will be able to contribute to the integrity and vitality of the teaching profession. These strategies include the role that a values-based approach to the teaching and learning of mathematics may have on influencing students' self-efficacy levels, both mathematically and within other subjects. We also believe that the findings of this study can contribute to the on-going debate about the process of quality teaching and learning in terms of the possible gains which could emerge from a values-based approach to teacher education.

The data suggest that, in spite of large classes of multi-cultural and diverse students with differing world views, the first author was able to encourage her students to commit themselves to devoting time and energy to the task of using values to develop an innovative educational approach, and to accept that such an approach has the potential to promote the development of their knowledge, competencies, and personal and professional values as they strive to reach their full potential and help shape a fairer, equitable and more just society in the 21st century (Department of Education, 2002; 2007). An underlying, but implicit intention of this study was to influence stakeholders at policy and other levels to consider and embrace the possibilities that a values-based approach to mathematics education may have on students' levels of mathematics self-efficacy, and to include these notions in the design of mathematics curricula for the 21st century. As such, we believe that the findings of this research study point to the potential to challenge teacher educators to re-assess, adapt and improve their teaching strategies, where necessary, and revise the assumptions about teaching and learning on which they are based.

Considering the current crisis in mathematics and science education in South Africa, as well as the crisis in responsible citizenship, it is imperative that we seriously consider the devastating consequences of teacher-education programmes which are not underpinned by sound value systems and effective approaches to teaching and learning. The challenge is to implement a curriculum which is relevant in content and context to South African educational demands for a strategic, but often controversial, reform processes. In attempting to promote such a paradigm shift, it is important that policy-makers, education departmental officials and teacher educators take cognisance of the reality that students consistently refer to the quality of their own teachers as the primary reason for their achievements and choice of careers.

Concluding remarks

Working in collaboration with students interrogating of her own practice was a new experience for the first author. She had always valued the opinions of others, but felt hesitant about giving her students the responsibility for holding her accountable for living out the values identified in this research study. However, she realised the importance of giving them a voice in finding meaningful ways of improving her teaching practice, which in turn would motivate their own mathematics self-efficacy levels. She was aware that students (as they did in this study) often refer to the quality of their own teachers as the primary reason for their achievements, or the lack thereof, and for the choice of their career, and was gratified that the findings indicated that they felt that her "good teaching" contributed to their positive feelings about mathematics; and hence, their self-efficacy levels.

For her the significance of this research study was largely that she had learned to live her values more fully in her own practice. She developed greater insights into the issues she was investigating, and came to understand how her work has the potential to influence her students in new ways. In the case of this study, the new way was through using a values-based approach to teaching and learning, an approach which positively influenced her students' levels of mathematics self-efficacy.

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Problem-centred teaching and modelling as bridges to the 21st century in primary school mathematics classrooms

P. Biccard
Stellenbosch University
pbiccard@yahoo.com

D.C.J. Wessels
Stellenbosch University
dirkwessels@sun.ac.za

Abstract

Moving mathematics classrooms away from traditional teaching is essential for preparing students for the 21st century. Rote learning of decontextualised rules and procedures as emphasized in traditional curricula and teaching approaches have proven to be unsuitable for the development of higher order thinking. The ‘dream’ is to have skills (that employer seek for the 21st century) such as being able to make sense of complex systems or working within diverse teams on projects [2: p. 316] fostered in mathematics classrooms, even at a primary school level. In this paper it will be shown that the problem solving perspective that modelling emphasizes includes competencies and skills that are essential in developing authentic mathematical thinking and understanding. Results of a study on modelling competencies [1] will be presented to highlight the growth of a problem solving mode of thinking. We will therefore explain that modelling achieves important aims for mathematics education in 21st century. Modelling fosters students’ abilities to actualise existing (but not yet explicit) knowledge and intuitions; to make inventions; to make sense and assign meanings; and to interact mathematically [10: p. 176], thereby developing authentic mathematical thinking. The aim is to provide a perspective that shows how modelling meets the challenge of changing mathematics classrooms.

Introduction

Students need to learn mathematics with understanding since ‘things learned with understanding can be used flexibly, adapted to new situations, and used to learn new things. Things learned with understanding are the most useful things to know in a changing and unpredictable world’ [11: p.1]. Adaptability and flexibility in using mathematical knowledge is particularly important when students solve contextual problems. Mathematical problem solving has many faces and requires definition. Schroeder and Lester’s [3: p. 32, 33] three main descriptions of problem solving are used for this paper. In a traditional sense, problem solving means solving ‘word’ problems as an extension of routine computational exercises. This can be seen as teaching *for* problem solving – teaching of procedures takes place first and then problems specifically related to the taught concepts are solved. In some progressive programs, students are taught *about* problem solving and are taught to employ various methods or heuristics as options when faced with a problem (e.g. drawing a table or graph etc). When students learn *via or through* problem solving, problems are used to teach important mathematical concepts. When students interact with modelling problems, they solve the problems in their own way with mental tools that they already have available to them. The teacher facilitates by connecting different ideas that allow students more sophisticated understandings through these connections. It is by solving problems first and then building by connections between student ideas and representations that students become adaptable and flexible and move toward a problem solving mode of thinking. Modelling allows students to learn via problem solving and can be appreciated as a significant mathematics teaching and learning opportunity.

The Problem-centred approach and modelling

Hiebert, Carpenter, Fennema, Fuson, Human, Murray, Olivier & Wearne [4] build a problem-centred approach on Dewey's principles of reflective inquiry. They work from the assumption that understanding is the goal of mathematics education. Students solve problems at the outset of a mathematics lesson and the process of solving, collaborating, negotiating and sharing leaves 'behind important residue' [4: p. 18]. As expressed by Human [5: p. 303] problems are used as vehicles for developing mathematical knowledge and proficiency together with teacher-led social interaction and classroom discourse. The problems are opportunities for students explore mathematics and come up with reasonable methods for solutions [11: p. 8]. The role of the teacher and the student changes which means that the classroom takes on a different culture. It is in the problem-centred classroom culture where the real benefits to student learning lie. Students have regular opportunity to discuss, evaluate, explain, and justify their interpretations and solutions [6: p. 6]. This can only be achieved if the teacher allows this discussion to take place without presenting or demonstrating set procedures to solve problems. It is this change in focus in the classroom away from 'teacher thoughts' to 'student thoughts' [13: p. 5] that epitomizes student-centred methods such as the problem-centred approach and modelling. The classroom culture now takes on what Brousseau [7: p. 30] termed an 'adidactical situation'. The teacher, in an adidactical situation, does not attempt to tell the students all. Brousseau also explains that the 'devolution' [7: p. 230] of a problem is fundamental in adidactical situations. This happens when the teacher provokes adaptation in the students by the choice of problems put to them. The problems must also be such that the student accepts them and wants to solve them. The teacher refrains from suggesting the knowledge, methods or procedures he/she is expecting or wanting to see. The teacher seeks to transfer part or all of the responsibility of solving the problem on the student. It is this adaptation to the adidactical situation that allows students to learn meaningfully. It is this adaptation to a problem-centred approach that allows a growth of mathematical understanding, adaptability and flexibility. This in turn promotes a problem solving mode of thinking that is so necessary in the 21st century workplace.

The problem-centred approach in mathematics education allows us to understand modelling and its place in effective mathematics teaching and learning. Modelling goes beyond problem solving in that the important questions of *when* and *why* problems are solved as well as *whose* thoughts ideas and constructs are used when solving problems. In defining what a problem solving capacity or mode of thinking entails, the constructs of [10] are used. When students solve problems, they should be provided with opportunities to: actualise existing (but not yet explicit) knowledge and intuitions; make inventions; make sense and assign meanings, and interact mathematically [10: p. 176]. These four constructs encompass what it means to solve problem with understanding and flexibility. It is often difficult for teachers to elicit existing knowledge from students since there is an array of different understandings and levels of thinking in a single classroom. Modelling allows students to verbalise their current ways of thinking and improve on these ways of thinking. In making inventions, students are able to use their own ways of thinking in constructing a response to the problem. Modelling tasks allow students to produce meaningful solutions that keep the context of the problem in sight. While students work collaboratively on modelling tasks they do make sense and assign meaning since they have to communicate their thinking and ideas while interacting mathematically with each other in order to make progress. These four constructs underline what it means to develop a problem solving mode of thinking since they encompass student understanding, adaptability and flexibility in solving problems. It also underlines what students need to learn meaningful mathematics in the 21st century.

Mathematical modelling goes beyond problem solving since students 'create a system of relationships' [12: p. 110] from the given situation that can be generalised and reused. Although students are solving problems when modelling, a modelling approach means that students must display a wider and deeper understanding of the problem. Modelling goes beyond problem solving because students structure and control the problem – not only solve it. The aim of this paper is to show that modelling tasks allow students and teachers access to significant problem solving that bridges student understanding and student problem solving abilities. The development of a problem solving mode of thinking that results from student involvement with modelling tasks is presented in this paper. Problem solving and modelling problems specifically hold a reciprocal interdisciplinary relationship with other knowledge fields. Modelling problems for mathematics classrooms are applicable to and can be sourced from fields outside mathematics such as engineering, architecture, commerce and medicine to name a few.

The study

The main study [1] investigated the development of modelling competencies in grade 7 students working in groups. Partial results will be presented in this paper. Twelve grade 7 students were selected to work in three groups of four students in each group. The results of only one of the groups working on the first (of three) task are presented in this paper. This group comprised students whose mathematics results in a traditional setting the previous year were considered "weak". The groups solved three model-eliciting tasks over a period of 12 weeks in weekly sessions of about one hour. The results from this group's discussions around the task – Big Foot is presented.

Task 1: Big Foot taken from [9: p. 123]. Example of footprint (size 24) given to students. Groups had to find the height/size of this person and also provide a 'toolkit' on how to find anyone's height/size from their footprint. Supporting material: rulers, tape measures, calculators

Table 1: Task Instruction for Big Foot

Students solved and presented their solution as a group with minimal teacher/researcher intervention. These students had not been exposed to a problem-centred approach nor had they solved modelling tasks before. Each group presented their solution to the other groups and students were encouraged to question each other's models. The contact sessions were audio recorded and transcribed. Transcriptions were coded for each competency for the main study and coded again for the results presented in this paper. The competencies identified for the main study were: understanding, simplifying, mathematising, working mathematically, interpreting, validating, presenting, using informal knowledge, planning and monitoring, a sense of direction, student beliefs and arguing. How students develop and refine a problem solving mode of thinking is highlighted in this paper. The four constructs [10: p. 176] were used to code the data from the transcriptions and to structure the discussion in the next section. This assisted in establishing to what extent students working in groups were engaging a problem-centred paradigm when solving model-eliciting problems.

Results

The results presented are from the group's solutions processes for Task 1: Big Foot. This was their very first modelling task so it exhibits the impact modelling tasks have on students thinking. Furthermore it highlights the mathematical learning opportunities that are implicit in a modelling task. In the transcripts R stands for researcher.

Actualising existing knowledge and intuitions

This group had an intuitive idea that there was a universal foot to height ratio although this took place in the second session. The first session was taken up by a seeming avoidance of mathematising the task. Once they had decided to take action on their own intuitions they were successful in producing a model for this task.

M: Ok wait, why don't I take my foot and divide it by my height, times a 100

R: why do you times it by 100?

M: Because that is how you find your percentage so we can find out ...I am saying that when we do his (Big Foot) *then we must get the same...*

The students introduced the idea of a percentage on their own accord, but it later transpired that they used the 'times by 100' to remove the decimal number that resulted from their division. This group also had an intuitive idea that Big Foot had to be very tall and they were able to use this to interpret and validate their progress which allowed them to make progress in their solution process. They had taken a number of measurements including across their hips which they called their 'width'.

M: 58 (a group member's height) *divided by 2 is 28.* Similar to his width - they measured 27 as this person's 'width'.)

N: So he is 30 inches!

M: No he can't be, that's too short.

If student do bring their own ideas and constructs forth and they act on these ideas it is clear to see how 'making inventions' is possible. This would not be possible if students are offered methods or procedures by the teacher.

Making inventions

This group 'invented' their model to assist them in resolving Big Foot.

M: I divided my feet to my height and I timesed it by 100 and I got 15.

N: Yes...

M: So now I have to try get 16 times by what to get 15 again because it is a human. Then that will be right.

On their presentation sheet (see Fig 1) they had written:

The solution is to take his foot size and divide it by an estimated number, multiply that by 100. The result should be 15-20.

Although they did not see a connection here between multiplying and dividing (surprising for this year of their schooling), they invented a way around this of 'estimating' the multiplicand so that the result would be 16. Once this group had found the foot length to height ratio of all their group members, they had four different (although very close) ratios. They then realized they needed more data and tried more people. After trying three more people they found that 16 seemed to be a common ratio. Although they never used the term 'mode', this is a construct that they 'invented' by understanding that they needed this from their set of data. When questioned:

M: R is 16 and N is 16. Then we must use 16.

R: why did you decide that N and I have the right measurements?

M: Because you are the most.

Making sense and assigning meaning

After calculating that 15 was one of the group member's foot/height ratio, they continued to work through the rest of the group, other people in the room as well as continuing this at home and with other students at school. They were clearly in control of this 'method' or model although it was not an elegant approach it was meaningful and they were able to assign meaning to other areas of the model.

M: Divided by (known height) *and times 100 and let hope it equals something nearby 15 and 20.*

N: 62 divided by 11, ag no sorry the other way; 11 divided by 62 times 100 is

M: Yes I told you. 17. So I equal 15 and you 17.

M: OK it (the quotient) *might be 15, 16 or 17. So he* (Big Foot) *might be: 98, 97 or 96.*

When looking at their presentation sheet- they understood that if the ratio was 15, then their estimated height was too short, or it the ratio was 19, then their estimated height was too tall. They were able to assign meaning to a fairly complicated model which is surprising since they achieved lower mathematics results than average in a traditional setting.

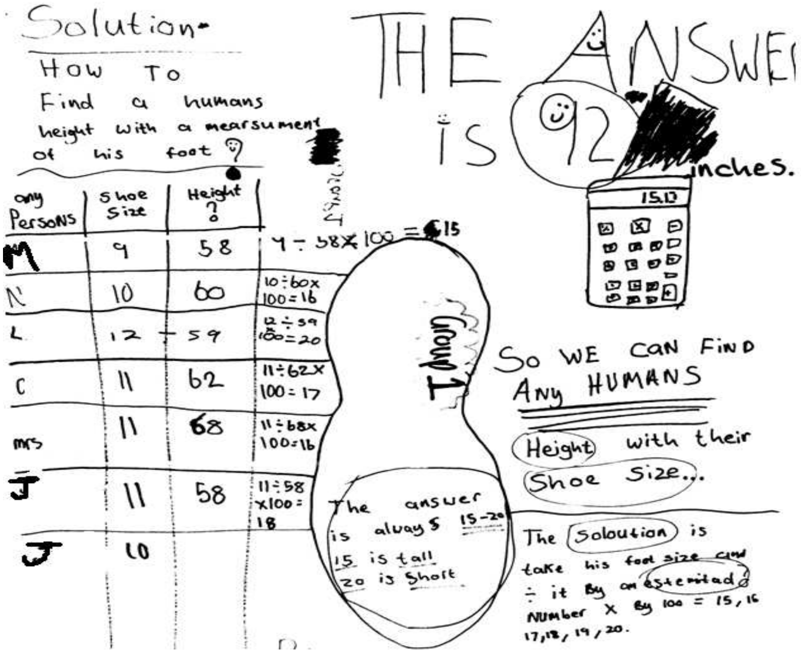

Fig 1: Group presentation sheet

Interacting mathematically

The following excerpt from the transcripts for this group shows how one group member explains a fairly inelegant yet complicated model for Big Foot to the other members.

M: *Look, I take your foot* (length) *right; the foot is 12* (inches)*, then I divide it by any* <u>*estimated number*</u>*, like I will take, a number will come in my head and I will divide it* (by the foot length) *and then multiply by 100. Probably* (the result will be) *over 20 or below 15. If it's below 15, it means the person is taller, if it's over 20 it means the person is a bit shorter. Then you estimate a bit lower until you get 15, 16 or 17.*

Conclusion

The confluence of the problem-centred environment and modelling tasks present mathematics education with a 'developmental space' for the learning of essential, meaningful mathematics [1: p. 37].

The data presented in [6] suggested that a problem-centered instructional approach in which the teacher and students engage in discourse that has mathematical meaning as its theme is feasible in the public school classroom [6: p. 25]. The results of [8] suggest that a problem-centered approach together with a change in teacher beliefs is a viable for reforming mathematics classrooms. Furthermore, a modelling approach assists in developing student competencies in problem solving, modelling and mathematics. Modelling tasks present an arena for teaching and learning that assists teachers in understanding a problem-centred approach and to simultaneously apply these principles in teaching. Modelling tasks can be used successfully by teachers and students unfamiliar to problem solving or a problem-centred approach.

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iMath- Reaching the iGeneration in the Mathematics Classroom

Norma J. Boakes, Ed.D.
Associate Professor of Education
Program Coordinator for the Teacher Education Program
School of Education, Richard Stockton College of New Jersey
Pomona, New Jersey, United States
Norma.Boakes@stockton.edu

Katie Juliani, MA
Professional Services Specialist
School of Education, Richard Stockton College of New Jersey
Pomona, New Jersey, United States
Kate.Juliani@stockton.edu

Abstract
The major aims of this paper are to: define a new generation of tech-savvy children that are in our math classrooms known as the iGeneration, discuss how instruction should be adapted to meet the needs of this new learner, and illustrate the potential of this generation's i-devices in the math classroom. Though educational technology is not a new concept, these i-devices, such as the iPod touch and iPhone, bring a new dimension to instruction that can offer a classroom greater "motivation, engagement, conceptual understanding, and problem solving skills" (Manzano, 2009, para. 11). This paper targets teachers and schools in an effort to shed light on these fairly inexpensive, readily available technologies that have tremendous instructional potential in mathematics.

Introduction
Generations in the American culture have been named since the early 1900's with the development of the Western World. Significant events like the boom in births in the 1940s, the Baby Boomers, or the creation and widespread use of the Internet, Millennial Generation, have defined groups of Americans based on cultural shifts. A new generation has cropped up that is having a major impact on how we think, teach, live, and learn. These individuals are known as the iGeneration. The formation of this generation came with the creation and widespread use of "i" devices such as iPods, iPhones, and iPads. This generation is now in elementary and secondary school. They are described "as being solidly committed to the ubiquitous use of mobile technologies, most commonly MP3 players, Smartphones, and similar devices." (Rosen, 2010, 229) This generation thrives with individualized technologies and has them as a daily part of their lives. They have had the Internet since they were old enough to read and never knew a time without cell phones.

Technology has distinctively influenced life for all ages, not just those in the iGeneration. We are more apt to Google a topic than looking in an encyclopedia. We book flights online rather than go through travel agencies. Information is emailed more often than sent hard copy through the postal service. With this undeniable shift to a more technology-saturated world, it is time to reflect on today's classroom and how we go about instruction. In the area of mathematics, the National Council of Teachers of Mathematics (NCTM), the largest math education association in the United States, highlights technology as a vehicle for the teaching and understanding of mathematics in their Principles and Standards of Mathematics (2000). Their vision of mathematics includes classrooms where 21st century technologies are readily available and used as a tool to engage and challenge children. The International Society for Technology in Education echoes these sentiments emphasizing the need for children to "think critically, solve problems, and make decisions" through the use of various technologies (ISTE, 2000, para. 2).

Technology is nothing new to the American classroom. Within the last ten years we have seen widespread use of such tools as interactive whiteboards, mathematics education software, laptops, and graphing calculators. The time has come, however, to make room for a new technology that is not necessarily seen in the classroom. The iGeneration uses them all the time. You might see a young boy with an iPod while mom shops or a teenager texting a friend on their iPhone. These i-devices provide many of the capabilities of a computer in the palm of the hand. This paper will focus on embracing this new generation and their technology in the mathematics classroom.

The Growing iGeneration

A striking reality is the research done on the boom of i-devices. Rosen (2010) in his book about the "iGen" speaks of research he did with 1,500 parents. Half of 9- to 12-years olds of these parents had cell phones and two thirds had some form of MP3 player or iPod. Even in the 5- to 12-year old category, more than half of them had video or handheld games in their bedrooms. These children are in our classrooms. With every passing day, the way these children are motivated and engaged changes.

The iGen are those born beginning in the 1990's. This generation has never known a time without technology and sees new technologies cropping up every year. Parents of these children are embracing this technology allowing more and more of it to be a part of their child's life. Rosen argues that this has bearing on why many children report school as boring or dull (2010). Though advances have been made with technology in classrooms, Rosen reports that it is not catching up with this generation. A teacher may use a Power Point presentation as part of a lecture but they don't necessarily utilize the kinds of technology tools they are motivated and engaged by. This paper focuses specifically on the i-devices that have helped characterize this generation including the iPhone, iPod, and iPad. These tools have many capabilities that can be educational and a meaningful addition to the mathematics classroom.

The Tools of the iGeneration

The focus of this paper is on just one of the many categories of technology that has quickly grown in popularity with the iGeneration or "iGen". The iPod Touch or iTouch, iPhone and iPad all fall within the category of wireless mobile devices (WMD). These i-devices are hand-held computers providing users with instant access to the Internet, communication tools, and a great variety of applications. They dominate the technology landscape of youth. For instance, Apple reports 160 million i-devices sold as of April 2011 (Ogasawara, 2011). Of those, 55 million are iPod Touch users. 46% of these users are under 18 years of age (Loechner, 2009). I-device features that make them so enticing to youth include touch screens, colorful graphics, real-time video, and quick access to Internet and applications. According to Rosen, i-devices have "great promise in education" (2010). However, these educational tools have not been used beyond small, isolated studies (Rosen, 2010).

One of the most popular and well known features of these i-devices is the applications, known by iGen simply as "apps". Apps are a piece of software like software that can be bought for a desktop computer or a videogame for a gaming system. These apps number in the hundred thousand with ten thousand plus specifically in the education category (Go Rumors, 2010). Applications are often free or inexpensive costing on average about $3-$5. They span a wide range of topics including mathematics. A quick search of the word "math" in the Apps Store on these devices will find anything from a simple flash card to graphing calculator app. Teachers are beginning to see the potential of these tools for the math classroom. A We Are Teachers blog article (2010), for example, offers educators a list of math education apps ranging from early learners to secondary school children. The popular McGraw-Hill education publishing company has created a collection of apps focusing on math teachers and their math curriculum (PadGadget, 2011). Even books published on apps such as Best iPad Apps feature apps suitable for the math classroom (Meyers, 2010).

Adapting Math Instruction for the iGeneration

With a new breed of learners comes a fresh perspective on how we teach them. Rosen's book Rewired (2010) has inspired many educators to reflect and consider how the classroom can be adapted to better meet the needs of the iGen learner. In the paragraphs to follow, major traits are considered that motivate the proposal to utilize i-devices in the math classroom.

iGens are said to be a "creative, multimedia" generation" (Rosen, 2010, p.218). They thrive in environments that blend sights, sounds, video, colors, etc. This doesn't mean that a clever Power Point presentation is enough to appease the iGen. Teachers need to consider varied digital modalities to reach them (Wood, 2010). Teachers shouldn't be afraid to use audio, video, or web-based information to teach with or to blend these approaches. iGens are multi-taskers so let them possibly use headphones while working on an assignment or have an iPod while they do a group activity. The more opportunities these learners have to be independent and active with digitally rich sources the more engaged they are likely to be.

Another area where iGens thrive is in creativity. These children "live to create" (Rosen, 2010). "Teachers can engage students by giving them choices in demonstrating content-area knowledge through different tools, whether it be a movie, podcast, digital poster, or webpage" (Wood, 2010, para. 1). iGens want access to all the tools they have, digital and traditional. Teachers should provide them with choice and flexibility when given a major assignment to work on.

Freedom to work at their own pace is preferred by iGens in school. It's said to be a product of their fast-paced lives. iGens can easily shift from task to task and work well under pressure. With a few carefully set guidelines, these learners want to be set free to approach the task (Rosen, 2010). Wood (2010, para. 1) warns in his blog that this would likely work best with check points to ensure learners are on target with their progress.

iGens have added a fourth R to the well known three R's- reading, writing, arithmetic, and realistic technology (Rosen, 2010; Wood, 2010). Teachers are encouraged to embrace technology as the fourth R and "use it to teach and augment the original three" (Rosen, 2010, p.225). There is also the "realistic" aspect of technology. Technology offers the chance for teachers to bring the real world into a lesson and the lesson into the real world (Shuler, 2009).

Teachers shouldn't fear this new generation of learners. However, there is an undeniable call to consider adapting the more traditional models of instruction. I-devices and their apps offer much of what the iGeneration seeks as part of their learning and is seen as a "key to the future of education" (Rosen, 2010, p.203). A professor who was sharing thoughts on a recently national report called *Pockets of Potential* (Manzo, 2009) compares the potential impact mobile devices can have on learning to that of Sesame Street on television. Just as this TV program convinced viewers that TV could be more than a tool for amusement, mobile devices are likely to some day be seen as a tool for engaging and activating learners in the classroom (Manzo, 2009).

Apps for the Mathematics Classroom

The iGeneration sits in today's classrooms. With a sense of who they are and what they need, the teacher has the task of determining how to adapt what they do to bring the "i" into their math instruction. I-devices feature inexpensive apps that can be loaded and played with only a few screen touches. These apps are a way to bring the characteristics of learning that an iGen hopes for in their classroom.

When looking at apps for i-devices in the area of mathematics education, the list grows daily. As of May 2011, a search done in the Apple App Store of an iPad using the key word “math” within the education category results in 2,533 apps for the iPhone and iPod as well as 1,006 for the iPad. These apps cover a wide variety of categories and each have their own unique spin on learning and practicing mathematics.

For this paper, apps were reviewed and categorized based on a number of features to reduce the thousands of apps listed down to a manageable set of quality math related apps for possible classroom use. One feature considered was the style of the app. iGens love multimedia so apps that offer a blend of strong audio and visuals are a good choice. These apps look and feel like a video game. HyperBlast, for example, offers an arcade like feel with 3-D graphics and three game play levels. While battling aliens, children can practice their basic facts in addition, subtraction, multiplication, and division. Other apps utilize mathematical thinking to play the game. Blokus, for example, has players using transformations to place geometric pieces on a game board. In Cut the Rope a player must decide how to cut swinging ropes so a piece of candy can be sent to the hungry animal waiting for it. These types of apps are “game-based” offering mathematical practice while playing a game.

Real world connection is another important iGen characteristic that apps can offer a classroom. Lemonade Tycoon is a perfect example. The player in this app manages his/her own lemonade business with the goal of making a profit. The player has lots of options including the choice of location, recipe, and weather. A more simplistic app for younger players is Pizza Fractions. This app uses pizza to illustrate fractional parts. In any of these real world apps, the player experiences real life application of mathematics in some form. i-Gens appreciate this connection because it establishes a link between what they are learning in school and its relationship to their everyday life.

Many of the apps that can be found offer children a chance to practice a math skill. There are also apps that teach skills and offer tutorials to help understand mathematical concepts. Fractions Helper is one such app offering the user a step-by-step walk through on how to solve addition problems with fractions. Another app called Cheater Pants can offer students a step-by-step visual of the solving process for doing basic arithmetic problems.

Other features that are cropping up among the many apps are the ability to reward the user and provide progress reports. In the app Cash Cow, for example, you can earn virtual money to purchase items and complete tasks on your own virtual farm. The app Academy rewards correct answers with stickers that can be used to decorate selectable backgrounds. Froggy Math app provides a “report card” to let the player know how they are progressing at each level of play. Both of theses types of features add a more personalized feel that appeals to the iGen learner.

Apps offer a wide variety of options and mathematical topics to the classroom as a whole. The chart, found at www.tinyurl.com/iMathBoakes, lists a total of 55 apps selected for the many features discussed. Beyond this analysis, each of these apps received a user approval rating of at least 4 out of 5 stars through the Apple App Store rating system. Apps are organized in alphabetical order and are categorized based on the NCTM (2000) mathematical standard area, grade level of the user (lower elementary, upper elementary, middle, and secondary), as well as features including price, game based/real world, practice/tutorial, reward, and progress report. With some creative new thinking, teachers, parents, and iGens themselves can use these apps to enhance and develop understanding of mathematical concepts and skills.

Conclusion

The focus of this paper was to highlight the ever-changing composition of the math classroom. When it comes to technology, the pace of advances is even more pronounced. This new generation of learners known as the iGeneration is cause for pause and reflection on how we approach instruction. iGens think and learn differently. They thrive in media-rich environments especially those that include the i-devices they are aptly named for. Studies over the past five years by the Research Center for Educational Technology at Kent State University in Ohio have found that using these hand-held devices in the classroom can improve students' motivation, engagement, conceptual understanding, and problem-solving skills. (Manzo, 2009, para. 11).

The applications of i-devices discussed are a simplistic way to begin capitalizing on the iGen's interests in the math classroom. Apps span all grade levels and mathematical standard areas. They can provide many of the characteristics that an iGen seeks in their learning from realistic applications to an engaging, multimedia style. With the prevalence of hand-held technologies and inexpensive versatile apps, it is time for all those who teach to consider this possibility as a way to reach today's learner.

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Physicists use mathematics to describe physical principles and mathematicians use physical phenomena to illustrate mathematical formula – Do they really mean the same?

Ulrike Böhm[1], Gesche Pospiech[2], Hermann Körndle[3], Susanne Narciss[4]
[1]*Center for Teacher Education and Research, Technische Universität Dresden, Germany*
Ulrike.Boehm@mailbox.tu-dresden.de
[2]*Department of Physics, Technische Universität Dresden, Germany*
Gesche.Pospiech@tu-dresden.de
[3,4]*Department of Psychology, Technische Universität Dresden, Germany*
Hermann.Koerndle@tu-dresden.de, Susanne. Narciss@tu-dresden.de

Abstract:
Physics and mathematics educators often find themselves in an absurd situation. Students have problems in thinking in an abstract way. In order to help them to overcome these problems mathematical formulae in math classes are often illustrated with physical phenomena which are introduced as authentic applications of mathematics in every day life. However, students in physics classes have also difficulties in gaining a deep understanding of physics. They try to overcome their lack of understanding by memorising facts or using formula (without understanding their meaning).

Given these problems in abstract thinking and understanding of physical phenomena the issue arises what might be the potential confusions regarding a physical phenomenon after having experienced mathematics and physics lessons on this topic? To address this issue we chose an example of geometrical optics – the mirror image.

We generate and evaluate a heuristic framework for describing and exploring the process of understanding a physical phenomenon. This heuristic framework differentiates several scientific models (e.g., physical, mathematical) which are necessary for understanding and explaining the phenomenon. Furthermore, it integrates these models in a multiperspective instructional model (i.e. didacticised model). Using this heuristic framework we analysed the problems in understanding which occur when students have to understand the mirror image.

Introduction

The mirror image is one of the phenomena most studied and most misunderstood in early physics education. Results of numerous studies (e.g. Blumör and Wiesner (1992a), Galili, Bendall, and Goldberg (1993), Jung (1981), La Rosa, Mayer, Patrizi, and Vicentini-Missoni (1984), Gropengießer(1997)) show that the understanding of the mirror image phenomenon is quite unsatisfactory. What is the problem?

At first we have to consider that the misunderstandings are not to be found in physics – but the learners think, that they are not able to understand a physical phenomenon! To explain the mirror image an optical and a non-optical (human) argumentation is needed – and have to be linked to each other. But both sides of this medal cause problems. Already early researchers like AlHazen and Euler said that for the explanation of the mirror image both different kinds of modelling are needed. The first one – the physical argumentation - means the explanation of the mirror image with the help of geometrical optics, and the second one - the non-physical argumentation - is our human interaction with light- the interpretation of the picture on the retina through our brain.

In this area of conflict one answer to the question for problems with the mirror image can be identified. The non-optical argumentation plays a marginal role in school lessons. But there is still another point of view – the mathematical modelling of the mirror image has an important influence on the understanding of the mirror image. This important role is being discussed in this paper.

The starting point is the question: “How do the pupils understand a physical phenomenon?”. We have to look at the process of constructing mental models of a phenomenon. In order to understand how students handle mathematical and physical knowledge it is necessary to examine their modelling process in physics in a more detailed way.

Understanding physical models

According to Stachowiak’s theory of modelling (1973) nobody can describe the real world – only a model of the real world. In particular novices have problems to understand science because they do not understand that teachers are talking about models of the reality and not about reality itself. This is one of the most important problems in modern science teaching – physical theories should be taught in a way acknowledging that these theories are models of the real world. In shaping this processes science teachers have to pay attention on insights in how students acquire scientific concepts (i.e. epistemic processes). The process of constructing mental models during the acquisition of physical knowledge plays an important role in understanding physical phenomena.

To describe and explore the process of understanding physical phenomena through science instruction we developed a heuristic framework which differentiates several scientific models (e.g., physical, mathematical). We assume that in order to help students understand a physical phenomon these different models have to be explained and integrated in science teaching. That is teachers have to develop a didacticised model of the phenomenon which addresses the various scientific models. Hence, the heuristic framework contains five main parts: (1) the phenomenon, (2) the scientific models necessary for explaining and understanding the phenomenon, (3) the teacher, (4) the didacticised model of the phenomenon and (5) the learner which interacts with both, the phenomenon and the didacticised model.

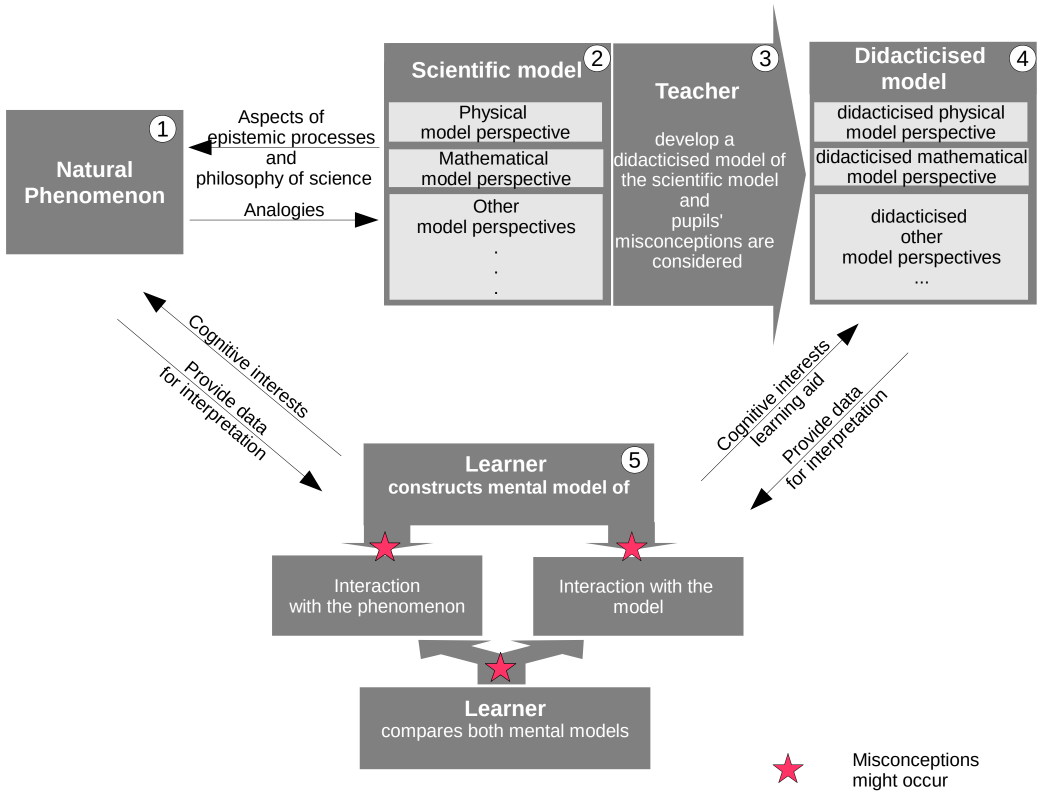


FIGURE 1: Heuristic framework for describing and exploring the process of understanding a physical phenomenon through science instruction (Böhm, Pospiech, Körndle, & Narciss, 2010b, p. 148)

According to this framework, in understanding a physical phenomenon the learner has two models to handle with: (1) the own mental model and (2) the didacticised model of this phenomenon. Hence, in science education a learner is not approaching a physical phenomenon in the way researchers do it. Researchers develop a scientific model, which describes in the phenomenon as detailed as possible. This scientific model is then examined by experiments and by applying it to the real world.

Learners are mostly taught a didacticised model which a teacher developed especially for the teaching and learning process. The challenging task of the learners is to combine these models with their own models of the phenomenon. This process includes a lot of interactions and is the main cause for misunderstandings in the learning process.

The new idea is to divide the scientific model into different parts of the model according to different science areas. Only all model parts together can explain the phenomenon correctly. If only some (not all) model perspectives are used, the learner is not able to understand the phenomenon in a correct way. With this model it is also possible to discuss the role of the mathematical model perspective in the process of understanding natural phenomena. According to Greca and Moreira (2002) the model of a natural phenomenon is divided into two model perspectives: (1) the physical and (2) the mathematical model perspective. For Greca and Moreira the physical model of a theory is described with linguistic symbols and the mathematical model is described with mathematical symbols; understanding physics in school is achieved if it is possible to predict a physical phenomenon from its physical models. To understand complex physical phenomena (like the mirror image) other perspectives besides the physical and mathematical perspective of modelling are necessary for understanding. The three model perspectives are shown in Figure 2.

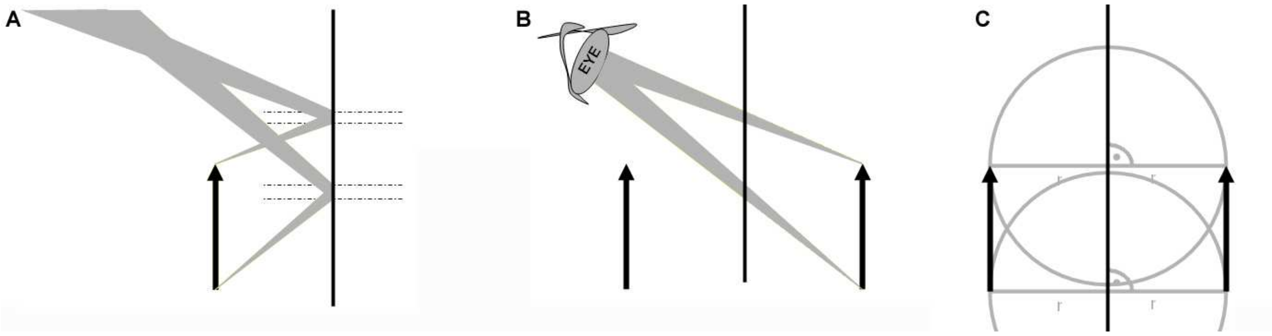


FIGURE 2: (A) physical model (B) “human” model, (C) mathematical model

Understanding mathematical models

The way of thinking in mathematics is totally different to the one in physics. Devlin (1994) describes the core of mathematics in recognizing a pattern. We define abstract objects and look for patterns. At this stage there is no connection to the real world, math is an abstract world. In mathematics abstract definitions and logical consequences of the definition are learned. Mathematicians formulate theorems and find arguments to prove them. Everybody can follow the strict logical rules (if he really wants to) in mathematics. Mathematics is abstract thinking without being linked to the real world.

This, however, does not hold true if mathematics is applied to real situations. This case is described by the modelling circle: (1) mathematical modelling of a physical system, (2) mathematical processing, (3) interpreting the mathematical representation and (4) evaluating the solution by comparing the physical system and the original system.

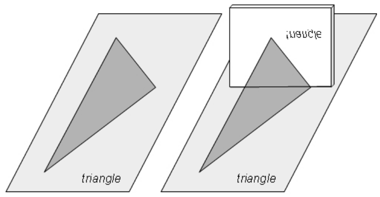
But we can also adapt the heuristic framework for describing and exploring the process of understanding (see Figure 1). Normally abstract problems have to be illustrated by the teacher, so that the learner wants to solve a given problem - the learner should develop a cognitive interest. Depicting a line reflection e.g. is motivated by folding tasks or using a mirror (see Figure 3). An often used mathematical explanation of the mirror symmetry in beginners’ lesson is: “A way to think about it is: if the shape of a figure were to be folded in half over the axis, the two halves would be identical: the two halves are each other’s mirror image.” However, if we think physically – the mirror image comes into being only in the mind of the observer. What does it then mean when we talk about ‘each other’s mirror image'? Mathematical knowledge itself, however, is not gained from the illustration of mathematical contents, but from a mathematical discourse.

Problems occurring when linking mathematical with physical models and the impact of prior knowledge

Normally learners have mathematics from the beginning of their school career and only some years later science teaching starts. Thus the learners have a lot of mathematical previous knowledge, which they can use in the physics lessons. How can we succeed in cross linking previous knowledge in Mathematics with new ways of physical thinking? The model of a light beam is used e.g. in beginners' classes to represent optical paths. The model is a single light ray – a half-line in geometry. In contrast to mathematics, models in physics are essentially needed to gain physical knowledge about reality.

The first step to solve this was to understand how students handle their previous knowledge in Mathematics when they start with lessons in Physics. We carried out an investigation to elaborate this process (Böhm, Pospiech, Körndle, & Narciss, 2010). During the process of evaluation we found two very interesting examples of fundamental problems by using the axial symmetry for modelling the mirror image: (1) understanding of the virtual image and (2) left and right conversion of the mirror image.

At first mathematics schoolbooks were analysed with respect to their definition of symmetry.



A figure is symmetrical if one half is the mirror image of the other half. That is why the axis of symmetry of a figure is called mirror axis.

FIGURE 3: The definition of axial symmetry by using the mirror image

This example does not only contain abstract mathematical content to define axial symmetry. It is not an exact mathematical definition like: "A plane figure is symmetrical if it has at least one identical image which is mapped on itself by e.g. line reflection and rotation.

In using the mirror image to define axial symmetry we forget that the mirror image is not existing – it is 'only' a stimulus on our retina and its interpretation through the brain. The mirror image is not really existing like the other image in line-reflection. But this is not mentioned in math lessons. Thus physics teachers should not be surprised that students have problems to understand the virtual image when axial symmetry for modelling the mirror image is used – without mentioning the eye as a mapping system.

The second problem is, that not in every case we are able to see the mirror image. In Figure 4 two cases are demonstrated: Only cases comparable to Figure 4, picture A, looking diagonally onto the mirror we can see an image. In case Figure 4, picture B, if we looking perpendicular at the axis of the mirror we can not see an image – but mostly exactly this case is used in physics lesson to model the mirror image by using the former knowledge of the learners from the math classes – the axial symmetry.

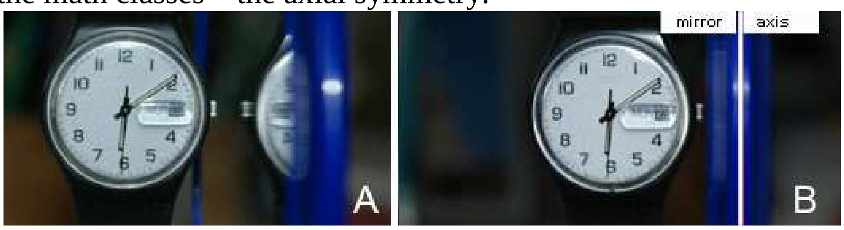


FIGURE 4: The definition of axial symmetry by using the mirror image

It is absolutely strange that for modelling the mirror image the only case of looking on the mirror is taken in which we can not see an image – and this is not told to the learners! When case B in Figure 4 is used pictures like the one in Figure 5 (in physics lessons) can be drawn.

In both drawings the eye does not play any role at all. But in this case, the mirror image does not exist – it only exists in the mathematical case – in line-reflection.

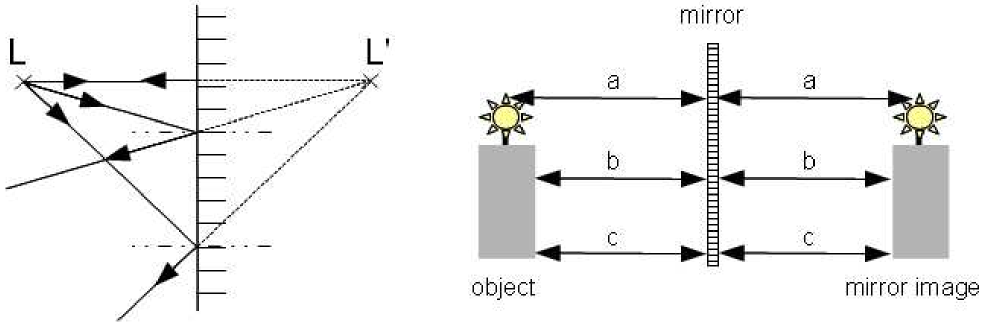


FIGURE 5: Drawings taken from physics text books to explain the mirror image.

If we look perpendicular at the drawing, we see that the left and right sides of the object and the mirror image are changed. But the mirror does not change left and right, the mirror changes front and back! This is easy to accept, if we know that we model the only case when no mirror image can be seen. We have to turn our head and look into the mirror – so the change of left and right becomes a change of front and back.

Conclusions

Every time models are used in physics education. The fact has to be taken in account that not reality itself, only a model of reality is being described. The model should fit the reality very well. We must pay attention to the role the used model has in its original meaning. On the other hand, if we are using examples to illustrate abstract structures in mathematics education we think about the physical understanding of the real phenomenon. In the case of the mirror image mathematical and physical explanations do not go hand in hand. The learner has no chance to construct adequate mental models in each subject.

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Moving from Diagnosis to Intervention in Numeracy – turning mathematical dreams into reality

George Booker
Griffith University
Brisbane, Australia
g.booker@griffith.edu.au

Abstract

When students experiences difficulties in mathematics, they require assistance to overcome the misconceptions or inappropriate ways of thinking they have developed. Such difficulties should be identified early and assistance provided to address the needs revealed by a *Diagnosis* of their ways of acting and thinking. *Intervention* can then focus on providing support within a teaching sequence as soon as a difficulty is noted or anticipated as an essential element of ongoing learning. In this way, students be can helped to develop the conceptual understanding and fluent processes needed to acquire and use mathematics. This paper will provide an overview to the intervention process, from determining underlying causes of any difficulties, leading a student to see inadequacies in their ways of proceeding and thus appreciate a need to change, to implementing means of building appropriate ways of thinking, generalising and applying mathematical ideas. A series of numeracy screening tests (Booker 2011) will be presented.

Introduction

> Children should have a robust sense of number … this includes an understanding of place value, meaning for the basic operations, computational facility and a knowledge of how to apply this to problem solving. A thorough understanding of fractions includes being able to locate them on a number line, represent and compare fractions, decimals and per cents, estimate their size and carry out operations confidently and efficiently.
>
> Final Report of the National Mathematics Advisory Board, 2008 pp 17 & 18

In a society "awash in numbers" and "drenched in data" (Orrill 2001), numeracy must be considered an essential goal of education for all (National Numeracy Review p.xi, 2008). It is no longer enough to simply study mathematics; mathematical knowledge needs to be able to be used in an ever-widening range of activities. Indeed, those who lack an ability to think mathematically will be disadvantaged, unable to participate in high-level work and at the mercy of other peoples' interpretation and manipulation of numbers and data. As Steen (1997) predicted "an innumerate citizen today is as vulnerable as the illiterate peasant of Gutenberg's time"; both numeracy and literacy are critical components for living full lives in the 3rd millennium. An ability to solve problems, communicate the results and methods used to obtain solutions, to interpret and use the results of mathematical processes and making sense are all essential to being numerate. However, many students fail to achieve even minimal standards of numeracy (DETYA, 2000; National Numeracy Review, 2008; OECD, 2004) and even those who do frequently say that they are 'no good at maths', feel inadequate and are unable to use the elementary mathematics that they have 'acquired'.

Diagnostic assessment

Assessment is integral to teaching and learning. While it can be used to grade students or measure them against National or International benchmarks, more importantly, well-focused assessment can reveal how students think and provide guidance to plan ongoing teaching. It also supplies information about how students are dealing with the mathematical tasks they are exposed to and feedback on particular programs or learning activities, whether they are suited to the students and content in question or whether they need to be modified to produce the expected learning. At the same time, assessment can provide information to the student, parents or caregiver and other teachers about the student's mathematical capabilities and potential. It may also highlight different outcomes across different groups of students, across different classes and schools when the results are discussed with other teachers.

While it is important to know *what* students know, of even more importance is *how* they know – is their knowledge simply memorised routines, or is there a deep understanding based on well-understood concepts and fluent, meaningful processes applied in appropriate ways? Hence the need for *Diagnostic assessment* that is designed to reveal not only what a student knows but also how they know, not to see what they do not know but to reveal what they need to know. Most importantly, it may reveal gaps in a student's mathematical knowledge – critical ideas may not have become part of a student's way of thinking or may not have been included in the steps used to build up a topic. For example, there are many programs where the aspects of renaming needed for computation and other number processes are not developed as an extension of place value or to provide a complete understanding of larger numbers.

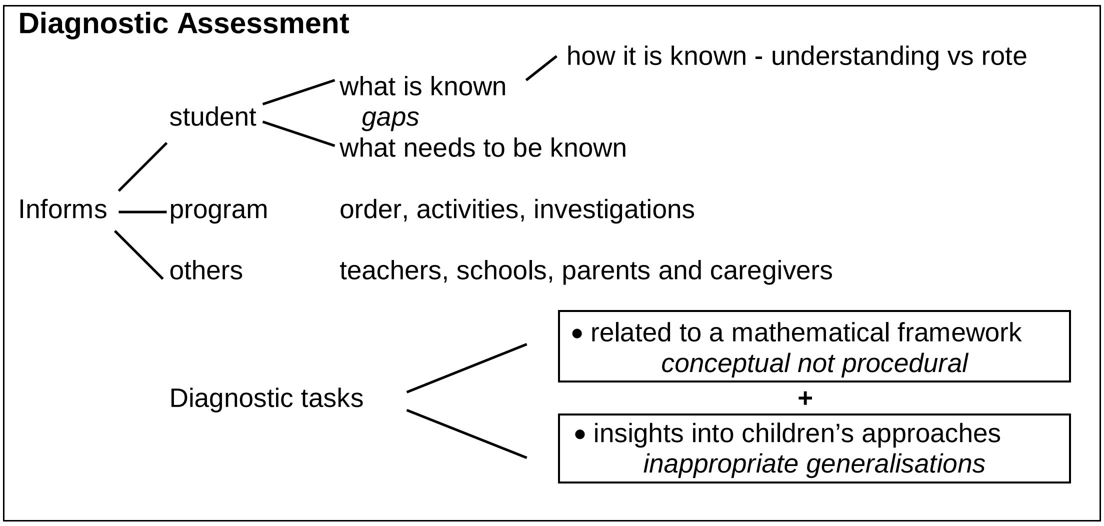


Figure 1: Components of Diagnostic Assessment

In this way, diagnostic assessment highlights the strengths that a leaner brings to a topic and also the weaknesses in their prior knowledge that may cause errors or difficulties.

A diagnostic test is different from other forms of testing. It needs to provide insight into all the steps required to develop facility with a topic and not just measure how well a final outcome or particular points in the development have been attained. However, the sequence of questions posed cannot simply mimic those used in teaching a topic as this may allow a student to refresh their knowledge and mask aspects that have not become central to their thinking. It is often difficulties with steps at the outset of developing facility with a topic that cause deficiencies rather than just an inability to or apply the final process. Misconceptions or errors might also arise from insufficient understanding of mathematical aspects that were developed outside of a particular topic but are essential to the processes. For example, many difficulties with computation are in fact due to underlying aspects of numeration such as zero, place value and renaming.

	8 7 - 4 0 4 0	8 6 + 4 2 1 4 8	7 0 2 - 3 4 8 2 6 4
Difficulties with	*zero*	*place value*	*renaming* & *zero*

Similarly, difficulties with measurement may be due to insufficient understanding of spatial properties or of the significance of zero or decimal fractions in the way measuring instruments are applied.

Many errors are based in confusion with and among the rules that have been acquired or else in insufficient understanding of the numbers being worked with. A further source of difficulties are *inappropriate generalisations*, where something that worked in one situation is taken to another setting where the conditions that permitted it no longer apply. For example, *additive thinking* is often used within multiplicative situations for computation, fractions or measurement.

	4 6 x 5 8 2 4 8	4 6 x 5 8 4 0 8	3 5 x 4 7 4 2 5 2 0 0 6 2 5
Multiplying only	ones by ones, tens by tens		*renaming* difficulties as well

Strengths and weaknesses revealed in diagnostic assessment then need to be stated in terms of the mathematical ideas that underlie them, reasons need to be proposed for why they came about and the underlying causes of the difficulties identified so that appropriate teaching strategies to overcome difficulties can be planned, Close observation is critical to provide insight into the ways of thinking being applied. Mostly it will require a task chosen to elicit the ways in which a student is acting then systematically exploring the possible forms this takes. Consequently, any initial attribution of reasons can only be an assumption, usually based on experience with this type of behaviour. Nonetheless it may not be the actual thinking being evinced and must be treated as only a possibility that needs to be investigated further. This will require further probing to determine what is in fact occurring and then analysis of the likely underlying causes.

assumptions

observations probing

↓
causes

Figure 2: Cycle of Diagnostic assessment

Often several possibilities for an error may need to be considered and the process of probing and observing continued in order to first dismiss one or more before the likely reason or reasons can be determined. In this way, a cycle of observations, assumptions and probing will eventually lead to an understanding of the underlying causes and suggest what is needed to overcome them.

From Diagnosis to Intervention

Diagnostic assessment is critical in planning how to teach or re-teach those essential aspects of mathematics that underpin numeracy. When misconceptions, difficulties and gaps in a student's knowledge have been identified, means to intervene in the learning can be planned and implemented in a manner appropriate to both the learner and the way in which concepts and processes are best established and consolidated. Both what is known and needs to be known must be identified and described in terms of the underlying mathematical ways of thinking rather than rules or procedures that may be followed in a less than meaningful way.

The most effective way of constructing new ways of thinking will mostly begin with the use of materials to draw out the patterns on which the ideas are developed, linking to a language that provides meaning and only moving to the symbolic expressions that express what is happening succinctly when the learner has adopted the way of thinking as his or her own. Simply showing a student what to do using a written procedure is rarely successful in replacing procedures that have led to errors. At best they will try to copy and remember a teacher's approach but it may be that the link to what they do know is not apparent in the purely recorded form. Rather, engaging and different practice activities, often in the form of games in which learners willingly participate, are an essential part of learning to bring a concept to the forefront of a learner's mind and enable a process to become fluent.

In this way, intervention can build from the understandings that are essential for the development of further concepts and processes and provide the links needed to extend the ideas to enable applications to new situations, means of solving a range of problems and make possible extensions to further mathematics. This process can be summarised:

The process of intervention

1. the identification of *understandings and errors* and the description of them in terms of the underlying mathematical concepts and processes
2. uncovering *sources of difficulties* – not only inappropriate thinking but also the degree of understanding of why processes and responses are correct
3. revealing *inadequacies in thinking* to a child in order to build an appreciation of a need for change
4. the implementation of means of *constructing or re-constructing* appropriate ways of thinking
5. *practice* that is focused and motivating to allow a way of thinking to become secure and provide a basis for generalisation to more complex problems and applications or to the development of further mathematics

Figure 3: The process of intervention

Case study

A student was observed to have difficulties with decimal fractions involving tenths:

Write the numbers that are 3 tenths more:

8.4 *11.4* **9.7** *12.7* **6** *9* **0.8** *3.8* **2.9** *5.9*

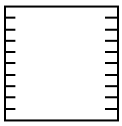
In this case the student has read the question as ‘write the numbers that are 3 more’ either because they have simply focused on the number 3 and used the cue word *more* when reading the question or because *place value* is not secure for decimal fractions. For instance, many students say ‘8 point 4’ for the number 8.4 without any emphasis given to the place values involved. 8 and 4 are simply seen as digits separated by ‘.’ rather than 8 ones 4 tenths which is then read as ‘8 and 4 tenths. In either case, no meaning for the ‘3 tenths’ requested in the question would have been present.

To increase a number by 3 tenths means that the digit in the tenths place is increased by 3 which may also involve renaming 10 tenths as 1 one:

8.4	*8.7*	**9.7**	*10*	**6**	*6.3*	**0.8**	*1.1*	**2.9**	*3.2*

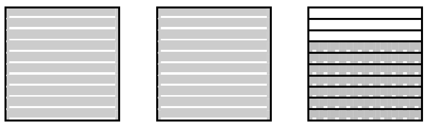
In order to provide intervention on the underlying difficulties the diagnosis has revealed, the fraction concept and place value for decimal fractions need to be built up. This requires teaching to

- revisit the model for fractions using rectangles to link model and names based on ordinal numbers
- use a square with the beginning lines and have the student connect the lines to see there are 10 equal parts - tenths



10 equal parts - tenths

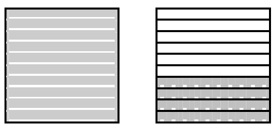
- name fractions with ones and tenths



2 ones and 7 tenths – 2 and 7 tenths

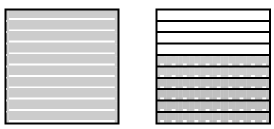
ones	tenths	
2	7	2.7

- shade ones and tenths diagrams to match symbols and read names



1 and 4 tenths 1.4

- shade decimal fractions that are 2 tenths more and name them



1 and 6 tenths 1.6

- write the symbols for the initial fractions and the fractions that are 2 tenths more

2 tenths more than 1.4 is 1.6

Note that the example chosen to (re-)establish the way of thinking is different to those that were completed incorrectly so as to focus on the underlying ideas rather than simply be seen to correct an example that was answered incorrectly. When several examples like these have been examined and the student has developed an understanding of what is needed, then the fractions originally answered can be looked at again to see that the student can not only obtain correct results but is able to see what was done incorrectly when he was first asked these questions.

Conclusion

Building numeracy in all students is a critical aspect of contemporary schooling. Understanding how concepts and processes are constructed and connected provides a basis for overcoming misconceptions and inappropriate ways of thinking that may have developed (Hiebert & Grouws, 2007, p.391). Appropriate intervention programs can then be planned and implemented to build students' competence and confidence with fundamental mathematical ideas. In this way, students will be prepared to engage with further mathematical ideas and be inclined to use their knowledge of mathematics in the many everyday and work contexts where reasoning and sense making will be required.

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PROFESSIONAL LEARNING COMMUNITIES AND TEACHER CHANGE

Karin Brodie

University of the Witwatersrand

Professional learning communities are increasingly seen as a sustainable and generative method of professional development in mathematics education. However links between the actual work of the community and changes in teachers' practices are rarely made. In this paper I examine part of a journey of one teacher in a professional development project that focused on teachers' learning to engage with learner errors in their teaching. I show how her classroom practice and the conversations in the community informed each other, and supported her to change her teaching.

Professional Learning Communities

Professional learning communities are increasingly seen as a sustainable and generative method of professional development in mathematics education (Clark & Borko, 2004; Jaworski, 2008). Professional learning communities support teachers to use their experience, evidence from their classrooms, their own and their colleagues' insights, and knowledge from research to decide what they need to learn and how they can learn it. Teachers monitor their own and their learners' learning in ongoing ways (Boudett, City, & Murnane, 2008), thus engaging in deepening cycles of analysis, reflection and action, interrogating current practice and exploring alternatives. The collective nature of professional learning communities is important - teachers collaborate and learn together about how their learners' needs can influence and improve their practice.

There are strong theoretical arguments for professional learning communities and some evidence that they do produce improved teaching practices and learner achievement (Boaler & Staples, 2008; Katz & Earl, 2010). There has been extensive research on how successful communities work and the difficulties in sustaining them (McLaughlin & Talbert, 2001). Less research has been done on explicitly connecting the actual work of the professional learning community to shifts in teachers practices, or as Kazemi and Hubbard (2008) argue, how teachers' development in both the professional community and the classroom co-evolve. They suggest (2008, p.453):

> One methodological entry point … is for researchers to identify a practice that is the focus of the PD effort, track how teachers reason and work with that practice as it travels to their classrooms, and track how they reason with that practice when they return to PD

Professional learning communities typically investigate artefacts of practice, such as tasks, student work, lesson plans and classroom teaching (Clark & Borko, 2004). However, we cannot assume that what teachers learn in the community, outside of the classroom, travels intact to the classroom and vice versa (Kazemi & Hubbard, 2008). Different aspects of the learning in the community will be salient in different ways for different teachers and links between teachers' experiences in the community and their developing practices need to be established.

The focus of our professional development project was learning to work with learner errors. Our position on errors and the main learning point of the program is that errors are evidence of reasonable and interesting mathematical thinking on the part of learner (Borasi, 1994; Prediger, 2010) and errors do not signal a deficit in teaching or learning, in fact they are a normal part of learning mathematics. At the same time, teachers can learn to work with learner errors in better ways. Many teachers tend to work away from errors: avoiding them through narrowing tasks; pretending to work with them through the use of leading questions; or re-teaching concepts with the assumption that errors mean that learners haven't learned "properly" the first time. We work with the notion that teachers can embrace learner errors in ways that can advance learners' mathematical thinking by understanding the validity in the reasoning behind the errors and the source of the errors as over-generalisations of previously successful ideas (Smith, DiSessa, & Roschelle, 1993).

In this paper I examine part of a journey of one teacher in a professional development project that focused on teachers' learning to engage with learner errors. I show how analyses of the teacher's classroom practice in the professional learning community informed her subsequent classroom practice.

The empirical site and data analysis

The data for this paper comes from a three-year professional development program, located in Johannesburg, South Africa. The project engaged teachers in a number of activities including test and curriculum analyses, interviews with learners, readings and discussions on learner errors in particular topics, lesson planning and lesson reflections. All of the activities focused on building teachers' understandings of and engagement with learner errors. The teachers worked in small grade-level groups of 3-4 teachers, with a group leader who was member of staff or post-graduate student at our university. At particular times in the programme, the small groups presented their ideas to a larger group that was facilitated by one of the project leaders, the author of this paper. In this paper I focus on the learning of one Grade 8 teacher, Andrea[1], through the first two years of the project, by analysing her teaching before the project began and through two cycles of the project.

In each of the two cycles, each small group planned and taught lessons to develop learners' understanding of the relational meaning of the equal sign or learners' use of visuals in solving problems. They had read and discussed papers on these topics and analysed tests results and the curriculum in these areas. One teacher from each group volunteered to teach the lessons and the teaching was videotaped. The small groups discussed the videotapes and then the teacher presented two episodes to the larger group: one where the small group thought the teacher had dealt well with learner errors and one where the small group thought the teacher had not dealt so well with learner errors. The presentation included a brief background to the episodes to contextualise them within the set of lessons and a justification for why the group had chosen each episode. Each group gave a 10-minute presentation and about 50 minutes were given for discussion where other groups could comment, question, challenge and give feedback.

The main data for this paper are the presentations on the two concepts and the teacher's lessons – two lessons before the project started, two on the equal sign and one on problem solving. Secondary data are the lesson plans, interviews with the teachers, and the teacher's written reflections after the presentations. Analyses of the lessons were done with the Mathematical Quality of Instruction (MQI Plus) instrument (Learning Mathematics for Teaching Project, 2010). Coding is done according to a set of categories in eight-minute episodes, on a scale of 1 to 3. A second, more qualitative analysis of the lessons listed all the learner errors made in the lessons and the teacher's responses to these. These analyses showed changes in the teaching and also confirmed that the episodes chosen for presentation to the larger group did in fact illuminate important issues relating to the teacher's engagement with learner errors that came up throughout the lessons. Detailed summaries of the presentations were analysed by looking for how the teachers accounted for their actions to each other and what counted for them as important to speak about at each point. This meant looking for who said what and when, presences and absences in what they said, and tracing comments back to previous comments and forward to subsequent ones.

[1] All names in the paper are pseudonyms.

Changes in teaching practice

The two MQI Plus categories that are most relevant to teacher learning in our project are: working with students and mathematics and student participation in meaning making and reasoning. Each main category has subcategories, the first has remediation of student errors and difficulties; and responding to student mathematical productions, while the second has students provide explanations; student mathematical questioning and reasoning; and enacted task cognitive demand. Table 1 below gives the number of episodes in each of the three sets of lessons that were coded 1, 2 or 3 for each subcategory.

	Prior to project 11 episodes coded			Equal sign 12 episodes coded			Problem solving 7 episodes coded		
	1	2	3	1	2	3	1	2	3
remediation of student errors and difficulties	11			12			5	2	
responding to student mathematical productions	8	3		11	1		3	2	2
students provide explanations	11			12			4	3	
student mathematical questioning	11			12			6		1
enacted task cognitive demand	9	2		12			6		1

Table 1: Shifts in Andrea's practice

Table 1 shows that Andrea's practice remained unchanged for the most part in the first set of lessons that she taught while in the project (equal sign) but shifted significantly in the second set of lessons (problem solving). The qualitative analysis of the shift shows that the teacher's responses to learner errors in the first two sets of lessons were predominantly leading questions, while in the third she moved towards asking probing questions.

Conversations in the professional community

The shifts in Andrea's practice can be linked to conversations about her practice in the two communities, the small group and the larger group. Her group chose the following episode to present on the equal sign. There were two expressions written on the board:

$$(+1) + (+1) + (+1) + (-1) + (-1) + (-1) = 0$$

$$(+1) + (+1) + (+1) + (+1) + (+1) + (+1) + (-1) + (-1) + (-1) + (-1) + (-1) + (-1) = 0$$

Andrea had asked the learners whether the left hand sides of both equations were equal to each other. The following interaction with one learner, Carol, occurred:

Andrea: Put up your hands if you say yes…[a number of learners put up their hands] put up your hands if you say no…[only Carol puts up her hand] Carol, very brave my darling well done, why do you say no

Carol: Mam, I said no because um, the numbers are different like there three negative ones there and there are six negative ones there, and there there are three negative ones there, I mean three positive ones, and there there are six positive ones, mam

Andrea:	Well, except that here I've got six negative ones and six positive ones and they equal to zero that's one number
Carol:	Yes mam
Andrea:	So does it matter how many numbers we've got on each side
Carol:	No
Andrea:	What matters
Carol:	Um, if they equal to zero
Andrea:	If they equal to zero, so is this top line gonna equal to zero
Carol:	Yes mam [nods head]
Andrea:	In this bottom line here, equal to zero
Carol:	Yes mam [nods head]
Andrea:	So we can say they are equal to each other

Initially the small group had thought that Andrea had dealt well with Carol's error but after some discussion with the group leader they came to see that she had led Carol to the answer without asking her to explain her thinking and there was no evidence that Carol had in fact understood the difference between what the expressions looked like and their values. The group named what Andrea had done as using "leading questions".

Two key questions structured the conversation in the larger group about this episode. The first question was asked about all the errors that teachers presented – what might the learner be thinking in order to make the error. The second question related specifically to Andrea's teaching - what is the role of leading questions in engaging with errors. After some discussion the teachers reached consensus about Carol's thinking, expressed here in Renee's words:

> I would imagine what was going through her mind was that she's thinking that equals means the same and that in some contexts that is correct but we we've got to be very careful that they've got to understand the difference between quantity and value, when we say equal we talking about the value of something.

Renee pinpointed both the validity in Carol's thinking and where her error lay. She showed that the error is sensible given the meaning of "the same" in some contexts but that a full understanding of equivalence required a different meaning – in relation to the value of the expressions rather than the quantity of terms. Andrea took the discussion further saying:

> The misconceptions are so deeply ingrained in them that actually they take them for granted and they don't think about, and it doesn't even cross their minds to change their thinking. I mean some of the stuff that I said, its pretty deeply ingrained in me as well. If I listen to some of the language that I use, I use things like the answer and all of that kind of stuff

In relation to the leading questions, Andrea argued that:

> I had not allowed her to let me know that she had fully grasped the concept, I just kind of barked out the orders … so I don't actually know that she can actually do these questions by herself now

So Andrea articulated a direct link between her practice and her engagement with learner errors, indicating her group's understanding of the teacher's role in working with learner errors. The notion of "leading questions" became a depiction of practice (Kazemi & Hubbard, 2008) that was spoken about in a number of ways throughout the program. In this session a number of teachers contested the idea. Tawana argued:

> I don't see why her questions are leading towards a solution. She was using a model previously and they've now left the model behind and are looking at the numbers and equating things, so if someone had a misconception, it was important for them to refer back to something that they'd agreed upon

Tawana's argument was that Andrea had explained the ideas previously, using conceptual resources, the model, and that it was appropriate for her to expect that this had been understood and she could therefore use leading questions to remind learners of what they had learned previously. Even though her leading questions were supported by other teachers, Andrea stood her ground and argued that she could act differently by asking more questions such as "why do you think that", "how do you come to that", take me through your thinking" to support learners to articulate their ideas. Such questions are referred to in the literature as probing questions or press questions (Kazemi & Stipek, 2001). Many of the teachers could not distinguish the difference between leading and probing questions at this point. Subsequent discussions articulated more clearly the differences between leading and probing questions. Andrea's word "ingrained" also became a shared term in the community. As the facilitator, I was able to pick up on the link that Andrea made, talking about teachers' "ingrained" practices such as leading questions, and how those are as difficult to shift as learners' misconceptions.

In Andrea's reflection she wrote that she had learned that leading questions were not useful in eliciting and engaging learner errors and that she was going to try to use them less in her teaching. Table 1 shows that she was more successful in engaging with learners' errors and reasoning in the next set of lessons. Her group chose one of a number of possible episodes, which shows some of her success. An analogue clock drawn with the hands at three o'clock was on the board and Andrea asked the question: how many ninety-degree angles will there be in twelve hours? A number of learners made conjectures. In the chosen episode, Rabia conjectured that "every hour can have a ninety degree angle" and when Andrea asked for an example she said "like um two o'clock and eleven o'clock". Millaine challenged Rabia's claim saying, "at two o'clock and eleven o'clock it won't necessarily be a right angle" which led Rabia to change her statement, saying that she meant five to two (i.e. the hands would be on eleven and two). Leanne listened to this interchange and then offered the possibility that "every right angle has two hours in between" and when Andrea asked her to elaborate she used the example of three o'clock on the board and Rabia's example to explain.

Andrea's responses were completely different from those in her interaction with Carol in the first episode. She asked for examples and elaborations and as she said in her presentation:

> I didn't refuse any of their answers, I put them on the board and I kind of left them there and I didn't say no you're wrong or wow you're amazing you're right, I just kind of put them there and we moved on, and talked about it a little bit and then put more up, and then some kids would say but these are wrong because of the following reasons, so the learners kind of deepened their understanding through their own questioning

This was not only the case in this one episode but in many others, and her teaching supported the learners to make key breakthroughs at later points – first that both hands move so that, for example, at five to two the hour hand will not be exactly on two, and second to get close to an answer, some arguing for 22 and some arguing for 24 and then deciding how to work out the answer. Andrea did not only work as she described above, she also made inputs where necessary, and asked some "leading questions" where appropriate, for example, she used leading questions to make the point that a ninety degree angle can be in a number of orientations. Her more flexible use of different kinds of questions supported me as the facilitator to make links between the different kinds of questions, to further support her learning and the learning of the other teachers.

Conclusions

The observed quantitative and qualitative shifts in Andrea's teaching can be linked to the discussion of her teaching in the small and larger group. The notion of "leading" questions came up in the smaller group and was further discussed in the larger group. As a "depiction of practice" these structured Andrea's work and the work of some of the other teachers who began to distinguish between these kinds of questions, try them out in their classrooms and bring them back to the community for discussion.

The project's methodological entry point (Kazemi & Hubbard, 2008), learner errors, supported this shift in Andrea's teaching. The discussions of her teaching of the equal sign pinpointed the validity of Carol's thinking that Andrea missed because of her use of leading questions. As she shifted her use of leading questions in the second set of teaching she also shifted her view of learner errors to conjectures (Borasi, 1994) and was happy to leave errors on the board as inputs to the discussion, to be taken up by other learners. Andrea's reflections on her own language and practices as "ingrained" also provided a point of focus for discussions about how to link new teaching practices to the teachers' developing understanding of errors.

So the two contexts, the community and the classroom came together and interacted with each other to produce learning in both contexts. Andrea was able to talk about her practice with others, build on or challenge their thinking, accept challenges from them and take forward what she learned into her classroom. At the same time, her classroom practice created possibilities for further discussion and for other teachers' learning.

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Numbers: a dream or reality?
A return to objects in number learning

Bruce J. L. Brown PhD
Department of Education
Rhodes University
Grahamstown
B.Brown@ru.ac.za

Abstract
The complexity of mathematical concepts and practice may easily lead to teaching practice that results in mathematical objects being seen as an abstract dream. This paper explores ways that mathematical objects may be seen as real, objective entities, while still acknowledging this complexity. It develops a systemic view of mathematical objects as mental objects constituted in the intersection of three major systems of the child's experience: the physical; social and technical (mathematical) systems. This development is fleshed out in examples relating to the teaching and learning of both whole numbers and rational numbers.

Introduction
A child's learning of numbers at school involves the mastering of a large number of technical details. Details that include different mathematical operations, different representations and models of numbers and operations, and different relationships between numbers, operations and representations. Together these form a complex technical system of facts and skills. In learning this system, children often get lost among the myriad of details, and come to see numbers as an abstract dream. Particularly when most of these details are expressed in terms of rules and symbols.
For this reason, is deemed important to teach in ways that develop conceptual understanding (Hiebert, 1986; Kilpatrick, Swafford and Findell, 2001), to lend meaning to what is learned and link these technical details into a web of meaningful relations. According to Kilpatrick et. al. (2001) conceptual understanding

> refers to an integrated and functional grasp of mathematical ideas. Students with conceptual understanding know more than isolated facts and methods. They understand why a mathematical idea is important and the kinds of contexts in which is it useful. They have organized their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know. (Page 118)

Yet, even with this, many children do not become proficient with this complex system and numbers remain a dream.
The Oxford English Dictionary (Oxford Dictionaries, 2008), sees a concept is an "idea of a class of objects". The importance of the mathematical idea is clearly evident in the above quote, but no mention is made of the class of objects to which these ideas refer. Without a referent, children may experience these ideas as meaningless, or imaginary. In this case, conceptual understanding may itself become meaningless, abstract and confusing. It is the reference object that provides a point of focus for the concept and grounds it in reality. This paper investigates the possibility of reinvigorating the status of mathematical objects, as objective entities that form the referents of mathematical concepts. Particularly in the case of school mathematics, when the child is still developing the capacity for mathematical abstraction at the level needed to enable effective engagement with the formal and abstract systems of higher mathematics . It proposes a framework for understanding mathematical objects as mental, or psychological objects that, for functional and structural reasons may transcend the boundaries of individual subjectivity and take on the status of objective entities.

This understanding will exemplified through examples from the learning of whole numbers and rational numbers.
This object framework arose as part of a research project into rational number learning and the examples from rational number learning will be drawn from the project. But the focus of this paper is the object framework — detailed research results will be reported elsewhere.

Informing Frameworks

Psychological theory of concepts: Prototype vs Definition

Gabora, Rosch and Aerts (2008) review the development of psychological theories of concepts since 1950. The classical view, is that a concept denotes a category and is mentally represented by specifying common attributes of the members of the category. Research into the psychological structure of concepts demonstrated that whether an object is seen as a member of a category is not generally seen in all or nothing terms, but rather in terms of graded degrees of membership and that these degrees of membership were strongly context dependent. This gave rise to the view that concepts are represented as rich, experiential prototypes, that are formed first by identifying and internalizing ‘basic level objects’ and then expanded through relations to more specific (subordinate) and more general (superordinate) objects. Conceptual prototypes are highly flexible and vary according to context and the manner in which the person wishes to interact with the context.

Cognitive Science: Distributed vs discrete models of thinking

Two major models of thinking and currently prevalent in cognitive science. The first (Newell, 1990) sees thinking systems as physical symbol systems and thinking as the manipulation of symbols. According to this view, concepts comprise of discrete symbols, that are linked to form complex networks. Thinking involves operating on these discrete symbols according to well defined rules. The power of this model lies in the possibility of identifying a concept by means of a single symbol and the control and insight this provides into manipulations and operations on the symbol/concept. The second approach (Rumelhart McClelland and the PDP Research Group, 1986) models thinking systems as neural networks and thinking as the spreading of patterns of activation in neural networks. A concept would thus relate to a pattern of activation that is distributed throughout the network. The mental effects of this concept would occur as a result of links between nodes activated in this pattern and other nodes in the network, whose activation is either stimulated or inhibited by these links. This model enables flexible and robust recognition of concepts, and an enhanced ability to coordinate thinking with detailed structural features of the environment and the interaction between person and environment.

Conceptual grounding: Everyday vs Scientific Concepts

Vygotsky (1987) identified two different types of concepts: everyday and scientific concepts. Everyday concepts arise spontaneously in everyday experience. They are closely linked to this specifically experience and attain personal meaning through this grounding. On the other hand, scientific concepts are general, abstract and organized to form a conceptual system. These concepts do not arise spontaneously, but are learned in response to some form of implicit, or explicit teaching. The generality and organization of scientific concepts gives them power and flexibility. But without appropriate experience to relate them to meaningful everyday concepts, this generality would be empty of meaning.
Building on Vygotsky’s work, three different perspectives on everyday experience were developed by Leontiev (1978). Two of which yield valuable insight for this work. **Activities** are active social experiences that are given significance because they enables the group (or individual) to satisfy some motivating need or desire. **Actions** involve the controlled

performance of an instrumental task in order to attain some goal. Attaining the goal requires one to understand the structure of both the situation and the task.
A number of recent issues in cognitive science relate to the interaction between everyday and scientific knowledge. For instance, Barsalou (2008) argues that semantic thinking is not merely amodal symbol processing, but is instead grounded in experience — occurring through the re-enactment of perceptual experience. Situated cognition (Clarke, 1997) also makes a strong case for the importance of the environment and one's interaction with the environment in structuring our thinking and understanding.

What constitutes a mental, mathematical object?
This paper draws on these different perspectives and formulates a mental object as an system of interrelated elements that incorporate the active experience of the individual, in both physical and social spheres. The system is given an characteristic identity through a specific way of viewing it as a whole. Examples of these elements will be given for two different number objects in a child's developing understanding of numbers: whole, or counting numbers; and rational numbers. To simplify our reference, we will abuse our notation and refer to rational numbers as 'fractions'.

1. Grounding instances
A grounding instance is an element of the person's experience that arises when engaging in personally significant activities, in the sense of Leontiev (1978). Similar instantiating activities give rise to the basic object of a prototype concept, which is grounded in the experiential context provided by the activities. A grounding instance emerges through repeated and varied participation in a particular activity. And an object will often emerge from multiple grounding instances in a number of different activities.
A number of different grounding instances can be identified for counting numbers:
- When a young child engages in the activity of counting in order to master the basic number sequence. Here the significance arises from the act of achieving competent social participation.
- When a child wishes to precisely describe the size of a collection of discrete objects, or the number of repeats of an action or event, and generates a number to do this by using 1-1 correspondence between counting sequence and object, action or event.
- When a child uses counting and the resulting cardinal number, to compare the sizes of two collections, or two sequences of events or actions.
- When a child accesses the passing of time, or forms and describes rhythms using counting and cardinal numbers or number patterns.

In a similar fashion, rational numbers arise from a number of grounding instances:
- Fair sharing, where a child wishes to share something (that can be broken up to share) in a fair and equal manner between a number of people. Fractions arise from a consideration of how much of the full amount each person received, or how much of a whole object each person received.
- Allocation, where a child wishes to subdivide and allocate something to a number of people, based on a reasonable, but unequal allocation principle. For example, when sharing a chocolate bar between three boys who were first, second and third in a race.
- Packing or filling involves the packing of a number of objects into containers which each have the same capacity. Here fractions arise from comparisons between the total number of objects, the number in a container and the number of containers packed.
- Exchanging occurs commonly in shopping, or swopping. We come into contact with fractions when we compare the quantities of the two things being exchanged.
- Converting or constructing involves starting with some raw material and then using it

to make something different. Fractions again are useful for comparing the quantities of input and output materials.

- Measuring may relate to selecting a desired amount of material, or to determining the amount of material present. Fractions may arise through comparing the total, the unit and the measurement, but also through the choice of fitting subdivisions of the unit and the effect of different subdivisions on the final value for the measurement.

2. Structural experience in the interaction

This relates to how we interact with the object in each instantiation, thus developing a more detailed knowledge of the object and our interactions with it. As we repeatedly engage in the grounding activities, certain regularities in the way the object emerges and in the way we interact with it become evident. Our knowledge of these regularities may be explicit, or we may deliberately focus on them (often through the mediation of others) and so develop an explicit understanding of this structure. Regularities which can be identified as units of structure or interaction, occurring across multiple grounding instances, become incorporated as basic structural elements of the object. Such unitary elements include:

a. Invariant elements in our interaction with the object, that occur in a number of instantiations. These provide different perspectives of the object in different contexts.

b. Relational elements that interrelate the different invariant elements and perspectives to form a coherent, integrated structure.

Some examples of structural elements of counting numbers are:

- The counting sequence itself.
- Setting up a 1-1 correspondence between numbers and objects, or actions..
- Using the last number in a count as a cardinal number to describe the ‘size’ of what was counted (seen in counting responses such as: “One, two, three. Three balls.”)
- The invariance of addition and subtraction bonds when combining, separating and comparing collections, and when calculating results both mentally and in writing.

Some structural elements that may be identified for fractions are:

- Equal subdivision of a whole into a number of equal parts.
- Forming composite groups containing a number of discrete items.
- Constructing or identifying units. Units can be either wholes (1 pie), parts (½ a pie) or composite (a 2-pie group, one ¾ pie).
- Linking chosen units of two quantities and comparing linked units, or quantities. For example. If 7 apples are needed to bake 2 pies and we have 21 apples, how many pies can we make? Forming linked groups of 7 apples and 2 pies, we get:

$$\begin{array}{rcr} 7 & \longrightarrow & 2 \\ 7 & \longrightarrow & 2 \\ \underline{7} & \longrightarrow & \underline{2} \\ 21 & \longrightarrow & 6 \end{array}$$

So we can make 6 pies. The comparison between the linked units results in the ratios 7:2, or 21:6; and fractions 7/2, 2/7, 21/6 and 6/21.

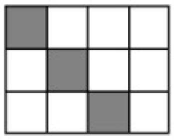
3. Presentational and representational tools

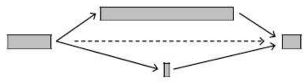
These are structured tools, signs and symbols that we use to physically and mentally present the object to ourselves and to represent the object for physical and mental analysis and manipulation. They include representational drawings, schematic diagrams, graphical models, mathematical symbols, and language, words and text. The majority of these representational tools are not developed by the individual, but are pre-existent in the community and are socially presented (in physical or verbal form) and aligned with the given practice. In the process of this social mediation, the person internalizes the tool and so constructs

corresponding mental symbols. Whether these symbols are seen as discrete and fundamental entities that can be simply ‘linked in’ by the individual, (as in the physical symbol system model) or as more complex constructions (for the distributed model), organized systems of symbols are important to both structure and enable our reflective capacity.
Some common representational tools for counting numbers are:

- Numerical representations using the base 10 place value system.
- The mathematical symbols for the four basic operations.
- The number line.
- Drawn collections of objects (or schematic dots), grouped by line borders.
- The standard symbolic formats for vertical performance of the four basic operations.

Representational tools for fractions include diagrams such as:





Descriptions such as: “Stretch by 3, shrink by 7”; and “Three out of 12”
and symbols such as: 3/7, 4:9 and 3.5

4. Constitutive / characteristic perspective

This element is the stable perception of an object which may be reliably distinguished within each instantiating activity; conforms to the identified structural elements; and is fittingly presented and represented by the learned cognitive tools. This element serves to objectify the concept by constitute it as an object in the person’s experience. Note that this is not a full description of the properties of the object, or a definition. Rather, it is a way of looking at things that allows the object to come to the fore as a coherent, discernable, entity. This element provides a strong, unifying focus, that enables the person to transcend the view of the interrelated elements as merely a conceptual system, and instead see them as a conception of a definite, identifiable object. In the case of numbers, a relational object, such as a father.

Because of its unifying and constituting function, there is only a single perspective for each object. For counting numbers, a strong candidate for this perspective would be:

- A number as a completed count.

For rational numbers, this perspective is no longer sufficient and a better candidate would be:

- A rational number as a rational comparison of two quantities.

Both of these perspectives are evident in the examples given above.

Mental objects and objectivity

The elements discussed above, combine to give weight to a person’s subjective experience that such a mental object has an objective status. This final section tentatively discusses possible relationships between this subjective status as a psychological object and the objective experience of the person.

Experience of the impersonal other, and objectivity

Here we take the impersonal other to refer to both the physical world, and also to interactions with other people where interpersonal relationships do not co-constitute the interaction. For example, a predominantly instrumental interchange, such as renewing a motor vehicle license at the Traffic Department, will be considered as experience of the impersonal other. An important aspect of our experience of the impersonal other in grounded contexts, is that our subjective experience is irrelevant to the response of the other — we are only able to influence the response through our instrumental actions in the interchange. This separation between our self and the other, warrants the experience of objectivity in these interactions. For it is more fitting with our experience to respond to this entity as if it were separate and objective, than as if it were connected and subjective.

For example, consider the activity of sharing a chocolate bar fairly between two people. Equivalent shares will only occur if the measuring and cutting are precisely done. The precision required to form equal halves thus becomes an important structural consideration in a child's developing understanding of halves and general fractions. And this precision is necessary for a fair sharing, irrespective of the child's subjective experience.

Social Presentations and Objectivity

It is important to note that experiences such as the above will often be guided and mediated by relating to the personal other in community — where interpersonal relationships do co-constitute the interaction. Social objectivity may be seen as arising through relating to the personal other individually, in small groups or in larger communities. These interactions mediate the presentations and representations that we develop for the object, our structuring of the object and our grounding experience of the object. In this way, conceptual metaphors, such as those described by Nunez (2006), arise. These are socially presented and socially shared perspectives on grounding experience, that bring out or preserve a certain structure. Objectivity is warranted through the achievement of a socially consensual perspective.

The two forms of objectivity balance each other. As demonstrated by Nunez, different communities may hold different perspectives on a single concept, resulting from incompatible metaphors that preserve different structural aspects of the object. A perspective that incorporates both aspects of the object will then unify these incompatible metaphors.

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CORRELATED SCIENCE AND MATHEMATICS: A NEW MODEL OF PROFESSIONAL DEVELOPMENT FOR TEACHERS

Sandra T. Browning, Ph. D.
Assistant Professor, Curriculum and Instruction
University of Houston-Clear Lake
Houston, TX USA
browning@uhcl.edu

Abstract
This article describes a new professional development (PD) model of linking science and mathematics instruction called *Correlated Science and Mathematics* (*CSM*), which enables teachers to integrate science and mathematics curriculum. Integration usually implies that science teachers use mathematics as a tool or mathematics teachers use a science as an application of a mathematics concept. Although both the science and mathematics standards recommend integration, unless there are effective PD models that will facilitate it, broad-scale integration is not likely to occur. The *CSM* model of PD links science and mathematics more thoroughly and uniquely than the traditional integration model. Each discipline is taught with seven fundamental goals: (a) teaching for conceptual understanding, (b) using each discipline's proper language, (c) using standards-based learning objectives, (d) identifying the natural links between the disciplines, (e) identifying language that is confusing to students, (f) identifying the parallel ideas between the disciplines when possible, and (g) using a 5E (Biological Science Curriculum Study [BSCS], 1989) inquiry format for science and mathematics when appropriate. Use of the *CSM* PD model resulted in grades 5-8 science and mathematics teachers planning and teaching integrated lessons and using the proper language of each discipline.

Introduction
Research has successfully demonstrated that teachers make a significant difference in how well students learn (Hiebert & Grouws, 2007; Loucks-Horsley, Stiles, Mundry, Love, & Hewson, 2010). Continued professional development (PD) provides teachers with an avenue to better understand the content of their discipline, as well as improve their pedagogical strategies that will enhance the students' conceptual understanding of the content. The literature on the importance of PD in improving the quality of teaching is substantial (Darling-Hammond, 1998; Office of Educational Research and Improvement, 1998; Porter, Grant, Desimone, Yoon, & Birman, 2000; Wenglinsky, 2000). Reform-based PD providing opportunities for active learning, attending as a group, study groups, mentoring relationships, and teacher networks have a positive impact on teaching practice (Cohen & Hill, 1998; Desimone, Porter, Garet, Suk Yoon, & Birman, 2002; Sowder, 2007).

Connecting Science and Mathematics
Both the National Council of Teachers of Mathematics (NCTM) and the National Academy of Sciences (NAS) advocate linking the disciplines of science and mathematics. NCTM (2000) states, "The process and content of science can inspire an approach to solving problems that applies to the study of mathematics" (p. 66). Likewise, NAS (1996) acknowledges the indispensable role that mathematics plays in scientific inquiry when they state that "mathematics is essential to asking and answering questions about the natural world" in scientific inquiry (p. 148). The integration of science and mathematics is one way to improve teacher achievement in

each discipline. Basista, Tomlin, Pennington, and Pugh (2001) report significant gains in the participating teachers' understanding of content and confidence to implement integrated science and mathematics. Similarly, Basista and Matthews (2002) report increased understanding of content and confidence.   Additionally, they reported increased pedagogical knowledge of teachers, increased administrator awareness of the science and mathematics standards, and increased administrative support for teachers.

The *Correlated Science and Mathematics* Model

Integrating science and mathematics traditionally means linking the two disciplines in some manner (Davidson, Miller & Metheny, 1995). Science integrates mathematics by using mathematics either as a tool to work science problems (e.g., solving genetics or rate problems) or to actually teach a science concept (e.g., using ratios in the equation to teach photosynthesis). Similarly, mathematics integrates science by using science applications to explain or practice mathematics concepts or to employ the science to reinforce students' interest in mathematics or to enable the students to recognize the broad utility of mathematics. The need to infuse mathematics and science more completely was recognized by West and Tooke (2001) and termed *Correlated Science and Mathematics* (*CSM*). The *CSM* PD model was developed in 2006 and has been continually revised and refined. The *CSM* model is unique in that it integrates science and mathematics in a more comprehensive manner than other integration models. Each discipline is taught with seven fundamental goals: (a) teaching for conceptual understanding, (b) using each discipline's proper language, (c) using standards-based learning objectives, (d) identifying the natural links between the disciplines, (e) identifying language that is confusing to students, (f) identifying the parallel ideas between the disciplines when possible, and (g) using 5E inquiry format in science and mathematics when appropriate (see Figure). The *CSM* approach to PD is "centered in the critical activities of the profession-that is, in and about the practices of teaching and learning" (Ball & Cohen, 1999, p. 13).
A new continuum is proposed that spans from pure mathematics to pure science with the midpoint now representing a correlated lesson including each of the seven goals of *CSM*.

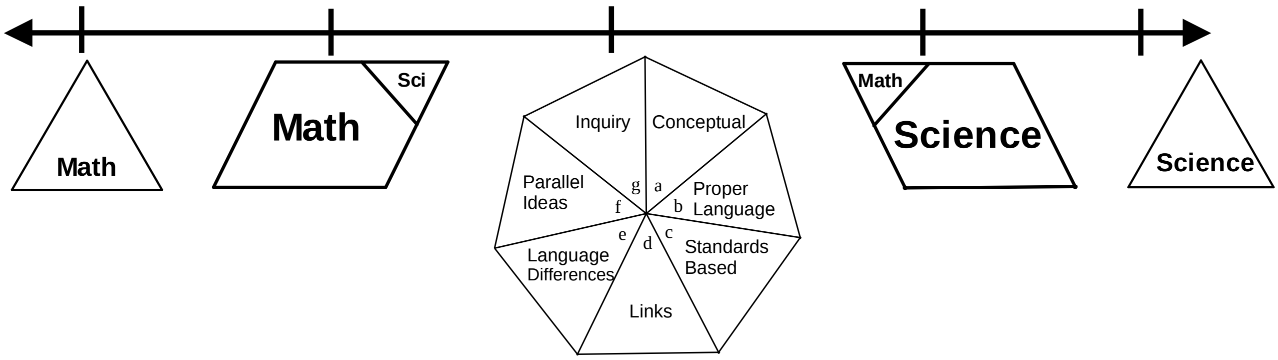

Figure. CSM Continuum of mathematics and science correlation.

Components of the *CSM* PD Model

The seven components of the *CSM* PD model are described as follows. (a) Conceptual understanding: Developing a deep understanding of a one's discipline is a critical attribute for teaching for conceptual understanding (Darling-Hammond, 1998; Devlin, 2007; National Research Council, 2001). Conceptual understanding is defined as the ability to apply concepts to new contexts or connect new concepts to existing information or to use general principles to explain and justify (Malloy, Steinthorsdottir, & Ellis, 2004; Wiske, 1997).; (b) Proper language: Each discipline has its own language or vocabulary (Jacobs, 1989). Content expertise, as well as content pedagogy expertise, is required of the instructional team to achieve the *CSM* goal of proper use of each discipline's language.; (c) Standards: National and state standards are purposefully considered as the *CSM* lessons are planned and the science and mathematics learning objectives are identified. Also, many of the national PD standards are met with the *CSM* model. As stated by the National Staff Development Council (NSDC), "It is essential that staff development leaders and providers select learning strategies based on the intended outcomes and their diagnosis of participants' prior knowledge and experience" (n. p.).; (d) Linkages between disciplines: Science uses some mathematical concepts and skills as tools, and mathematics uses science to explain mathematics concepts.; (e) Confusing language: In the *CSM* model, language that may be confusing to students is identified and clarified. An example of confusing language is the difference between the meaning of distance in mathematics and science. In mathematics, distance is the linear length from one point to another, whereas in science the same concept is called displacement; (f) Parallel skills, ideas, processes or concepts: Concepts, etc. that are similar because they behave similarly in both science and mathematics or share many characteristics.; (g) 5E: The *CSM* science lessons developed generally follow the 5E model for inquiry. The *CSM* instruction is team-taught by a science expert and a mathematics expert, both of whom are well versed in both content and content pedagogy. The instructors design lessons purposefully to meet the defined goals for the PD participant. *CSM* is implemented with middle school science and mathematics teacher teams and their principals. Each project consists of 70-hour summer institutes and 30 hours of academic year (AY) trainings for the teachers. The principals received 12 hours of summer training and an optional 30 hours of AY training. According to the NSDC (2001), "An extended summer institute with follow-up sessions throughout the school year will deepen teachers' content knowledge and is likely to have the desired effect" (n. p.). To follow the *CSM* model, the first five goals must be addressed whether the lesson or project is primarily focused on science or primarily focused on mathematics. The last two goals of identifying parallel ideas and using inquiry are met when appropriate since not every lesson contains parallel ideas between science and mathematics or is suitable for inquiry. For example, a skill in science, such as using a piece of equipment, is more effectively taught using a direct instruction model rather than inquiry and does not share parallel ideas with mathematics.

Research Questions and Methods

Several research questions concerning the *CSM* projects arose as the *CSM* projects were developed and revised over time. This paper contains data from the 2009-2010 project. Data concerning improved student achievement is under analysis but not available at this point in time. However data concerning improved teacher content knowledge and integration of lessons is available and will be addressed in this paper. Research questions: (1) would CSM training increase teachers' content knowledge in physics and mathematical reasoning, (2) would *CSM* training enable science and mathematics teachers to plan integrated science and mathematics lessons, (3) would *CSM* training enable science and mathematics teachers to implement integrated science and mathematics lessons?

Teachers' content knowledge was accessed with pre and post-tests in physics and mathematical reasoning at the beginning and end of the two-week summer institute. Teachers' increase use of integrated lessons in their classrooms was assessed using level four of Gusky's (2002) critical levels of professional development evaluation. Three data sources concerned with Guskey's level four, a change in professional practices, were utilized: (1) AY classroom observations in conjunction with teacher and principal interviews, (2) observations of sample integrated lessons taught by the participants during the AY Saturday sessions, and (3) examples of integrated lessons reported by teachers during interviews and during AY sessions. Twenty teachers (ten mathematics, ten science) participated in this study. The teachers worked in teams to create lessons that included science and mathematics concepts. A minimum of two classroom observations of each participant were conducted during the fall and spring of the AY following the summer institute. Interviews of both the teacher and their principal were conducted after each set of classroom observations. During each AY session, teachers taught lessons to their fellow participants and used planning time to create lessons.

Data Analysis

Teacher Pre and Post-tests: Participant teachers were administered pre and post-tests in physics and mathematical reasoning at the beginning and end of the two-week summer institute. A related-samples t-test was conducted for each content test to determine whether teacher content knowledge of physics and mathematics significantly improved as a result of the summer institute. Results indicate that participant teachers did significantly improve their knowledge of both physics and mathematical reasoning. The mean score on the physics post-test (M = 62.29, SD = 12.970) was significantly higher than the mean score on the physics pre-test (M = 45.76, SD = 11.377, $t(16)$ = 4.992, p = .000). The effect size was large (d = 1.211), and 61% of the variance was accounted for by the treatment (r^2 = 0.609). The 95% confidence interval for the mean difference between pre and post-test physics scores was 9.51 to 23.55. The mean score on the mathematical reasoning post-test (M = 82.00, SD = 12.817) was also significantly higher than the mean score on the mathematical reasoning pre-test (M = 73.50, SD = 13.135, $t(15)$ = 2.984, p = .005). The effect size was medium (d = 0.746), and 37% of the variance was accounted for by the treatment (r^2 = 0.372). The 95% confidence interval for the mean difference between pre and post-test mathematical reasoning scores was 2.43 to 14.57.

Through observations and interviews, three major findings were revealed. Classroom observations disclosed an increased ability to create and teach integrated science and mathematics lessons and to use the proper language of each discipline. Teachers reported that prior to the training they never even thought about teaching integrated lessons, whereas after training they were teaching from one to eight integrated lessons per month. Teachers reported that the integrated lessons went well and they were planning to use the lesson again the next year. During each afternoon of AY training the teams were given time to design an integrated science lesson and an integrated mathematics lesson. During the planning, each teacher taught their partner the content and proper language of the lesson. At the end of the day, the teams shared their written plans with the entire group. Providing joint planning time is an integral part of *CSM* PD. As stated by NSDC, "teachers are likely to adapt their instruction to new standards-based curriculum frameworks through the joint planning of lessons".

Principals reported seeing science and mathematics teachers collaborating more and creating integrated lessons. Also, the principals stated that prior to training they looked for general instructional strategies, not science or mathematics specific ones. They reported that they now see the connections between mathematics and science so that when observing a science or mathematics lesson they ask themselves whether the lesson could be integrated. Moreover, they are asking themselves if some manipulatives or science equipment could be used in hands-on strategies during the lesson.

Contribution to Teaching and Learning of Science

The integration of science and mathematics has long been advocated by both disciplines and their national standards, *National Science Education Standards* and the NCTM *Principals and Standards for School Mathematics*. Although the discussion for developing an integrated science and mathematics curriculum has been ongoing, teachers have not been trained to develop an integrated curriculum. Moreover, no PD model was structured and described well enough to support trainers in facilitating teachers to integrate science and mathematics. Research indicates that teachers are better able to help their students learn mathematics when they have opportunities to work together to improve their practice, time for personal reflection, and strong support from colleagues and other qualified professionals (Brown & Smith, 1997; Putnam & Borko, 2000). The critical attributes of *CSM* can enable teacher teams to effectively teach integrated science and mathematics. The CSM model of PD seems to support teachers in the integration of mathematics and science. As stated by a recent *CSM participant*,

> Through intense hands-on instruction, numerous investigations, use of manipulatives, and the instructors' questioning strategies and modeling of collaboration, I was able to work side-by-side with a fellow teacher to develop wonderful integrated lessons for our students. This team-teaching approach to lesson planning and construction will help to keep my students' interest in math and science alive. (*CSM* participant, 2008)

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Mathematical Practices and the Role of Interactive Dynamic Technology

Gail Burrill, Michigan State University, USA
burrill@msu.edu

Abstract:
The Common Core State Standards in Mathematics adopted by most of the states in the United States offer a set of mathematical practice standards as part of the expectations for all students. The practice standards suggest mathematical "habits of mind" teachers should cultivate in their students about ways of thinking and doing mathematics. With the teacher as a facilitator and using the right questions, dynamic interactive technology can be an effective tool in providing opportunities for students to engage in tasks that make these practices central in reasoning and doing the mathematics.

Introduction
Until 2010, each state in the United States had its own version of what mathematics was important to teach in K-12 classrooms and its own ways of measuring whether students were meeting their standards. The consequences of this were vast: enormous text books designed to meet every standard from any state but providing teachers little direction for making the text applicable in their state; assessments varying from multiple choice questions focused on procedures to using college entrance examinations as high school exit examinations. This variability in standards and assessments resulted in very different benchmarks for quality performance - one state's "A" was another state's "D". Comparing student performance on state assessments to the results of the National Assessment of Educational Progress, given to randomly selected students across the United States, makes this clear. Overall, individual states' level of satisfactory performance for students has been consistently lower than that of NAEP but varies greatly from state to state; for example in 2009, the difference in the percent of eighth grade (age 13-14) students who achieved a basic level of proficiency on the two tests was 11% in Massachusetts (43% to 32%) and 45% in New York (71% to 26%).

Deeming these results unacceptable and responding to the poor showing of United States students on TIMMS and the Programme for International Student Assessment (PISA), the Council of Chief State School Officers, the directors of education in each of the states, commissioned a set of national standards, the Common Core State Standards in Mathematics (CCSS), written by a small team of mathematicians and educators and released in 2010. As of April 2011, the standards were accepted by 42 of the 50 states. In addition to a set of content standards, one central and potentially significant component of the CCSS is a set of standards that focus on the mathematical "habits of mind" teachers should cultivate in their students about ways of thinking and doing mathematics, beginning in primary grades and continuing throughout secondary school. If implemented, these standards have the potential to significantly change how teachers in the United States approach teaching mathematics and what students take from their mathematics education. The eight practices are:

- MP1: Make sense of problems and persevere in solving them
- MP2: Reason abstractly and quantitatively
- MP3: Construct viable arguments and critique the reasoning of others
- MP4: Model with mathematics

MP5: Use appropriate tools strategically
MP6: Attend to precision
MP7: Look for and make use of structure
MP8: Look for and express regularity in repeated reasoning

Elaborations of the practice standards can be found at www.corestandards.org/assets/CCSSI_Math%20Standards.pdf. While the practices are part of the standards in the United States, in essence they seem universal. Cuoco and colleagues (1996) claim that the central aim of mathematics instruction should be to give students mental habits that allow them to develop a repertoire of general ways of thinking, strategies and tools they will need to use and understand in order to do mathematics and approaches that can be applied in many different situations. The mathematical practices are an attempt to make this visible to teachers – to call attention to ways of reasoning and sense making in mathematics.

The focus of this paper is to suggest how the practices can be used to leverage deeper understanding of core mathematical content by elaborating on two examples, one from calculus and one from mathematics. The discussion below describes some of the research about learning and how the use of dynamic interactive technology can support the implementation of the examples in classrooms.

Research and Dynamic Interactive Technology

Research suggests learning takes place when students engage in concrete experiences, observe reflectively, develop abstract conceptualizations based upon the reflection, and actively experiment or test the abstraction (Zull, 2002). The kind of learning that supports the transfer of concepts and ways of thinking occurs when students are actively involved in choosing and evaluating strategies, considering assumptions, and receiving feedback. They should encounter contrasting cases, noticing new features and identifying important ones (NRC, 1999). Dynamic interactive technology provides an environment in which these kinds of learning opportunities can take place. They allow students to deliberately take a mathematically meaningful action on a mathematical object, observe the consequences of that action, and reflect on the mathematical implications of those consequences, an "Action/Consequence" principle (Dick & Burrill, 2008).

In such environments, the task and role of the teacher are central as interactive technology alone is not sufficient for students to learn. Research about effective use of interactive applets in learning statistical concepts suggests teachers should engage students in activities that help them confront their misconceptions and provide them with feedback (delMas, Garfield, & Chance, 1999). Students' work needs to be both structured and unstructured to maximize learning opportunities; even a well-designed simulation is unlikely to be an effective teaching tool unless students' interaction with it is carefully structured (Lane & Peres 2006). In addition, students need to discuss observations after an activity to focus on important observations, become aware of missed observations, and reflect on how important observations are connected (Chance et al, 2007).

The following suggests how the mathematical practices might be realized in a calculus task. (Note that the figures were produced on a TI Nspire; could also be reproduced in any applet.)

Example 1: The Urn

Students are given an interactive file displaying an urn that can be filled with water using a clicker. A side panel displays a graph of the height vs. the volume as the urn is filled (Figure 1). To help students make sense of the problem and become familiar with the context (MP1: make sense of problems), they are asked questions such as, "Will the graph ever be a straight line? Why or why not?"; "If the urn were in the shape of a cone, what do you think the graph would look like? Why?"; "How would the graph of the height vs. the volume for a cone differ from the graph for an inverted cone?" Where is

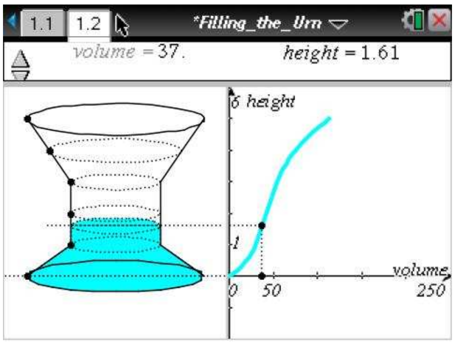

Figure 1 Filling the urn

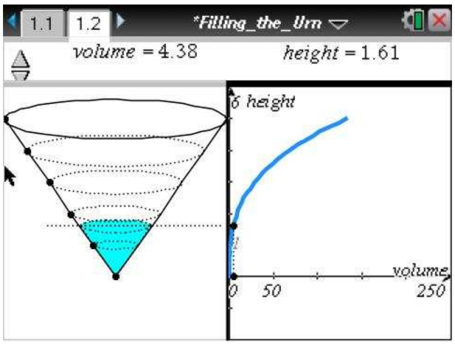

Figure 2 Filling a cone

the graph of height vs. volume the steepest? Explain your thinking." Once students have considered how the shape of the urn might affect the volume and made some conjectures (MP2: Reasoning), they "fill" the urn with water and check their reasoning. Students can be given time to experiment with different shapes and by describing them to their peers, can strengthen their understanding and ability to talk about the critical points of a function and the characteristics of its first and second derivatives.

Evidence from an analysis of student assessments in calculus indicated that many students struggle with several core calculus concepts. One of these seems to be the relationship among the characteristics of a function, its derivative and its second derivative, in particular with respect to multiple roots. The problem below, answered correctly by 28% of the students including 55% of those who earned top marks, is typical (College Board, 2003).

> If $f''(x) = x(x+1)(x-2)$, then the graph of f has inflection points when x =
> A) -1 only B) 2 only C) -1 and 0 only D) -1 and 2 only E) -1 0, and 2 only

To develop and support understanding, questions such as the following can be posed focus students on "contrasting cases, noticing new features and identifying important ones" (NRC, 1999): "What shape will produce a graph that has two points of inflection? Is it possible to have more than one shape that will produce such a graph?"; "What shape, if any, will produce a graph that is concave up with one point if inflection?"; "Predict the shape of the graph given an urn that is shaped like an hourglass, describe its characteristics, and defend your reasoning."

To provide students with feedback from both teachers and peers, a central feature of formative assessment (Black et al, 2004), classroom discussions should engage students in defending their solutions and sharing their graphs, preferably using a display that allows screen captures of student handhelds or computer screens (for example, the TI

Navigator). Students can be called on to look for similarities and differences in solutions, explain how another student might have been thinking about the problem, or decide which graphs are appropriate for a given question and why. The discussion can provide opportunities for students to engage in a variety of mathematical practices including:

- MP1: using visual representations to solve a problem; explain correspondences between tables and graphs; comparing approaches
- MP2: Stop and think about what the symbols represent in context
- MP3: Make conjectures; distinguish correct reasoning from that which is flawed
- MP4: interpret mathematical results in the context of the situation and reflect on whether the results make sense; identify important quantities in a practical situation
- MP5: use technology to visualize the results of varying assumptions, explore consequences, and compare predictions with data; use technological tools to explore and deepen understanding of concepts
- MP6: Communicate precisely to others.

Example 2: An Optimization Problem

An analysis of high stakes exit or end of course tests at the high school level (Dick & Burrill, 2009) suggests that students struggle with reasoning about graphs and about the mathematics. Problematic areas for students as opposed to experts include making connections among representations (Pierce, 2004; Stacey, 2005; Ramirez et al, 2005) and recognizing when to use certain techniques (NRC, 1999). The following task involves some of these elements: A sailboat has two masts, 5m and 12m. They are 24m apart. They must be secured to the same location using one continuous length of rigging. What is the least amount of rigging that can be used (figure 3)?

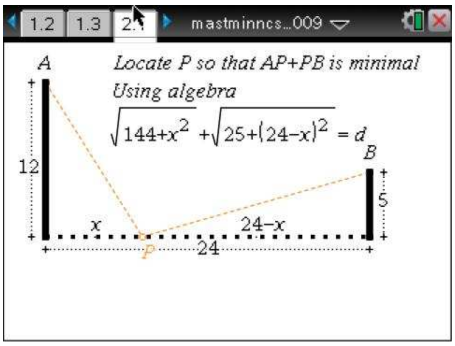

Figure 3 The Mast

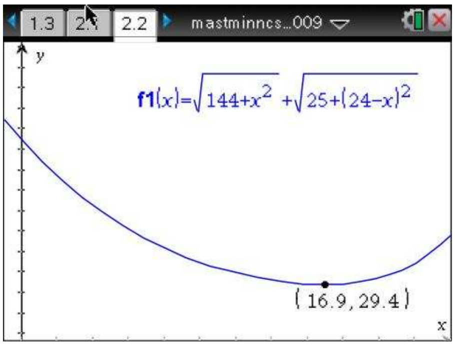

Figure 4 Using the Pythagorean Theorem

Students investigate the problem (MP1: making sense of the mathematics) by dragging point P making conjectures about possible locations for P that will minimize the amount of rigging. While different mathematical approaches to a solution are possible, fairly consistently in our work, students approach the solution algebraically using the Pythagorean theorem. Also fairly consistently, many of them graph the equation and trace to find a minimum point, which they suggest, without additional support for the claim, identifies the location of point P (figure 4). In less than half of the situations do students or teachers in workshops refer to calculus as a strategy to find the solution.

MP2 suggests mathematically proficient students should reason abstractly and quantitatively:

- make conjectures and build a logical progression of ideas
- determine domains to which an argument applies
- analyze situations by breaking them into cases
- recognize and use counterexamples
- compare effectiveness of two plausible arguments
- distinguish correct reasoning from that which is flawed and explain any flaws.

As students work through the strategies to find a solution, with the teacher as facilitator, they have the opportunity to engage in all of these mathematical practices. Graphing the equation results in a plausible suggestion for the location of P that will be the minimum but without further justification, it remains a conjecture. Students with a background in calculus should recognize the situation as one in which analyzing the derivative will yield critical points, which in conjunction with the second derivative, will lead to the solution. But it is also possible for students without a calculus background to reason from the context and implied domain for the function.

To reason about the situation means students have to recognize the difference between the *x-coordinate* of point P and the *distance* of point P from the first mast to the second (MP6: use clear definitions in discussion and in reasoning; state the meaning of symbols used), defining the total length to arrive at the same conclusion. The function defined by $L=\sqrt{x^2+25}+\sqrt{(24-x)^2+144}$ is constrained by the domain: $0 \leq x \leq 24$, which suggests that the length of the rigging is between the two extremes determined by the length of the deck and the length of a mast; the length is $\leq 12+24=36$ and the length is $\geq 5+24=29$. Reasoning about the sum of the parts of the graphs of the two hyperbolas over the domain can lead to the conclusion that the sum from $5 \leq x \leq 24$ can have only one minimum, and thus, the one illustrated on the graph is that point.

The task can be done geometrically by reflecting one of the masts over the horizontal deck line and connecting the top of one mast to the bottom of the reflected one (figure 5). Point P lies at the intersection of this line and the deck, reasoning that the shortest distance between two points is a straight line and the two small triangles are congruent.

Configuring the problem on a grid, dynamic interactive geometry software led one student to examine the point of intersection of the two diagonals (figure 6). She

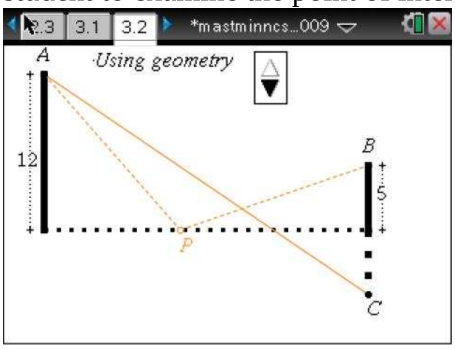

Figure 5 A transformation approach

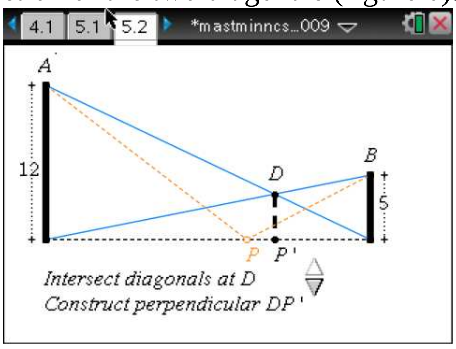

Figure 6 Diagonals

hypothesized that the *x*-value of the point of intersection might be that of point P, which has a nice proof that can be generalized (MP1: check answers by a different method; MP5: analyze graphs, functions and solutions generated by technology).

Conclusion

The two examples illustrate how mathematical "habits of mind" can emerge in a task, the way it is posed, and the way in which it is implemented, engaging students in thinking and reasoning about core mathematics. A task should be judged by the number of opportunities it affords to engage students in the mathematical practices. The role of the teacher is to frame the tasks and their implementation to maximize opportunities for learning. The questions teachers ask are central in bringing the practices – and ways of mathematical thinking and reasoning – to the foreground in the discussion. Technology is a tool that supports this work, providing students with opportunities to visualize the results of varying assumptions, explore the consequences, and compare predictions, explore and deepen understanding of concepts, and analyze graphs, functions and solutions generated by technology (MP5).

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HANDS ON WORKSHOPS by Douglas Butler
Douglas Butler debutler@argonet.co.uk
iCT Training Centre, Oundle (UK)
www.tsm-resources.com

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A rare chance to indulge in what the web has to offer on the TSM Resources website: web broadcasting (straight into the classroom), resources for the busy teacher, calendar of training events, useful excel file for mathematics, integer lists, resources for dynamic software, mathematics links and data from around the world ...

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Autograph for 11-16 age

A chance to work through some lesson plans that work for the youngers students, and inspire them to understand through visualisation. Topics will include transformation of shapes and graphs, fitting graphs to interesting images, and introductory trigonometry.

HANDS ON WORKSHOP – computer lab or laptops with Wi-Fi

Autograph for Statistics and Data Handling

Statistics is increasingly being seen as an important and vital part of school mathematics. This workshop will look at how Autograph can generate and analyse amazing data sets, and how the basic principles can be illustrated with ease. This will include the tricky topics of continuous and discrete variables, and frequency density.

HANDS ON WORKSHOP – computer lab or laptops with Wi-Fi

Autograph for 16-19 age

As the topics get harder, a firm understanding of the basic underlying principles becomes more important. This workshop will look at how dynamic software can bring topics to life, and will include vectors (2D and 3D), trigonometry, the principles of differentiation and integration, and logs and exponentials.

Mathematics Teachers' Knowledge Growth in a Professional Learning Community

Million Chauraya
School of Education,University of the Witwatersrand, South Africa
Million.Chauraya@wits.ac.za; Mchauraya04@yahoo.com

Abstract

Professional learning communities are regarded as a viable teacher professional development model, but knowledge about how teachers learn in such communities is not yet well developed. This paper reports on how a conversation about ratio in a professional learning community of mathematics teachers supported the teachers' conceptual understanding of the concept. Features of a professional learning community that were found to support teacher learning include: diversity of opinion; challenging each others' ideas; voicing uncertainties; collective construction of meaning; and regarding learners' learning as the focus of the group's activities. The paper argues that professional learning communities that reflect these characteristics can support significant teacher learning.

Introduction

Contemporary teacher professional development models are mostly underpinned by the notion of professional learning communities. Professional learning communities are regarded as effective organizational structures in which teachers can learn and develop their instructional practices. Such communities enable teachers to learn from one another, and have the capacity to promote and sustain teachers' learning with the ultimate focus of improving learners' learning (Stoll & Louis, 2007). The effectiveness of professional learning communities in fostering teacher learning lies in ensuring that the nature of the collaboration can produce shared understandings, focus on problems of curriculum and instruction, and that they are of sufficient duration to ensure progressive gains in knowledge (Little, 1993). However not all professional learning communities manage to succeed in achieving these ideals. Little (1999) highlights the distinction between 'traditional communities' that coordinate to reinforce traditions, and 'teacher learning communities' where teachers collaborate to reinvent practice and share professional growth. Katz, et al. (2009) point out that "together can be worse", for example when collective activities, that may be well-intentioned, lack a clear needs-based focus and fail to address real teaching problems. These observations about the effectiveness of professional learning communities raise the need for more research-based knowledge about how professional learning communities foster real professional learning among teachers. In this paper I draw from an on-going research study to address the question: How do professional learning communities foster teachers' knowledge growth?

Learning in Professional Learning Communities

The conception of a professional learning community adopted in the study is that of 'a group of teachers sharing and critically interrogating their practice in an on-going, reflective, collaborative, inclusive, learning-oriented, growth-promoting way' (Stoll & Louis, 2007, p. 2). Learning in professional learning communities is conceptualized from a situative perspective, which regards learning as increased participation in communities of practice (Koellner-Clark & Borko, 2004; Lave & Wenger, 1991). Learning and participation are inseparable as they involve both the 'interpersonal and informational aspects of an activity' (Greeno & Gresalfi, 2008, p. 171). In professional learning communities, learning 'involves working together towards a common understanding of concepts and practices' (Stoll & Louis, 2007, p. 3). Understanding how professional learning communities foster teacher learning therefore entails paying particular attention to the interactions within the professional learning communities, and how new understandings are created and negotiated. Such interactions and learning are enhanced if teachers view the professional learning community as a safe, non-threatening environment in which they can raise uncertainties or difficulties, with the confidence and trust that they will be supported and helped by others (Koellner-Clark & Borko, 2004). Open acknowledgement of what one does not know leads to possibilities for real new learning (Katz, et al., 2009).

Learning in a professional learning community is enhanced by: having a supportive leadership; collaborative inquiry that involves questioning and reflecting on practice (Jaworski, 2003b); and having a shared and clear learning focus that guides activities (Katz, et al., 2009). Such learning involves: challenging each others' assumptions about teaching and learning; discussion of new ideas that challenge existing knowledge; diversity of opinion; discussion of points of disagreement rather than of agreement; maintenance of the mathematical substance of the conversations; treating participants' ideas as objects of inquiry; and joint sense-making (Katz, et al., 2009). Analysing how these characteristics are manifested in the activities of a professional learning community helps determine the quality of the teacher learning that occurs. In what follows I use these concepts to show how a conversation among teachers helps them to develop a conceptual understanding of ratio and give meaning to an algorithm related to ratio.

Methodology

The qualitative case study extended over a period of nine months in 2010. Five mathematics teachers (Grades 7-10) from one school and the researcher (the author), met once a week for two hours at the school to work on activities which included: analysing the conceptual and skill demands of test items relating to ratio; interviewing learners about errors they made on the test items; analysing the errors made by the learners; planning and teaching lessons for dealing with learning needs based on the errors; and reflecting on the impact of those lessons. The activities were adopted from the Data Informed Practice Improvement Project (DIPIP), an on-going teacher professional development project based at Wits University. The researcher acted as a 'critical friend' in the group.

The activities described above supported conversations which sought to understand the errors made by learners on each test item with a view to collectively develop innovative lessons for dealing with observed errors and misconceptions. In this paper I focus on the conversations of the professional learning community in analysing learners' errors on the concept of ratio. I analyse how the conversations fostered the teachers' deeper understanding of the learners' errors, and their own subject content and pedagogical content knowledge about the concept of ratio. I also show how the teachers' developing confidence and trust in the professional learning community enabled them to voice some uncertainties; critically reflect on their knowledge about the concept of ratio; and jointly construct new understandings of the concept.

Findings: Developing Meaning for Ratio

The following two episodes drawn from the teachers' conversations illustrate some of the different ways in which the professional community led the teachers to a deeper understanding of some aspects of the concept of ratio, as well as addressing some of the uncertainties that they had. In the episodes the teachers' names are all pseudonyms, while 'Re' refers to the researcher.

Episode 1: Raising uncertainties in the group
In this episode the conversation was on the test item: 'Simplify the ratio 6:5'. In the conversation one teacher, Tsepo, raised his discomfort with the item. The following are some excerpts from the conversation:

Tsepo:	Don't you think that that ah giving learners questions like this is very much confusing because the learner will now try to work out this when it is already in its simplest form?
Re:	It depends on what we are testing. In this case I think we are testing understanding of a ratio and when a ratio is in its simplest form. But Tsepo is right, you have to think like a learner sometimes, because the moment you see simplify, in the mind you have to do something, and this is what some of the learners did.
Mandla:	And by doing that something they messing up everything.
Tsepo:	Ja, so they just have to leave it like that?
Re:	Yes, because it's already in its simplest form. How do they realize that, you can't have, six and five do not have a common factor.
Tsepo:	Uh because now the learner will think that maybe they have to give, I don't know out of how many marks is, is that, this question.
Karabo:	Two marks.
Tsepo:	So in the case like this we need to be careful with the marks also.
Karabo:	If they see the marks, they become scared the way I realize, wow, seven marks, it means I have got to work out here, whereas I am going to use a long method.
Tsepo:	I think there is a confusion with the total parts of a ratio, where you are given something like six as to five and then the total parts is six plus five which is eleven and then
Re:	Did they need to do that? They did not need to do that because they are not dividing a quantity into that ratio.

At this point the researcher pointed out that this item was testing more of the conceptual understanding of a ratio than the method of dividing a quantity in a given ratio, and marks allocated were not an issue. There was consensus that the learners who did some working as shown in the errors did not understand when a ratio is in its simplest form. It also became evident that Tsepo had a conception of ratio that was limited to dividing some quantity in a given ratio, but this became more apparent in subsequent conversations (see Episode 2 below).
In the episode Tsepo openly raised his discomfort with a test item, which could be an indication of his developing confidence and trust in the professional learning community. His public voicing of the discomfort initiated a conversation that supported an understanding that some items could test conceptual understanding only and did not necessarily involve some working or an algorithm.

Episode 2: Giving meaning to algorithms
The excerpts in this episode were drawn from an extended conversation in which the focus was on the conceptual meanings of a ratio and the algorithm for dividing a quantity in a given ratio. The conversation occurred after analyzing all the errors in the test and the professional learning

community was now identifying what they thought were the critical concepts for the learners in order to inform the lesson plans.

Karabo:	Ratio is the problem.
Re:	Karabo thinks ratio is a big issue.
Kholiwe:	It’s a big issue when they move to grade ten, but when they are in grade nine
Re:	What do you mean, can you explain?
Kholiwe:	They understand when they are in grade nine but when they move to grade ten
Re:	What is it that they understand about ratios in grade nine, what is a ratio? I think lets talk about this, what is a ratio?
Tsepo:	Is dividing in parts, ratio is a total. The total parts of a whole.
Kholiwe:	If you say, if maybe we say you are three, and I give you twenty rand to share it
Re:	I think that’s an application of ratio in a sharing context, but what is ratio?
Karabo:	We are sharing, it’s a share.
Kholiwe:	It’s a way of sharing
Re:	I beg to differ when you say it’s sharing because we also talk of ratio, for example what is the ratio of men to women, are we sharing?
Kholiwe:	*Laughs.* You are sharing but not in equal parts.
Mandla:	You are distributing.
Re:	What is the ratio of men to women among the educators in this school?
Kholiwe:	Ten to fifteen.
Re:	What are we doing?
Tsepo:	We are comparing.
Re:	Yes, ratio is a way of comparing quantities.

The episode shows that the teachers initially understood ratio as a way of sharing, with Tsepo specifically alluding to the algorithm for dividing quantities in a given ratio. The researcher then explained that we use ratio as a way of comparison without necessarily knowing the size of the whole, for example using a ratio of 1 teacher to 35 learners to compare teachers to learners without necessarily talking about how many teachers or learners there are altogether.
The researcher then raised the issue of dividing a given quantity in a given ratio. From the transcript below it was evident that the teachers were familiar with the algorithm for dividing a quantity in a given ratio, but could not articulate a conceptual understanding of the meaning behind the algorithm nor of the concept of ratio. Excerpts from the ensuing conversation illustrate how a conceptual understanding of the algorithm was developed.

Re:	If you are dividing a quantity in a given ratio what basically are you doing? For example divide eight hundred in the ratio five to three, why do they need to add five and three?
Tsepo:	They need to get the total.
Re:	Why, what is the meaning of that total? If you were to do it physically what would you do?
Tsepo:	The total is the meaning of a whole; the whole consists of eight parts, five and three.

Mandla:	Are you asking for the process? This is what is not clear.
Re:	Yes. If we can explain that process, that gives meaning to five as to three in a sharing context, remember we are trying to give meaning to the algorithms that we use.
Tsepo:	It doesn't mean to bring them to their equal sizes? Ja, now we are saying in terms of the denominators when we are multiplying
Re:	Remember my question, how would you do it physically?
Mandla:	I will say this one you are five, you are three, and then I give you your five hundred and you get your three hundred *(laughter from everybody)*

Mandla could not explain how or why he would do that. The researcher then explained that sharing in the ratio 5:3 meant that for every 5 given to one person, 3 is given to the other one. Using this understanding the teachers were able to work out that each time a total of 8 is given out and the process would need to be repeated 100 times in order to exhaust the 800, and one person would get 500 while the other would get 300. This led to the following remarks:

Mandla:	Mister, I can start tomorrow, delivering to the learners *(Laughter from everybody)*
Kholiwe:	But this is a nice way of teaching.
Mandla:	No, I am going to apply the very same way.

The episode shows how, in the conversation, the teachers' algorithmic understanding of ratio shifted to a more conceptual understanding. The teachers' initial understandings about ratio were used as objects of inquiry in the conversation to develop a conceptual meaning of ratio and the algorithm for dividing a quantity in a given ratio. The conversation shows how challenging the teachers to explain their initial understandings about ratio facilitated collaborative reflection on their knowledge. The conversation also illuminates how the teachers' participation supported joint sense-making in developing conceptual understanding of the concept of ratio. The two teachers' remarks at the end of the conversation could be indicators of shifts in understanding. The two teachers' reference to their teaching could be evidence of their developing confidence in teaching the topic as a result of shifts in their understanding. The remarks also allude to the teachers' conscious awareness of learners' learning as the focus of the activities.

Conclusion

From the conversations it was evident that the teachers' initial conceptions of ratio reflected a procedural understanding. This is consistent with research findings which highlight that mathematics teachers generally depend on algorithms and memorization of formulae, and do not offer conceptual explanations for those procedures (Zakaria & Zaini, 2009). Ratio as a mathematical topic is considered a difficult topic to teach and learn (Misilidou & Williams, 2002). Deepening teachers' understanding in such topics improves their subject content and pedagogical content knowledge (Shulman, 1987). The analysis shows that the teachers' conceptions of ratio shifted to a more conceptual understanding through their engagement in the conversations, and they were able to develop meaning for an algorithm. A conversation about learners' errors supported the teachers' deepening of their subject content knowledge and pedagogical content knowledge. The teachers developed the beginnings of a conceptual understanding of ratio and how to teach ratio meaningfully.

The results highlight a number of features of a professional learning community that can support teacher learning. The researcher challenged the teachers' initial understandings of ratio and these conceptions were treated as objects of inquiry in a conversation that deepened their understandings of the concept. Diversity of opinions contributed to the inquiry process in this community. The teachers' explanations and justifications of their different viewpoints supported their collective interrogation of their understanding of ratio, leading to joint sense-making and collective construction of the new understandings. The teachers' frequent reference to 'they' (learners) in the conversations could be an awareness of learners' learning as the focus of the activities. The teachers' freely expressed their uncertainties and understandings without feeling threatened, which could be an indicator of their developing confidence and trust in others. Professional learning communities which reflect these features, among others, have been found to result in shared understanding of concepts (Stoll & Louis, 2007). Although data analysis is on-going, from episodes like those cited in this paper I argue that professional learning communities as a model of professional development can support growth of teachers' professional knowledge.

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Using Online Textbooks and Homework Systems: In Particular MyMathLab and WebAssign

Wil Clarke, MS, PhD
Professor of Mathematics
College of Arts and Sciences, La Sierra University
Riverside, CA 92515, USA
wclarke@lasierra.edu

Abstract
Major textbook companies are providing homework grading and testing services to educational institutions for a very nominal fee. These services can be used to provide the teacher with relief from long hours of tedium grading assignments, quizzes, and tests. They also give the student instant feedback on the correctness of their responses. The jury is still out as to whether or not these services actually improve learning.

Introduction

In an effort to meet the ever growing threat of digital pirating and illegal distribution, textbook companies are racing against time to find ways to market textbooks in a profitable way. Some of the larger textbook companies are producing extensive Internet services to try and lure teachers to adopt their books and to stay with them. These online services include e-texts, online homework programs as well as quizzes, tests, and examinations, online grade programs, and online communication options to use between teachers and students and between students and their classmates. I have used two of these services from competing companies, and this talk explores some of the features as well as frustrations I have encountered.

The first such service I tried, MyMathLab, I used in teaching hybrid classes in Elementary Algebra and College Algebra at Riverside Community College. Later I used WebAssign, a service from a different company, because of its online homework program for an Introduction to Linear Algebra at La Sierra University.

Common features of MyMathLab and WebAssign

One of the features of both MyMathLab and WebAssign is an e-text. If students are willing to use an e-text instead of a printed textbook, they can sign up for the program for approximately one-third of the cost of a printed textbook. If they choose to buy a new text, they can register for the service for a very nominal fee extra. If they have a textbook or buy a used textbook, then registration costs the same as for the e-text.

Another common feature it that the services include places for the teacher to upload syllabi and other resources for the course. There is also provision to display announcements the teacher—and the textbook company—want the student to see.

In both of these services students can do their homework, quizzes, tests, and exams online. The programs include algorithms so that each student’s assignments are enough different that simply copying from another student is impossible. For example if one student has to solve the equation $3x + 4 = 7$, another student might be asked to solve $5x + 2 = 12$ for the same numbered problem.

What I did

At the community college I had the opportunity to teach both Elementary Algebra and College Algebra in an accelerated format. Accelerated meant teaching each class in 8 weeks rather than the usual 16 weeks using approximately 5 hours of lecture a week. Hybrid in theory meant I was to lecture on every second section in the text and require paper and pencil homework, and the students were expected to cover the other sections independently and submit homework electronically using the features of MyMathLab (Pearson). In actuality we found that briefly covering all of the course sections in the lecture proved better for most students. I taught each of these classes in three different semesters from August 2009 through December 2010 at Riverside Community College.

My full time work is teaching at La Sierra University. Recently, my department chair encouraged me to try using an online system for a class at LSU. Because of the multiple sections of each course offered at the elementary levels, he did not want me experimenting at those levels. So from September to December of 2010 I chose to teach a non-accelerated Introduction to Linear Algebra course using WebAssign (Cengage) electronic homework. Naturally, comparing the two systems at such different levels is not fully possible. What I can do is give an overview of the major similarities and differences between the two.

Similarities between the systems

Exploring the similarities, I noted that in both programs students had the choice of using an e-text for the course. Another feature that is the same in both systems is the ability to submit homework electronically. Students can submit some or all of their homework electronically. Since the answers are all open ended, they can submit each question electronically and get immediate feedback on their answers. I typically allowed them to make five attempts at getting their answer correct. (The teacher has the option of setting how many attempts students are allowed.)

There are plusses and minuses for allowing multiple attempts. On the positive side, this encourages a student to stick with the question until he or she gets it right. Since the responses are open-ended, a particular response might not be in the format that the website was expecting. Multiple attempts mean students can experiment with getting their format to match the one the website is expecting. On the negative side, this allows students to simply guess at the right answer until they get it right.

Another similarity between the two systems was the computer enforced submission deadlines for assignments, which I found particularly helpful. While I might have a soft heart toward a particular student's hardships, the computer program enforces deadlines impartially. I set the due dates to correspond to the beginning of the next class. This meant that trying to finish the homework was not an excuse for being tardy.

Each system has a very limited palette or toolbar of symbols that allows users to submit, fractions, radicals, exponents, matrices, etc. Yet both are quite exacting about how answers are entered. This proved frustrating at times and became the reason why I allowed up to five attempts to submit an answer. The toolbar in WebAssign seemed a little bit more intuitive than the one in MyMathLab, although both provided enough tools so the student could write an appropriate answer (see Figure 1).

Figure 1. Tool bars for MyMathLab and WebAssign.

The MyMathLab toolbar is on the left and that of WebAssign is on the right. Each named bar on the WebAssign toolbar opened further mathematical symbols, as did the "More" button on the MyMathLab toolbar on the left.

In both systems students appreciated the immediate feedback. When given a choice, the vast majority of students opted for--and in fact begged for--the opportunity to submit their work electronically. On the other hand, they were often frustrated because the computer did not accept some equivalent forms of a correct answer. For example if the question asks for the equation of a line it might accept $y = 3/2\ x + 7$ but would not accept $y = 1.5\ x + 7$.

In terms of support, sales representatives from both companies were initially very helpful. Yet when problems arose or when we discovered errors in the program, it was almost totally impossible to get any response out of the textbook company. I finally I quit trying to phone or e-mail them because I could never get a person on the phone nor any response to my messages.

Both textbook companies advertised that the student could print out the portions of the textbook they needed. In practice however, these companies made it very difficult do anything more than simply view the text on the screen.

When I was lecturing on a portion of the book, I liked to project it onto a screen since most students did not possess a printed copy of the text, nor was there sufficient room in the classroom for all students to bring their laptops. However, it was impossible to enlarge the text to make it easily readable from the back of the classroom. I found this seriously restrictive. Another disadvantage connected to this is that in most of the classrooms where I used the computer, the screen covered a portion of the white board. This limited the amount of writing I could have in front of the students at any one time.

Related to the homework systems, both seemed to do much the same thing. They created individualized homework assignments and quizzes. They kept a grade book that was limited because one couldn't upload scores into it. On the other hand, one could download the scores from the online grade book into a spreadsheet such as Microsoft Excel.

Differences between MyMathLab (by Pearson) and WebAssign (by Cengage)

When looking at the differences between the two services, MyMathLab came across as cumbersome. The interface is uninspiring and very non-intuitive to me. I could easily "paint myself into a corner." For example, I would be working with the e-text, and then when I wanted to switch to the Homework/Test Manager. There was no way I could seem to get there short of going back almost to the login screen. Probably this is because everything is done with pop-ups, which I normally keep switched off. Yet an advantage it offered was that as a teacher I could easily see what the students see when they log in.

To me WebAssign seemed simpler to operate. It was not driven by pop-ups. I could also enlarge the font sufficiently so it could be read from anywhere in the classroom. However, in neither program can the view of the textbook be widened enough to see the full width of the page once it has been enlarged so that students can read it. Probably the most annoying feature with WebAssign was that the graph "paper" on which the student had to draw graphs was fixed for values of x and y between -11 and 11 (see Figure 2). But the random question generator had no such limitations. Therefore, if a graph was part of the answer, it would often be impossible for the student to get it correct. This is a feature I kept trying to bring to the attention of the sales rep or one of the tech's, to no avail.

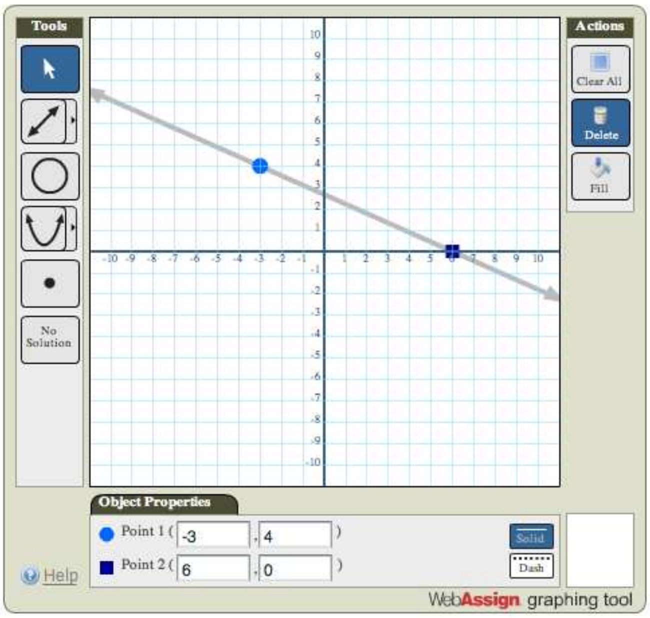

Figure 2. WebAssign graph template

Both programs gave hints as to how to do exercises. When I used it, the WebAssign version for my linear algebra text, however, must have still been in the programming stage because, after we got past the first chapter, the hints were non-existent. They would simply refer the student back to the beginning of that chapter in the text. MyMathLab, on the other hand, provided hints for completing exercises throughout the text.

Pedagogical Concerns

Looking back, I see that allowing students multiple attempts at getting the right answer may have been counterproductive. On tests, which were all pencil and paper, they had only one chance, of course, to get the correct answer.

While immediate feedback is very helpful for the students, I am concerned that it might encourage sloppy thinking. Rather than learning to actually critically evaluate their own thinking, they may only wait to see what the omniscient computer gives as a reply. They become willing to accept whatever the computer tells them as actually being true.

The literature I found about how well students learn using this type of service is divided in its evaluation. Some scholars indicate that, as far as the students are concerned, on-line homework does not hurt them. LaRose writes, “We find that students working homework on-line appear to do no worse in the course than those with pencil-and-paper homework, and may do better.”[i] Allain and Williams concur: “Results show that there are no significant differences in conceptual understanding or test scores.”[ii] Their study was conducted in a science class. Brewer also found little difference between traditional and online students: “The results of the study found that while the treatment group generally scored higher on the final exam, no significant difference existed between the mathematical achievement of the control and treatment groups.”[iii] In a study conducted on “web-based versus paper-based homework” done for physics classes, Demirci states, “In general, statistically no significant differences were found.”[iv] Lenz[v], Bonham[vi], and Hauk and Sequalla[vii] also all weigh in on the side of there being no significant difference in performance between students who submit web-based homework and those who turn in paper-and-pencil homework.

On the other hand, some researchers found that the online students did not do as well. Moosavi, for example, states “Regardless of whether achievement is measured in terms of a single semester test, comprehensive final exam, course average, or test performance across the semester the results presented here indicate that students perform better in traditional classes than in CAI classes regardless of the CAI curriculum used.”[viii] Furthermore, he goes on to indicate that students using Thinkwell Computer Aided Instruction (CAI) did better than those using MyMathLab.

One would be surprised, of course, if there were no findings supporting online homework. Affouf finds a strong correlation between achievement on the web-based homework assignments and achievement in the final examination.[ix] Mendicino likewise reports that with a “group of 28 students, students learned significantly more when given computer feedback than when doing traditional paper-and-pencil homework.”[x]

So the jury is still out as to whether or not there is any real benefit to the students in using online homework systems. Why then, should we spend the extra time it takes to set up the system for our courses? The answer is because of the much larger amount of time, especially in subsequent courses, we as teachers can save when it comes to grading homework and quizzes.

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Hearing the teacher voice: teachers' views of their needs for professional development

Els De Geest
The Open University, Milton Keynes, United Kingdom, e.n.f.degeest@open.ac.uk

Abstract
Research literature (Little, 1993; Goodall et al, 2005; Joubert and Sutherland, 2008) suggests that for professional development (PD) to be effective the aims of the PD should fit the perceived needs of the teacher. This paper reports on a study, which investigated teachers' views of their needs for PD and how these needs were being addressed when partaking in PD. Data consisted of responses to semi-structured interviews with 38 teachers who were involved in different types of PD initiatives. The data was analysed using a process of constant comparison (grounded theory). The analysis offers descriptive categories of teachers' perceived needs for PD which seem to resonate with Perry's (1999) descriptions of learner development. The findings corroborate a model of teacher change as growth or learning where teachers are themselves learners who work in a learning community and support models of empowerment for professional development, not models of deficiency (Clarke and Hollingsworth, 2002).
Keywords: teachers' needs, professional development, teacher voice

Introduction
Several authors in the research literature (Little, 1993; Goodall, 2005; Joubert and Sutherland, 2008) have reported that for PD to be effective the aims of the PD should fit the perceived needs of the teacher. Joubert and Sutherland (2008) report in their review of recent literature on professional development of mathematics teachers that formal professional development can arise from changes in government policy and innovation, from identified student-performance and from under-qualification of teachers. This suggests a deficiency of skills and knowledge leading to the adaptation of a deficiency model of PD with the needs of the teachers perceived as needs for prescribed knowledge and skills. At the same time, research indicates that PD initiatives based on deficiency models are ineffective (for example Clarke and Hollingworth, 2002; Guskey 1986) questioning the rationale for supporting such models of PD.

The study
The study reported on in this paper was part of the Researching Effective CPD in Mathematics Education (RECME) project, a large research project funded by the National Centre for Excellence in the Teaching of Mathematics (NCETM) and explored factors of effective PD for mathematics teachers. It was a short term (15 months) non-interventionist project investigating 30 ongoing PD initiatives representing different models of PD for teachers of mathematics in England. Overall, about 250 teachers in pre-primary, primary, secondary, further and adult education settings were involved in these initiatives. The project adopted the theoretical framework that all human activity, including the learning of teachers, is historically, socially, culturally and temporally situated (Vygotsky 1978). This suggests that the experiences and contexts of teachers will have a major influence on their learning and implies a need to pay attention not only to the situation, the opportunities and the context of sites of learning (in this case initiatives of professional development), but also to the individuals taking part in professional

development. Importantly, the philosophical underpinning of the project was one of co-constructing meaning with teachers, researchers and other stakeholders. The data we obtained for the study we report on in this paper reflects this and contains self-reported data from teachers. Details and descriptions of the full data set, the research design and case studies can be found in the final RECME report and other publications (NCETM 2009; De Geest *et al* 2008).
Joubert and Sutherland (2008) report in their review that the voice of the teacher is under-represented in most research on PD of teachers. The RECME project offered us the opportunity to hear the teacher's voice on what they perceived to be their needs for professional development. We addressed the following research questions:

1. How do teachers perceive their needs to be addresses in the PD they are involved in?
2. Are there common characteristics in their perceived needs, independent of the model of PD?

The study was conducted using a constant comparison approach from grounded theory looked at from a social constructivist theoretical perspective (Vygotsky 1978). Insights into the perceived needs are given through descriptive categorisation of empirical evidence. Data consisted of 38 responses from interviews with case study teachers representing the participating PD initiatives in the RECME study. We asked the questions: "Does this PD in which you are currently involved address your needs? Could you please explain why?"

Findings

Of the 38 teachers, two teachers replied 'no', four said 'partly' and 32 said 'yes' to the question "Does this PD address your needs? Could you please explain why?"

Some of the interviewed teachers voiced that they were not sure what their needs were, or made a distinction between professional and personal needs, for example:
"In some ways the PD addresses my needs in that I like meeting up with others in the field. However, I do not feel that I have direct needs concerning my immediate work and I am not looking to the PD to fill any particular needs at the moment. However, applying standards unit principles to Key Skills does have a useful role in validating my ongoing classroom practices in a college environment where learner training is sometimes seen as more important than educating – the network is also an excellent place to get support if needed"
"Yes – my NQT targets were all in maths as area for development. Ironically this is where I have made the most success. All my professional targets relate to raising attainment in mathematics with specific improvement grades for my children. My personal need for PD is that I want to be inspired. I feel a bit like a groupie when involved in all this academic debate – I am hanging on to every word. I find the talk so interesting. "

The PD addressed the needs of the teacher

Teachers reported factors or a combination of factors of how the PD *had* addressed their needs (n=32). Through constant comparison of the responses of the teachers who considered their PD to address their needs, we identified six descriptive categories:

1. The teachers felt enabled to respond, and at times find solutions, to issues they had identified as problematic, or not knowing how to do address, in their classroom practice. For example:
"It was clear that there was also a need for a good assessment system and for recording assessment for learning and to see where support is needed on a day to day basis and that has come from [the PD initiative] as well"
2. The PD made them *think* at a high level, had *challenged* them which had made it interesting. For example:
"It seems a long time since I last engaged in research at this level – reading about what current thinking is in maths. It is interesting. [the PD organiser] fed a few articles that started me going"
3. The PD made them look afresh at things and had inspired them with new ideas. For example:
"Really inspired to do lots of things] "
"It challenges me, makes me think, makes me look afresh"
4. They had been able to follow/satisfy their own interests within the PD. For example:
"Yes we have the opportunity to say if there are things we'd like to do so we can shape it. I think it's invaluable"
"I am really pursuing my own interests"
5. They felt strengthened in their views, their opinions, their thinking because the input into the PD had been theirs, based in their needs. Some teachers reported this had made them feel stronger, enabled them to argue their views better. For example:
"Yes because we run it ourselves and it follows the needs of the group like the problem solving" "The more we've had to stand up for ourselves the more we feel we're not alone"
6. They knew how to put the theory of their professional development into their classrooms by having concrete examples and practical skills, and had experienced positive responses from the students. For example:
"Yes because you are seeing someone work in a practical context and seeing someone work in your class and that is amazing. Because a lot of training you have is theoretical rather than practical. We all want ideas that you can apply straight away and that's what [the PD organizer] provides really."
"Yes exactly so you are using that as well as in action so you are using the theory and you know how to apply it because you have been shown that".

The PD *partly* addressed the needs of the teacher
The teachers (n=4) who said the PD was only *partly* addressing their needs lacked some of the factors identified above, and these missing factors were considered *important* by these teachers. They included some, but not necessarily all of:
The teachers could not see a pragmatic use of what was learned for the teacher's specific classroom in *some* parts of the PD, and as a result had no interest in that aspect of the PD. This referred to the learning of subject knowledge. For example:
"Some of the things we don't even touch on well I can't really use any of this and I'm not interested in learning that. It depends how interested in maths you are yourself. A lot of us just wanted to find different ways of presenting things to children and resources and so on. Some of the earlier stuff like the algebra we couldn't really use with our children"

The teachers acknowledged a personal enriching of new pedagogies and tools for teaching, but experienced the mathematics learned as too advanced for him/her. The PD also did create an awareness of what alternatives might have worked better. Again, this concerned PD on subject knowledge. For example:

"It has certainly given me lots of tools and allowed me to consider different strategies for teaching.   I was not confident going in, I maybe could have used a refresher course in GCSE maths.   I know you can do level 3 numeracy courses and they are more on personal skills, so I wondered whether that would have been good for me to do that first. Then do this higher level course …I think it is just confidence for me, I know that for some I can remember how to do it, but for some I was reading it before the lesson trying to work out how do I re-arrange the formulae. That is not always the best position to be in"

One teacher considered the PD very useful, but felt nothing new was learned because he/she had worked on the same issues at university the year before:

"I think it has been very useful but I think, as I am new out of university quite recently I think it backs up what I have already learnt at university".

One teacher was missing seeing what was worked on at the PD in a pragmatic setting of the classroom:

"I need to see other people using the resources and have the chance to speak with some of the kids about how they experience it or would they rather have the book back?"

The PD did *not* address the needs of the teacher

The teachers (n=2) who said the PD was *not* addressing their needs lacked some of the factors identified, and these missing factors were considered *crucial* by these teachers:

One teacher reported that the course on subject knowledge did "its best to cater for what they saw as our needs", however did not address the teacher's need because the teacher could not perceive a pragmatic use of what was learned in the classroom.

"I am teacher of less able children and I'm doing proof theory but practically it is of no use to me at all. Maybe something to go on for the hands on, nitty gritty, in the classroom ideas and resources. Whereas modulo arithmetic is of less use".

The other teacher acknowledged that she liked the discussions that were taking place during the PD, however she personally would like to do a higher degree

"I am thinking of doing something else – maybe a doctorate. We've got, what I like is having a discussion about the mathematics and we have two good NQTs and I can have really good discussions"

Discussion and conclusion

The needs the teachers perceived and reported as being met in the PD seem to concern one or more of: finding answers and solutions to issues they had identified as problematic, made to think at a high level, being inspired with new ideas, being able to follow their own interests, feeling strengthened in their thinking and views on teaching and learning, being able to put theory into practice. The teachers who felt the PD had met their needs only partly or not at all identified a shortcoming in one or more of the above factors.

Several researchers consider the professional development of teachers as professional learning, and see the process of developing professionally similar to a process of learning (Eraut, 2007; Franke et al.,1998; Clark and Hollingworth, 2002).

What struck us is how the descriptive categories of this study seem to resonate with ideas of learner development as described by Perry (1999/1968) confirming views that PD is similar to the process of learning and development. Perry developed a framework for characterizing student development at university level.

Our findings seem to resonate with his ideas of student development in terms of self-trust and commitment. To think, be challenged, be inspired and find solutions to problems, teachers are exposed to views and interpretations, which can new, different or a re-phrasing of existing views, allowing for learning. This resonates Perry's description of his 'relativism' phase, when there is *"a plurability of points of view, interpretations, frames of reference, value systems, and contingencies in which the structural properties of contexts and forms allow of various sorts of analysis, comparison, and evaluation of multiplicity"* [Perry, in glossary]. The views of the teachers in our study were strengthened by following their own needs and interests, using their input, and being able to apply what was learned into their own classrooms. Their own personal values and choices were affirmed.

This study wanted to hear the teacher voice on what they perceived to be their needs for professional development. Findings and analysis suggest their needs are similar to those in adult learner development. We suggest professional development should not be based on a model of deficiency seeking to address perceived inadequacies but on a model of professional growth, which builds on existing knowledge, and expertise of the practitioner.

Acknowledgements

I am indebted to my colleagues on the RECME project, Jenni Back, Christine Hirst, Marie Joubert and Ros Sutherland, for their contributions to the study.

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Using A Computer Pen to Investigate Students' Use of Metacognition during Mathematical Problem-Solving

Iris DeLoach Johnson, PhD
Professor of the Teacher Education Department
&
Nirmala Naresh, PhD
Assistant Professor of the Mathematics Department
Miami University, Oxford, Ohio, USA
johnsoid@muohio.edu and nareshn2@muohio.edu

Abstract

The major aim of this paper is to present the computerized Smartpen as a tool for capturing and exploring students' metacognitive processes as they solve mathematical problems. After sharing their thinking through self-talk or group-talk students worked with others to share their strategies and reflect upon the process. An additional aim for this paper is to share any generalizations that may be helpful for teachers who are helping students strengthen their metacognitive and mathematical problem-solving skills. In this study the Smartpen was used to listen in on undergraduate preservice teachers' problem solving as they explored the Problem of Points, a classic probability problem. Capturing each stroke of the pen as the students wrote while simultaneously engaged in self-/group-talk, each Smartpen session produced a pdf document and accompanying audio for further analysis. Problem solvers who lacked confidence in their problem-solving ability were more reserved in their self-talk and often solved problems unrecorded in advance to appear more successful later during the recorded problem-solving session. It would appear that direct, real-time monitoring by the teacher is needed to capture significant information about students' metacognitive reasoning.

Introduction

The more successful mathematical problem solvers are those who think more deeply *during* the problem-solving process and *after* they have achieved a solution (Pugalee, 2001; Teong, 2003). The term metacognition best characterizes this phenomenon (Legg & Locker Jr., 2009). When teachers attempt to help students become better problem-solvers by employing questioning strategies to tease out critical steps in their thinking verbal or written communication is often a barrier. Students find it difficult to articulate what they were thinking previously when attempting to solve a problem (Hennessey, 2003). The major aims of this paper are to present the Smartpen as a tool for capturing and exploring students' metacognitive processes as they solve mathematical problems and to share generalizations about students' mathematical processing of a classic probability problem known as the Problem of Points.

A computer Smartpen (hereafter simply referred to as a Smartpen) may be useful in helping students to think out-loud or engage in self-talk while they solve problems. The Smartpen stores the written strokes—the image of a student's writing in real-time—and any oral communication that occurs simultaneously during the recording session. Consequently, the communication barrier between the student and the teacher, or between the student and him/herself could become less of a problem. When a student's written work in solving a mathematical problem is completed while using a Smartpen, his/her metacognitive work can receive a boost as the student listens in on any self-talk or group-talk, monitors the solution as it unfolds, and reflects on the process to determine decisions that should be made. Teachers can listen, make observations, note strengths, and determine weaknesses to support intervention as needed.

This paper will share highlights of specific phenomena associated with problem solving for which the Smartpen may assist in monitoring and improving students' metacognitive and mathematical problem solving skills. Specifically we will share a brief introduction to metacognition and problem-solving then move to the context of the classic probability problem—the Problem of Points (Berlinghoff & Gouvêa, 2004). We will also share our findings and generalizations from the investigation of the problem-solving attempts of preservice teachers who worked on the Problem of Points.

Metacognition and Problem Solving: In A Nutshell
The process of thinking about one's own thinking takes place internally—in one's mind—yet the results should also be manifested externally to lead to productive problem solving and to support sharpening of this skill through personal reflection and external intervention by teachers, peers, and learned experiences (Hacker, 2009; Lee, Teo, & Bergin, 2009; Legg & Locker Jr., 2009; Ponnusamy, 2009; Pugalee, 2001; Steif, Lobue, Kara, & Fay, 2010). As early researchers highlighted the importance of metacognition for student learning four key components of the process were identified: (a) verbal reports as data (telling what you know); (b) executive control (directing/managing what you know); (c) self-regulation (monitoring and making decisions about how you will use what you know); and (d) other regulation (internalizing feedback from interactions with others and external experiences) (Brown, Bransford, Ferrara & Campione, 1983 as cited in McKeown & Beck, 2009).

Metacognition has actually been identified as an essential skill in reading and in problem solving (Lee et al., 2009). Problem solving is one of the mathematical process standards of the National Council of Teachers of Mathematics (2000) and an essential component of the Standards of Mathematical Practice of the new Common Core State Standards (2010) in the United States of America. Stronger problem solvers "start by explaining to themselves the meaning of a problem…analyze givens, constraints, relationships, and goals… make conjectures…monitor and evaluate their progress and change course if necessary" (Common Core State Standards Initiative, 2010, p. 6).

Students are often taught specific strategies for problem-solving such as Polya's four-step problem solving model: understand the problem; devise a plan; implement the plan; and looking back (Taylor & McDonald, 2006). However, actual next-steps in the problem-solving process are not easily tracked when students are solving complex problems. Thus the role of metacognition along with the associated skill of communication (NCTM, 2000) can be particularly helpful. Artz and Armour-Thomas (1992) recognized this connection between problem-solving, metacognition, and communication (e.g., thinking out-loud) when they suggested an eight-step problem solving process (with the predominant cognitive levels shown in parentheses): "read (cognitive); understand (metacognitive); analyze (metacognitive); explore (cognitive or metacognitive); plan (metacognitive); implement (cognitive or metacognitive); verify (cognitive or metacognitive); and watch and listen" (p. 142).

Probability and the Problem of Points
One of the main implications for instruction in probability is that students must be engaged in activity-based, experimental investigations to support their conceptual understanding. The prevalence of misconceptions with regards to probability and the persistence of those misconceptions despite traditional instruction are well-documented (Fischbein & Schnarch, 1997; Khazanov, 2005; Polaki, Lefoka, & Jones, 2000; Shaughnessy, 1992). Metacognitive processes, combined with effective instruction, should help to increase students' awareness and

understanding of their misconceptions as well as help to identify ways to overcome them (Fiscbhein & Schnarch, 1997).

Using the probabilistic thinking framework developed by Polaki, Lefoka, and Jones (2000) one might investigate students' understanding of probability with regards to five major constructs (sample space, probability of an event, probability comparisons, conditional probability, and independence) at four different levels, numbered from 1 to 4 (subjective, transitional, informal quantitative, and numeric), respectively. Students operating at the lower level (Level 1) have difficulty appropriately operating in probability situations with anything more than subjective judgments. However, on the high end of the scale (Level 4) where deeper metacognitive thought is applicable, students are able to identify a sample space, predict and assign numerical values for the probability of events, and engage in numerically-supported probability comparisons and generalizations with conditional as well as independent events. The Problem of Points was specifically chosen to engage students in this higher end of the framework.

Subjects

During the spring semesters of 2010 and 2011, convenient samples of preservice teachers preparing to teach children of ages 9-14, consented to participate in this study. The preservice teachers were enrolled in a combined mathematics history-technology course in a medium-sized university (approximately 16,000 students) in the Midwest (USA). The group included 42 students in their second to fourth year of undergraduate study: 23 in spring of 2010 (22 females, 1 male) and 19 in spring of 2011 (14 females, 5 males).

Method

During a four-week period of 12 hours of in-class instruction, with additional instructional assistance provided in an online learning environment, supported by Smartpen pencasts, students explored various probability problems in various multicultural and historical contexts. In 2010 a Pulse™ Smartpen was only distributed to groups of students when engaged in hands-on probability activities in class. However, the need to be better acquainted with the technology dictated distribution of a Smartpen to each student for at least eight weeks (including four weeks prior to the study). Practice engaging in self-talk while writing and doing mathematical problem solving was suggested in specific instructions to students to communicate their inner thoughts as if tutoring a peer or sharing their procedures with the classroom teacher.

The Problem of Points was posed in class (as shared in brief below).

> Our story begins around 1654 as the Chevalier de Méré, a wealthy gambler, requested that Blaise Pascal provide a mathematical validation for a specific dice problem. As Pascal collaborated with Fermat on the solution the study of probability as a field of mathematics began! … Actually there is some confusion about the actual dice problem that the Chevalier first proposed, so the problem we will explore is a simple version of one of his problems that became known as the **Problem of Points**.
>
> > Xavier and Yvon staked $10 each on a coin-tossing game. Each player tosses the coin in turn. If it lands heads up, the player tossing the coin gets a point; if not, the other player gets a point. The first player to get three points wins the $20. Now suppose the game has to be called off when Xavier has 2 points, Yvon has 1 point, and Xavier is about to toss the coin. What is a fair way to divide the $20? (Berlinghoff & Gouvêa, 2004, p. 207)

After giving some initial thought to understanding the problem [while engaging in self-talk] and determining a plan for a possible solution you will join a team of four to provide solutions to Question 1 taken directly from of our course text.

a. Let's call the interrupted-game case described there *(X2, Y1)*, meaning that Xavier has two points and Yvon has one. How many other cases are there? What are they?
b. Analyze the rest of the cases from part (a). (Hint: Only two others are "interesting;" you might begin by disposing of the rest quickly.)
c. State your answers for part (b) as fractions or percentages that apply to any amount of money at stake.
d. Suppose the game required 4 points to win. What unfinished case is analogous to the *(X2, Y1)* case of the 3-point game? Generalize your answer to describe cases that result in a $\frac{3}{4}$-to-$\frac{1}{4}$ division of stakes for an unfinished *n*-point game.

... Berlinghoff & Gouvêa (2004, p. 213)

After approximately 10 minutes, students moved into groups to share their individual perspectives about the problem and become informed or inform others about a possible solution. The instructor monitored the groups as they engaged in group-talk, with one member of the group serving as a recorder while writing notes using the Smartpen. Once consensus was reached each group member asked questions of the group as needed to build his/her confidence in sharing the solution to another group or to the entire class.

All pencasts (i.e., the oral recordings and the accompanying pdf documents) were uploaded to a special class email address for analysis by the instructor. A mixed qualitative-quantitative design was used to analyze the data gathered. A comparison of the means of the pre-/post-assessment on probability was conducted using a *t*-test (in 2010, but a post-assessment was not given in 2011). Constant comparative and content analysis qualitative methods were used to investigate the categories of responses regarding metacognitive thought and the alignment of comments with the probability framework. The findings are shared in brief here.

Results

Self-talk sessions indicated that additional targeted training would be needed to help students truly share the details of their thoughts as they solved a problem. Most students appeared to lack confidence in sharing a "work in progress," and preferred to share their more polished results in a recording *after* attempting to solve the problem one or more times without the pen. Students with better than average scores on the pre-assessment of probability content knowledge were more likely to share more details of their thinking process or engage in the verbal reports of telling with such statements as "What I did was...I assumed that… The approach I'm taking is… I came up with …I thought there were 3 coin tosses at first rather than 3 [points] to win. That's the way I understood it. … I always looked at what the next toss would be." However, students who shared such comments often engaged in self-talk much more than they wrote. (See Figure 1.) For example, one student shared her metacognitive thoughts quite eloquently for almost 6 minutes before writing anything with the pen!

Group conversations featured much better sharing of metacognitive thoughts as students explained their positions, questioned the thoughts of others, and contributed to the solution. Since each student had expectations for later sharing with the class, each student clearly wanted to be prepared and often sought to confirm their understanding of the problem and the solution more than once. For example, one student who shared her thinking quite well with the group often asked at interim steps: “Do you understand that?” Another student, who often responded affirmatively to those questions as she indicated that she understood, waited until the last step of the solution and asked again, “Is it [i.e., the goal] the first to get to 3, or is it three (3) rolls?”

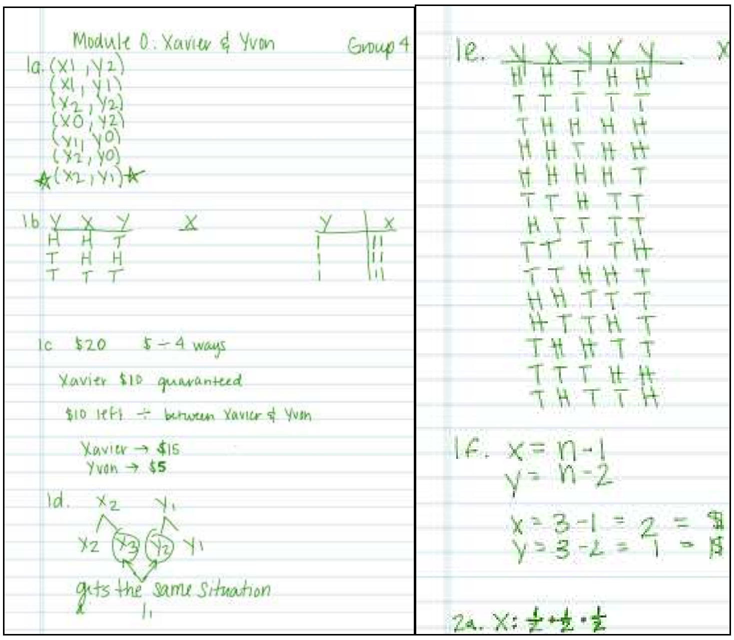

Figure 1. Two-page copy of pdf document resulting from Group 4’s pencast.

Conclusion

It appears that using a Smartpen to support students’ metacognitive thought while engaging in mathematical problem-solving may increase their awareness of metacognitive processes, but will do less to share it with others. Many of the students we encountered were often too self-conscious to share their unpolished thoughts. Perhaps they believe that the classroom teacher or the high achievers in class do not have moments of uncertainty during their attempts to solve mathematical problems. Perhaps we should be sure to have students share portions of their decision making process as they report their solutions to class. Of the four historical roots of metacognition, we believe we approached three: the verbal reports of telling, a hint of self-regulation, and metacognition that comes from interacting with other sources and experiences. However, most difficult to capture—and perhaps a source of greatest insight for helping novice problem solvers—was the executive control (directing/managing thoughts). Our informal experiences with classroom teachers, undergraduates, and high school students seems to indicate support our findings in this study: a pattern of solving problems “off the record” and sharing only best attempts with others. Most of the decisions about what to do and think next is still happening in the minds of students—virtually uncaptured, for the most part—because of fear of imperfection or appearing cognitively disheveled. We would like to suggest that students need many more experiences working in groups to share and defend solutions (through each step of the problem-solving process) to increase the awareness of the many moments of uncertainty during a problem-solving process that may eventually lead to a beautiful solution to a problem. To tap into the metacognitive thought that occurs throughout the problem-solving process we would suggest teachers or researchers sit and observe/question students while they are solving a problem and students should be encouraged to prepare to share a solution—along with accompanying decisions (executive control).

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Conceptualization – a necessity for effective learning of mathematics at school

Gawie du Toit

Faculty of Education University of the Free State, Bloemfontein, South Africa

dutoitgf@ufs.ac.za

Abstract

This paper examined reasons why effective learning does not always materialize in mathematics and more specific in algebra at school level. In an attempt to identify possible reasons why effective learning evades learners a qualitative investigation was performed on students enrolled for mathematics education courses as well as on teachers furthering their studies in mathematics education. The outcomes were compared to possible reasons as portrayed in literature. In this paper the responses of the participants are discussed by analysing their responses to some of the questions posed to them.

Introduction

Teaching mathematics for effective learning was and still is a big challenge to mathematics teachers. Various reasons contribute to this phenomenon. It can be the way in which teachers execute their roles; it can be the confidence systems of learners based on their perspectives of mathematics; it can be the teaching methods used in the mathematics classroom; it can be misplaced outcomes; it can be the text books (incorrect content; way of unpacking the content; etc.) used to teach mathematics.

At first, effective learning and how it fits into the paradigm of social constructivism will be discussed. An investigation into the dichotomy between algorithms and heuristics; procedural knowledge and conceptual knowledge; inductive and deductive reasoning and concept definition and concept image will be address.

There is a saying that states that teachers must research their teaching and then teach what they have researched. This contribution can serve as an example of this saying.

Theoretical background

- **Effective learning**

De Corte and Weinert (1996) identified a series of characteristics of effective and meaningful learning processes which emerged from research that constitute building blocks that can serve as an educational learning theory. Those characteristics about which there is a rather broad consensus in the literature can be summarized in the following definition of learning:

> Learning is a constructive, cumulative, self-regulated, goal-directed, situated, collaborative, and individually different process of meaning construction and knowledge building (De Corte & Weinert 1996: 35-37).

The characteristics of this definition relates to the principles of constructivist teaching as discussed by Muijs and Reynolds (2005). It thus fits perfectly into the framework of reference of social constructivism.

Various authors (Cobb 1988; Hiebert & Wearne 1988; Nieuwoudt 1989; Schoenfeld 1988) stated that research has shown that teachers can formulate good goals, but despite of that there were still core problems that existed in the teaching of Mathematics at school. Learners are not seen by educators as constructors of their own knowledge; Learners cannot relate procedures of manipulating symbols with reality; Learners accept methods taught by educators without any criticism and apply it just like that; The over emphasizing of the answer; The teaching suppresses divergent thinking activities and creativity and problem-solving strategies are not established; are some of the core problems highlighted in the literature.

If this is true, then no effective learning took place if measured against the definition of effective learning by de Corte and Weinert. The dominant role of the teacher, the perspective of learners, misplaced objectives and the teaching methodology used to teach mathematics were identified by these authors as possible reasons that contributed to the existence of the mentioned core problems.

- **Algorithms and heuristics**

Researchers (Suydam 1980) distinguish between two methods of problem solving, namely the algorithmic and heuristic methods. An algorithm is defined as "...a recursive specification of a procedure by which a given type of problem can be solved in a finite number of mechanical steps" (Borowski & Borwein 1989:13). The aim of heuristic is to study the methods and rules of discovery and invention (Polya 1985:112-113). It is evident that self-discovery plays an important role in this method (Schultze 1982:44-45).

Heuristic methods are not rigid frameworks of fixed procedures which provide a guarantee for the obtaining of a solution. The purpose of value thereof lies mainly therein that you search purposefully and systematically for a solution (De Villiers 1986).

These two methods of problem solving clearly differ from one another. For instance, an algorithm ensures success if it is used correctly and also if the correct algorithm is selected and used. Algorithms are problem-specific, while the heuristic method is not problem-specific, because it is normally a combination of strategies. This leads to the fact that a heuristic method is applicable to all types of problems. A heuristic method provides the "road map", a blue print, which leads a person to the solution of a certain problem situation. In contrast to algorithms the heuristic method does not necessarily lead to immediate success (Krulik & Rudnick 1984).

It is important to note that, although the heuristic method could serve as guideline in the solution of relatively unknown problems, it cannot replace knowledge of subject content. Quite often the successful implementation of a heuristic strategy is based on the fixed foundations of subject-specific knowledge (Schoenfeld 1985).

The heuristic way of doing problem solving should play an ever-increasingly important role in the teaching learning situation where problem solving is the focus of teaching. Algorithms, on the other hand, form part of the subject content and are therefore also important. What is of cardinal importance, however, is that an algorithm should be part of the package of knowledge only after it was constructed in a heuristic manner.

Students will empower themselves if they are capable to apply a range of problem solving strategies when confronted with a problem (Schoenfeld 1988). The heuristic method should not be viewed as a goal in itself but must rather be seen as a way in which a certain goal is achieved. Drawing diagrams, for example, should not be taught as a unit in the mathematics classroom, but must rather be used to solve problems where applicable.

Groves and Stacey (1988) consider the strategies as important, especially at the beginning when actual problems are tackled. These strategies give the pupils a degree of control in the process of problem solving and it is important that they should be able to apply it spontaneously without being dependent on the teacher's support (see also Roux 2009).

- **Procedural and conceptual knowledge**

Students and even some teachers have a limited conceptual knowledge span of algebra and it was further found that there conceptual knowledge does not correlate with their procedural knowledge (O'Callaghan 1998; Hollar & Norwood 1999; Roux 2009). Procedural knowledge focuses on the development of skills and can it therefore be deduced that it relates more to the use and application of algorithms (O'Callaghan 1998). Conceptual knowledge on the other hand is characterised by knowledge that is rich in relationships between variables and also including the ability to convert between various forms of presenting functions, i.e. in table format or in graph format, etc. (Hiebert & Lefevre 1986). Conceptual knowledge lends it more to self discovery which relates more to the use of heuristics and inductive and deductive strategies.

Developmental, reinforcement, drill and practice as well as problem solving activities are generally used in the teaching and learning of mathematics (Troutman and Lichternberg 1995). Developmental and problem solving activities lends it more towards the development of conceptual knowledge whereas reinforcement and drill and practice activities lend it more towards the development of procedural knowledge. The advantage to first expose learners to developmental and problem solving activities is that they are challenged to develop conceptual knowledge before being exposed to procedural knowledge (Davis 2005).

RESEARCH QUESTIONS

The following research questions were investigated:

- Is the procedural knowledge which teachers use the outcome of conceptual knowledge?
- Is the procedural knowledge which student teachers use the outcome of conceptual knowledge?

RESEARCH METHODOLOGY

Qualitative research methods were used to find answers to the mentioned research questions. At first mathematics school textbooks and examination papers were analysed to determine to what extend the focus was placed on procedural and/or conceptual knowledge. A questionnaire consisting of mainly algebraic statements was designed based on these findings. These questionnaires were administered by the researcher. The target population consist of different groups of fourth year mathematics education students over the period 2005-2009 as well as practicing teachers who have enrolled for an advanced certificate in mathematics education and/or who have participated in mathematics education workshops (2005-2009). This was thus not a longitudinal study because each year different groups of students were involved in the study. The questionnaire consisted of ten questions. The respondents completed the questionnaire in class and it took them more or less 15 minutes to do so. The responses to each of the questions were either true or false. These responses were noted and the questionnaire was thereafter discussed and debated which contributed towards the reliability and validity of the questions posed in the questionnaire. During these discussions the researcher continually posed questions, obtained answers, and critiques the answers, to obtain a deeper understanding of the thought processes of the respondents.

RESULTS AND DISCUSSION

The outcome of two of the questions will briefly be discussed in the following paragraphs.

- Question 1: If $x^2 = 4$, then $x = 2$

Most of the respondents over the years indicated that $x = \pm 2$ should actually be the answer. The answer $x = 2$ was also indicated as the correct answer by quite a few participants. They arrived at this answer by substituting $x = 2$ into the equation and then found 4 as answer. It was also clear from the discussions that the procedure 'taking roots on both sides' was applied by most of the respondents in solving the equation. This procedure is also advocated by some text books. The participants did not realise that this procedure are actually lowering the grade of the equation from a quadratic equation to a linear equation and by doing that there can only be one answer, namely $x = 2$. The concept of solving $x^2 = 4$ was visualized by representing $y = x^2$(using excel) as a graph. The two values $x = \pm 2$ were identified as the solutions to the equation. The algebraic solution of the quadratic equation $x^2 = 4$ was discussed by solving the equation by means of factorization.

- Question 2: If $x^{\frac{2}{3}} = 4$, then $x = \pm 8$

Most of the respondents guessed the answer, but a few substituted $x = \pm 8$ into

$x^{\frac{2}{3}} = 4$ and concluded that the statement is true. In solving the equation algebraically – as is done in some text books - the statement was found to be true, but when represented as a graph, it was clear that $x = 8$ was the only solution. This once more was an indication that learners are in a framework of mind to follow procedures rather than conceptualise the problem. In question 1 they saw that the solving of the equation provide the solution and therefore argued that it must work in this instance as well. The solving of the equation does provide the solution if the restriction $x > 0$ is applied.

Over the mentioned period (2005-2009) only one student (mathematics on third year level) demonstrated a clear correlation between his procedural and conceptual knowledge. He applied algorithms where applicable but work heuristically when confronted with a situation that seemed unfamiliar to him. The majority of the students applied rules mechanically without reflecting on their answers. Rare evidence of conceptual knowledge was noted. It can thus be concluded that the students focused mainly on concept definitions and did not demonstrated concept images. Conceptualised knowledge was thus not the outcome of the application of procedural knowledge. The demonstration of subject content knowledge was not on a desired level and the same applied to their professional pedagogical knowledge.

The situation was even worse in the case of the teachers. Each group clearly demonstrated that they apply rules mechanically without applying problem solving strategies at all. They in other words did not work heuristically or inductively. A possible reason for this phenomenon could be that these teachers did not receive any training in mathematics at post grade 12 level. Their training in mathematical content was restricted to grade 12 level because they were all trained at Colleges of Education. Their mathematical factual knowledge was at a substandard. It is evident that they will not be able to apply their professional pedagogical knowledge in full when teaching mathematics due to the lack of mathematical content knowledge. It can thus also be concluded that conceptualised knowledge was not the outcome of the application of procedural knowledge for these groups of teachers.

CONCLUSION

Teachers and mathematics education students who participated in this research apply mainly algorithms when solving problems involving algebra. They work deductively and demonstrate procedural knowledge. It can be concluded that the same core problems discussed previously still exist in the teaching of mathematics and more particularly in the teaching of algebra.

It was evident that students and teachers who have participated in this endeavour have a limited conceptual knowledge span of algebra and it was further found that there conceptual knowledge is not in line with their procedural knowledge.

The questionnaire used in this investigation was not discussed in full in this written report because it is the intention to actively involve those who will attend this presentation by exposing them to the questionnaire. In the discussion of each of the questions the aspects as discussed above will be unpacked and debated.

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Meeting under the "Omei" Tree in the Torres Strait Islands: Networks and Funds of Knowledge of Mathematical Ideas

Bronwyn Ewing PhD
YuMi Deadly Centre, Faculty of Education, Queensland University of Technology, Brisbane, Queensland
Bf.ewing@qut.edu.au

This paper focuses on the turning point experiences that worked to transform the researcher during a preliminary consultation process to seek permission to conduct of a small pilot project on one Torres Strait Island. The project aimed to learn from parents how they support their children in their mathematics learning. Drawing on a community research design, a consultative meeting was held with one Torres Strait Islander community to discuss the possibility of piloting a small project that focused on working with parents and children to learn about early mathematics processes. Preliminary data indicated that parents use networks in their community. It highlighted the funds of knowledge of mathematics that exist in the community and which are used to teach their children. Such knowledges are situated within a community's unique histories, culture and the voices of the people. "Omei" tree means the Tree of Wisdom in the Island community.

Background

A recurring theme in government policy and literature on Indigenous parents engagement in education is the importance of parents actively engaging in their children's learning (Department of Education and Training, 2010; Lester, 2004). Today, an important shift is occurring. A critical factor in this shift has been the establishment of community network spaces in Aboriginal and Torres Strait Islander communities where this engagement can occur. Communally, this has resulted in knowledge and learning that is situated within community networks that can be used to support Indigenous parents and their children. Through networks Indigenous parents are wanting to develop new discourses about learning and engaging in education that is community-based but which is also linked to contemporary environments (see for example, O'Connor, 2009). They have used community spaces as a way of locating themselves in their community to learn the knowledge, languages and protocols of their culture.

Indigenous Networks

The idea of networks has not previously been examined extensively in the literature (see for example, McIntyre, McGaw & McGlew, 2008) but is important to investigate. This is because Indigenous parents and networking are interconnected and rely on networks. That is, Indigenous parents belong to community networks, and, in turn, community networks are comprised of parents (Ewing, 2009; Martin, 2007).

Community networks are not static homogenous entities. They reflect values, beliefs, as well as hopes and accomplishments, through the people that members associate and those who are avoided. They are complex and can be mobilizing forces for social justice and the redistribution of power and material advantage (Sefa Dei, 2005). They can exist in a range of combinations: 1) as spatial localised settings that are defined for the pursuit of socially meaningful interactions; 2) as affective and relational communities where members draw on bonds of affinity, and shared experiences of values, attitudes, beliefs, concerns and aspirations and; 3) as moral communities where participation and belonging in a citizenry work to achieve common goals defined as a collective good. Within these communities, tensions, struggles, ambiguities and contradictions are captured; however, the integrity of a collective membership is maintained. Local knowledges are nurtured and made relevant for daily life (Lahn, 2006; Sefa Dei, 2005).

Networks as Funds of Knowledge of Mathematics

Networks build capacity in Indigenous parents (Makuwira, 2007). They validate the parents' own definitions of maths as it exists in their communities—"funds of knowledges" that have been historically and culturally accumulated into a body of knowledge and skills essential for people's functioning and well-being (Moll, 1992, p. 133). The idea of funds of knowledge views that people are competent and have knowledge that has been grown and developed through their life experiences that have given them that knowledge. The claim in this paper is that through working with communities, such knowledge and competence leads to possibilities for Indigenous parents engaging in teaching and learning to support their children. A funds of knowledge approach provides a powerful and rich way to learn about communities in terms of their resources, competence, the wherewithal they possess and the way they utilise these resources to support them with the education of their children.

If funds of knowledge of mathematics are those that reflect the unique histories and culture of Indigenous communities, they provide an effective entry point into engagement in learning because it is connected with and situated in communities and the voices of the people. The process of learning then is owned and framed by the community and its people which then works to build a sense of pride and self worth in individuals (Khamaganova, 2005). Indigenous parents' identities are shaped in their distinct ways in a distinct physical space such as networks, with their knowledges, stories and relationships an integral part and tied to physical locations (Khamaganova, 2005). This relationship ensures that maths is emerging from communities, their networks and funds of knowledge because they are taught and learned in such contexts.

Methodology

The project adopted a qualitative design approach: community research (Smith, 1999). Community research is described as an approach that "conveys a much more intimate, human and self-defined space" (Smith, 1999, p. 127). It relies upon and validates the community's own definitions. As the project is informed by the social at a community level, it is described as "community action research or emancipatory research", that is, it seeks to demonstrate benefit to the community, making positive differences in the lives of one Torres Strait Islander community. The researcher established strong working relationships with the parents and community members over time. This paper focuses on the illuminative moments that worked to transform the researcher and provide turning point experiences that resulted in a cumulative sense of awareness before consent to conduct the project was given. Embarking on this preliminary process in close collaboration with the community was a challenge intellectually, linguistically and geographically.

A geographic excursion

The Torres Strait Islands consist of eighteen island and two Northern Peninsula Area communities (Torres Strait Regional Authority, 2010). They are geographically situated from the tip of Cape York north to the borders of Papua New Guinea and Indonesia and scattered over an area of 48, 000 square kilometres. There are five traditional island clusters in the Torres Strait: top western, western, central, eastern and inner islands. The research project was conducted at one site in the eastern cluster.

Participants

Twenty adults and eight children took part in the community consultation meeting. All live at the site where the meeting was held. For ethical reasons pseudonyms have been used to protect the identity of participants. The location is referred to as the Island.

Data collection

For the purposes of this paper, the data collection techniques included: digital photography, field notes and email documents. Digital photography as a non-written source of data allowed for the capturing of visual images that were central to the preliminary process and which served as a reminder for the researcher (Stringer, 2004). They also assist audiences to more clearly visualise settings and events. Field notes provide descriptions of places and events as they occur. They provide ongoing records of important elements of the setting and assist with reporting and reflecting back over events. Email documents provided an efficient and easy form of community between participants who then networked with their community. Each technique afforded the value of insight into the important preliminary planning of the project.

Analysis and discussion: learning about the community's networks and funds of knowledge

In recent years, building on what communities bring to particular contexts and on their strengths has been shown to be effective with engaging with communities (GonzÃ¡lez & Moll, 2002). How does this occur? A way to engage community was to draw them in with knowledge that was already familiar to them, and which then served as a basis for further discussion and learning. However, with this process there was the challenge and dilemma. How did the researcher know about the knowledge that they brought to the meeting without falling into stereotyping their cultural practices? How did the researcher address the dynamic process of the lived experiences of the community? The responses to these questions have emerged from community-based research that relies on the community's definitions.

The Community Meeting

Prior to the official commencement of the project, preliminary community meetings were held to discuss the project's intention. The process of networking within the community developed through several steps: a) discussion of the project with school campus leader, b) discussion with the Island Councillor and to seek permission to meet under the Omei Tree, c) a chance meeting with the Radio Announcer for the Island radio that resulted in an interview that was broadcast to the Island community, d) with support from one Senior Woman, Denise, and a parent from the community, a paper-based flyer was delivered in person to the homes of Island parents to inform them face-to-face about a proposed community meeting (Figure 1) and, d) community meeting held under the Omei Tree (Figure 2).

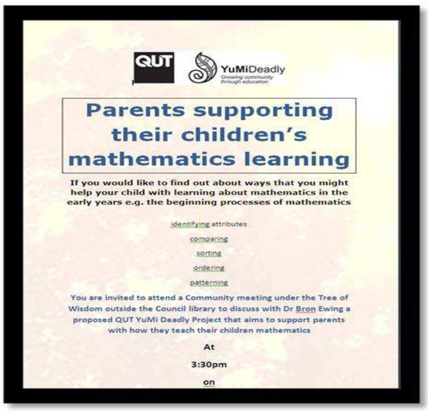

Figure 1: Proposed community meeting flyer

Figure 2: Community meeting held under the Omei Tree

The content of the flyer was brief and aimed to provide succinct information. The tree is a large fig tree believed to be over 100 years old and has been a significant meeting place for the Island community.

During the meeting the researcher explained the project and how participants might be involved. Gaining consent was respectful of the community's place and environment as also was that as a visitor, the researcher needed to be mindful of her actions and presence in the community and conduct herself in an ethical manner. She explained some of the early number ideas using shells, sticks, leaves, and poinciana pods gathered from the community. Verbal permission was sought from one senior woman, Julia, in the community to collect some shells from the beach front near her home to be used in the meeting.

As the meeting progressed, early number ideas were discussed, for example the beginning process of sorting or classification. Children learn to sort objects into sets, such as shells, from their environments into groups. They learn to identify sameness that defines the characteristics of groupings (Sperry Smith, 2009). In the meeting, Denise volunteered to sort shells into groups (see Figure 3). The larger community group and the researcher had to identify what criteria were used for the groupings.

The idea of creating and naming sets continues throughout life and is a way of creating and organising information and making connections with peoples' experiences. Before young children can learn to count sets, they begin the process of defining a collection using the objects in their daily lives (Baroody & Benson, 2001). Denise established the features of each of the sets of shells. If the criteria for membership to a set are vague, it is more challenging to decide whether the shells belong to a particular set. Meeting members talked amongst themselves, with the Denise allowing them time to identify the features of the sets.

Figure 3: Sorting shells into sets

The researcher could not identify the criteria that defined the sets; however, there was consensus amongst community members that criteria had been established—edible and non-edible creatures that live in the shells. In this example, the community used their daily lives and activities as an opportunity to talk about sorting using their home language—Creole, and English. When asked when children learn about edible and non-edible shells there was consensus that this occurs very young, for example, one to two years of age, and during times when families walk along the shores of the Island and when fishing or playing in the water. This example reinforces what Nakata (2007) and Moll (2002) state, that learning can be rich and purposeful when it is situated within that which already exists— the culture, community and home-language of the group.

As the meeting progressed, discussions lead by the researcher focused on an introduction to early algebra—patterning. Patterns are a way for people to recognise and organise their lives. In the early years, two particular pattern types are explored: repeating patterns and growing patterns. They are used to find generalisations within the elements themselves (Warren & Cooper, 2006). What comes next? Which part is repeating? Which part is missing? Repeating patterns are patterns where the core elements are repeated as the pattern extends. Young children recognise patterns when singing songs, when dancing, learning how to weave and when playing. Some examples of repeating patterns include:

ABABAB

AABAAB

Repeating patterns can be represented with actions: jump, hop, jump, hop; as sounds: bell ring, clap hands, bell ring, clap hands; as geometric shapes: triangle, square, triangle, square; and as feel: soft, hard, soft, hard. Generally children explore patterns in a sequence: copy a pattern, continue the pattern, identify the elements repeating, complete the pattern, translate the pattern to a different medium.

Using two poinciana pods, the researcher tapped them together to create a repeating sound pattern. The community was then asked to continue the pattern and identify the repeating elements, using clapping. They were then asked if they would like to offer a repeating pattern. One community member clapped a pattern to which the remaining members responded. The community were then asked where they might see or use patterns in their communities. Responses included: the seed pattern inside the poinciana pod (ABABAB), the weaving pattern used to weave coconut leaves together (ABABAB), when singing songs to the children, the seasons of the year and how the winds, seas, sea life, plant life, bird life work in a repeating cycle with many core elements. It was this final response that reminded the researcher of her place within the community, a visitor who had a great deal to learn from community about patterns and how they are evident in their natural environment. It also reminded the researcher that the community have extensive knowledge of patterns because they exist in their everyday lives in very rich ways—funds of knowledge, knowledge that researcher did not possess.

In concluding the meeting, the community were asked if they would like the researcher to plan and trial some early mathematics workshops with the parents and children. In doing so, she also stated that she had a lot to learn from community, their funds of knowledge—learning was to become two-ways for the researcher as well as the community. Of importance was that the community needed time to network and discuss whether they wanted the researcher to return and work with parents and children on the Island.

Conclusion

As outlined in this paper, the process of networks within the Island community has been significant for the researcher's engagement with the community and to respond to research as described by Smith (1999). The researcher found this imperative to be somewhat of a challenge because she had to rely on the community's networks which the researcher did not belong. However, what did become evident was that the networks comprise of parents and they use such networks for communication. What was also evident was the networks were used as a valuable source for nurturing local knowledges and how these are used in the Islanders' daily lives. At the meeting, the community validated their own definitions of knowledge—sorting and patterning. In doing so, this process provided a rich way to represent their knowledge, competence, and wherewithal that they possess to support them with educating their children.

At this stage, the researcher can report that after providing time for the community to network and consult about whether to permit her to conduct the parents and children mathematics workshops, the process has allowed her to:

1. Have confidence in the way that the researcher has consulted with community and the community with the researcher;
2. Continue to work with community;
3. Continue to work with the materials similar to those gathered for the meeting;

4. Frame the project's agenda that situates research as reflexive engagement with the real—the community's funds of knowledge as well as her own;
5. Understand how pluralism is about respect for diversity and a willingness to explore and change in ways that continues to remain diverse for situated learning.

These points relate to the processes of engagement and community consultation and research which are envisaged as continuing once the project officially commences.

Acknowledgment

The author respectfully acknowledges the community from where this paper has emerged. She is duly thankful for their support and enthusiasm with such a project.

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Problem solving: A psycho-pragmatic approach

Paul Giannakopoulos, BSc, HEd, BEd, MEd
Faculty of Eengineering, University of Johannesburg
Johannesburg, South Africa
paulg@uj.ac.za
Sheryl B. Buckley, BEd, MEd, D Litt Et Phil
Faculty of Business Management, University of Johannesburg,
Johannesburg, South Africa
sherbuck@gmail.com

Abstract

The aim of this paper is to present an alternative approach to problem solving in general and mathematics in particular. Problem solving has become very prominent especially in the last two decades due to a number of reasons: the move from an industrial society to a knowledge society; globalisation; complexity in management systems; and technological innovations, to name just a few (Halpern, 1997; Pascarella & Terenzini, 1991; Bérubé & Nelson, 1995). An analysis of the various definitions and research indicates that problem solving: involves a number of cognitive processes (e.g. analysis, synthesis, comprehension and so on); it requires a certain mode of thinking for different problematic situations; and it is a skill that needs to be developed over a period of time (Green & Gillhooly, 2005:347; Halpern, 1997:219).

Behaviourism and cognitivism learning theories have been used in the teaching and learning situation but with mixed results. Pragmatism on the other hand has started 're-emerging" due to a number of reasons. The most important reason that was "rejected": Emphasis on the practical application of knowledge. Mathematics, unlike other subjects, by its nature is a subject whereby problem solving forms its essence but paradoxically is "accused" of being too abstract. However, there can be no mathematics without problem solving. A generic model is developed where by the psychological theories as well as pragmatism are used in the teaching and learning situation.

Introduction

As the world economy moved from an industrial economy to a knowledge economy, it can be argued that the nature of many problems also changed and new problems have arisen which may require a different approach to overcome them. Certain problems remained unchanged for the human being. For example, human beings that are born, they have to learn how to walk, run, speak and so on. Educational institutions and governments have recognised long ago the importance of problem solving and volumes of research have been written about problem solving (Pushkin, 2007; Astin, 1993; Halpern, 1997; Pascarella & Terenzini, 1991; National Education Goals Panel, 1991; South African Qualifications Authority (SAQA), 1998; National Council of Teachers of Mathematics [NCTM], 2005). Universities and other higher learning institutions are entrusted with the task of producing graduates that have such higher order thinking skills among other skills (Pushkin, 2007; Astin, 1993).

Theories about learning in general and problem solving in particular have also taken cognizance of these changes and old theories have been revisited and modified if necessary while new ones have come to existence. Behaviourism, cognitivism and their variations dominated and still dominate education where depending on the style of the teacher he or she uses more than one and less of the other. Since learning is about knowledge, the difference between the two theories lies on the fact that the one views learning as acquisition of knowledge while the other as knowledge construction. Both theories recognise that problem solving is essential for all learners (Johansen, 2003) especially complex problems (DeCorte,

Verchaffel & Masui, 2004; Gareth, Weiner & Lesgold, 1993 cited by Sigler & Talent-Runnels, 2006). However, the 'how' to teach and 'what to teach' to achieve this is still debatable.
Pragmatism on the other hand has started 're-emerging" due to a number of reasons. The most important one being the main reason that was "rejected": Emphasis on the practical application of knowledge. Mathematics, unlike other subjects, by its nature is a subject whereby problem solving forms its essence but paradoxically is "accused" of being too abstract. Howeve,r there can be no mathematics without problem solving. There is evidence (Luneta, 2008; Pushkin, 2007; Williams, 2005) that success in mathematics is directly related to academic success as mathematics is contained in other subjects to a greater (e.g. physics, strength of materials, thermodynamics and so on) or a lesser extent (e.g. history, geography, psychology and so on). A generic model is developed where by the psychological theories as well as pragmatism are used in the teaching and learning situation to improve problem solving and academic success.

Learning theories

For the last twenty years behaviourism, cognitivism and variations of them have shed some light to the phenomenon of learning in general and in mathematics in particular. Hergenhahn & Olson (1997) discuss the various learning theories in detail. Briefly, according to behaviourism learning brings change in behaviour as we learn through experiencing the world. By responding to the environment we learn (stimulus-response phenomenon). However, there are internal processes that are involved in learning. It is cognitive psychology that is concerned with these various mental activities (such as perception, thinking, knowledge representations and memory) related to human information processing and problem solving (Shuell, 1986 cited in Hergenhahn & Olson, 1997).
Cognitive psychology like behavioural psychology has given rise to a number of different perspectives such as constructivism and variations of it such as, personal (Kelly (1955) and Piaget (1972), radical (Von Glasersfeld, 1985, 1987a, 2002), social (Vygotsky (1978), critical (Taylor) and contextual (Cobern) (all cited in Venter, 2003). Parsons, Hinson & Sardo-Brown, (2001:431) define constructivism as a "cognitive theory emphasising learner interest in and accountability for their own learning which manifests in student self-questioning and discovery." In a way it was a reaction to the traditional way of teaching that gave rise to it. Constructivism is a theory of how learning occurs (Henson, 1996 cited by Parsons et al., 2001) rather than the product of such learning. This learning theory is student-centred where learners are actively involved in constructing their own knowledge and making use of past experience or prior learning or pre-existing schemas.
However, John Dewey saw existing theories to be too theoretical and lacked practicality, applicability, usefulness. So pragmatism was born. It endorses practical theory (theory that informs effective practice; praxis). And according to pragmatism knowledge is validated by its usefulness: What can we do with it! Of late pragmatism (Johnson & Onwuegbuzie, 2004; Schaffler, 1999) has started gaining ground again as it is felt by some educationists we have become too theoretical again. The pragmatic view stresses the experimental character of the empirical science, emphasising the active phases of the experimentation. It promotes an inquiring mind with respect to physical laws. "Inquiry itself is action, but action regulated by logic, sparked by theory, and issuing answers to motivating problems of practice" states (Scheffler, 1999:4). For pragmatism knowledge is viewed as being both constructed *and* based on the reality of the world we experience and live in. Learning from experiences is an active process. The mind is viewed as a capacity for active generation of ideas whose function is to resolve the problems imposed to the organism (the human being) by the environment. Pragmatism encourages imaginative

theorising by the student but at the same time insists upon control of such theorizing by the outcomes of active experimentation (Scheffler, 1999).

The psycho-pragmatic approach to teaching and learning in problem solving

This approach combines behaviourism and cognitivism (neocognitivism, the psychological aspect) and pragmatism. Pragmatism could be seen as the missing link between the other two theories as it promotes learning from experiences which is an active process. For this reason, this new paradigm gets its name "Act of learning" (see Figure 1), which gives rise to 'thinking while doing' and 'doing while thinking'. Thus the constructs 'thinking' and 'doing' form the two pillars of the learning. So the first step in the teaching – learning situation is the promotion of these two actions. But thinking can be of low (concrete) or high (abstract) order. Through thinking and doing, concepts are formed in a conscious as well as subconscious way. But concepts can be primary or secondary. Primary concepts give rise also to other secondary concepts. Concepts can be concrete or abstract and as concepts are connected and form a web of connections, principles are formed. This is the second step in teaching – learning situation, concept formation, a very important prerequisite to problem solving. Various principles make up the structures of knowledge as knowledge is either acquired and assimilated in the existing cognitive structure or knowledge is constructed and a re-structuring of the cognitive structure takes place. Experience adds a new dimension to existing knowledge as a result tacit knowledge also becomes part of the cognitive structure. Knowledge can be of different types such as procedural (knowing how), declarative or conceptual (knowing 'that'), schematic (knowing 'why'), and strategic (knowing 'where, when and how') (Hiebert & Lefevre, 1986; Shavelson, Ruiz-Primo & Wiley, 2005; Stolovitch & Keeps, 2002).

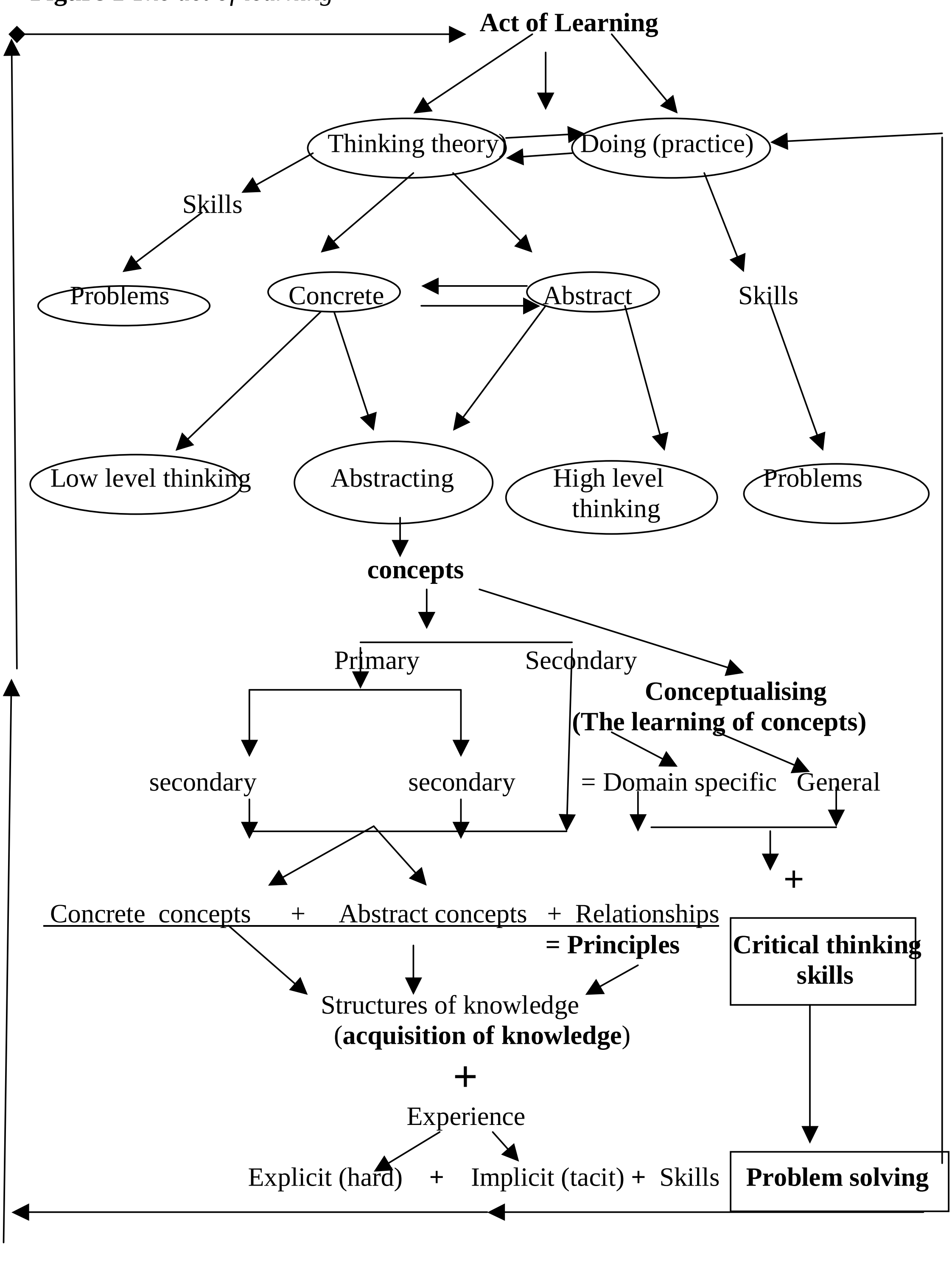
It can be said knowledge remains dormant in the cognitive structure and it comes to life when a problem is encountered, when the knowledge has to be applied into a real situation, the pragmatic aspect. If a problem is defined as an 'obstacle' on the path of an individual to a goal, then problem solving is finding a way out of a difficulty, a way around an obstacle, attaining an aim that was not immediately understandable (Polya, 1973). Green and Gilhooly (2005:347) state that 'problem solving in all its manifestations is an activity that structures everyday life in a meaningful way.'

Knowledge can be viewed as a 'tool' used to solve problems. But possessing any tool (be it a drill, a spade, a welding machine and so on) is not sufficient. One has to have the skill to use it. A skill is defined as expertise developed in the course of training and development (Malone, 2003). "A skill is an ability to do something well, to competently perform certain tasks" and "they (skills) consist, in part, of methods and strategies that have been incorporated into a performance routine" (Smith, 2002). Ferrett (2008:467) associates skills with capabilities that have been learned and developed. Skills include trade or craft skills, professional skills, social and sporting skills or more broadly motor skills and cognitive skills. And one of the most important cognitive skills is use of critical thinking (see Figure 1) which precedes problem solving. This is the third step in the teaching – learning situation. A literature review though on critical thinking and problem solving revealed that there is no conclusive evidence that critical thinking is a prerequisite to problem solving or vice versa (Papastephanou & Angeli, 2007). From a mathematics perspective, irrespective of the type of problem to be solved, in every step of the solution, given information needs to be critically examined.

The pragmatic approach then to problem solving creates a new approach to problem solving (see Figure 2).

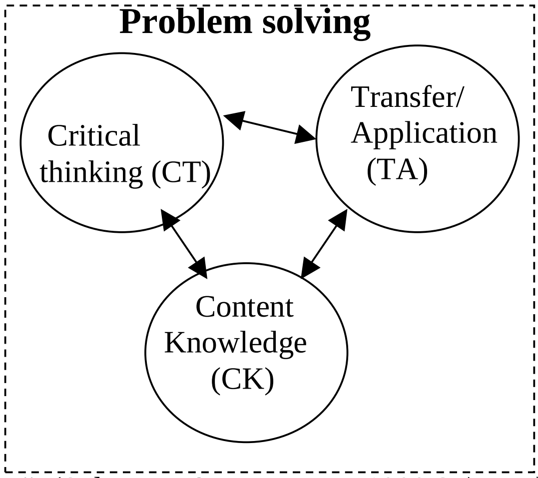
Problem solving then is an outcome of a number of co-ordinated cognitive processes. Existing knowledge combined with critical thinking skills are applied to a real problematic situation. Irrespective of the problem solving method to be used, from the simplistic Polya's approach (understanding the problem, devising of a plan, carrying out the plan, looking back (evaluating)) to a more comprehensive model of Sternberg (2007) (see Figure 3) every step requires a certain type of knowledge and information to be critically examined.

Figure 1 *The act of learning*



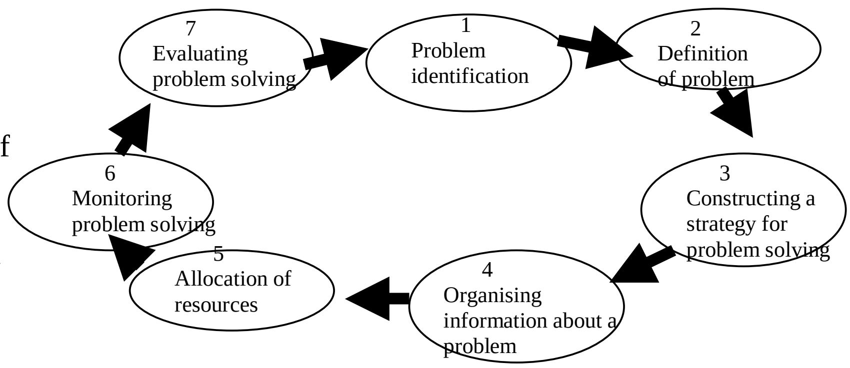
However, the success of implementing such teaching-learning approach depends on the two main role players, the teacher and the learner. Briefly speaking, the teacher has to become what Solomon and Morocco (1999) call a diagnostic teacher. Diagnosing goes beyond finding out what the learners know. It is about understanding students' "*particular* thinking patterns, current understandings, or misconceptions" (Solomon & Morocco, 1999:234). A teacher who concentrates on diagnosing focuses on the individual rather than the class. Diagnosing is non judgemental. It concentrates on trying to understand student's understanding and "assessment is part of a recursive cycle of observation, selection of teaching strategies, reflection, and readjustment of one's strategies" (Solomon & Morocco, 1999:34). The learner, has to be willing to learn and if necessary to restructure his or her cognitive structure (Vosniadou, 1999). Furthermore the teacher has to become a researcher. Structural equation modelling (SEM) (as discussed in detail by Ullman and Bentler (2004) can be of great help when he or she evaluates their teaching.

Figure 2. *A problem solving conceptual model*



Finally, the 'Act of learning' is of cyclic as well as of a spiral nature. Cyclic, by the virtue that the learning of concepts and problem solving go through the various steps; spiral, due to the fact that concepts are acquired in higher levels in each cycle.

Figure 3: *Problem solving model, Sternberg(2003: 361)*



Conclusion

The above exposition introduced a new approach to problem solving, which can be applied in any situation, any subject. This approach complements the behaviourist and the cognitivist approach to learning by combining them with pragmatism. Pragmatism brings problem solving to life by applying knowledge to real situations. However, the success of it depends mainly on the teacher becoming a diagnostic teacher.

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Reflecting Problem Orientation in Mathematics Education within Teacher Education

Günter Graumann, Prof. Dr.
Retired Professor of Mathematics and Didactic of Mathematics
Faculty of Mathematics, University of Bielefeld
Universitätsstrasse 25, D-33615 Bielefeld, Germany
graumann@mathematik.uni-bielefeld.de

Summary: Problem orientation is an important aspect within mathematics education we all know. But we also know that problem orientated mathematics teaching is practices in school reality rarely. To get a change the paradigm of mathematics teaching need a transformation.
I think besides theoretical discussions within the didactical community and the presentation of interesting proposals for different classroom conditions - which are important - first of all the teachers must be familiar with and get a positive belief about problem orientation in mathematics education.
In my presentation I will report about experiences in respect to problem orientation I did make with teacher students at our university in Bielefeld.

Problem orientation is an important aspect within mathematics education we all know. It is demanded since more then fifty years. The students first can learn a lot of mathematics on this way. Secondly they learn mathematics as a process. But most important is that they can train several general goals as
- how to handle a problem, how to make investigations,
- ability of problem solving and finding analogies
- why and how to order and systematize a given situation
- positive belief about mathematics and self-confidence .

But in most countries its realisation in school leaves a great deal to be desired. Therefore it is important to support a teaching which contains a lot of acting with problems.
For this - besides the necessary theoretical discussions within the didactical community and the presentation of interesting proposals for different classroom conditions - ***first of all the teachers must be familiar with and get a positive belief about problem orientation in mathematics education***. The teachers must have own experiences with working on single problems as well as problem fields. Moreover they must know about and reflect upon fundamental ideas about heuristics, problem solving and problem finding or variation of problems. Moreover they should be able to find different ways of working with a problem and be open for new approaches used by the students.
Here I will report about experiences in respect to problem orientation I did make with teacher students at our university in Bielefeld.

Survey of four seminars

Besides integrating some theoretical and practical aspects about problem orientation into all my lectures and seminars in the last time I offered four special seminars concentrating on problem orientation in mathematics education. Three of these seminars have been seminars for preparing teacher students for writing a final Bachelor paper. For all of these about 20 students per seminar I determined that the theme of each paper should refer to problem orientation. The other seminar was a normal one for senior teacher students within their Bachelor study. In all of these three seminars on one hand the students had to work by themselves with different problems and find new problems within the discussed problem field – I here already will mention that the last task was very hard for the students – and on the other hand I made some inputs with copies out of literature. Of course they also had to reflect on mathematics teaching with special concern to problem orientation.

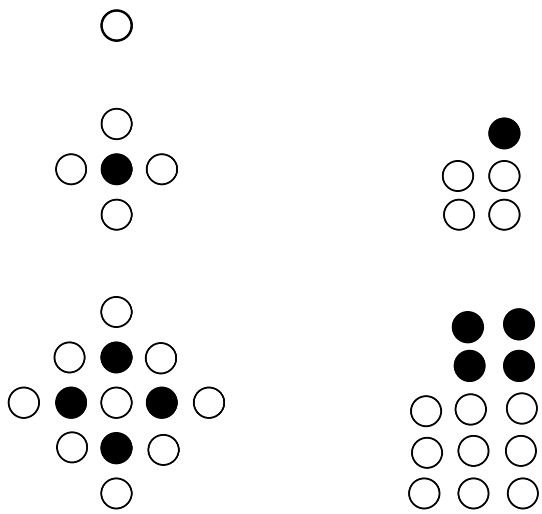
Before going into details I would like to give an overlook of the topics we discussed. In all of the seminars in the first session (lasting one and a half hour) I presented *three different problems to work on by themselves* (together with their neighbours) without giving any help or hint. I will report on the results of this session later on.
Lateron we discussed with help of literature and internet the following themes: *History of problem orientation as well as learning by discovery, learning by doing and self regulation, definitions of "What is a problem", methological hints for problem solving, heuristics and problem orientation* and also *types of tasks, types of problems and ways of developing tasks by your own, self-activity and self-regulation in the discussion of didactics of mathematics within the last ten years* (cf. e.g. Pehkonen & Graumann 2007, Büchter & Leuders 2005 and Polya 1949).
We also deepened some theoretical aspects like *variation of taks* (according to Schupp 2004), "*logic of failure*" (according to Dörner 1989), *mathematical learning from constructive view, beliefs about problem orientation, barriers in respect to changing mathematics teaching, aims of and motivation for problem orientation, statements according to problem orientation in official guidelines*.
Mathematical topics considering the aspect of problem orientation have been e.g.:
- *Figured numbers and special sums, Partition of sets and representations of numbers as sums in grade 1, sequences and chains of numbers, number trains, number walls* (cf. Graumann 2009)
- *Magic squares and sudoku - a topic for grade 2 and 4, Polyominos – a geometrical problem field for grade 3, distribution of prime numbers in grade 3, division with rest in grade 4*
- *Triangles with integers as side length, regular polygons and polygons in space, Pythagorean triples,*
- *problems from PISA, Fermi tasks*

The "Mason problem" as inspirer
Some years ago I did hear from a seminar in Debrecen where John Mason as guest was present. In this seminar John Mason asked the participants (mathematics teacher students and secondary mathematics teacher) to solve the following problem.

$1 =$

$1 + 3 + 1 =$

$1 + 3 + 5 + 3 + 1 =$

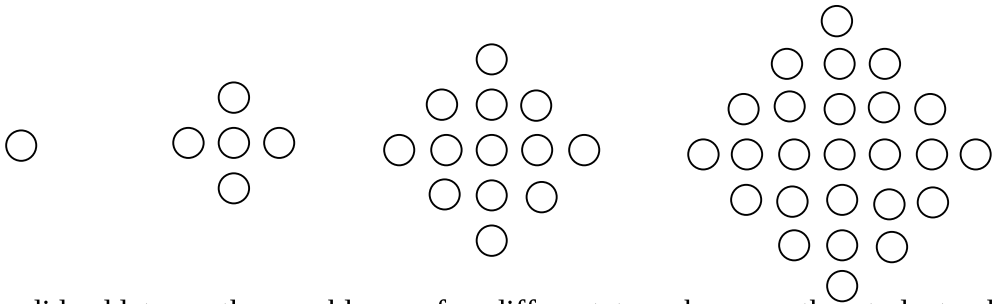


Task: "*Formulate a general question for this problem! Try to formulate a conjecture to your question! Prove the conjecture!*"

After 10 to 15 minutes it was clear that such open problems are very uncommon to Hungarian students and teacher, most of them could not do anything. This caused me to present the "Mason problem" to my students at the beginning of the seminar so that they are not influenced by discussions within the seminar (even though the title of the seminar may have influence and in their first semester they have already seen simple figured numbers like square numbers and triangle numbers). I wanted to know how they will act on this problem.

I varied the presentation of the Mason problem a little bit in that way not giving symbolic hints in respect to sums of numbers and not painting the little circles different because I wanted to see how the students will do it by themselves and whether they will discover different structures.

Two rows of the figures shown below have been given with the following text: "*Draw the next two figures of this sequence. Look out for partial figures and mark them. Which arithmetical representation do you can find on this way?*"



I also did add two other problems of a different type because the students should make experiences with different types of problems too. These two other problems have been given in text in the following way.

Problem concerning a ghost of a river: *A ghost of a river says to a walker who just will cross a bridge: "If you cross the bridge I will double the money you have in your pocket; but if you go back across the bridge I will take 8 Euros away from your pocket." When the walker came back the third time the money in his pocket was gone away (exactly 0 Euros).*

Problem concerning small animals: *Grandfather Miller has in his yard hens and rabbits. Once upon a time he counted 7 heads and 20 legs. (Variation: He did count only 20 legs).*

Different results of these two problems given in text

First of all I can tell that all of the students except one in the discussion at the end of the first double hour reported that an open problem like that from Mason was very new for them. But all of them (at least together with a neighbour) started to work on that problem and most of them got at least one special result. And it could be noticed that in any of the seminars There appeared different ways of working with these three problems.

In the following I will present all different ways of working with these problems; in doing so I will start with the two problems given in text.

Problem concerning a ghost of a river

1. Method of trial and error with variation: We start with 6. The transformation from this are $6 \rightarrow 12 \rightarrow 4 \rightarrow 8 \rightarrow 0 \rightarrow 0 \rightarrow$ not possible. We see that we have to get 6 after the first crossing and the way back. Thus we try it with two more Euros and get $8 \rightarrow 16 \rightarrow 8$

$\rightarrow 16 \rightarrow 8 \rightarrow 16$ and find an endless sequence. Now we try the number between and get the solution $7 \rightarrow 14 \rightarrow 6 \rightarrow 12 \rightarrow 4 \rightarrow 8 \rightarrow 0$.

(By varying our thoughts we could start with 5 or 4 and see that all numbers of the sequence decrease just like starting with 9, 10, … will let increase all numbers of the sequence. That means we may find a functional relation.)

2. Method of working backwards: $0 \leftarrow 8 \leftarrow 4 \leftarrow 12 \leftarrow 6 \leftarrow 14 \leftarrow 7$.
3. Method with algebraic formula: $2 \cdot (2 \cdot (2 \cdot x - 8) - 8) - 8 = 0$ or

$x \rightarrow 2x \rightarrow 2x\text{-}8 \rightarrow 2\cdot(2x\text{-}8) \rightarrow 2\cdot(2x\text{-}8)\text{-}8 \rightarrow 2\cdot(2\cdot(2x\text{-}8)\text{-}8) \rightarrow 2\cdot(2\cdot(2x\text{-}8)\text{-}8)\text{-}8$

From $2\cdot(2\cdot(2x\text{-}8)\text{-}8)\text{-}8 = 0$ we will get $x = 7$.

We can get a generalisation via this method with $2 \rightarrow a$, $8 \rightarrow b$ and $3 \rightarrow n$:

$x \rightarrow ax \rightarrow ax\text{-}b \rightarrow a\cdot(ax\text{-}b) \rightarrow a\cdot(ax\text{-}b)\text{-}b \rightarrow a\cdot(a\cdot(ax\text{-}b)\text{-}b) \rightarrow a\cdot(a\cdot(ax\text{-}b)\text{-}b)\text{-}b$

$\rightarrow a\cdot(a\cdot(a\cdot(ax\text{-}b)\text{-}b)\text{-}b) \rightarrow a\cdot(a\cdot(a\cdot(ax\text{-}b)\text{-}b)\text{-}b)\text{-}b \ldots \rightarrow a^{n}\cdot x - (a^{n-1} + a^{n-2} + \ldots + 1)\cdot b$

Problem concerning small animals

1. Method of trial and error with variation: We try 4 rabbits → 16 feet, with the left 4 feet we get 2 hens; that make together 6 heads. Because one head is undercharged we have to increase the number of hens respectively decrease the number of rabbits. 3 rabbits → 12 feet and 4 hens → 8 feet gives 7 heads and 20 feet as desired.
2. Method of working backward from the heads: Any animal has at least 2 feet, so with 7 heads we have at least 14 feet. The rest of 6 feet is going in pairs to 3 rabbits, so we get 3 rabbits and 4 hens.
3. Method with algebraic formula: x = number of rabbits, y = number of hens. $4x + 2y = 20$ (number of feet) and $x + y = 7$ (number of heads) . Then solving with algebraic instruments gives $x = 3$, $y = 4$.

Variation of this Problem

The variation shall show the students that we also can get a problem that has more than one solution and we have to find a systematic for finding all solutions.

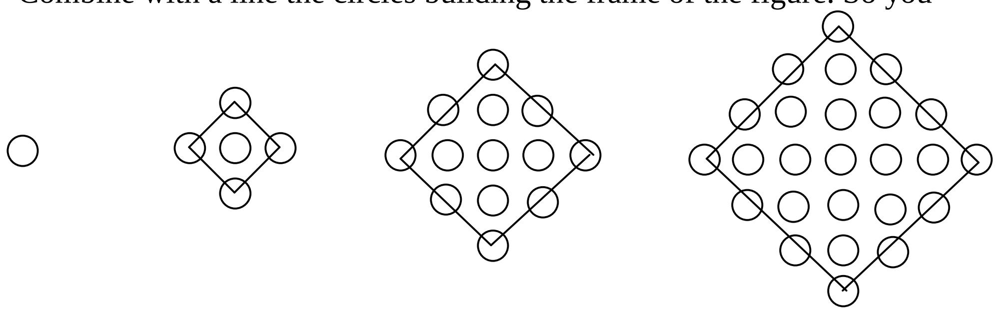
Here the minimal number of heads is 5 because 5 rabbits and 0 hens makes 20 feet. Reducing the number of rabbits step by step you will get 6 heads (4 rabbits and 2 hens), 7 heads (3 rabbits and 4 hens), 8 heads (2 rabbits and 6 hens), 9 heads (1 rabbit and 8 hens), 10 heads (0 rabbits and 10 hens). The solution with 0 hen and that one with 0 rabbit probably does not fit to the text and thus these solutions have to be erased.

In addition we can make investigations in respect to functional relations like "Reducing the number of rabbits by one causes increasing the number of hens with two" or "Reducing the number of heads by one causes decreasing the number of hens with two".

Different ways the students worked with the "Mason Problem"

The following different groupings by colouring some circles or combining some circles with a line and symbolic descriptions have been the following:

1. Combine with a line the circles building the frame of the figure. So you

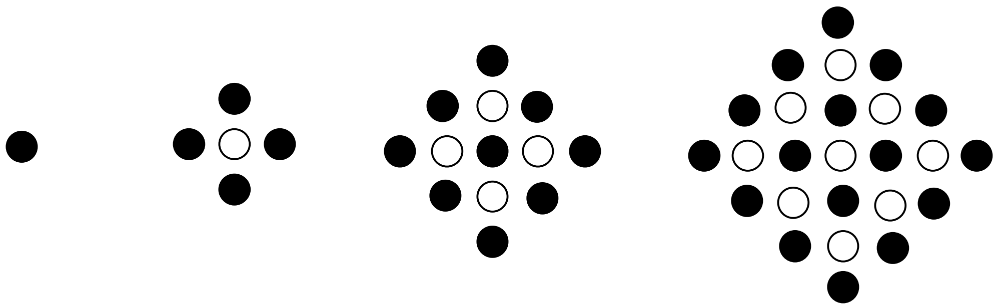


get the description 1, 1+4, 1+4+8, 1+4+8+12, … and in general $1 + 4 \cdot [1+2+3+\ldots+(n-1)]$.

If we already know that $1+2+3+\ldots+(n-1) = \frac{1}{2} \cdot (n-1) \cdot n$ we will get the general symbolic description $1 + 2\cdot(n^2-n)$ [resp. $2n^2-2n+1$].

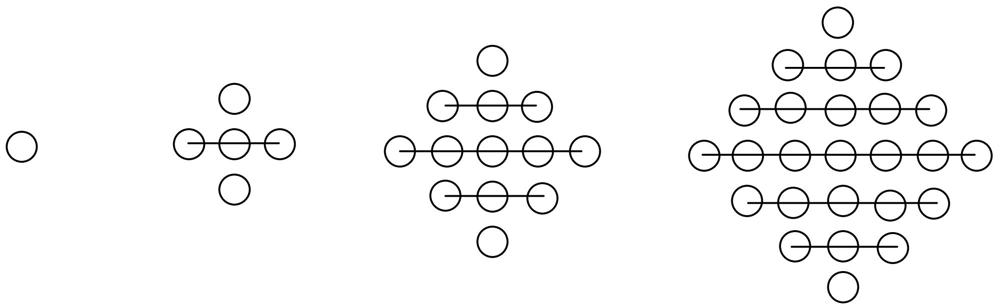
2. In a more arithmetical view on these figures some students looked at the respective number of circles: 1, 5, 13, 25, … . From this they detected that the difference sequence is built by the multiple of 4. On this way they got the same general description as above.

3. Some students coloured the circles in the frame together with inner circles for getting a squared number. The non-coloured circles then built a squared number too but a smaller one, more precisely the length of the side is one less than the length of the side of the coloured square.



The symbolic description thus came out as 1^2, 2^2+1^2, 3^2+2^2, 4^2+3^2, … or in general $(n-1)^2 + n^2$ [respectively $2n^2 - 2n + 1$].

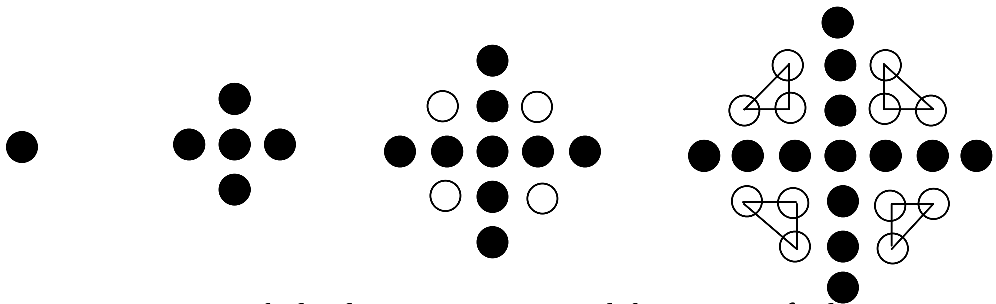
4. A fourth group of students looked at the horizontal (or vertical) rows and got the symbolic description 1, 1+3+1, 1+3+5+3+1, 1+3+5+7+5+3+1, … .



If we already know that the numbers 1, 1+3, 1+3+5, 1+3+5+7, … describe a square number we can see the identicalness with the symbolic descriptions above.

5. One student coloured the vertical and horizontal middle lines building a cross. The non-coloured circles then build four triangle figures and we get 1, $1 + 4\cdot1$, $1+ 4\cdot2 + 4\cdot1$, $1 + 4\cdot3 + 4\cdot[1+2]$, … .

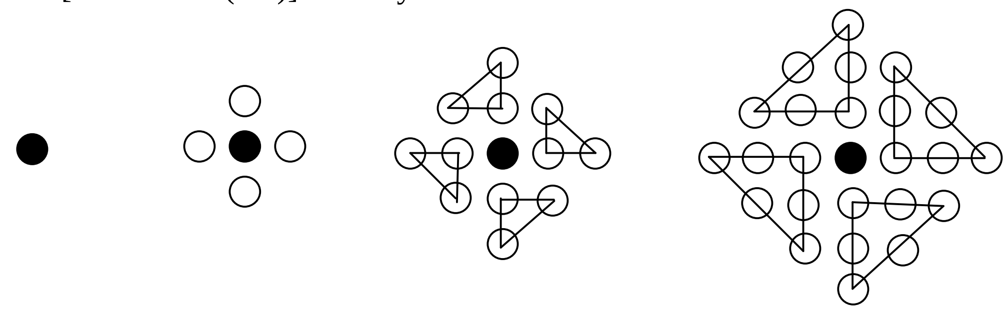
6.



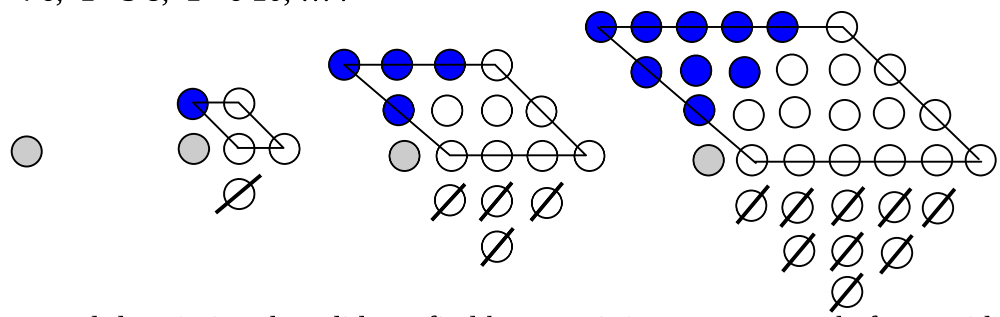
If we want to get a symbolic description in general then we can find

1+4·(n-1)+4·[1+2+3+…+(n-2)]   resp. 1+4·[1+2+3+…+(n-1)].

7. Two students also saw four triangle numbers but including always one branch of the cross so that only the middle point is extra standing. This leads to the formula 1+4·[1+2+3+…+(n-1)] directly.



8. Another group of students resorted the figures of circles by erasing the rows below the horizontal middle line and then added these circles on the left side of the remaining circles so that after that all horizontal line have the same length without the bottom line which has one circle more. So they got the sequence 1, 1+ 2·2, 1+ 3·4, 1+ 4·6, 1+ 5·8, 1+ 6·10, … .



A general description they did not find because it is no so easy as before. With some considerations you can find the formula $1+ n\cdot(2\cdot(n-1))$ resp. $1+ 2n^2 - 2n$.

In the discussion with the whole group the different solutions were presented with adding the missing symbolic descriptions. For teacher students this is very important because they can see the large variety of working on such a problem. Later on as teacher in school on one hand it is important to be open for different ideas of pupils and on the other hand the arrangement for working with problems should include working in small groups as well as reflecting different approaches and their connections.

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A Good Instruction in Mathematics Education should be Open but Structured

Olga Graumann, Prof. Dr. Dr. h.c.
Retired Professor of Education
Institute of Education, University of Hildesheim
Marienburger Platz 22, D-31141 Hildesheim, Germany
jaugrau@uni-hildesheim.de

That we have to change the teaching method away from the traditional cramming school was demanded already about hundred years ago and is uncontested in pedagogy today. But the reality in school often is different from the theory. Even though the conception “Self Directed Learning” / “Independend Study” / “Minimally Invasive Education” (Offener Unterricht) is practiced a lot in primary school in Germany the pedagogical cardinal points are not understood by many of the teachers; they often think one can let work the pupils by themselves without those instructions the pupils need to become successful.

Self Directed Learning is amongst others characterized by free work, week-plan, project work and differentiating teaching. These teaching methods trace back to conceptions of reform pedagogy arising from different theories in different times and different countries (cf. e.g. Montessori, Petersen, Dewey, Kilpatrick). Therefore Self Directed Learning is defined by an assortment of several elements of different theories. Thus one runs the risk to lose sight of the theoretical background which can lead to a frivolous simplification and non-reflected acting.

Some critical remarks about a reduced setting of open instruction are the following:

- One assumes in respect to Self Directed Learning that children of today need more possibilities of own decisions and more free space for self-activity and creativity. But do become children more autonomous if they get the choice to take one working sheet or another one? And don’t have children of today already too much stress of decisions in their every day life so that it would be better to relieve them from nonsensical stress in school?
- The change from commanding style to bargaining style in household escalates today and also captures school. Many children already start with bargaining instead of looking at the task. This keeps energies away from struggling with the objects. Shall we strengthen this trend?
- Self Directed Learning can mislead a teacher to relieve him/her from handling and responsibility in course processing. In this situation children can feel left alone which may underline their experiences from at home.
- In underprivileged families and families with migration background the bargain style often is not infiltrated in household. Most decisions there will be made by the parents or the personal living condition. Children of such families will get problems with open instruction if they do not get help from the teacher or a classmate.
- Children of underprivileged families and families with migration background as well as children with mental handicaps often do not have the ability for motivation and creativity. Can we see here the limitation of Self Directed Learning?

Another question is: How shall we react on the existing contradictions, which are caused besides others from the ascending demands by the big heterogeneity of the learning groups and the achievements ask from society with parallel necessity of differentiated and individual learning. To make plain this I first will paint a situation as example.

Jonas *is in grade 3 of a primary school. He has two sisters and lives in a one family house. His father is a technician and his mother is a teacher. He has an own Computer. When starting school he already could read and count until thousand.*
Simon *in the same class. He has several brothers and sisters and lives in a flat of a high-rise building. His father and mother both go for working so that the children on the afternoon are mostly by themselves. Simon has interest for general science.*
After Christmas holydays the children of this class first discuss their experience during their holydays. ***Jonas*** *was skiing with his father.* ***Simon*** *did not have any special adventures. On Christmas Eve his parents did have a struggle with each other.*
In the second part of this lesson the children get two different work sheets about symmetry – the subject they did deal with before Christmas holyday. They can decide by themselves with which one they will start. One sheet bears reference to a Christmas tree. The children shall put different figures- they can decide by themselves - on the tree so that the tree is symmetric still.
In the third part of this lesson the children get a working sheet with tasks for a week-plan covering exercises of arithmetic, an essay about holiday adventures and a plan to find different connections for an electric circuit. In the classroom there is a corner the children can go to read something or to make experiences with material like electric circuits.
This described situation in school has a conceptual design of an Self Directed Learning. But you may imagine that Simon and Jonas can learn in this situation less and more.

School cannot afford to have so much idle time as in the example described above. What did **Simon** learn in this lesson? The working sheet with the Christmas tree for him is reminiscent of the bad situation at home on Christmas Eve so that he did not find figures to put at the tree. With the arithmetic tasks on the week-plan he did have had problems already before Christmas. Because there was nobody in the class to help him it again came out that he is not able to overcome given tasks. The essay also was nothing for his situation. And the electric circuit he already knew before.
And what did **Jonas** learn in this lesson? The given tasks he could carry out very fast because they did include nothing new for him. He was used to do arts and handicrafts at home. And about his adventures on Christmas holydays he reported on his classmate in the talking circle in the morning. So he did overcome the time somehow or other but did not learn anything new.
Self Directed Learning runs the risk to degenerate to a mere actionism if not each step, each task is prepared for different previous knowledge as well as different situations of life of the children.

In a good Self Directed Learning there must be given possibilities for each child to find an access to any of the topics dealt with in class. There must be a basis for each child to take a step forward in his/her development.

This is not a simple task for the teacher and often it is not possible to realize this in total. But during preparing a lesson the teacher has to look out for different possibilities of access and take account of the different situations of life of his/her children in class.

There is a second point of good instruction we have to bear in mind. Children do not only learn in school. Therefore it is an important task of school to help children to order and structure their diffuse and unstructured knowledge and experience which deliver the children in school.

Week-plan is besides of free work and project work the most popular form of Self Directed Learning. But it cannot be put on the level as differentiated teaching. An empirical research of Huf (2000) e.g. has shown that children see week-plans not as possibility for individual work but more as a set task which should be fulfilled as fast as possible with using time-saving tricks. This means that undifferentiated week-plans can be seen only as a little variation of tradition exercise courses. Jürgens (1996, 1965) for example could find out by an inspection of teaching research concerning Self Directed Learning that Self Directed Learning cannot do without a clear structured and task orientated arrangement of learning. Hereby it is important to find a balance between structured openness and course orientated closeness, between spontaneity and checking, between distance and closeness, etc. In principle also for Self Directed Learning we have to respect the findings from Weinert and Helmke (1997) in their SCHOLASTIK-study that for an achievement promoting education clearness of demonstration, efficient management, control of the class and positive social climate in the class is important.

In this way I will speak about "Self Directed Learning" or "Developmentally Appropriate Teaching" or "Minimally Invasive Education". ***I would define "Self Directed Learning" as a teaching which is doing what the child does need.*** If the child is able to learn by themselves so it must get the possibility to do so. If the child is not able to learn by themselves the teacher must give him the help which the child does need. Self Directed Learning means in my comprehension that the teacher always is ready to help or to go in the background. Self Directed Learning in my comprehension first and foremost does not mean openness in respect to the methods and social forms but ***openness for the child*** with its special circumstances.

Very important in this way of thinking is that the teachers must be capable to diagnose the learning ability of each child. This is very difficult and demands knowledge, endurance and attentiveness. Often the teachers do not learn this in the field of study and so they can not help the children really. Thus in the preserves as well as in the in-service study of teachers the diagnosis of children and the main points of Self Directed Learning in the above named way have to be discussed – and this means in the pedagogical part as well as in the part of didactics of mathematics.

In the following the necessity of structuring Self Directed Learning will be made clear with some different points of view.

1. A clear structure of courses of events at school is required as already seen by Weinert and Helmke. This is no contradiction to Self Directed Learning. Clear structures, a noticeable thread (developed by the teacher or/and the pupils) make instruction efficient and comprehensible for all pupils. Clear structures reduce incertitude on which children are suffering on often and they bring calmness to the courses of events.

   The pupils need guidance for systematic learning. Systematic thinking and acting is for the future of the children important more than ever. The learning with a special systematic makes it possible new contents of learning to integrate into the existing knowledge and understanding in a new wholeness. This is necessary for children with problems of learning in particular but also the others need help to sort and integrate the plenitude of knowledge and experiences.

   Also several children sometimes need a course orientated and teacher centred instruction. In an Self Directed Learning the teacher can work with one child a while whereas the others work by themselves / in groups if the structure of working is clear for all of the pupils.

2. A solid preparation and postprocessing of instruction is necessary for open instruction. During scheduling the Self Directed Learning the teachers have to look into the object of instruction as well as discuss different learning proposals, ways of structuring approaches to solve the possibly appearing tasks and solutions. If the teacher in our situation named above did ask him/herself which conceptions, hints, situations, detections, narrations, experiments, models etc (cf. Klafki 1963, 141) he/she would have found out that for Simon and Jonas then his/her given tasks would be different. Self Directed Learning requires a good preparation with reflection on aims and topics as well as social and institutional conditions. This is not in contradiction to Self Directed Learning because once during the performance you can let go the pupils their own ways and another time a good preparation avoids excessive demands as well as under-challenge, idle time and blind actionism. Nevertheless the pupils will not be restrained in their freedom to look out for own focuses, planning of time to work on a special problem, to develop own ideas, to manage team working, etc.
3. Self Directed Learning needs a personal responsibility of the teacher more than ever. The teacher can not just give a task and then let work the children. The teacher can not behave laissez faire. He/She has to bring in him/herself much more than in traditional teaching namely not so intensive in respect to the object but more intensive in respect to the subjective demands. Thus a teacher in Self Directed Learning must have the ability of noticing and making a diagnosis.
4. Self Directed Learning is no grab box where you can pull out this or that. Moreover Self Directed Learning does not mean “take any working sheet you want” out of a pile of different working sheets concerning one topic. Unfortunately often this will be mixed up with Self Directed Learning. Do you really call the decision between some trivial working sheets a free decision whereby the children can learn about decision doing?

   The Question how self-determined a child can work on a course can be answered only individual. One child may open oneself for a special topic only after the teacher did give guidance for a small walk on this way and another child does see an access immediately. Learning self-determined does not mean without adults.

   Self Directed Learning can not be measured by the extending of guidance through the teacher. The degree depends on the amount of alternatives the children do have to develop themselves.

If one bears in mind all named conditions of Self Directed Learning then Self Directed Learning leads to a good instruction with intellectual, emotional and social development of all children. But of course it is not easy to perform such a good open instruction and not at all times a teacher can perform a good instruction. But most important is that at any time the teacher goes to great length with such good Self Directed Learning. This does include that the teacher reflects on his method and social conditions.

This in particular demands a teacher training with reflection on these points as well as good mentoring while making experiences in school. Let me demonstrate this with an example about a “poor” teacher.
A teacher-student was watching a class while the children did handicrafts with paper. She asked the teacher why the children should do so. After a while the teacher answered that the children like to do that.

If the teacher would have answered that he/she did not have time for preparation (of course an answer which would not be good with an inspection) so it would have been a candid answer which shows that he/she exactly knows what he/she performs and why he/she performs like this. No teacher is free to do anything anytime anyhow.

Return to our example:
What should the teacher do with Jonas and Simon? He/She could give Jonas new special tasks about symmetry to challenge him but not overtax him. And Jonas could write a fairy tale that is playing in Christmas Eve to show his fantasy. On the other hand the teacher could take time to explain Simon the arithmetic tasks while the other children are working by themselves. Later on Jonas could discuss with Simon a plan to construct an electric equipment in a little house. So both boys can win new knowledge and social competences

As summing up I would say:
For any instruction - open or not open - it is important that pedagogical and didactical acting does have a good basis with passable aims as well as a thread and a clear structure.

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Do South African Mathematics teachers need narrative therapy?

Professor Mellony Graven, South African Numeracy Chair, Rhodes University
Faculty of Education, P O Box 94, Grahamstown, 6140, South Africa, m.graven@ru.ac.za

Abstract This paper will argue that ubiquitous stories of mathematics teachers as the root cause of the crisis in mathematics education shuts down the space for meaningful teacher learning. There are many statistics used to 'back up' these stories, for example: South African learners performed worst on the TIMMS 1999 and TIMSS 2003 study. Newspapers are quick to pick up on this in headlines such as: "Teachers flunk maths"(Mail and Guardian, 03/08/08); "Teachers battle with Maths" (news.africa.com 05/04/11 – from SACMEC111).These stories create a cycle of ongoing failure. Of course there are always many other stories that don't get told such as "mathematics teachers have the experience and classroom knowledge needed to inform curriculum change". In this paper I argue that a primary concern of in-service mathematics teacher learning should be 'narrative therapy' – that is a focus on supporting teachers to actively construct preferred realities (Freedman and Combs, 1996). Such construction requires the formation of supportive communities of practice (with a focus on inquiry into mathematics teaching and learning through partnership between 'teacher educators' and teachers) where teachers are supported through active participation in the community to challenge negative stories and to develop and foreground new stories.

Introduction

Do mathematics teachers need teacher educators or do we as teacher educators need to construct teachers as 'needing' through deficit discourses in order to justify our work? This is indeed a challenging question and one teacher educators need to reflect on. Breen (1999) illuminates the interdependence between teacher educators and teachers in his referenc e to 'fix it' approaches - teacher educators need someone to fix and teachers need fixing. He highlights that such approaches tend to ignore what teachers are actually doing and look for solutions outside of the practice of teaching. The problematic nature of this relationship seems clear. Yet the dominance of teacher development models in which 'teacher educators' (e.g. Department of Education (DoE) district officers, NGO or university employees) position themselves as knowledge-authorities bringing knowledge to less knowledgeable teachers, would suggest that teacher educators have not reflected sufficiently on the nature of their own need in the relationship. Indeed in order to understand the nature of mathematics teaching and learning and to advance this field of knowledge we (teacher educators and researchers) need access to the world of teachers and their classrooms. The right to access is sometimes taken for granted as if participation in what we have to offer will automatically provide rewards for teachers. Yet in service 'development' is often experienced by teachers as disempowering and teachers complain of unprofessional treatment (OECD, 2008).

I would like to argue that this relationship must be changed and that as teacher educators we need to change our stories from 'development on teachers' to forming supportive communities of inquiry into mathematics teaching and learning in South African classrooms where ‘equal’ partnerships with teachers are established. The negative stories that teachers themselves often buy into need to be re-narrated as stories that foreground teachers as experienced and of teachers as life- long learners, willing and able to partner with policy makers, the department of education, teacher educators and so forth to find solutions to the challenges faced in mathematics education. Of course ‘equal’ partnerships are not simply the result of naming the partnership equal but will involve practices that lead to the experience of the partnership as equal and this will take conscious work and time to develop. Equality of course does not mean that the knowledge each partner brings is the same – indeed the reason for the partnership is precisely because each has knowledge (through their participation in differing professional practices and landscapes) that the other does not. The knowledge should however have equal status (at least within the practices of the evolving partnership).

Drawing on Sfard and Prusak’s (2005) definition of identity I will argue that deficit stories of Mathematics teachers result in self- fulfilling prophecies and that teacher educators need to become the significant narrators that narrate teachers as experienced professionals, critical partners and life-long learners.

Providing an operational definition of identity

In previous work (Graven 2003, 2004, 2005) I built on Wenger’s (1998) notion of identity to analyse teacher learning, but the work of Sfard & Prusak (2005)goes further to operationalise the definition. In doing so, they equate identity with reifying, endorsable and significant stories about a person. While Sfard and Prusak (2005) concur with Wenger’s work in terms of linking learning with the construction of identities they argue that the ‘notion of identity cannot become truly useful unless it is provided an operational definition.’ (15). They highlight that notions of identity as being a kind of person “sound timeless and agentless”; and therefore reject such definitions as “potentially harmful because the reified version of one’s former actions that comes in the form of nouns and adjectives describing the person’s “identity” acts as a self fulfilling prophecy” (Sfard & Prusak, 2005, 16). Sfard and Prusak (2005, 16) thus choose to define identities as “collections of stories about persons or, more specifically, as those narratives about individuals that are reifying, endorsable, and significant”. Reification comes with verbs such as ‘have’. (E.g. “Teachers have mathematical weaknesses”. Stories are considered endorsable if the identity builder can answer to them being a faithful reflection of a state of affairs (E.g. we use the headline“teachers flunk maths” because a study showed…). Stories are significant if a change in the story changes the storyteller’s feelings about the identified person. (E.g. ‘teachers are unqualified’ to ‘teachers are experienced’).

Within their definition identities are human made, collectively shaped by authors and recipients. They explicitly highlight that their definition presents identities as the discursive counterparts of lived experiences whereas Wenger (1998, p151) sees such words as only a part of “the full, lived experience of engagement in practice”. Sfard and Prusak thus stress “No, no mistake here: We did not say that identities were finding their expression in stories – we said they were stories" (p.14).

This definition gives increased agency to the learner as it opens the space for the re-authoring of identities. It also opens the space for significant narrators, such as mathematics teacher educators, to deliberately challenge existing negative stories and to reflect on their own authoring of mathematics/ numeracy teacher identities. Reflection should lead to the re-authoring of negative stories that may be obstacles to learning into stories that enhance teacher learning. My assumption here is that every negative story can be countered with a different story that is more conducive to stimulating learning. For example: 'Teachers are poorly trained to cope with the new curriculum' can be countered with 'Teachers, with their wealth of teaching experience, are best placed to make sense of the curriculum and provide feedback'. It is this space for re-authoring that appealed to me.

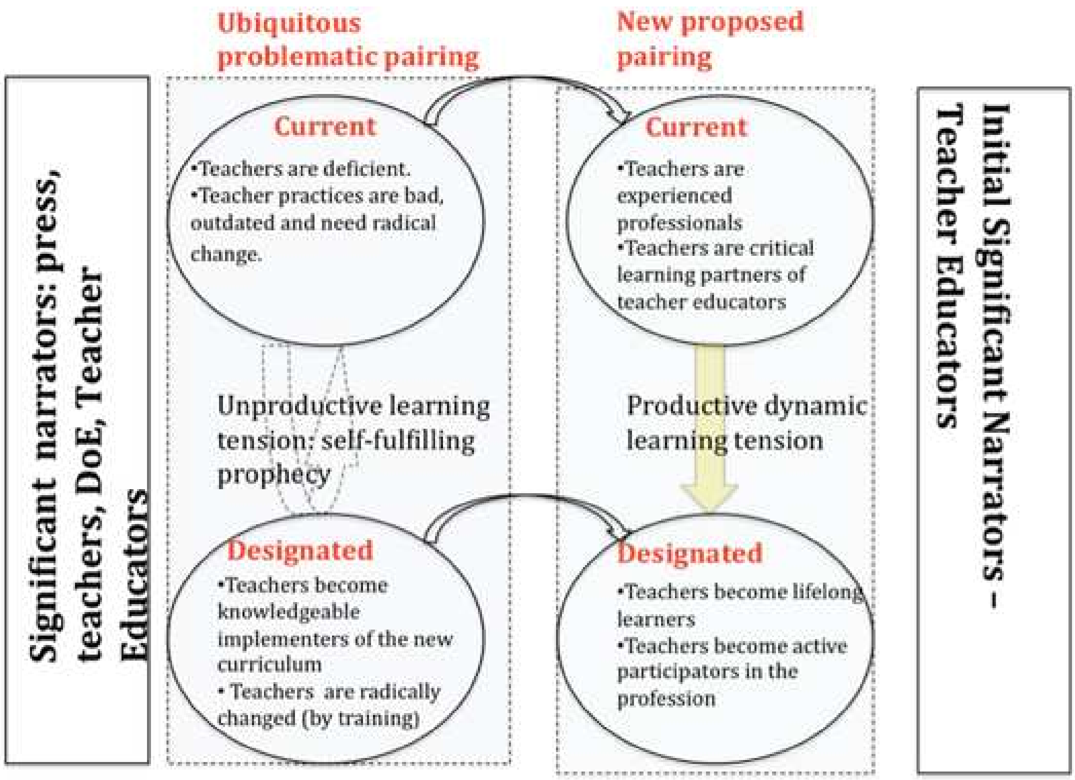
Sfard & Prusak (2005) continue to identify two sub categories of stories: *current* identities (e-mail correspondence with Anna Sfard (2009) suggests a move away from the term 'actual' identities to 'current'), told in the present tense and formulated as actual assertions, and *designated* identities (narratives expected to be the case – now or in the future). Learning is then conceptualized as closing the gap between *current* and *designated* identities. With this definition of identity as discursive counterparts of one's lived experiences, the re-authoring of identities is not only possible but could enable and give momentum to learning. This is especially important in cases where identities have been negatively constructed. Indeed this is precisely what narrative therapists enable people to do:

> A key to this therapy is that in any life there are always more events that don't get "storied" than there are ones that do… this means that when life narratives carry hurtful meanings or seem to offer only unpleasant choices, they can be changed by highlighting different previously un-storied events, thereby constructing new narratives. Or when dominant cultures carry stories that are oppressive, people can resist their dictates and find support in subcultures that are living different stories (Freedman & Combs, 1996, p32-33).

The above quote highlights that narrative therapy is not restricted to the domain of individuals and their therapists but extends the opportunity to groups of people in supportive communities or 'communities of practice' (Wenger, 1998) which enable 'living different stories'. In a similar vein Sfard & Prusak (2005, 18) note that "A person may be led to endorse certain narratives about herself without realizing that these are "just stories" and that there are alternatives". A supportive community of practice, such as those formed in in-service teacher education programs can and should open up these alternatives especially when existing stories 'carry hurtful meanings', undermine professional identities or impede learning.

Thus part of what drew me to Sfard & Prusak's definition of both identity and learning is the increased agency afforded to learners and the opportunity for both learners and significant narrators to deliberately reject negative stories (thus breaking down the stumbling blocks to learning) and re-author more productive stories which will give momentum to learning. In my experience with in-service mathematics teacher education over the past 15 years, 'substantial' teacher learning requires re-authoring of certain negative current teacher identities and counter productive designated teacher identities. It requires the creation of supportive communities which can provide the space and the 'subculture' where teachers can challenge these stories and live out more productive stories.

The pairing of 'deficient' *current* identities with designated 'curriculum knowledge authority' leaves teachers trapped between two conflicting and imposed identities. The hypothesis here is that widespread low morale and retention (of South African mathematics teachers in particular) are partly a result of teachers feeling trapped between two conflicting identities where one's history and experiences are negated and one's designated future is unattainable (especially in the context of constantly changing curricula). This gap results in the removal of stimulus for learning. In contexts where teacher morale is low and negative stories predominate, teacher education must involve deliberate re-authoring so as to construct a productive learning tension (gap) between teachers' current and designated identities. Of course these current and designated identities and the gap between them are dynamic and changes will emerge through the learning process. That is stories become modified, new stories emerge, negative stories become a stimulus for learning (e.g. I have little knowledge of probability and I need to learn about it to teach it), and new designated identities might be added (e.g. I'm a lifelong learner constantly making sense of curriculum initiatives'). A diagram is useful to highlight key *deficiency* narratives that need to be re-authored as *proficiency* narratives by teacher educators:



This re-authored pairing requires mathematics teacher educators to narrate teacher *current* identities as experienced teachers and learners (acknowledging the value of bringing experience and existing knowledge to the learning process) and critical partners in the process of making sense of and reviewing the curriculum. Teacher educators narrate the *designated* identities as life long reflective learners. Linked to this emphasis on life long learning and the value of teachers' experience is the designation as active participators in a range of professional activities such as: engaging with others, providing feedback on the curriculum, attending professional conferences and so on. While teacher educators are the initial significant narrators tasked with the job of challenging previous stories of teachers, with time and through successful learning within in-service programs, other significant narrators (fellow teachers, community members, learners, parents) are likely to reinforce these stories.

From what has been discussed above it is argued that the way forward in working with teachers is to form supportive inquiry communities where teachers and teacher educators partner to both reflect on and learn from teaching practices and to look towards finding innovative ways to strengthen teaching and learning in Mathematics classrooms. The South African Numeracy Chair, Rhodes University is aimed at improving the quality of learning and teaching of numeracy at primary level. This development aspect of the chair is dialectically connected to the aims of researching sustainable and practical solutions to the challenges of improving numeracy in schools as laid down in the concept document of the Chairs Initiative. In my work as the South African Numeracy Chair, Rhodes University, it has been important to develop a conceptualization of 'teacher development' that challenges deficit discourses of teachers and works with a conceptualization of teachers as critical partners. While this was always the intention the need for this became increasingly clear when my colleague, Zonia Jooste, and I visited schools in the Grahamstown area to invite numeracy teacher participation. Teacher histories of 'teacher development' and 'workshops' had not led to a 'we want more' response but rather a skepticism of the value of participation. Explanations that we wanted to partner with them and that in this partnership we would not tell teachers what to do were well received but skepticism persisted. Our launch and orientation focused on what working as partners might mean and the importance that the path for the way forward must be carved from the perspective of numeracy classroom practices in collaboration with researchers/teacher educators rather than the other way round.

Thus we named our partnership with teachers the Numeracy Inquiry Community of Leader Educators (NICLE) where all participants, teacher educators, professors, and researchers would be learners in the community and through participation would provide leadership within their sphere of influence and their overlapping communities. Thus NICLE was conceptualized as a community of practice based community of inquirers where teachers, lecturers, researchers and professors partnered to inquire into looking towards finding solutions to challenges faced in primary mathematics education from a classroom based perspective. In this partnership all are co-learners. By working together each brings different experiences and expertise to share in the community. Through active participation each member of the community will increasingly take on leader roles in primary mathematics education relating to their sphere of influence. For example teachers will run workshops or mathematics competitions in their community of schools, researchers will publish and engage in panels as conferences, teachers and researchers will present their work at teacher and research conferences, teachers will publish their classroom reflections in teacher focused journals and so forth.

The start of NICLE

Fifteen school were initially invited to participate in NICLE. Six schools and 19 teachers participated in this launch held on the 26th March 2011. Following this word seems to have spread that this might indeed be a different type of learning endeavor and some schools that had declined participation have since committed participation. By the second NICLE session (12th April 2011) fifteen schools and 51 teachers attended indicating willingness of teachers to become life-long learners provided their views and experiences are taken seriously. Indeed there will be the challenge of sustaining teacher involvement as well as developing practices that truly support equal partnerships where learning is led from the basis of teacher experiences. Such practices do not follow automatically from the naming of a learning community in this way nor from the removal of deficit discourses. The notion of partnership must come alive in the practices of NICLE. Further research into the nature of learning evolving within this community for all participants will hopefully reveal key elements of NICLE practices that enable or constrain learning so that these might inform future endeavors with numeracy teachers.

Acknowledgements:

Thanks to the SA Numeracy Chair team, Rhodes University: Zonia Jooste, Varonique Sias, Peter Pausigere, Debbie Stott and to the newly participating teachers and schools for their collaboration in the work of the chair. This work is based upon research supported by the South African Numeracy Chair Initiative of the FirstRand Foundation (with the RMB), Anglo American Chairman's fund, Department of Science and Technology and the National Research Foundation.

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Horizontal and Vertical Concept Transitions

May Hamdan, PhD
Associate Professor of Mathematics, Lebanese American University
Beirut, Lebanon, mhamdan@lau.edu.lb

Abstract:
Transfer of concepts, ideas and procedures learned in mathematics to a new and unanticipated situation or domain is one of the biggest challenges for teachers to communicate and for students to learn because it involves high cognitive skills. This study is an attempt to find ways for driving students to generalize and expand mathematical results from one domain to another in a natural way, and to promote that mathematics is not a collection of isolated facts by providing meaningful ways for students to construct, explain, describe, manipulate or predict patterns and regularities associated with a given system of theorems and mathematical behavior. One would wish there were a universal genetic decomposition for generalization and for the abstraction of properties from a given structure and applying it to a new domain. In this study I plan to focus on particular cases of generalizations in calculus and distinguish between two different types of examples.
Keywords: genetic decomposition, abstraction, generalization, Calculus, RME

Many instructors choose to informally display concept transfer by just drawing the students' attention to the connection between the initial result and the new transferred result so as to make the extension seem natural in retrospect. This helps the learner gain ownership over the mathematical content. One could say that even if the students are not able to deduce the generalized concepts on their own, at least they are convinced that those mathematical facts are the result of a human activity (Streefland, 1991, p. 15). To facilitate this, it helps to view effective learning as a series of processes of horizontal and vertical mathematization that together result in the reinvention of mathematical ideas. Realistic Mathematics Education (RME) is about how mathematics is learned and at the same time about how it should be taught: students must be guided and encouraged to create their own systems or internalize the process of such creations; because by doing that they learn best or even reinvent mathematics. RME distinguishes between two types of "mathematization": horizontal and vertical mathematization. Horizontal mathematization involves set patterns, rules and models that should be learned and applied with given principles. Vertical mathematization, on the other hand, requires flexibility and allows students to shape and manipulate mathematical results. The first type relies primarily on memorization; whereas the second requires higher cognitive skills and activities. Students will develop attitudes to mathematics more in line with those preferred by mathematicians while standard mathematics lectures designed to "get through the material" may force them into rote-learning habits that mathematicians hate.

Unfortunately, many teachers still choose to teach mathematics "as a set of rules of processing or…algorithms" because "it is the way they learned it themselves" (Freudenthal, 1991, p. 3). Those instructors, I believe, are suspected to be the ones who usually tend to overemphasize details and conditions earlier on, at the expense of suppressing mathematical intuition and free “guessing”.

In the examples below I distinguish between two types of generalizations:

1. The chain rule for the case of $y = f(u(x))$ is $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$. In the case of higher dimensions it translates into $\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$, when $z = f(x, y)$ and $x = x(t), y = y(t)$.

2. Total differential: $dy = f'(x)dx$ for a function of one variable translates into: $dz = f_x dx + f_y dy$. For the case of a function of two variables. A plausible question would be: why not the average of the last two addends? Specially that the error’s upper bound involves an average. $|E| \leq \frac{1}{2} Max(f_{xx}, f_{xy}, f_{yy})$, where the maximum is taken over a certain domain.

3. Taylor theorem as an extension of linearization.

4. Independence of path in the case of line integrals as an expansion of the Fundamental theorem of calculus (into line integrals).

5. Fourier series as an inspiration from Taylor theorem: while Taylor theorem models certain functions as power series, Fourier’s theorem models certain periodic functions as trigonometric series.

There is an obvious distinction between two different types of generalizations in the examples above: when a result A is a generalization of a result B, then A could be seen as a special case of B. However when a result D is an abstraction of a result C, then it means that they both share properties. Or that are applied to different fields altogether, or that the properties themselves are extensions of one another. I refer to this distinction by vertical and horizontal transition:

- The chain rule for the case of a function of a single variable can be seen as a special case of the chain rule for a function of several variables; likewise, linearization can be perceived as a special case of Taylors theorem for $n = 1$. Moreover, the Fundamental Theorem of Calculus $\int_a^b f'(x)dx = f(b) - f(a)$ can be perceived as a special case of the path independence theorem in the case of line integral.
  According to Niss (1999) one of the major findings of research in mathematics education is the key role of domain specificity. The student’s conception of a mathematical concept is determined by the set of specific domains in which that concept has been introduced for the student. By expanding the domain the problem of domain specificity would be transcended.
- On the other hand, Taylor’s theorem or Taylor approximation may not be thought of as a special case of Fourier Theorem. But the similarity is clear in that both of

them approximate particular types of functions into either a power series (in the case of Taylor's theorem) or a trigonometric series (in the case of Fourier theorem), with adequate conditions in both cases. Also in both cases approximations are made for the purpose of simplifying a function. The "transition" from Taylor series approximation to Fourier series approximation can be considered as a (stretched) abstraction, where specific properties have been abstracted and applied to a different context for similar purposes.
Recall that generalization is the process of forming general conclusions from particular instances while abstraction is the isolation of specific attributes of a concept so that they can be considered separately from the other attributes. Yet, abstraction is often coupled with generalization, but the two are by no means synonymous. Any arguments which apply to the abstracted properties apply to other instances where the abstracted properties hold, so (provided that there are other instances) the arguments are more general (Tall, 1988).

In lieu of a proof?

As instructors one should ask: is a mere display of the natural transition a substitute for a thorough proof or justification along the lines of a "scientific proof"? To what degree shall we be satisfied with "observing" a similarity in the results and just accept the extensions in all those listed illustrations as new natural facts? In the case of engineering students whose programs do not require knowledge of thorough justification in the form of formal proof, is observing the smoothness of the extensions between cases of functions of one variable and functions of several variables considered as a good enough justification of the new result as a substitute for what is referred to as "scientific proof". Is it enough to draw students' attention to the fact that Green's theorem is a generalization of the Fundamental Theorem of Calculus, and Stokes theorem is a generalization of Green's Theorem for different dimensions so to speak? No need to mention that some students are able to see the connection between those results with no help from the instructor. Once the generalized result is presented to them, they look back at what could have been its source and figure out in retrospect how it could have been anticipated. In all cases, isn't it an innate instinctive drive to want to connect and look at things as part of one big whole entity or behavior?

General questions:

1. To what extent does drawing the students' attention to the transition in a natural way a substitute for a formal thorough proof that highlights conditions and particularities of the two domains?
2. How much emphasis should be placed on the conditions and the particularities of both domains without menacing mathematical intuition and free "guessing" skills?
3. What are the cognitive skills involved in the generalizations and or abstraction? And are the students ready for that level of cognition at that stage?
4. Should transition be solicited or just lightly pointed out to (in retrospect)?

5. What instructional procedures should be followed that guide students to come to terms with the second type of transition on their own?

Conclusion:

One disadvantage of this behavior would be creating overconfident students. For instance, in probability a student might get tempted to expand the case of binary outcomes of a coin to the case of a die. As for most of us, we must have learned such transitions mostly in retrospect, since the traditional textbooks we used do not bother pointing to those connectives: once we saw the generalized result, we must have figured out its source of inspiration hoping that next time around, we can anticipate it ourselves. One positive outcome is bound to happen, and that is seeing mathematics as a collection of connected facts.

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The importance of using representations to help primary pupils give meaning to numerical concepts.

Tony Harries, David Bolden, Patrick Barmby, Durham University (UK)

Abstract

The workshop will be a practical one in which participants will have the opportunity to work on a suite of computer programmes which aims to help primary pupils and teacher trainees, in particular, to make sense of numerical concepts through an exploration of representations of these concepts. During the workshop we will also look at some data which illustrates the way in which both primary pupils and trainees have responded to the use of these ideas. There are 4 sets of programmes: early mathematics, addition and subtraction, multiplication and division, fractions.
In total the suite contains about 60 programmes. The writing below gives some background to the programmes and explains the philosophy underpinning their development.

Introduction

During their primary education, pupils are introduced to a number of 'big' ideas – for example addition and subtraction in early primary and multiplication and division later on. In teaching addition and subtraction, there has been a clear use of visual representations such as number squares and number lines, with number lines being viewed as the most appropriate representation for demonstrating the characteristics of these operations, for visualising the calculations and for developing flexible ways of executing the operations. It would appear that as pupils progress within the primary sector and explore other "big ideas", the use of representations is less prominent.
In developing this suite of programmes we have explored the use of visual representations in facilitating the understanding of other big ideas such as multiplication/division and fractions. In doing so, we examine the different aspects of these concepts that we can access through different representations.

The Importance of Representations

Shulman (1986) identified representations as being part of teachers' pedagogical knowledge. He defined these representations as "including analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others" (p.9). Specifically in mathematics, Ball *et al*. (2008) also highlighted representations as being part of the 'specialised content knowledge' of mathematics unique to teaching. This specialised knowledge included selecting representations for particular purposes, recognising what is involved in using a particular representation, and linking representations to underlying ideas and other representations. Teachers need to be able to draw on a variety of representations as there is "no single most powerful forms of representation" (Shulman, 1986, p. 9).
In particular, researchers have highlighted the role that representations play in the explanations of mathematical concepts by teachers (Leinhardt *et al*., 1991; Brophy, 1991; Fennema & Franke,1992).

> "Skilled teachers have a repertoire of such representations available for use when needed to elaborate their instruction in response to student comments or questions or to provide alternative explanations for students who were unable to follow the initial instruction" (Brophy, 1991, p. 352)

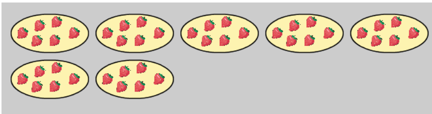
Leinhardt *et al*. (1991) also identified the skill and knowledge required by teachers in considering the suitability of particular representations, as "certain representations will take an instructor farther in his or her attempts to explain the to-be-learned material and still remain consistent and useful" (p.108). The effective use of representations therefore require that teachers have 'deep understanding' of the topics that they are teaching.

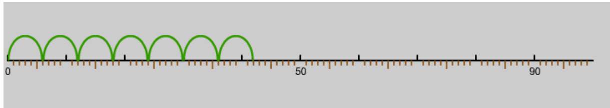
Representations also play an important role in the learning of mathematics by students: "An important educational goal is for students to learn to use multiple forms of representation in communicating with one another." (Greeno & Hall, 1997, p. 363) More specifically, researchers have outlined the role that representations play in linking the abstract mathematics to the concrete experiences of learners (Bruner & Kenney, 1965; Post & Cramer, 1989; Fennema & Franke, 1992; Duval, 1999).

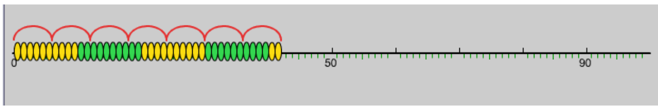
> "Mathematics is composed of a large set of highly related abstractions, and if teachers do not know how to translate those abstractions into a form that enables learners to relate the mathematics to what they already know, they will not learn with understanding." (Fennema & Franke, 1992, p. 153)

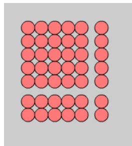
In addition, representations can support the working memory of learners (Paivio, 1969; Perkins & Unger, 1994), for example through 'offloading' elements of a given computation to externalized representations (Ainsworth, 2006). Related to the issue of explanation of mathematical concepts highlighted above, representations can be designed in order to constrain interpretation and to highlight particular properties of a mathematical concept (Kaput, 1991; Ainsworth, 1999).
More broadly, multiple representations play an important role in the development of learners' mathematical understanding: "They can be considered as useful tools for constructing understanding and for communicating information and understanding." (Greeno & Hall, 1997, p.362) In considering the role of representations within understanding, we make the distinction between *internal* and *external* manifestations of representations (Pape & Tchoshanov, 2001), or 'mental structures' and 'notation systems' respectively as referred to by Kaput (1991). Understanding of a mathematical concept is based on the internal representations of a concept, which are influenced by the external representations of the concept that are presented to learners (Hiebert & Wearne, 1992). Wood (1999) stated that conceptual understanding rests on a multiple system of 'signs' or representations. Lesh *et al*. (1983) used the definition that a student understands a mathematical concept if he or she could 'translate' or move between multiple representations. Hiebert & Carpenter (1992) defined mathematical understanding as being a network of internal representations, with more and stronger connections denoting greater understanding.

The medium for exploring representations
As a medium for exploring ideas on representation, we developed a suite of programmes which allowed representations to be explored in a dynamic and interactive way. Through the programmes the characteristics of these representations were explored. For example some of the representations for multiplication/division were:



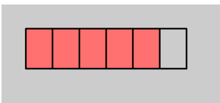


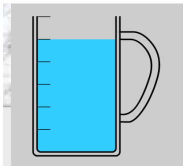


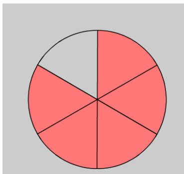


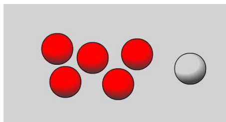
Using these representations we can ask what characteristics of multiplication are emphasised by a particular representation and can consider the possibility of there being a key representation. Further we can then explore how the representations could be used to make sense of the various procedures that need to be understood.

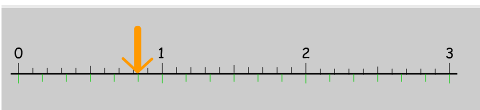
Similarly for exploring fractions the following representations were used:

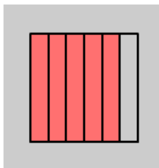












Through exploring these representations and the relationship between them the pupils/trainees are encouraged to build up a language which facilitated a discussion about the nature/characteristics of fractions.

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The programmes were created as a stimulus and scaffold for class discussion. The main guidelines for their design and development were:

- The diagrams and animation should illustrate concepts in a way which is impossible in any other medium;
- There should be a minimum of distraction;
- Pupils should be offered a choice of representation. By using several representations, they are encouraged to realise each way of looking at a problem has its strengths and limitations;
- When a mistake is made the computer should provide a clue (or clues) to the correct answer.

With animated visual representations it is first a question of seeing what is happening, then working out why this is happening, and finally developing robust procedures which will work without the diagrams. In this way the computer programs are envisaged as a bridge between physical manipulatives and abstract figures and symbols on paper. We conjecture that the use of animated visual representations with substantial discussion will allow more both pupils and trainees to see and understand their mathematics more deeply.

Anyone involved in educational research should make explicit their underlying assumptions and beliefs about the nature of knowledge and what conditions facilitate learning. There are fundamental philosophical questions here about the human mind and what exactly is involved in knowing. In a discussion of technology- enhanced learning, Derry (2009) highlights two crucial aspects of learning: its social nature and the central importance of knowledge. Offering a critique of some of the claims of technology- enhanced learning, she says "focus on the learners without recognition of knowledge domains offers no way forward" (p.153). She then quotes Balacheff to show that knowledge domains vary greatly:

> "The characteristics of the milieu for the learning of mathematics, of surgery or of foreign languages are fundamentally different. . . . One may say that the milieu of surgery is part of the 'material world' (here the human body), for foreign languages it includes human beings, for mathematics already a theoretical system." (p.154)

In conclusion Derry says that technology- enhanced learning should "turn attention away from technology to the knowledge domain, from here to questions of pedagogy and from there one step further back to epistemology" (p.154).

Conclusion

The process that pupils are encouraged to follow through using the programmes - of questioning or interrogating a representation- resonates with what Mason (2005) has called "structures of attention". He identifies 5 ways in which we can interrogate the representation: gazing (looking at the whole), discerning details, recognising relationships, perceiving properties, reasoning on the basis of the properties.

Within a representation this idea leads to such questions as:

- *What do you notice about the image/representation*
- *What are the characteristics of the image/representation?*
- *Can you explain how this image/representation shows us …..?*

When working across representations we have questions such as:

- *Why these representations show the same mathematical idea?*
- *What is the same about the different representations?*
- *What is different about the representations?*
- *What are the particular characteristics of the various representations?*
- *What aspects of the structure of fractions are emphasised by the representations?*
- *Can you explain how we move from one representation to another?*
- *What are the most useful characteristics of a particular representation?*

The workshop will allow participants to explore how the programmes can be used to help pupils/trainees to build up a language which facilitates discussion about the nature/characteristics of key numerical concepts within the primary Mathematics curriculum.

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“Shuffle and Shake” and “Pay as you go” - The VG grade 8 experiment

Ms Nicci Hayes (on behalf of the team including Sarah Abel, Susan Richards and Soosan Babu)

Head of Mathematics, Victoria Girls’ High School,

Grahamstown, South Africa, nhayes@vghs.co.za

Abstract

The major aims of this paper is to present a new methodology of classroom practice, analyse the extent to which this new methodology was successful in terms of increasing the effectiveness of the teaching and learning of grade 8 Mathematics at our school; as well as analyzing the extent to which we have been successful in our aim of creating a paradigm shift in the minds of the learners to them seeing Mathematics as a dynamic exciting subject intrinsically associated with the 21st Century and 21st Century skills. The new methodology focuses primarily on the twinned concepts of changing the approach of the teachers and changing the approach of the learners. Educator changes include a lead teacher taking responsibility for each section; the introduction of more technology-based, games-based and kinesthetic-based approaches, and the changing of classes and teachers per section. Learner approaches and attitudes would (hopefully) be changed both by the constant classroom reshuffle, and changes in the way additional support is delivered. The success of implementing this system as well as the success of the system itself is discussed. Finally the transferability of this system is evaluated.

Introduction

The paradigm shift that we were hoping to achieve was essentially a psychological one: Mathematics is viewed by many in society as talent rather than a skill or set of skills. Many of our learners thus suffer from a double curse of a lack of basic Mathematical skills coupled with a societal excuse not to overcome this lack of skill. (The ‘No-one in my family is a “Maths person”’ syndrome.) In addition to, and ironically **despite** this perception, it is also common for both learners and families to blame the teacher if the child is struggling with Maths. Moreover, Mathematics is generally viewed in our society as something that is boring, difficult and only for the minority, a special elite of “Maths people”. More subtle societal prejudices are the concepts that “girls can’t do Maths” and that “black people can’t do Maths”. These are perhaps more insidious as it is politically incorrect to say such a thing in the 21st Century. The result is that the prejudice continues, largely unspoken; and because it is unspoken it cannot be rebutted; and because it cannot be rebutted, it continues.

As a result of combination of all of these factors, many learners struggle with Mathematics; struggling learners, need constant assistance, which causes a drag on the system, thus disadvantaging stronger learners.

Traditionally streaming has been used to address this problem. In South Africa, the Department of Education is against streaming, and for good reason: children who end up in “bottom” classes soon begin to identify with the stigma of being “bottom”. However, the advantages of being able to move at different paces with different classes mean that this approach is, in reality, often still used.

Aware though we are, as a department, of these factors, it is often hard to know how to redress them. We have for many years had, at the school, an **extensive** academic support system including extra lessons from staff, peer tutoring and computer-based support amounting to an average of 15hours of “extra Maths” on offer per week. However, many learners do not utilize the help on offer, and of those who do, many use the support as a continuous crutch, thus not taking responsibility for their own development, and underutilizing class time. Thus potential solutions become rather part of the problem.

In seeking ways to address these issues, in a way that would be beneficial to both staff and learners, bearing in mind the time constraints on staff, we came up with the “Shuffle and Shake”[i] and “Pay as you go” strategies.

In short we wanted to

- create a feeling of Maths being fun, relevant, and accessible to all
- tailor learning to be as individually paced as possible
- ensure that our lesson planning was as dynamic as possible, and in particular
  - ensure the inclusion of extension work to stimulate top learners
  - ensure the inclusion of technology in our teaching
  - ensure the inclusion of kinesthetic and model based learning[ii]
- make learners more accountable for their own successes and failures in Maths

The System (in theory)

“Shuffle and Shake” -instead dividing learners into classes assigned to a specific teacher, teachers take it in turns to introduce each new topic to the whole grade, then (after a short test following the introduction) learners are divided into working groups for that topic. So for each topic, learners will be with different classmates and have different teachers. Introductions are taped and made available on our network for revision. The “lead teacher” for the topic is also responsible for providing extension work and the end of topic test. Interns work as teacher assistants on certain days to assist individuals who are really struggling. At the end of the topic learners write a control test before returning to the lecture group for the next topic.

At the same time a “pay as you go” system has been introduced for extra lessons facilitated by teachers. Payment” is in the form of 10 mark worksheets generated using Microsoft Worksheet generator (or similar online versions) on the specific example that gave trouble. This approach is designed both to reinforce the correct methodology once remediation has happened, and also to encourage a more pro-active use of class time for solving problems rather than relying unnecessarily on extra lessons.

The advantages we anticipated were

- **Appropriate pacing without stigma**- dividing classes per topic rather than per year, provides all the advantages of traditional “streaming” in terms of being able to go at a different paces with different classes, without the disadvantages of the stigma of being in the “bottom class” as the classes are continuously in flux. In addition to this class sizes can be altered per topic to match the “natural break” in the test results. The extension work aspect, the use of computer lab (for Cami-Maths, online worksheets, World Maths day entries etc), as well as the use of teacher assistants would also play into our ability to pace work individually.
- **Label breaking -** by keeping the groups in flux children will be able to let go of labels like “Artsy not Mathsy” by realizing that Maths has many faces and that they can excel in some even if they struggle in others.

- **Learner accountability –** as the learners will largely have different teachers for each topic, they will not be able to attach either success or the lack of success to the teacher and will have to own their own triumphs and difficulties. Having the introductions available digitally means that learners can take the initiative to revise the section from the start in their own time. "Pay –as-you-go" results in less misuse of extra help, and good consolidation for those in genuine need.
- **Fun by association –** the Audio-Visual room (the venue for the introductory lectures) is associated with watching videos , we hoped that the positive association would rub off!
- **Teacher creativity -** as each teacher is responsible for only one in four topics, each has more time to be creative in preparation. Also, because of the experimental nature of the new system , teachers are given the freedom/permission (even the expectation) to experiment and be creative.
- **Teacher accountability** - introducing the topic means being watched by the rest of the department, you are accountable for delivering a dynamic lesson and extension work for your section, and your success or failure is very visible –thus ensuring the followed through on good intentions of making lessons dynamic; including more kinesthetic activities and more technology; and providing extension work.
- **Teacher in-service development -** teachers will learn from and critique each other in an informal and constructive manner, providing support and development

The disadvantages that we anticipated were

- **The personal touch -** not getting to know the learners as well as with a traditional class system.
- **Preparation time-** increased time in preparation in preparing dynamic introductory lessons for the whole grade
- **Class division pressure and admin –**the pressure of having to mark the whole grade overnight in order to be ready for new class divisions.
- **Multiple classes unsettling –**children might struggle to adapt to new teachers, and find the class swopping unsettling

The execution of the system in reality

At the time of going to print, certain aspects of the system have not been implemented because of practical/technical difficulties

- Videoing of lessons – video equipment failures, and videographer (i.e. me) failures have meant that this has not happened. Most of the introductory lessons have involved PowerPoint presentation, however, and these have been made available on the network, along with other resources and links to sites online.
- Because we have wanted to give ourselves flexibility in terms of when it is best to move on to the next section, rather stick to a rigid ("equations will take two weeks") time-table, our ability to utilize the interns has been restricted because of time-table clashes. Also some of the interns were daunted by the prospect of tutoring Maths .
- We have not included as many kinesthetic activites as we would have hoped (although this may be a more natural part of the geometry sections still to come).
- After the second session we realized that it would often be better to split the lead teachers'

lectures with a few days of small classes, to give learners a chance to master the basics before moving on, as there was a tendency for lead teachers to try to fit too much into the given time.

- We also needed to add a “correction and extension” lesson between the test and the returning to lecture group, in order for tests to be returned and corrections and remediation to happen before the next topic was started.
- The logistics of collating end of term marks, learners’ portfolios and report comment slips was a stumbling block that we had only half anticipated.

Evaluation of the system after a term and a half

It is impossible to analyze the success of this system in terms of results. We have deliberately used some easy tests working on the principle that success breeds success, and we have reshuffled the order in which we teach the syllabus, so that tests from last year are no longer covering the same range of materials as we have covered. Also there is no way of knowing how strong this group would have been in the conventional system, as they are new to our school in grade 8 and we have multiple feeder schools so no “control” is available. Moreover the motivation behind the change of system was more about changing the way learners approach and think about Maths, than about improving marks. (Although we certainly hope that the long term effect of an improved attitude will be improved results!) Perhaps by the end of the year, we will be able to get a fair sense, by comparing exam results to those of the last few years. For now however, all we have to go on is impressions. To ascertain these, in addition to the organic daily and weekly discussions amongst staff , learners and staff were asked to respond to a questionnaire after the first term and a half. By this time 6 topics had been covered and classes had swopped around 5 times (learners stayed in the same classes for topics 4 and 5 as these were relatively short sections).

(1) Maps; direction and bearing; angles of inclination and declination; Cartesian planes and plotting

(2) Exponents

(3) Introduction to negative integers

(4) Factors LCM HCF

(5) Pattern

(6) Introduction to algebra

The learners’ questionnaires were kept deliberately vague, and generalized, not referring directly to the new system, but trying go gauge their feelings in general while allowing enough space for the new system to be included in their responses. The teacher questionnaire was specific.

Learner questions:

In terms of my Maths marks so far this year I feel...
The Main difference between Maths this year and Maths last year is ...
What I really enjoy about Maths is ....
Sometimes I wish...

Teacher questions:

1. What has worked?

2. What hasn’t worked?

3. a. Should we continue next term?

   a. If so with what modifications?

4. a. Should we do this system again with next years’ grade 8s?

   b. If so with what modifications?

5. Other comments

Analysis

Learners were generally disappointed with their results. This is not surprising as the jump between primary and high school is always a shock. However, what was encouraging was that most learners were still enjoying Maths and /or were strongly motivated to improve their marks despite being disappointed by the mark itself.

18 out of a total of 63 questioned mentioned the ‘class changing’ system at all, 4 specified that they liked being taught be the different teachers; 7 mentioned without comment that the classes changed; 7 students wished that they could have the same teacher.

23 mentioned the level of complexity the detail and the speed of High School Maths. 8 said that Maths this year was easier.16 commented on Maths being more fun and enjoyable this year. (There was a overlap here of 9 with those commenting on complexity.) (There was no correlation between marks and any of the tendencies/preferences.)

The technological aspect was only mentioned by one learner who commented that she enjoyed Cami-Maths, though I have no doubt that this contributes generally to their sense of Maths being “fun”

The issue of favourite/preferred teacher was only raised by 3 learners. 1 learner commented that their current teacher didn’t explain clearly enough (the only blaming the teacher type comment, although arguably the 7 wanting 1 teacher may be transferring blaming the teacher to blaming the system)

The response from the staff was strongly positive with the following pros and cons being identified.

Pros: general positive attitude about Maths, general desire to improve their marks; generally taking responsibility for their successes and failures; the diversity of teaching styles is positive; staff learn from each other, and feel supported and encouraged; staff pick favourite topics or areas of expertise which enhances teaching and satisfaction; development of staff skills facilitated organically; staff get to know the whole grade, and see the whole spectrum of ability levels within the grade

Cons: individual attention hard in lecture venue; time is sometimes lost setting up the data projector; centralized admin is needed and is hard to manage; knowledge of individual learners is reduced.

3/4 of us said we would like to use this system again next year with slight modifications, and the remaining person saying she wasn’t sure. 4/4 of us opted to continue with the system this year, with modifications.

From my subjective perspective the system has been enormously beneficial despite its problems. The pros far outweigh the cons to my mind. The number of student who are clearly unsettled by the system is not as dramatic as we anticipated, and the generally positive effect is apparent. It is ideally suited to grade 8 which is a stage at which both the Maths, and the learners themselves change dramatically, so it is an ideal moment for a paradigm shift. Learners are also more open to a new way of doing things because they accept that High School is different from Primary School; and for many a paradigm shift has certainly been achieved.

In addition, I believe that the teaching staff have also undergone a paradigm shift. We have developed wonderful resources, enjoyed each other's lessons, and help each other to constantly improve our teaching styles and methodologies. We have seen both how easy and how powerful the inclusion of technology, games, and kinesthetic activities are, and we ourselves are less likely to label individuals learners as "strugglers".

Adaptability and applicability elsewhere

I have broken this experiment down into 5 components to analyze its adaptability, as this isn't an all or nothing system.**Class reshuffling:** adaptable provided the department is large enough and staff are willing, however I think that grade 8 is the ideal place for such a methodology. I am not convinced that it would be as successful in higher grades at school. At university level this could be effectively implemented in certain tutorial settings. Effective in terms of de-stigmatizing Maths ability and encouraging learners to take responsibility for their own learning. **Team teaching to introduction topics**: adaptable anywhere. Extremely valuable both for its developmental aspect and for encouraging top class creative work. Expectations need to be firmly laid out from the beginning.**Use of games, modeling and kinesthetic exercises**: adaptable anywhere. Definitely had a positive effect on learners perception of Maths.**Technological component**: adaptability depends on resources but powerpoint element requires only one computer and one data projector. Extremely valuable especially for visual learners and attention deficit learners. **"Pay as you go"** Easily adaptable at school level (though easier with the technology available) Adaptable to university. Extremely effective.

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[i] This system is an adaptation of a similar system was used a Westerford a number of years ago.
[ii] Based on Howard Gardner's "Multiple Intelligences" theory.

Left to their own devices: Student-led inquiry into mathematical ideas in kindergarten

Marjorie Henningsen, Ed.D.
Head of School, Wellspring Learning Community, Beirut, Lebanon
marjhenningsen@gmail.com

Absract
In this workshop, participants engaged with and reflected on authentic artifacts from student inquiry into measurement (4 years old) and data (5 years old). Participants analyzed and reflected on these authentic examples in order to discuss the children's learning processes that arose and what were the respective roles of the classroom environment and teachers in affording the students' self-directed inquiry. Participants were also invited to reflect on the radical implications of this for curriculum development and mathematics learning in school, even at the pre-primary level, as well as implications for teacher development.

Student-led Inquiry with Young Children
The mathematics education community is still engaged in a lively debate about the nature of mathematics, what it means to do mathematics, and how to best help students to become doers of mathematics in school. Many educators still believe that such a view applies only to older students who have already "mastered the basics" in order to move on to higher-level mathematical activity. The examples we explored in this workshop put the spotlight on how very young children might come to do mathematics and actively, purposefully and appropriately use it as a tool for learning on their own, without direct instruction from a teacher and well before having "mastered the basics."

Very recent studies in educational psychology by Bonawitz, et al. (in press) and Buchsbaum et al. (2011) have called into question the appropriateness of direct instruction for young children and also the role the teacher should have in the learning process. It turns out the very young children are capable of engaging more fully and fruitfully in authentic inquiry and exploration on their own than with explicit guidance from a teacher. Their work suggests that a more appropriate role for the teacher is to be a learner side by side with the children, a more implicit role as a learning partner in a process of inquiry. The Primary Years Program of the International Baccalaureate, if practiced as the documentation of it envisions, provides a potentially ideal programmatic school setting in which to explore these notions in practice because of the emphasis on student-led inquiry.

Indeed, the students in the present examples were able to pursue logical lines of inquiry on their own with little explicit teacher intervention. The children were working in the context of a pre-primary program housed within a larger international school in Lebanon that follows the International Baccalaureate Primary Years Program with some significant features of the Reggio Emilia educational approach. The program, if practiced as the documentation of it envisions, provides a potentially ideal programmatic school setting in which to explore these notions in practice because of the emphasis on student-led inquiry.

The four-year old students' inquiry into measurement began with noticing something placed in the environment by the teacher, namely a measuring tape posted vertically on the wall (the kind often used to keep track of changes in children's height). One student noticed it and began trying to make sense of it. He then called over other students until eventually several students were standing next to the measuring tape, comparing themselves to the lines drawn making the units, and comparing themselves to one another. The teacher stayed at a distance only taking photos and noting down what children were doing and saying. Later the same day there was an opportunity for students to recount what they had done and discovered. The teacher then elicited questions from the students about what they thought they should do next to find out more. Eventually this led to the students exploring the use of non-standard units for measuring length and making some connections about the concept of a measuring unit that we often think of as beyond their capability, such as discovering the idea of iteration of a unit and the relation between the unit size and the measure of the attribute. Student engagement in such an inquiry process constitutes significant learning and can be viewed as an essential foundation for students conceptual development in mathematics (Gravemeijer, 1999).

In the second set of examples, five-year old students engaged in a variety of authentic data collection and data analysis activities in order to pursue their own questions and to make sense of things in their environment. It is often hard for teachers to imagine how they can use data modeling (Lehrer & Schauble, 2002) with young children given all the traditional constraints and the tendency to view younger children as unable to deal with complex reasoning and problem solving. In our program, we have found that working with data is essential and comes very naturally for young children if they are truly engaged in trying to make sense of phenomena and understand how the world works, thus developing a better sense of the nature of mathematics and science and their role in our lives.

Implications for Curriculum and Teacher Development

Allowing children from such an early age to engage in more self-directed learning implies a radical shift in thinking about what mathematics learning should look like in school. The idea of engaging students in inquiry processes or in an inquiry cycle in school is aligned well with reform-oriented approaches to mathematics education as advocated for the past 20 years is not very radical by itself. However, I believe that taking it a step further and thinking about inquiry not as an instructional method, but as a *stance* toward mathematics curriculum (Short, 2009) represents a radical shift. Another aspect of the shift is from thinking of mathematics as a school subject toward thinking of it as a tool for learning about oneself, the human condition, and about the world in general. Such a shift has great implications for the design of school learning environments and materials, as well as for teacher education and ongoing development.

First, with respect to curriculum development, it is important to realize that inquiry is not just a way to make learning more fun; it is the embodiment of doing mathematics and science and the way children naturally learn and make sense of things. Any earnest attempt to engage children with authentic inquiry highlights the obvious tension between thinking of mathematics in school as a body of content knowledge that needs to be acquired by the child vs. mathematics as a tool for systematically understanding how the world works and imagining/inventing ways to improve upon it. This tension raises all the questions about the utility and role of textbooks as resources (and for whom should they be a resource) and what children "should" learn and how that should be determined and what really are "the basics" when it comes to mathematics in the 21st Century.

Second with respect to teacher development, the major challenge is helping teachers to get in touch with their own deep-seated beliefs about what mathematics is and why it is important to worry about whether children learn it and their beliefs about what children are capable of. Children do not know that they cannot direct their own learning and pursue their own ideas until they learn that through the schooling process—an assumption worth questioning. Teachers first need to believe it is possible for themselves and their students to generate legitimate questions that can lead to fruitful investigations and learning without always being told what the questions should be. Professional development programs need to address this explicitly and to engage teachers in authentic problem solving and inquiry processes as learners. Professional development experiences must be contextualized in teachers' practice as much as possible in order to impact on their practice (Smith, 2001) and it is further helpful if teachers actually come to view teaching as an important inquiry in and of itself. In other words, adopting inquiry as a stance toward curriculum implies a need to also adopt an inquiry stance toward one's teaching practice.

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Adjusting the Mathematics Curriculum Into the 21st Century

Classroom Examples

Hoffmann R. Ph.D., Klein R. Ph.D.
Kibbutzim College of Education,
Tel Aviv, Israel

Abstract

In our efforts to improve the mathematics curriculum and adjust the mathematics education into the 21^{st} century, we have developed a technology based course at a teacher education college. Technology enables us to integrate into the school curriculum topics which were until now only taught in higher math education courses. Raising students' awareness to the mutual relationship between math and technology will broaden the students' view of the nature of mathematics and its applications in the real world.
This paper focuses on one of the topics taught in the course- finding roots of various kinds of equations. We begin with computing square and cubic roots using the intuitive 'trial and error' method followed by Heron's method (100 a.d.). Then we generalize both to first and second order numerical methods which enable to solve even equations which have no analytic solving formulas.
We use the GeoGebra software to obtain the graphs of the functions or to check student's answers which were obtained using calculus. The students define the number of the solutions (if any). They get acquainted with the numerical methods, write their own algorithms, translate them into computer programs and get the solution by using Excel.
We hope that adding new and vital subjects, which are ordinarily absent from the regular school programs (in Israel), using technology and rich learning tasks, will make the shift in mathematics education.

Introduction

The last decades are characterized by rapid changes in mathematics teaching all over the world. Technology changes the way students think and learn (Dori, Barnea, Herschkovitz, Barak, Kaberman, and Sason, 2002) and demands adequate changes in the curriculum as well. "In mathematics instruction programs, technology should be used widely and responsibly, with the goal of enriching students' learning of mathematics (NCTM, 1991).
In our efforts to improve the mathematics curriculum, students' math understanding and adjust the mathematics education into the 21^{st} century, we have developed a technology based course**.** This course is taught to mathematics B.Ed and M.Ed students in a teacher training college.
Technology enables us to integrate into the school curriculum topics which were until now only taught in higher math education courses. Raising students' awareness to the mutual relationship between math and technology will broaden the students' view of the nature of mathematics and its applications in the real world. Lecturing at a teacher education college, we believe that after participating in the course, our students will incorporate its topics into schools.

This paper describes one of the topics taught in the course - finding roots of various kinds of equations. This is one of the issues mathematicians dealt with from the dawn of history, in order to solve various kinds of mathematical problems.
Students are not aware that (according to Abel, Galois and Lie) there does not exist and will never be found a closed formula for solving polynomial equations of an order greater than 4, and for other non algebraic equations such as exponential or trigonometric (Arbel, 2009). Working with advanced software (such as Matlab) one can get solutions to those problems. We Raise students' mathematical curiosity as to how the computer functions. In other words, they learn "the story behind the key" of calculators, graphic calculators and behind computer built in library functions.
During the course the students get acquainted with the mathematical ideas and numerical methods absent from the school curriculum (in Israel). At first the students use calculators, then they write an algorithm and translate it to a computer program using Excel. Realizing the "strength" of computers they construct permanent software that is both efficient and fully automatic.

Computing square and cubic roots

1. <u>The intuitive 'trial and error' method</u>

This method is based on finding two sequences of upper and lower bounds which get closer and closer to the root (until the desired accuracy is reached).
For example:
To compute √5 we start by intuitively finding two numbers a, b as lower and upper bounds for √5. In our example 2<√5<3 (first approximation). Knowing that √5 is greater than 2 and smaller than 3, let us take 2.5 (the average) as our next intuitive approximation. Now we calculate the value of x^2.If x^2 is smaller than 5 the new approximation is the new lower bound a, otherwise it is the upper bound b. We continue until a is close enough to b. The algorithm is shown in figure 1.

COMPUTING THE SQUARE ROOT OF S *Trial and Error*
1. input s (positive number)
2. input a, b a< s^0.5 <b
3. <u>while</u> a = b (to a desired accuracy) <u>do</u>
3.1 x ◄── (a+b)/2
3.2 if x^2 <s then a ◄── x
else b ◄── x
4. print x
5. end

Figure 1

The next stage is translating the algorithm to a computer program (using Excel) –see figure 2 below.

2. <u>Heron's (100 a.d) iterative formula for computing the square root of s (a given positive number)</u>

Heron's method is based on creating a sequence of rectangles, all with area S. In each new rectangle both sides are getting closer to each other. As a limit of the sequence we get a square. The sides of this square are the desired square root of S.

trial and error					Heron	
a	b	x	x2	S	a	b
2	3	2.5	6.25	5	1	5
2	2.5	2.25	5.0625		3	1.666666667
2	2.25	2.125	4.515625		2.333333333	2.142857143
2.125	2.25	2.1875	4.78515625		2.238095238	2.234042553
2.1875	2.25	2.21875	4.922851563		2.236068896	2.236067059
2.21875	2.25	2.234375	4.992431641		2.236067977	2.236067977
2.234375	2.25	2.2421875	5.027404785		2.236067977	2.236067977
2.234375	2.2421875	2.23828125	5.009902954			
2.234375	2.23828125	2.236328125	5.001163483			
2.234375	2.236328125	2.235351563	4.996796608			
2.235351563	2.236328125	2.235839844	4.998979807			
2.235839844	2.236328125	2.236083984	5.000071585			
2.235839844	2.236083984	2.235961914	4.999525681			
2.235961914	2.236083984	2.236022949	4.999798629			
2.236022949	2.236083984	2.236053467	4.999935106			
2.236053467	2.236083984	2.236068726	5.000003346			
2.236053467	2.236068726	2.236061096	4.999969226			
2.236061096	2.236068726	2.236064911	4.999986286			
2.236064911	2.236068726	2.236066818	4.999994816			
2.236066818	2.236068726	2.236067772	4.999999081			
2.236067772	2.236068726	2.236068249	5.000001213			
2.236067772	2.236068249	2.23606801	5.000000147			
2.236067772	2.23606801	2.236067891	4.999999614			
2.236067891	2.23606801	2.236067951	4.99999988			
2.236067951	2.23606801	2.236067981	5.000000014			
2.236067951	2.236067981	2.236067966	4.999999947			
2.236067966	2.236067981	2.236067973	4.99999998			
2.236067973	2.236067981	2.236067977	4.999999997			
2.236067977	2.236067981	2.236067979	5.000000005			
2.236067977	2.236067979	2.236067978	5.000000001			
2.236067977	2.236067978	2.236067977	4.999999999			
2.236067977	2.236067978	2.236067978	5			
2.236067977	2.236067978	2.236067977	4.999999999			
2.236067977	2.236067978	2.236067977	5			

COMPUTING THE SQUARE ROOT OF S –

HERON OF ALEXANDRIA

1. input q (desired no. of correct figures)

2. input s (positive no.)

3. a ⟵ 1

4. b ⟵ s

5. E ⟵ abs(a-b)

6. <u>while</u> E>10^ -q <u>do</u>

6.1 a ⟵ (a+b)/2

6.2 b ⟵ s/a

7. <u>else</u> print a, b

8. end.

Figure 2

In our presentation we will show how the students expand Heron's algorithm to a fully automatic one that is stopped when a desired accuracy of q significant digits is obtained. We will also compute the cubic root similarly.
Another way of calculating the digits of the square root of a given number s, is by finding the roots of the equation x^2-s=0.

Solving equations

In order to solve f(x)=0, in other words to find the real roots of the equation, we look at the function y= f(x) and solve $\begin{cases} y = f(x) \\ y=0 \end{cases}$ (points of intersection with the x axis).

To investigate the given continuous function y=f(x) the students are asked to:

- Plot the graph of the given function (using calculus and/or GeoGebra software). The software is used either for checking students' answers which were obtained using calculus, or to obtain the graphs of the functions.
- Decide the number of zeros (if any).

We will now present two methods that enable to solve (numerically) all kinds of equations, even those which have no analytic solving formulas (exponential, trigonometric or polynomial of a degree greater than 4). These methods are a generalization of both methods discussed previously.

1. Bisection method
For each of the zeros of an increasing function

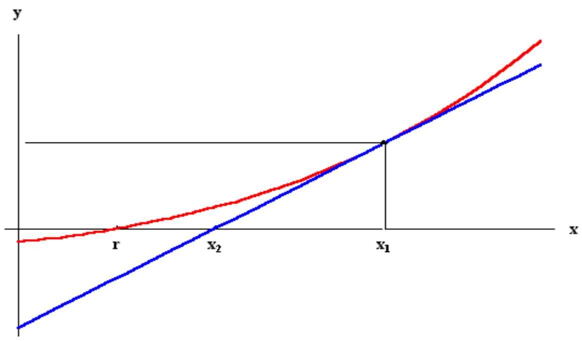
- Choose a relevant interval [a,b] where f(a) <0 and f(b) >0 (switch f(a) with f(b) for a decreasing function). The required root lies between a and b (Cauchy's mean value theorem), precisely where the graph intersects the x axis.
- Let x_m= (a+b)/2 be the midpoint of the interval. Compute y= f(x_m)
  If y<o take x_m as the new a or else take x_m as the new b.
- Continue until the desired accuracy is reached. (Breuer and Zwas, 1993).

The algorithm and the program for calculating the square root:

1. Input s
2. Input a, b, f(a)<0, f(b)> 0
3. x $\leftarrow$ (a+b)/2
4. f(x) $\leftarrow$ x^2 -s
5. While f(x)$\neq$0 do
   5.1 if f(x)>0 x $\leftarrow$ b
       Else x $\leftarrow$ a
   5.2 x $\leftarrow$ (a+b)/2
   5.3 f(x) $\leftarrow$ x^2 -s
6. print x

Another example solving xe^{-x} -0.25=0, will be presented during the presentation.

3. Newton Raphson Method- using the tangent line for a differentiable function



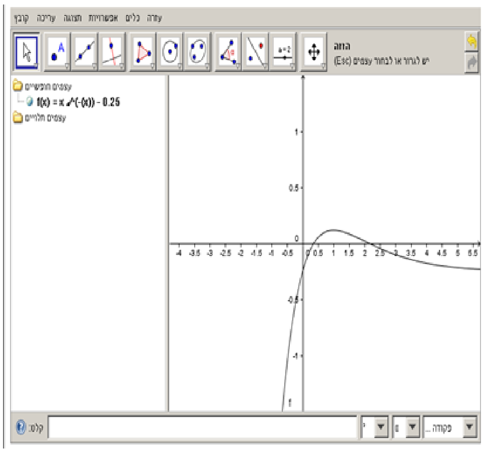
- Choose x_1 to be the first approximation for the root r. $y_1=f(x_1)$.
- Find $f'(x_1)$.
- At (x_1, y_1) calculate the tangent line.
- Find x_2, the point where the tangent line intersects the x axis. x_2 lies closer to r, therefore x_2 is chosen to be the next approximation of r.
- Continue similarly until f(x)=0 (the desired accuracy is obtained).
- For each iteration: Find $y=f'(x_n)$ and compute
  $\boxed{x_{n+1}=x_n-f(x_n)/f'(x_n) \qquad f'(x_n)\neq 0}$ **Newton Raphson** formula (1690).

Newton Raphson Method **$f(x)=xe^{-x}-0.25$**

$\boxed{\begin{array}{l} x_n+1=x_n-f(x_n)/f'(x_n) \\ f'(x_n)\neq 0 \end{array}}$

x	f(x)	f'(x)
0.5	0.05326533	0.3032653
0.324360635	-0.01549068	0.4884801
0.35607263	-0.00059899	0.4510207
0.357400699	-1.01E-06	0.4494932
0.357402956	-2.93E-12	0.4494906
0.357402956	0	0.4494906
0.357402956	0	0.4494906

increasing



x	f(x)	f'(x)
2.5	-0.0447875	-0.1231275
2.136251007	0.002284318	-0.1341876
2.153274331	2.41E-06	-0.133899
2.153292364	2.89E-12	-0.1338987
2.153292364	0	-0.1338987
2.153292364	0	-0.1338987

decreasing

the algorithm

```
1. a ← x1
2 b ← f(a)
3. c ← f'(a)
4. while f(x)≠0 do
  4.1 print a
  4.2 a ← a-b/c
  4.3 b ← f(a)
  4.4 c ← f'(a)
5. end.
```

One can see:

- When dealing with the same equation $xe^{-x}-0.25=0$, the Newton Raphson Method (of the second order) gives the solution much quicker than the bisection method (of the first order).
- For $x^2-s=0$ Newton Raphson's method yields the same formula (and result) as Heron got without using calculus.

During the course, the students have learned many and varied numerical methods taken from different branches of mathematics. Emphasis is given to the mathematical knowledge and to accompanying justifications.
We believe that technological developments make it possible to incorporate selected chapters of this course in high school or even in the upper grades of the elementary school curriculum, by adapting the topics to students' knowledge.

We hope that these topics will be integrated into the curriculum and our students will be the agents who incorporate it into schools.

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Intervening for Success

Marilyn Holmes & Viv Thompson
University of Otago College of Education, New Zealand
marilyn.holmes@otago.ac.   viv.thompson@otago.ac.nz

Abstract

Although New Zealand has been named in the top six countries in the world for achieving consistently highly in Mathematics, Reading and Writing according to PISA findings, 2009, there remains concern that a significant number of children are underachieving in Mathematics. Whilst it could be argued about the exact numbers it is patently clear that Maori and Pasfika children are over represented in the bottom 20%. This paper outlines a New Zealand pilot ‘Accelerating Learning in Mathematics’ (ALiM), trialled in 2010. The purpose was to support children by using a variety of approaches to accelerate children’s mathematical learning over a short term period, thereby giving the children an equitable opportunity to achieve at their appropriate level.

Within the context of this paper the use of the word children is used to define primary school age children (5 – 13 years) as opposed to the older secondary students.

Introduction

It was not until the results from the Third International Mathematics and Science Studies were released in 1996 showing New Zealand children (in the studies) were found to be below the International average in mathematics that the government sprang into action by forming a Mathematics and Science Taskforce group, in 1997, to address the problem. Of special concern were the lower results of Maori and Pasifika children who are destined to become a significant component of the future workforce. Also, at that time, it was noted that teachers in New Zealand were experiencing difficulties with implementing problem solving reforms in mathematics teaching.

Although there were some strong mathematics educators in New Zealand they were few in number and disparate. Four of the Auckland College of Education mathematics lecturers and mathematics advisers from each of the regions in New Zealand under the guidance of the Ministry of Education formed the basis of the first co-ordinated numeracy community covering the whole of the country. The New South Wales Department of Education and Training initiative ‘Count Me In Too’ was seen as a worthwhile project to trial. An experienced team was brought out from Australia to demonstrate the work they had introduced to teachers of junior classes. That particular project provided a well researched base from which to develop the New Zealand Numeracy pilot in 2000, and later the growth of the Numeracy Development Project (NDP). Its aim from 2001 – 2009 was to raise children’s achievements in mathematics by improving teachers’ professional knowledge, skills and confidence. During those years every teacher was allocated 12 hours professional development time as part of NDP. Facilitators, principals, and teachers were interdependent in effectuating the successful implementation of the Numeracy Project. Schools were encouraged to either work in the project by syndicates or as whole school professional development. Having several teachers involved at once meant the commitment was easier to maintain, through the support they gave each other.

From 2000 to 2010 milestone reports and evaluations annually documented evidence from teachers, facilitators, researchers and policy analysts and those continued to inform any further development of the Numeracy Development Project and beyond 2009. Today the project has moved from phase 1 which was implementing the project into phase 2; sustainability. Whilst funding limits the number of school facilitators work in, a substantial website means teachers have support at their fingertips (refer www.nzmaths.co.nz for further information).

Why Accelerating Learning in Mathematics (ALiM)?

Although there have been great successes in the Numeracy Project in New Zealand there remains an important issue. One of the aims was to close the gap, (identified through TIMSS and PISA reports) of the lowest performing students (NZ) as compared to the top performers (NZ) and this has not been sufficiently attended to. Maori and Pasifika students, after ten years in the numeracy development project, still remain a significant number of the 'below expectations' group. In an attempt to focus on the issue the government has released funding solely to address such a problem. Along with the funding National Standards were introduced in 2010 to highlight at each school year (Year 1-8) the expected levels of achievement for Mathematics, Reading and Writing. National standards were put in place to help counteract the slowly diminishing tail with urgency. The numeracy development project was about raising children's achievement through improved teacher capability whereas ALiM is about attending to the child.

ALiM

The purpose of the pilot was to see if the children's learning could be accelerated with targeted teaching and learning for a period of ten weeks. The project involved thirty nine schools with funding from the Ministry of Education. Numeracy facilitators were given twenty hours to offer support and guidance per teacher. In the Otago/Southland region four schools were involved in the pilot and were selected by facilitators on the basis that the classroom teachers were good classroom teachers but still had small groups of children who were not achieving to expectations.

The children were selected for this project by principals and classroom teachers in consultation with numeracy facilitators. Factors that were taken into consideration were; regular school attendance, status in relation to the national standards, behavioural issues and a desire to learn. The teachers were selected for their effective classroom teaching practice, and had demonstrated a willingness to be part of the project.

At a national meeting, mathematics education researchers were called to present research ideas about accelerating learners who were not achieving to expectations. Several aspects arose from the discussion. Alton-Lee mentioned the emotional environment, "children will not learn if they do not feel safe and that we need to strengthen valued social outcomes and that an ethic of care needs to be created" (A. Alton-Lee, personal communication, April 26, 2010). Bullying is a huge issue in primary and secondary schools and scared children often think of the next break outside the classroom when they are at their most vulnerable rather than on the tasks at hand. Anthony shared her thoughts that making connections is central to the learning process and learning mathematics is not a linear process as children do not move along the same path (G. Anthony, personal communication, April 26, 2010). Attention was drawn to other researchers outside the group such as Mulligan's work on patterning which can lead to a significant improvement in mathematical outcomes, the importance of building harmonious relationships between school, families and communities which can have reciprocal benefits for all concerned (Ministry of Education, 2008, p.3) and reference to Wright's address at a numeracy conference in Auckland where he stated any teaching/tasks must develop their number knowledge to support non-count-by-ones strategies (R. Wright, personal communication, February 20, 2008).

With those considerations in mind facilitators wrote 16 small resources that could be used as a basis for planning with teachers. An initial two day ministry funded seminar was held where teachers and facilitators, from all over New Zealand, worked together to look at effective pedagogy and select an appropriate intervention which was then tailored to meet the particular needs of the targeted children. Throughout the months teachers and facilitators were mainly guided by ten principles:

- An ethic of care
- Arranging for Learning
- Building on students' thinking
- Worthwhile mathematical tasks
- Making connections
- Assessment for learning
- Mathematical communication
- Mathematical language
- Tools and representations
- Teacher knowledge (Anthony & Walmshaw, 2009)

The principles focus on effective teaching and although intended to be 'nested within a larger network', in general teachers found some practices were more applicable to their children's needs.

The most common mathematical concepts covered were place value in the senior classes and addition/subtraction in junior classes. Most teachers chose to run an extra four or five sessions for the targeted children, over and above the normal classroom mathematics programme. The sessions ranged from an extra 20 – 30 minutes depending on teachers' release time. Teachers were assigned a facilitator who would mentor and support them, and on their return to school began to implement their planned intervention.

Success of ALiM

The New Zealand Council for Educational Research (NZCER) was contracted to supervise the data gathering, analysis and evaluation of the exploratory study. The children were assessed using NumPA (a diagnostic interview), a Progressive Achievement Test (norm-referenced assessment tool) and an attitudinal survey. These assessments were administered prior to the intervention and at the end of the intervention (usually about ten weeks later). The PAT assessment was debated as to the appropriateness of having the children, who usually were at the lowest percentile; sit a test in exam like conditions. PAT involves a significant amount of reading therefore teachers were encouraged to read the questions for the children.

Findings showed that achievement levels increased (NZCER, 2010). Most of the children moved at least one numeracy stage, and surprisingly in the PAT Mathematics test, scores increased by an average of 80% of a year's growth over the ten weeks involvement in the pilot. When comparing ethnic groups results Pasifika children also averaged 80% with Maori children increasing by 40%. An achievement gain for Maori children but the questions remaining unanswered were; why was it that they didn't make the same gains and what were some of the variables that impacted on their lesser achievement?

Children's attitudes to mathematics also showed a positive shift, their confidence and self-belief showed gains.

Reasons identified by teachers and facilitators for these successes were; regular well-structured sessions which made effective use of concrete materials, repetition of material learned at previous sessions, connections between knowledge and strategy, knowledge gaps identified and addressed, small groups where the children were actively involved and could take risks and be free from other children's/ teachers preconceived notions of their ability in mathematics, more complex mathematical language being explored and used by the children to explain their thinking, teachers reflecting on their practice and having an active involvement from their facilitator, communication and support from whanau (family).

ALiM as Seen Through One School's Success

Throughout New Zealand thirty nine schools implemented the project in 2010 but for the purpose of this paper the picture can be seen through the story of one of the four schools in the southern reaches of the country and how they implemented ALiM (further stories can be found on the website www.nzmaths.co.nz).

One School's Story

The small school is situated about one and a half hour's drive north of Dunedin with a population of approximately13,000. It is a decile 4 school (decile 10 relates to a high socio-economic area), with 156 pupils, 25% of the students identifying as Maori and 20% identifying as Pasifika. It is a contributing school which has children from five years old to eleven years old. The school has a high emphasis on family values and has very strong connections with the community. With the downturn in the economic climate in NZ there has been a greater transience of pupils in the last three years.

Participants and Data Gathering

Two groups of students were chosen. Six children who were five years old in one group (Group A) and five children, seven and eight year olds, were in the other group (Group B). The small sized groups were deliberately chosen as the teacher felt they were more likely to take risks and were able to make better interactions with the teacher and other children.

Group A was taken by a teacher aide and happened for 38 of the sessions over the same period. This teacher aide (TA) has a TA certificate and was guided by the teacher. Those sessions were held from 2.30pm to 3.00pm, therefore, the children received extra mathematics lessons as they had their own class session daily from 12.40pm to 1.40pm.

Group B was taken by the teacher responsible for this project and happened on 40 x 30 minutes sessions. This was the highest possible number of sessions that could be taken over the eight week period. The sessions were held from 9.00am to 9.30am with those children also receiving extra mathematics lessons as they had their own class session daily from 12.40pm to 1.40pm.

Initial and final data was collected, as set out by NZCER and teachers administered the Numeracy Diagnostic Interview which they were very familiar with because of their involvement in the Numeracy Development Project. PAT assessment was not administered in this school due to the age of the young children. Records taken were put onto the school's files for future reference and target setting.

Facilitation of the Exploratory Study

The teacher, an experienced teacher of nearly 40 years chosen by the Principal, was coupled with a numeracy facilitator who also had many years in the classroom environment. The teacher was able to offer many ideas and worked well with the facilitator to fine tune the plans for both groups. Assistance to the teacher was also given through visits, email and phone.

As the support resource for 'children after one month at school' was well structured the experienced teacher felt that her teacher aide would be able to follow the format with encouragement from the teacher. The focus for that work was a) to increase students' every day and mathematical language through exploratory activities and b) to develop an understanding of numbers to 5. The lesson structure was based around six short 5 minute periods and always included: counting, matching, physical, sorting and copying activities as well as action rhymes (see nzmaths.co.nz\ALiM for further information).

Four resources for Group B children were used: after 6 months, after 9 months, after 18 months 'moving children to simple additive strategies' which the group had just been introduced to when they finished in the study.

The teacher and the teacher aide started each lesson with tasks the children had already experienced success with and made opportunities to connect their knowledge with other 'harder' aspects. An example was to use their knowledge of counting backwards 9-0 to connect with bigger numbers. If the children started at 27 the teacher would point to the 7 then the 6 with 26 and 5 with 25 and so on. A variety of activities were required to be adapted and made for the children's use. This proved to be a time consuming effort but worthwhile as the end products were able to be used time and time again

In the 7th week both the teacher and the teacher aide were videoed working with the children and gave interviews. The DVD has proved an inspiration to many other teachers who have used and adapted many of the ideas the teacher, teacher aide and facilitator worked on.

Results

The results (see Table 1) show where the children were at the start of the programme on the New Zealand Framework, and where they were after 8 weeks. All stages are counting stages from Stage 0 where children cannot one to one count, to Stage 4 where they can 'hold a set in their head' and count on. For example, to solve 8 + 5 they would say 8 and count on 9, 10, 11, 12, 13 to solve the problem. In New Zealand children are expected to reach Stage 2 or 3 by the end of Year 1 and Stage 4 by the end of Year 2.

Table 1

Number of students	Year	Initial stage Add/Sub	Final stage Add/Sub	Time in programme	Predominant Focus
Group A 6	**0/1**	3 students - stage 0 3 students - stage 1	6 students - stage 2	5x30mins weekly 7.5 weeks	Language development & Number Knowledge to 5
Group B 5	**2/3**	1 student - stage1 4 students - stage 2	5 students - stage 4	5x30mins weekly 8 weeks	Number Knowledge & Strategy

The group A children in 7.5 weeks were able to reach a stage on the New Zealand Framework which is an expectation for a child who has been at school for a year. The group B children who were well below expectation for their year group managed to achieve at a comparative stage with their peers. In 8 weeks they made the equivalent progress of 12 months at school

Reasons for success

The teacher identified several factors for success:

- Extremely supportive principal and facilitator
- Parental involvement with close communication between the teacher and the parents
- The interest, involvement and support of other teachers in the school. They observed some lessons and were able to apply some of the concepts to their own classroom practice
- The environment where the sessions took place was set up specifically for mathematics and showed mathematics was valued
- The group size was ideal, children were able to take risks without other children making value judgements. Problem solving and working together as a group increased the children's confidence in their own ability. The children's image of themselves as being able to solve problems became much more positive

- Connections between known material and new material were made explicit. The children became much more confident to share their thinking using more complex mathematical language.
- Having a variety of materials was essential so that children had a great deal of repetition of the same concept but in a variety of ways so concepts were able to be generalized

Conclusion

The teacher gave several reasons why ALiM was successful in the small school but failed to mention the pivotal factor and that was, of course, herself. She is a highly effective, experienced classroom teacher and was prepared to take on a challenge to improve outcomes for children in mathematics.

The exploratory study was so successful in her school that the principal has decided that all five year olds will be given the opportunity of 'after one month at school' sessions and the ALiM resources will be made available for other teachers on their staff. The TA has decided to enter teacher training through an online course with practicuum in the school where she is familiar. The teacher has since incorporated many of the ideas into her teaching approaches. The school's story is only one small part of the wider exploratory study but it is a story that is repeated in many other schools.

Following the successful report from NZCER the Ministry of Education through the government has secured funding for a pilot to be held in 2011 involving 160 schools working in ALiM. A new component is the appointment of specialist mathematics teachers in each region who are appointed 0.5 time in their schools and will work with the 'well below expectations' children who have had few or no opportunities for mathematics interventions. New approaches need to be found for those children as opposed to serving the same 'diet' year after year with little success.

As ALiM showed that children's learning could be accelerated after interventions it is with interest that mathematics educators await the next evaluations. From the work in 2010 one factor stands out: When principals, teachers, parents, and outside facilitators work together, alongside the children, achievement improves.

Acknowledgements

Further information on ALiM exploratory study, 2010 and ALiM pilot, 2011 can be found on www.nzmaths.co.nz The views expressed in this paper do not necessarily reflect that of University of Otago College of Education or the Ministry of Education. Thanks are conveyed to the school involved, the staff and the children whose story has been told in this paper.

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What can be Learned from Comparing Performance of Mathematical Knowledge for Teaching Items found in Norway and in the U.S.?

Arne Jakobsen, Janne Fauskanger, Reidar Mosvold, & Raymond Bjuland
Faculty of Arts and Education, University of Stavanger
4036 Stavanger, Norway
arne.jakobsen@uis.no; janne.fauskanger@uis.no;
reidar.mosvold@uis.no; raymond.bjuland@uis.no

Abstract
This paper reports from a Norwegian research project, where a U.S. developed model for teachers' mathematical knowledge for teaching (MKT) was studied. Part of this project included the adaption of MKT measures developed in the U.S. to gauge teachers' MKT. We present results from a pilot study where 149 Norwegian teachers were tested, and where 10 teachers were interviewed in 5 focus group interviews. We discuss how these measures can be used as a tool in relation to professional development of teachers in Norway.

Introduction

Teachers play an important role in determining the quality of graduates, and there is widespread agreement that teachers' understanding of content matter is important for their teaching (Askew, 2008). Still, exactly what knowledge teachers need to have in order to teach is continually discussed (e.g. Hill, Schilling, & Ball, 2004; Rowland & Ruthven, 2011). Researchers at the University of Michigan in the U.S. have contributed to this discussion by developing a framework referred to as "mathematical knowledge for teaching" (MKT. Ball, Thames, & Phelps, 2008). From studies of mathematics classrooms they have identified specific tasks that are involved in mathematics teaching and the mathematical demands behind those tasks (ibid.). Based on this, they have developed measures of teachers' MKT. Their studies have shown that a high MKT score among teachers can be positively associated with increased learning by their students and with higher quality of instruction (Hill et al., 2008; Hill, Rowan, & Ball, 2005).

Knowledge about the topics that teachers struggle with is useful when preparing professional development programs (Hill, 2010). Some research has already been done within this area in the U.S., but there is a need for more research concerning in-service teachers' MKT in other countries. Investigations of how the MKT measures can be used in professional development of teachers in other countries will be an important contribution to this field of research. This paper aims to contribute to an investigation of how the MKT measures can be used in professional development of teachers in Norway. When attempting at using these measures in connection with professional development of teachers, it is interesting to learn more about the connection between teachers' MKT score and their teaching experience. It is also interesting to learn more about the connection between the amount of professional development that teachers have had and their MKT ability. Our research question for this paper is:

> What is the connection between teachers' MKT, their experience and professional development?

This question is virtually impossible to approach on a general level, but we investigate these connections in a sample of Norwegian teachers and discuss possible implications of these findings.

Theoretical Background

After having developed the MKT framework, Ball and her colleagues (2008) have developed items that can be used to measure teachers' MKT. These measures include forms of teaching-specific knowledge (Hill, 2010). One such aspect of MKT is related to purely mathematical knowledge which is specific to the work of teaching or "specialized content knowledge" (Ball, Thames, & Phelps, 2008). Hill (2010) recommends that specialized and pedagogical content knowledge (Shulman, 1986) have particular focus in professional development.

The MKT framework, is a further development of Shulman's (1986) model of teacher knowledge. The MKT model (see Figure 1), which is still in development, consists of a number of knowledge domains describing two of Shulman's initial categories in more detail.

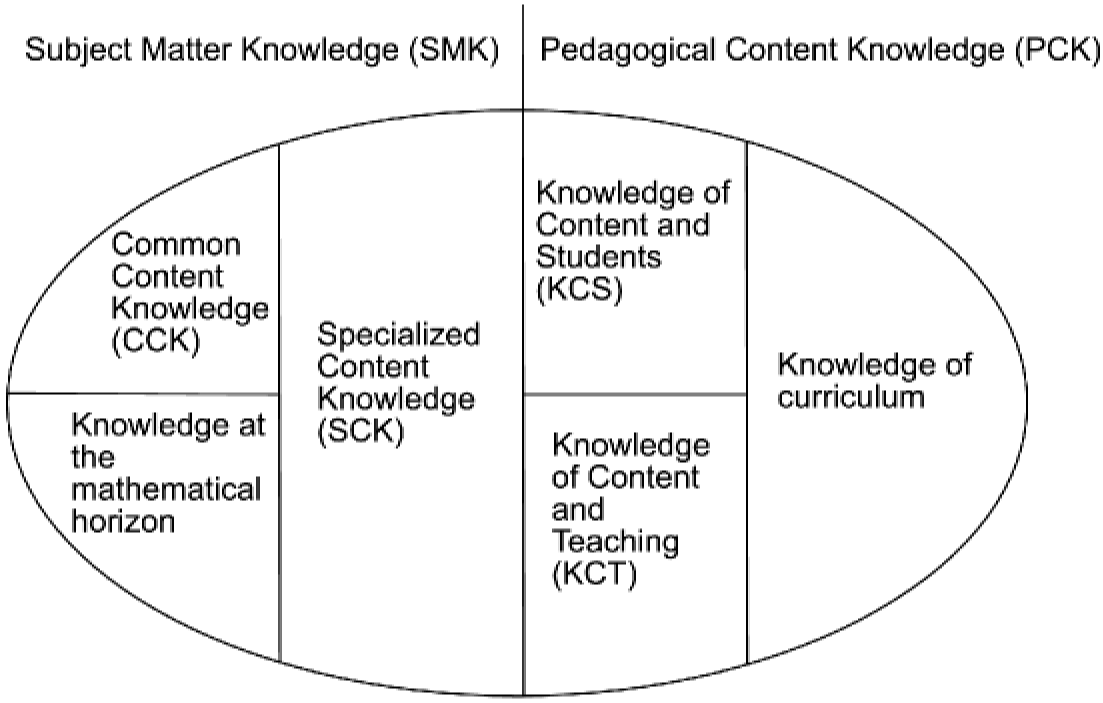

Figure 1: Domains of MKT (see Ball et al., 2008, p. 403 for a definition and discussion of the domains)

The assessment of practicing teachers' knowledge is not a widely accepted practice (e.g. Hill, Sleep, Lewis, & Ball, 2007), at least not in Norway (Lysne, 2006). The goal of Hill and her colleagues (2007) is to move the debate concerning assessment of teachers "... from one of argument and opinion to one of professional responsibility and evidence" (ibid., p. 112). To make advances in developing instruments to study teachers' knowledge, a set of agreed-upon, reliable and valid methods for assessing teachers' MKT is required (Hill, et al., 2007). This is in line with Shulman's (1986) initial aim, which was to develop tests where those educated to teach would get high scores.

Hill, Sleep, Lewis and Ball (2007) argue that assessing teachers' knowledge:

> "... can be done in ways that honor and define the work of teaching, ratify teachers' expertise, and help to ensure that every child has a qualified teacher. Doing so requires carefully constructed instruments that take seriously the work of teaching and that can be used at scale" (ibid., p. 150).

Hill and colleagues (ibid.) see further development of the MKT measures as one attempt to attain this goal. These measures can, to a certain extent, help in-service educators to identify teachers' lack of knowledge (e.g. Hill, 2010) and thus identify opportunities for teachers to learn (Hiebert & Grouws, 2007). Research so far has indicated that the MKT instrument can be relevant for use in professional development in the U.S., but little has been done to investigate its use in other countries. Norwegian teacher education has until recently certified teachers to teach any subject in grades 1-10, and this differs from the situation in many countries. It is therefore interesting to investigate if the U.S. developed measures can give the same useful information when used in a Norwegian context and if the measures can be a relevant tool for in service educators planning professional development.

Methods

After having decided to use the MKT measures in Norway, our first step was to translate and adapt measures for use in a Norwegian context. The 2004 elementary form A (MSP_A04) from the LMT project[1] was translated and adapted (Mosvold, Fauskanger, Jakobsen, & Melhus, 2009). The process of translating and adapting items was conducted based on recommendations from Delaney and colleagues (2008). When the entire set of items was translated, a pilot study was conducted. The pilot study included a quantitative as well as a qualitative part. Mathematics teachers at our partner schools were invited to participate (grade 1 to 10), and 142 teachers from 17 schools participated in the initial phase. In a second phase two new partner schools were added, and the number of participating teachers was extended to 149. In the quantitative part of the study, all participating teachers completed the test individually. All tests were conducted at the teachers' respective schools, and the testing situation was organized in order to be as similar as possible. Among the participating schools, teachers at five schools were selected for participation in semi-structured focus group interviews (FGIs). These interviews were held directly after the teachers had completed the test, and ten teachers participated in the interviews altogether.

The final form used consisted of two parts. Part 1 included the translated and adapted MKT items[2], a total of 61 items (30 item stems). Of the 61 items, 26 items were from the content domain number concept and operations (NCOP), 19 from geometry (GEOM), and 16 from the domain patterns functions and algebra (PFA). In Figure 2, one of the released items is shown in order to illustrate the nature of the items.[3] This item asks teachers to respond to a mathematical task situated in a teaching context. In part 2 of the form, teachers were asked about factual information concerning their gender, their teaching experience, their mathematical background, and their participation in professional development courses.

The MKT items are meant to relate to the underlying MKT construct and can be viewed as one possible operationalization of the construct. An item response theory (IRT) model can serve as a link to the observed latent world (Edwards, 2009). A basic idea in IRT is that an observed item response is a function of person properties and item properties (ibid.). To estimate teachers' MKT score and item characteristics, we have, in the same manner as initially done in the U.S., used a two parameter IRT model.

[1] See http://sitemaker.umich.edu/lmt

[2] For sake of simplicity, we refer to items from the LMT project as "MKT items" in this paper.

[3] The items used in the test are not released and not available for publication.

2. Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
35 x 25 125 +75 875	35 x25 175 +700 875	35 x25 25 150 100 +600 875

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

	Method would work for all whole numbers	Method would NOT work for all whole numbers	I'm not sure
a) Method A	1	2	3
b) Method B	1	2	3
c) Method C	1	2	3

Figure 2: Example from the set of released items (Ball & Hill, 2008).

We have used the program BILOG-MG (Zimowski, Muraki, Islevy, & Bock, 2003) for the estimation of teachers' MKT score and testing of IRT models. For the calculation of correlations, we have used PASW Statistic 18 (formerly known as SPSS statistics).

Results

In our analyses of the data, we looked for correlations between the teachers' MKT score and answers in Part 2 of the form. First, we did not find any significant correlation between teachers' MKT and their experience. When taking a closer look at the number of years they had worked as teachers, however, we found out that our data sample consisted of a rather experienced group of teachers with 80 percent of the teachers having more than six years of work experience. Only 1.4 percent from this convenience sample of teachers was in their first year of teaching.

Second, we studied correlations between teachers' MKT and the grades in which they where teaching. Here we found that there was a significant correlation between the level in which the teachers had teaching experience and their MKT score. Teachers with experience from grades 5-7 or 8-10 (or both) had significant higher MKT score than those with experience only from grades 1-4 (p-value < 0.0005), but the correlation factor was low (Pearson correlation 0.462). If we looked at 1-7 teachers as one group and compared to teachers with experience in grades 8-10, the latest group had significant higher MKT (1.005) and with higher correlation factor (Pearson correlation 0.522, p-value < 0.005).

Third, we studied the correlation between teachers' MKT score and the number of days they had participated in professional development in the years they had worked as teachers. This variable only informed about the total number of days with professional development, and did not say anything about when this professional development took place or what kind of professional development this was. First we considered teachers that had participated in professional development as one big group and compared their MKT score with teachers that had never participated in any professional development program. We did find that this group had a significantly higher MKT score (0.349 higher) compared to teachers without any professional development, but the correlation was weak (Pearson correlation 0.194, p-value < 0.05). Second we grouped the respondents into 6 subgroups: Those who had a) 1-5 days of professional development; b) 6-10 days; c) 11-15 days, d) 16-20 days; e) 21-25 days and e) more than 25 days of professional development. For all of these subgroups we found that the correlation between teachers MKT score and professional development was close to what was found for the big group. However, the subgroups containing teachers with more professional development (e.g. group d), e) and f)) had on average a higher MKT score (0.541 for group d), e) and f)).

Concluding Discussion

In previous analyses, we found that the Norwegian adapted item characteristics are strongly correlated to what is reported in the U.S. (Jakobsen, Fauskanger, Mosvold, & Bjuland, 2011). For item difficulty, the correlation was strong (0.804, p-value < 0.0005). We have also found strong correlation between teachers' MKT in the three content areas (ibid.). Building upon these results we have now analyzed the correlation between the teachers' experience and their MKT score.

We studied a convenience sample of relatively experienced teachers. Despite the limitations of this present study, it has given some indications of issues that should be further investigated. A larger and representative sample of teachers should then be studied.

The results from our study indicate that teachers with experience in teaching higher grade levels have stronger MKT. It should be emphasized, however, that the results from our study cannot be used to argue that experience from higher or varied grade levels produces higher MKT score, only that there is a correlation.

In our study we did not gather information about what kind of professional development courses the teachers had taken or when. The weak correlation between professional development and MKT can thus be interpreted as an area that needs to be investigated further. From our data, we cannot say if there was a change in teachers' MKT after taking part in professional development. Future studies should be conducted in order to learn more about what kind of professional development courses produce stronger MKT, and more generally to investigate the connection between professional development and the development of MKT.

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A Comprehensive Model for Examining Pre-Service Teachers' Knowledge of Technology Tools for Mathematical Learning: *The T-MATH Framework*

Christopher J. Johnston, PhD
ChristopherJohnston1976@gmail.com
Patricia Moyer-Packenham, PhD
Utah State University
Patricia.moyer-packenham@usu.edu

Abstract
This paper proposes a comprehensive model for understanding the multiple dimensions of knowledge employed by pre-service elementary teachers' when they choose technology for teaching mathematics (Johnston & Moyer-Packenham, in press). The *T-MATH Framework (Teachers' Mathematics and Technology Holistic Framework)* integrates several frameworks, including TPACK (Mishra & Koehler, 2006), MKT (Ball, Thames, & Phelps, 2008), and technology evaluation criteria (Battey, Kafai, & Franke, 2005). This model, which can be used to examine the manner in which pre-service elementary teachers rank and evaluate technology tools for mathematical learning, suggests that there are multiple dimensions to understanding teachers' knowledge of technology for teaching mathematics. The paper reports recommendations for mathematics teacher educators and researchers.

Introduction

This paper posits an integrated model of teachers' *technology knowledge for teaching mathematics*. The model is based on the integration of the relevant literature on technology for teaching mathematics including: TPACK (Technological Pedagogical Content Knowledge) as it applies to the teaching and learning of mathematics (Mishra & Koehler, 2006; Niess, Suharwoto, Lee, & Sadri, 2006); the TPACK framework proposed by Mishra and Koehler (2007); the Domains of Mathematical Knowledge for Teaching (MKT) framework proposed by Ball, Thames, and Phelps (2008); and, the mathematics and technology evaluation criteria proposed by Battey, Kafai, and Franke (2005). By integrating these frameworks and evaluation criteria, this model can be used to investigate pre-service teachers' knowledge as they develop in their evaluation and use of technological tools for mathematics teaching.

Frameworks, Models and Constructs Used to Build the Teachers' Mathematics and Technology Holistic Framework (T-MATH Framework)

The literature includes several theoretical and graphical frameworks which are used to explain the relationships among technology knowledge, pedagogical knowledge, and mathematical content knowledge within the context of mathematics education. Elements in all of these frameworks are valuable in understanding the multi-dimensional nature of teachers' technology knowledge for teaching mathematics. For that reason, we have chosen to integrate these various models in a comprehensive model that is specific to the use of technology for mathematics teaching, which we call the *Teachers' Mathematics and Technology Holistic (T-MATH) Framework.*

The first framework used to develop the T-MATH Framework proposed in this paper is *technology, content, and pedagogical content knowledge,* or simply TPACK. TPACK for mathematics is defined as "the intersection of the knowledge of mathematics with the knowledge of technology and with the knowledge of teaching and learning" (Niess et al., 2006, p. 3750). Mishra and Koehler (2007) represent this intersection of knowledge as a Venn diagram, where each of three circles contains pedagogical knowledge, technological knowledge, and content knowledge, with each circle intersecting the others. At first glance, one might assume that these three components are distinct entities, but Mishra and Koehler remind researchers that "TP[A]CK is different from knowledge of all three concepts individually" (2007, p. 8). The representation of concepts using the technology, based on pedagogical strategies for teaching the content, and an understanding of how to use the technology to develop those concepts in children demonstrates the complexity of the integrated nature of TPACK.

Early research on TPACK referred to "content" in general; Niess (2008) suggested identifying how TPACK can be expressed within mathematics education. Specifically, she identified various components of TPACK within mathematics education, namely the importance of teachers' knowledge of students, curriculum, and instructional strategies for teaching and learning mathematics with technology. Niess notes that TPACK requires teachers to consider "what the teacher knows and believes about the nature of mathematics, what is important for students to learn, and how technology supports learning mathematics" (p. 2-3). To support this view, Niess (2005) designed a course where pre-service teachers identified technology tools for mathematical learning as well corresponding mathematics and technology standards which could be supported by the technology tools. The results of this study suggest that the pre-service teachers meaningfully engaged in reflection while considering appropriate technology tools for mathematical learning. Although the TPACK model is an integrated model for pedagogical knowledge, technological knowledge, and content knowledge, the T-MATH Framework proposed in this paper goes beyond the TPACK model to make the mathematics in the TPACK model more explicit by aligning types of mathematical knowledge and fidelity with the elements in the TPACK model.

A second framework used to develop the T-MATH Framework proposed in this paper is the Domains of Mathematical Knowledge for Teaching (MKT) graphical framework described by Ball et al. (2008). In describing this framework they note that, based on their empirical results, "content knowledge for teaching is multidimensional" (p. 403). Within the context of their work, three of their six domains are important for the T-MATH Framework that we propose. These three domains are Common Content Knowledge (CCK; "the mathematical knowledge and skill used in settings other than teaching," p. 399), Specialized Content Knowledge (SCK; "the mathematical knowledge and skill unique to teaching," p. 400), and Knowledge of Content and Teaching (KCT; "combines knowing about teaching and knowing about mathematics," p. 401). These three domains of MKT are important for the T-MATH Framework because they provide much greater explication of the type of mathematical knowledge than that which is described in the TPACK framework (where mathematical knowledge is not described at this level of specificity). What the MKT domains bring to the framework is that there are different types of mathematical knowledge in interaction with the different elements of technological, pedagogical, and content knowledge proposed by Niess.

The third model used to develop the T-MATH Framework proposed in this paper is centered around the criteria proposed by Battey et al. (2005). In their study of pre-service elementary teachers, they identified four main criteria used by the teachers for evaluating mathematics software for use with students: software features, mathematics features, learning features, and motivation features. Further studies which used these four criteria among pre-service elementary teachers noted similar results. These studies found that pre-service teachers emphasized Software Features most often over all other criteria. This finding is important because it suggests that pre-service teachers should consider technology use in mathematics teaching situations as a *mathematical instrument*, not simply as a stand-alone tool. It further demonstrates the challenge that this type of integrative thinking poses for pre-service elementary teachers. In the T-MATH Framework, these criteria highlight a teachers' focus, and that focus reveals the complexity of a teachers' thinking on the use of technology in mathematics teaching. For example, focusing on a motivation feature indicates less complexity because the teacher is considering pedagogy only, while focusing on a mathematics feature indicates more complexity because the teacher is simultaneously considering pedagogy, technology and the mathematical content.

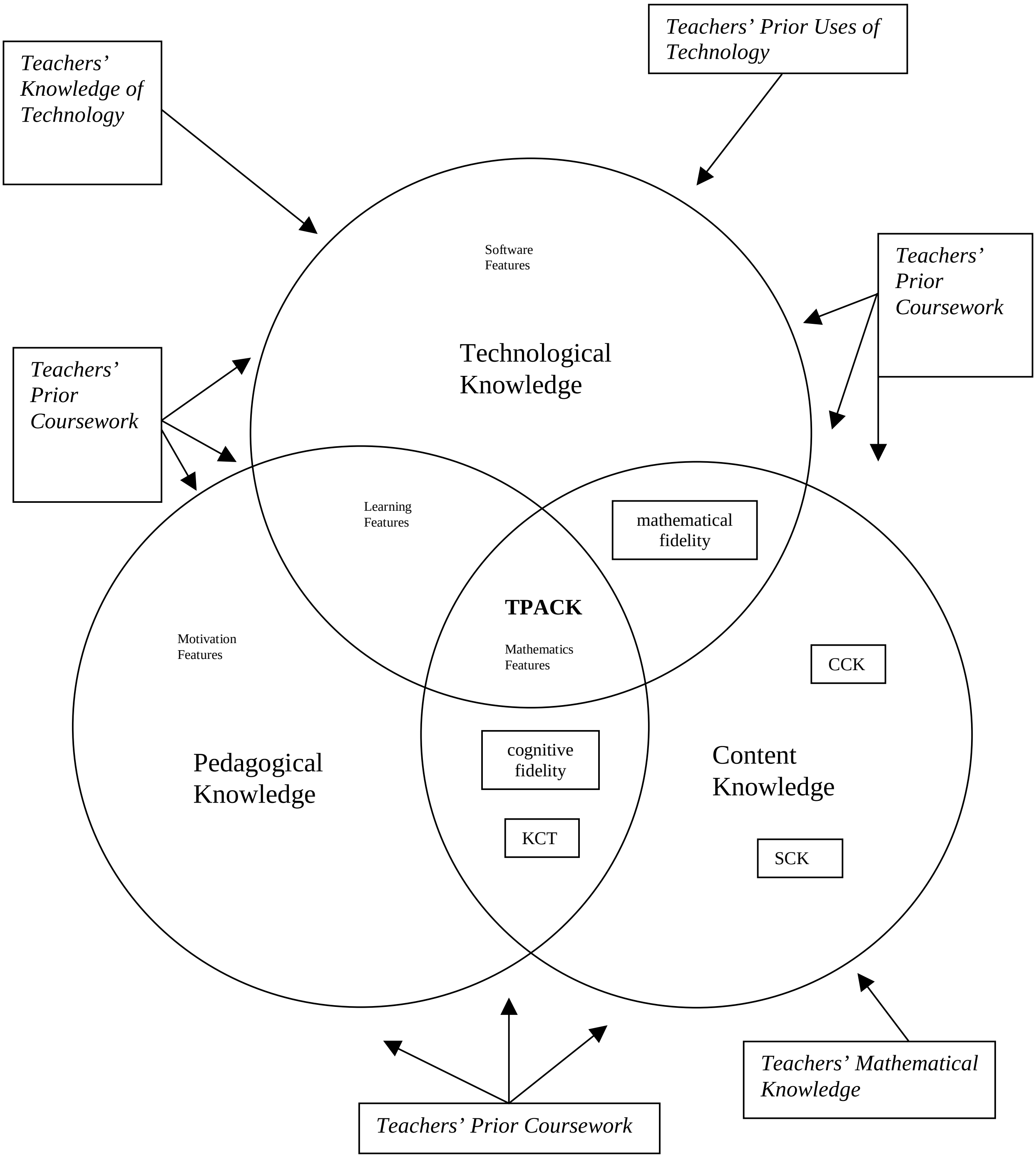
An additional construct used to develop the T-MATH Framework proposed in this paper is *fidelity*. If a tool has high *mathematical fidelity*, “the characteristics of a technology-generated external representation must be faithful to the underlying mathematical properties of that object” (Zbiek, Heid, Blume, & Dick, 2007, p. 1174). Thus, the technology should model procedures and structures of the mathematical system, and be mathematically accurate. A tool is considered to have high *cognitive fidelity* “if the external representations afforded by a cognitive tool are meant to provide a glimpse into the mental representations of the learner, then the cognitive fidelity of the tool reflects the faithfulness of the match between the two” (Zbiek et al., 2007, p. 1176). Thus, a tool which matches the thought processes and procedures of the user has high cognitive fidelity.

The Teachers’ Mathematics & Technology Holistic Framework (T-MATH Framework)

The frameworks and constructs discussed in the previous section demonstrate the *integrated* nature of knowledge and the *complexity* of knowledge specifically as it is needed by teachers who want to use technology to teach mathematics effectively. Because each framework and construct informs different aspects of teachers learning to use technology for teaching mathematics, we propose a comprehensive model that takes into account elements of each framework in an integrated way which we call the *Technology Knowledge for Teaching Mathematics (T-MATH) Framework*.

This model in Figure 1 is a specific extension of TPACK, forming a model of teachers’ knowledge of mathematical TPACK. The proposed model begins with the Mishra and Koehler TPACK framework (2007), which includes Technological Knowledge, Pedagogical Knowledge, and Content Knowledge in three circles in a Venn diagram. Next we map onto this framework Ball et al.’s (2008) three domains of knowledge including: Common Content Knowledge (Content Knowledge circle), Specialized Content Knowledge (Content Knowledge circle), and Knowledge of Content and Teaching (intersection of the Pedagogical and Content Knowledge circles). Finally we consider the constructs of mathematical fidelity (intersection of the Technological and Content Knowledge circles) and cognitive fidelity (intersection of the Pedagogical and Content Knowledge circles).

Figure 1. Teachers' Mathematics and Technology Holistic (T-MATH) Framework



Interpreting a Teacher's Location in the T-MATH Framework

In the T-MATH Framework, we propose that when prior researchers (Battey et al., 2005; Johnston, 2008; 2009) reported that pre-service elementary teachers focused on various features of teaching mathematics with technology (e.g., software, mathematics, learning, motivation), their focus on these features reflects important information about the dimensions of their technology knowledge for teaching mathematics. For example, when pre-service teachers focus their selection of technology tools for mathematics teaching on Software Features (which shows their Technological Knowledge) or Motivation Features (which shows their Pedagogical Knowledge), this is a "one dimensional" focus. That is, they are interested solely in an aspect of the technology tool that is not explicitly linked to students' learning or the learning of mathematics. For example, identifying features such as “has clear directions” (Software Feature) or “is fun for students to use” (Motivation Feature) could apply to many different technologies or learning situations and does not consider how the features are related to teaching and learning mathematics concepts. On Figure 1, we have positioned Software Features and Motivation features in the Technological Knowledge and Pedagogical Knowledge circles, respectively. Pre-service teachers who focus on these features are exhibiting a singular focus and no intersection with other knowledge areas in the model, thus reflecting a less integrated knowledge.

Within the T-MATH Framework, we propose that when pre-service teachers focus their selection of technology tools for mathematics teaching on Learning Features (which shows their knowledge of how the technology is related to student learning), this is a “two dimensional” or integrated focus. That is, they are connecting features of the technology to students’ learning with the technology. For example, identifying a feature such as “applicable to what we are learning in the classroom” connects the technology with the pedagogy of the classroom, considering the implications of both the technology and the pedagogy. On Figure 1, we have positioned Learning Features in the intersection of the two circles for Technological Knowledge and Pedagogical Knowledge. We propose that identifying a Learning Feature requires teachers to make a connection between technology and pedagogy, and thus reflects a more integrated type of knowledge with respect to learning and the use of technology. For example, when pre-service teachers consider the use of embedded buttons on an applet which allow the selection of “easy” or “difficult” mathematics problem items, this allows for learning differentiation afforded by the technology.

Finally, in the T-MATH Framework, we propose that when pre-service teachers focus their selection of technology tools for mathematics teaching on Mathematics Features, this is highly complex, representing a “multi-dimensional” focus and requiring highly specialized and integrated knowledge. On Figure 1, we have positioned Mathematics Features at the intersection of the three circles. We propose that identifying a Mathematics Feature requires teachers to make multiple connections among technology, pedagogy, mathematics (including CCK, KCT, and SCK), and mathematical and cognitive fidelity. Let us further examine the complexity of this placement. The circle of Content Knowledge itself, specific to our model, includes Common Content Knowledge and Specialized Content Knowledge (i.e., mathematical knowledge and skill; Ball et al., 2008). At the intersection of Technological and Content Knowledge is mathematical fidelity. The intersection of the Content Knowledge circle and the Pedagogical Knowledge circle integrates knowledge, and is reflective of Knowledge of Content and Teaching (i.e., combines knowing about teaching and knowing about mathematics; Ball et al., 2008) and cognitive fidelity. The intersection of the Content, Pedagogy, and Technology circles are at the highest levels of complexity because they integrate each of the types of knowledge discussed here, thereby forming the total package.

The positioning of Mathematics Features in the three-circle intersection indicates the complexity of this focus for teachers. We propose that when pre-service teachers focus on Mathematics Features, they must have a deep understanding of technology, pedagogy, *and* mathematics. Specifically, Mathematics Features can focus on three primary areas, namely: *Provides multiple representations of mathematical concepts; Links conceptual understanding with procedural knowledge*; and *Connects multiple mathematical concepts.* In addition, their mathematical knowledge can be the type of knowledge that is used in settings other than teaching (CCK), the mathematical knowledge unique to teaching (SCK), or the knowledge that combines teaching and knowing about mathematics (KCT) (Ball et al., 2008). The identification of a teacher located in the proposed framework and focused on the mathematics features would indicate much more complexity in that teacher’s technology knowledge for teaching mathematics.

Conclusion

In this paper we have proposed the *Technology Knowledge for Teaching Mathematics (T-MATH) Framework.* This framework is a specific extension of TPACK that integrates MKT, fidelity (i.e., cognitive and mathematical), and criteria for evaluating technology (i.e., motivation, learning, software, and mathematics features). This framework demonstrates the complex, multi-dimensional nature of a teacher using technology to teach mathematics. It can be used to examine teachers' mathematics lessons or to observe mathematics instruction that integrates technology to determine the complexity of a teacher's knowledge in planning and teaching mathematics with technology. It can also be a framework for teaching pre-service teachers to design technology experiences for their K-12 students that integrate technology in mathematics teaching. Our hope is that this framework illuminates that complexity, and by making these complex elements explicit, helps to focus on the critical elements necessary for consideration and integration when teaching mathematics with technology.

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Using Large-Scale Datasets to Teach Abstract Statistical Concepts: Sampling Distribution

Gibbs Y. Kanyongo, PhD

Associate Professor of Educational Statistics and Research

Duquesne University, Pittsburgh, PA 15282, USA

Abstract

With the advancement in computer technology, more and more statistics instructors now rely on simulations, web-based statistical applets and other artificially-generated datasets to teach abstract statistical concepts. While this approach may be useful, this paper shows how instructors can use large-scale datasets to make these concepts more real for students thereby facilitating their understanding of the concepts. The paper uses the Education Longitudinal Study of 2002 (ELS: 2002), a large-scale real dataset to demonstrate the concept of sampling distribution and standard errors. The paper focuses on the distribution of sample means and sample standard deviations.

Introduction

Statistics is used in almost our everyday lives and it has applications in a wide variety of settings, for example in weather reports, in business, in crime reports, in finance, and in education. This wide application of statistics makes it essential for students' to have a good understanding of statistical concepts and become statistically literate. However, students often do not grasp certain statistical concepts due to the abstract nature of the subject matter. According to the Guidelines for Assessment and Instruction in Statistics Education (GAISE), an introductory statistics course should: (1) promote statistical literacy and statistical thinking; (2) use real data; (3) promote conceptual understanding; (4) foster active learning; (5) use technology for developing conceptual understanding and analyzing data; (6) use assessments to improve and evaluate student learning. This paper will focus on the second guideline, the use of real data in teaching statistics.

One of the ways to enhance the understanding of statistical concepts, one needs to run real experiments that generate reliable data (Akram, Siddiqui, & Yasmeen, 2004). In most cases, the main limitation for running real experiments is the lack of reliable real data. Even if in cases where real data may be available, the challenge then becomes how to effectively integrate those data in a statistics curriculum. It is difficult to run real experiments during the teaching period in the university. Because of the challenges in using real datasets in teaching statistics, one option that is commonly used by statistics instructors is the use of artificial data in conducting statistical experiments. Statisticians developed simple and very economical experiments, which can be performed in the class by the students (Akram, Siddiqui, & Yasmeen, 2004). A number of studies have been conducted over the years that demonstrate improved students' learning through the use of statistical experiments and simulations. For example, Larwin & Larwin (2010) conducted an experimental study in which they examined the effect of teaching undergraduate business statistics students using random distribution and bootstrapping simulations. Their results indicate that students in the experimental group-where random distribution and bootstrapping simulations were used to reinforce learning demonstrated significantly greater gains in learning as indicated by both gain scores on the Assessment of Statistical Inference and Reasoning Ability and final course grade point averages, relative to students in the control group.

In another study, Raffle & Brooks (2005) reported results of the effectiveness of a Monte Carlo simulation method in teaching a graduate-level statistics course. In their paper, they reported how computer software (MC4G, Brooks 2004) can be used to teach the concepts of violations of assumptions, inflated Type I error rates, and robustness in an introductory graduate course statistics course.

While simulations and simulated data are valuable tools in teaching statistical concepts, the Guidelines for Assessment and Instruction in Statistics Education (GAISE) calls for the use of real datasets to teach these concepts. In line with this guideline, there are several studies that report results for using real-life situations or examples in teaching statistical concepts. In one such study, Leech (2008) used the game of poker and small group activities to teach basic statistical concepts. He noted that this approach helped to reinforce the understanding of basic statistical concepts and helped students who had high anxiety and it made learning these concepts more interesting and fun. In another study, Connor (2003) illustrated statistical concepts using students' bodies and the physical space in the classroom. The concepts covered included: central tendency, variability, correlation, and regression are among those illustrated. In this study, these exercises encouraged both active learning and the visual-spatial representation of data and quantitative relations. Connor (2003) reported that students evaluated the exercises as both interesting and useful. Makar & Rubin (2009) presented a framework for students to think about informal statistical inferential reasoning covering three key principles of informal inference--generalizations "beyond the data," probabilistic language, and data as evidence. In their study, they used primary school classroom episodes and excerpts of interviews with the teachers to illustrate the framework and reiterate the importance of embedding statistical learning within the context of statistical inquiry.

In their paper, Schumm, Webb, Castelo, Akagi, Jensen, Ditto, Spencer Carver, & Brown (2002) use of historical events as *examples* for *teaching* college level *statistics* courses. They focused on *examples* of the space shuttle Challenger, Pearl Harbor (Hawaii), and the RMS Titanic. Finds *real life examples* can bridge a link to short term experiential learning and provide a means for long term understanding of *statistics*. Stork (2003) describes a pedagogy project that keeps students interest. He used student survey to gather data on students' university experience and demographics. The class used the Statistical Package for the Social Sciences (SPSS) data set for demonstration of a range of statistical techniques. The student survey instrument and data set provided examples and problems for each of the major topics of the course. He concluded that student comments and performance demonstrated the project's positive results.

In most cases when talking about sampling distribution, most students think this only applies to the distribution of sample means. The main reason for this is that most textbooks explain this concept using the distribution of sample means. Yet, in actual fact sampling distribution applies to any sample statistic that we calculate to estimate population parameters. It is important for students to understand that when talking about sampling distribution, this applies to standard deviation, variance, correlation coefficient, and many other statistics that we can calculate. What is also important for students to understand is that the sampling distribution is a model of a distribution of scores, just like the population distribution, except that the scores are not raw scores but statistics. The resulting distribution of statistics is called the sampling distribution of that statistic. For example, if standard deviation is the statistic of interest, the distribution is known as the sampling distribution of standard deviation. In this paper, sampling distribution is illustrated for sample means and standard deviations. While this may not be new, what's unique is that a large-scale real dataset is used do illustrate these concepts.

Method

Data used in this paper came from the Education Longitudinal Study of 2002 (ELS: 2002). This survey was designed to monitor the transition of a national sample of young people as they progress from tenth grade through high school and on to postsecondary education and/or the world of work (National Center for Education Statistics 2008). During the 2002 base year, students were measured on achievement, and information was also obtained about their attitudes and experiences. These same students were surveyed and tested again, two years later in 2004 to measure their achievement gains in mathematics, as well as changes in their status, such as transfer to another high school, early completion of high school, or leaving high school before graduation, and also in 2006. This cohort will be interviewed again in 2012 to measure their transition into the job market (National Center for Education Statistics 2008). For the purpose of this paper, only data collected during the base year (2002) was be used.

Procedure

One of the several variables contained in ELS: 2002 is a measure of mathematics achievement of the cohort of 2002 10^{th} graders using a standardized mathematics score (MathScore). This variable was selected for use in this paper. The entire ELS: 2002 dataset contains 10, 094 cases, and this represents the population (N=10,094) for the purpose of this paper. Initially, the population distribution of MathScore was obtained, and this is displayed using a histogram (μ= 51.40, σ = 10.094) in Figure 1. Next, using an SPSS syntax, 100 random samples each of size 100 were drawn from the population and an SPSS dataset with n=100 was created with mean and standard deviation for each sample produced, resulting in 100 means and 100 standard deviations. This was repeated for 500 samples, and 1000 samples, and in each case, the size of each sample drawn was 100 generating means and standard deviations.

Results

The population distribution is shown in Figure 1 while the sampling distribution of the means and standard deviations for the samples on n=100, n=500, and n=1000 are presented in Figures 2 to 4. Population parameters, expected values and standard errors are presented in Tables 1 below. The mean of the sample means in these distributions is the expected value of the sampling distribution of the mean, and the mean of the sampling distribution of the standard deviation is the expected value of the sampling distribution of the standard deviation. Both of these parameters are estimators of the corresponding parameters. When the expected value of a statistic equals a population parameter, the statistic is called an unbiased estimator of that parameter. Students should note that the standard deviation of the sampling distribution of the mean is called the *standard error of the mean*. Similarly, the standard deviation of the sampling distribution of the standard deviation is called the *standard error of the standard deviation*. Students will have the opportunity to learn that sample statistics are most likely to be different from the population parameters due to random sampling errors. However, as sample size increases the sample statistics approach the population parameters, thereby reducing the discrepancy between the statistic and the parameter. This discrepancy is known as the sampling error, and students will be able to see that large samples yield small sampling error. In addition, the histograms show that sampling distribution of the mean will approach a normal distribution as sample size increases.

Figure 1. Population Distribution

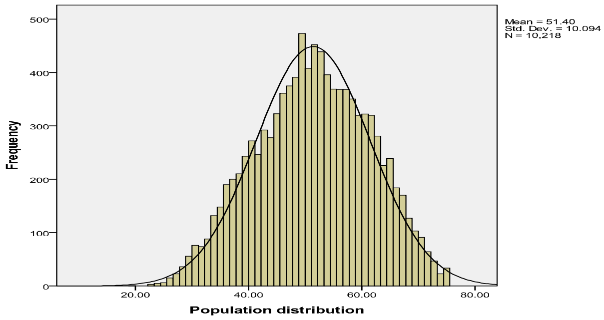


Figure 2a. Distribution of Sample Means (*n*=100)	Figure 2b. Distribution of Sample Standard Deviations (*n*=100)

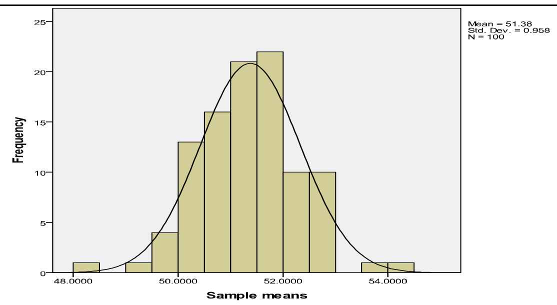
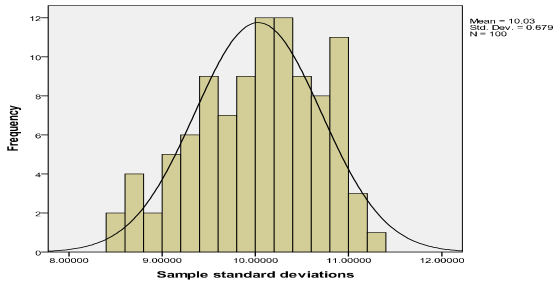

	

Figure 3a. Distribution of Sample Means (*n*=500)	Figure 3b. Distribution of Sample Standard Deviations (*n*=500)

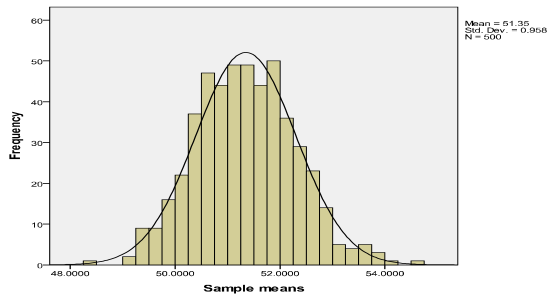
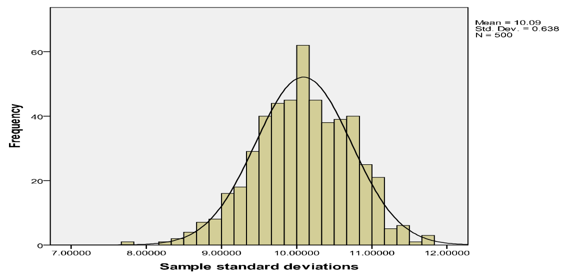
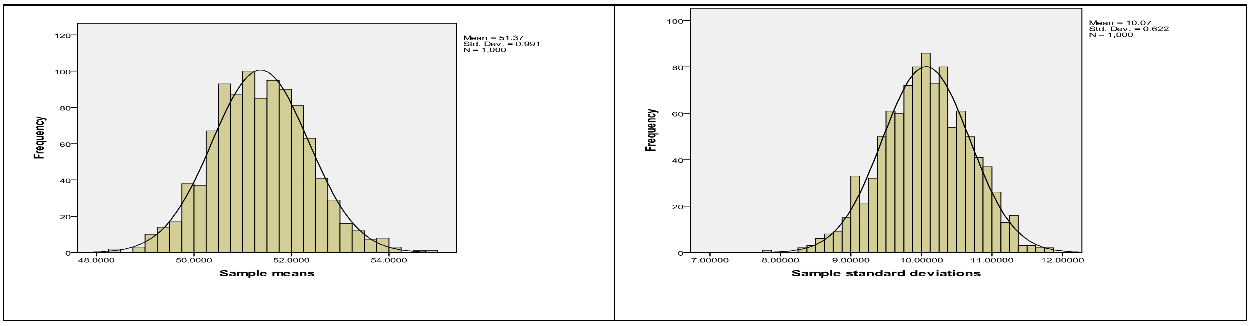

	

Figure 4a. Distribution of Sample Means (*n*=1000)	Figure 4b. Distribution of Sample Standard Deviations (*n*=1000)



Conclusion

In this paper, I have demonstrated the use of a real dataset to teach sampling distribution. While artificial data is usually used to demonstrate these concepts, this paper showed that it's possible to use real data to achieve the same objective with the added advantages that real data bring to students' learning experiences. The dataset ELS: 2002 contains thousands of variables, and students will be able to select variable(s) of their choice, that make sense to them. The fact that ELS: 2002 and other large-scale datasets are publicly available should be an incentive for statistics instructors to use them as teaching tools.

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Transforming Instruction and Assessment Using Student-created Video

Virginia (Ginny) Keen
University of Dayton, Dayton, OH USA, keenvirl@notes.udayton.edu

Abstract of Workshop: The major aim of this workshop is to provide an opportunity for participants to assess the potential for the use of student-created videos in their instruction and assessment activities. This will be accomplished through direct experience with sample student videos. Information about the development of the student-video project will also be shared. **It is recommended that attendees bring a computer with video capability to the session, if at all possible.**

Introduction

The transformation of students, with narrow views of what mathematics is and how to learn it, into thoughtful mathematics learners is a major focus of mathematics for prospective elementary teachers courses. Mathematics teacher educators have used a variety of teaching strategies and provided many traditional and alternative learning and assessment opportunities for students with the goal of enhancing the learning of meaningful mathematics by prospective teachers of young children. Video technology has been used in a supportive role, with videos created by knowledgeable-others shown for this purpose.

Building on the work presented in my Dresden paper (Keen 2009), this session serves as an opportunity for others to observe how I use *student*-created video for both instructional and assessment purposes. The dream of creating knowledgeable teachers becomes a reality as the students begin to think like teachers, developing teaching video vignettes that provide opportunities for peers to learn about the mathematics that is the subject of each video. The videos serve as evidence of the student-creator's learning and understanding in very rich, personal and candid ways.

The video project is designed as a way to invite students into the study of mathematics while experimenting with the role of teacher. So, rather than using videos commercially produced, students experience first-hand the preparation and understanding required to present their thinking about a mathematical concept. The concepts to be "taught" are part of the regular course content and can then serve as tools for later study and revisiting of content by all class members.

Workshop Activities

In this workshop, participants view student-created videos. By viewing examples of videos showing a variety of evidence of content knowledge for teaching of mathematics, attendees will determine first-hand whether this type of activity appears to have value as a vehicle for student learning and assessment. Collegial discussion of the knowledge, skills, and mathematical dispositions in evidence on videos will serve as the basis for a professional analysis of the potential for the tool as a transformational vehicle for the student *and* the mathematics teacher educator.

Using the scoring rubric used for the student videos, participants will both scrutinize the video evidence for student understanding of the mathematics presented as well as critique the rubric designed to assess the students' videos. Of interest is whether this form of student learning and assessment appears to serve as a useful and transformational tool.

Extensions

Participants will be asked to use their on-board video capability to create a short video vignette about a topic that students often find confusing or in some way problematic. They will gain a clearer sense of the potential this assignment has for enhancing both student understanding of mathematics as well as assessment alternatives for the instructor.

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A case study of a teacher professional development programme for rural teachers

Dr Herbert Khuzwayo (UNIZULU); Dr S Bansilal (UKZN), Dr Angela James (UKZN), Dr Lyn Webb (NMMU), Ms Busisiwe Goba (UKZN),

Institution: University of Zululand(UNIZUL),University of KwaZuluNatal (UKZN), Nelson Mandela Metropolitan University (NMMU)

Email: hbkhuzw@pan.uzulu.ac.zabansilals@ukzn.ac.za; jamesa1@ukzn.ac.za; lyn.webb@nmmu.ac.za**;** gobab@ukzn.ac.za;

Introduction

Since 1994, the South African school education system has experienced many curriculum waves. Several universities have configured their curriculum to suit the needs of the country as it moves through these curriculum waves. Many universities developed Advanced Certificate in Education (ACE) programmes in various specializations to help teachers prepare for the demands of the new curriculum. Teachers join the ACE programmes expecting that their acceptance into the programme will start the process of them turning the dreams into reality. They have dreams of improving their qualifications, or re-skilling themselves so that they are more relevant to the education system. A further incentive for the teachers is that the programmes have been funded by the Education department, thus removing from them the financial burden of covering the costs of the programme.

However for many teachers these dreams are shattered early in their journey and they drop out without completing the programme. Most full time undergraduate students in South Africa who drop out of their studies, do so because of an inability to pay for their fees and living costsThe high drop-out rate of students enrolled in various programmes offered by higher education (HE) institutions in South Africa has been a concern for many years. For instance, South Africa's graduation rate of 15% is one of the lowest in the world, according the National Plan for Higher Education compiled by the Department of Education (DOE) in 2001. In 2005 the DOE reported that of the 120 000 students who enrolled in HE in 2000, 36 000 (30%) dropped out in their first year of study. A further 24 000 (20%) dropped out during their second and third years. Of the remaining 60 000, 22% graduated within the specified three years duration for a generic Bachelor's degree (Letseka & Maile, 2005).

In this case of funded ACE programmes, the teachers' studies are funded, so they do not have the financial burden of paying for their fees. However the dropout rate for the ACE programmes has not been investigated since it was introduced in 2000. We felt that it was important for us to investigate why teachers drop out of such funded programmes. We engaged in a preliminary drop out analysis of the ACE programmes, and it emerged that the ACE (Mathematics, Science and Technology) programmes offered by the University of Zululand (UNIZULU), a historically black university[1] had a very low drop put rate. This ACE (MST) programme, is offered to teachers from rural areas north of the province of KZN, who teach in the General Education and Training (GET) phase.

The purpose of this small scale study was to explore the effectiveness of the programme, with particular reference to the low drop out rate. The research question guiding this study is : 1. What are some factors that contribute to the low drop out rate?

History of ACE programmes in South Africa

Prior to the introduction of ACE, a similar programme known as Further Diploma in Education (FDE) existed. The FDE was the state's first major intervention to re-skill or upgrade teachers from their initial qualification in teaching. Teachers with a three-year teaching diploma were eligible to enroll. The FDE was introduced in the late 1980's and early 1990s by the old distance education college of education such as the South African College of Education as a way of upgrading teachers to a qualified teacher status. With the introduction of the NSE (Norms and Standards for Educators; DoE, 2000) framework, the FDE was renamed the ACE. The ACE's purposes were similar to those of the FDE and it soon became the multi-purpose qualification for teacher upgrading, re-skilling and access to higher level programmes. During the review process instituted by the Council for Higher Education (CHE) in 2006, it became clear that there were 69 different kinds of ACE's in the country and that over 290 specializations were being offered (CHE, 2010).

Profile of the ACE (MST) at the University of Zululand

The three ACE programmes offered at the University of Zululand (UNIZULU) are a dual combination: Maths/Science; Science/Technology & Maths/Technology. These programmes were designed to deepen the knowledge of Senior Phase educators in the three areas; Mathematics, Science and Technology. The contact sessions were delivered on weekends or during block sessions during school holidays to cater for the needs of the teachers who came from rural areas situated far away from the campus.

The study

The design of this study was a naturalistic, qualitative, interpretive case study. A naturalistic inquiry was used as it has an emphasis on interpretive dimensions where the goal of the researcher is to understand reality (Cohen, Manion & Morrison, 2007). In this case we wanted to understand the experiences of the ACE teachers to find out why they persevered in their studies. A qualitative case study approach provided the opportunity to concentrate on a specific instance or situation (Cohen *et al*., 2007), namely the particular ACE programme. Data for this study was generated from the student's examinations results (from the university archives), an interview with the programme coordinator, questionnaires to 8 past students and a focus group interview with the same students. These 8 students were selected because they have continued with their studies after completing the ACE programme.

Results

In this section we first report on the throughput rate for these ACE students, before discussing some factors accounting for the high perseverance rate. Thereafter we share some suggestions made by the teachers on how the programme could be improved.

Throughput rate.

The original group consisted of 234 students which was reduced to 217 after the students who were registered for another qualification were excluded. Of these 181 (83%) students graduated within the minimum time of 2 years while 50 (23%) are still in the system. There were six students who dropped our three of whom passed away. This is a very high throughput rate as compared to another ACE programme which had a 50% graduation rate in minimum time rising to 60% over a 3 year period (Bansilal, 2010). When compared to the reported 22% throughput rate for the undergraduate students reported by Moketsi and Maile (2005), the 83% graduation rate in minimum time is impressive.

Factors which contributed to the high throughput rate

Residential Nature of Contact Sessions

The coordinators of the programme decided to run the lectures in block sessions during holidays or weekends. Students (who are practicing teachers) were transported from their local areas to a hotel close to the university. They were accommodated during the sessions and were also transported to the university for the lectures. The reasoning behind this arrangement was that since the teachers on the programme were from rural areas, public transport was notoriously unreliable and the teachers would have spent much of their time just travelling to the lectures. Although regular, day–long or shorter sessions would have been preferable to avoid mental fatigue, the residential block sessions were selected as the mode of delivery to alleviate the transport and isolation problems of the students.

This decision seems to have paid off in many positive ways, some of which were identified by the coordinators as well as the teachers. We outline two of the benefits below.

Contact sessions were intensive learning opportunities for the teachers:

Because of the residential nature of the sessions, lectures and tutorials were spread out throughout the day from 8 in the morning to 7 in the night. All eight respondents indicated that the intensity of the contact sessions helped them to succeed in their studies. The planning of the sessions was such that the teachers made optimal use of their available time. In fact the teachers indicated that as adults, they had family duties and these often interfered with their studies. By taking them away from their everyday situations, studying was made easier because they did not have to worry about their family responsibilities while they attended the sessions. The teachers spoke about the value of working in groups. After dinner, they would meet in groups and go over the work that they learnt. Ambiguities and misunderstanding of content was often cleared up by colleagues during these informal evening discussions.

The student representatives were able to meet often to discuss issues of concern across the different classes as well as to disseminate information to people in the classes.

The teachers were comfortable during their contact sessions

This theme may seem to be out of place in a study of an academic programme. However the coordinator and the teachers both stressed that the comfortable accommodation served as inspiration for the teachers to work harder. Because they did not have to stress about arranging or paying for their own accommodation, all their efforts could be concentrated on their studies. The coordinator commented that the teachers were very appreciative that the university provided the accommodation for them.

Student support

The analysis of the data also revealed many different support mechanisms which improved the students' experiences. We discuss five of these:

Academic Support

The coordinator stressed that at least 50% of the people who taught on the programme, were full time university staff because of the stipulations of the Council for Higher Education (CHE). This meant that the teachers were taught by experienced and well qualified staff, who were available for consultations during the block sessions. The particpants were positive about the academic support offered by the staff and named particular lecturers who made a big impact on them.

The teachers also felt good that the university considered them as important by ensuring that the teaching was done mainly by permanent academic staff.

Attitudes of staff

The drop out study by Letseka and Maile (2005) revealed that some of the reasons for students dropping out was linked the institutional culture of the HEI's which did made them feel ill at ease. They spoke about students' problems being dismissed as 'someone else's problem". In this case the teachers spoke about the warmth and respect they experienced during their studies which made them feel at home and comfortable. One teacher was impressed at the personal attention they got when the "coordinator sat with all [the teachers] who were going to be involved [in the programme].

One student said that in his first year, they found that certain staff had an "attitude" towards them and they found it very hard to work under those circumstances. However after approaching the lecturer as a group, he was willing to hear their concerns and thereafter they had no problems. This solution helped them to become more confident as well.

The teachers considered this approach to the problem as a learning curve. One student stated that " so if ever there are any problems, we can come to the lecturer and then we can try and discuss the problem". These comments suggest that the students felt welcome and respected at the institution.

Tutorial Support

One innovation of this programme was the use of full- time young university students as student tutors in the afternoons. As mentioned earlier, the programme ran until 7p.m. This was because the formal lectures lasted until 4pm, and thereafter the student tutors were available to work with the teachers in an informal basis, by helping them to work through particular concepts, practice exercises and tasks. The teachers were very pleased with this support. One teacher commented that the students "explain everything". She felt that the students explanation were so good that they should "come to our area to teach". This additional facet of support was seen as a reason for the high pass rate in the modules because teachers were given the opportunity of working through the materials while having students available for their assistance.

Financial support

The teachers were appreciative of the government's commitment to improving their skills by finding their studies. The teachers are mature students who have their own families and funding for their studies is not prioritized in the hierarchy of their family needs, so this was a welcome opportunity of many. In the words of one teacher: " When entering this course, I am fully capacitated, we are bread winners, we are having children who need to go to university and it is not easy for me to pay for my child at university and also to pay myself and upgrade myself" This comment captures the students' gratitude about being offered a funded opportunity to upgrade herself. The other benefits of free transport, accommodation and food made the teachers even more determined to make the most of the opportunity.

Classroom support visits

The programme had a classroom support component, where lecturers visited the teachers at their schools and watched them teach and thereafter held individual discussions about their practices. All the teachers found this aspect very helpful and useful. Although one teacher said that she was nervous initially when she knew the lecturer was going to watch her teach, but the lecturer was friendly and helpful and "there just to help you improve". They found the discussion after the lessons useful. Some mentioned that the discussions helped them to concretise the outcomes that they were expected to plan for, and to explore alternate strategies.

Relevance of curriculum and materials

The teachers found the curriculum relevant and all expressed the opinion that the programme had changed the way they teach. They learnt alternative teaching strategies. Two teachers mentioned that the section they struggled with was drawings in technology and this was addressed by the programme. Another spoke glowingly about how the lecturer for science "gave us tips on the designing material in how she was doing it". The teachers also mentioned that what they learnt was relevant to their teaching and they found the materials and some activities useful in their classrooms.

Flexibility of programme

The university policy for such funded joint programmes was that money generated from such programmes would be used for the programmes only. Thus the university deducted a minimal percentage for administration fees. This left the planning and administration of the programme to the departments. The coordinator emphasized that this approach reduced the bureacratic load as well. The department was able to plan the delivery of the programme around the needs of their students. They had the freedom to allocate the funds in an optimal manner and chose to organise the transport and accommodation for the students, a decision that led to much of the success. In addition sufficient staff were employed to mange particular aspects of the programme such as transport and accommodation, administration coordinator an dan academic coordinator. This meant that the academics were freed up to concentrate on the academic delivery of the programme.

An additional advantage of the institutional arrangement was that the coordinators worked directly and closely with the national department of education who sponsored the programme. The department representative met often with lecturers and teachers. Complaints from teachers were taken up quickly and resolved together. The department representative was also pleased with the attention given by the programme coordinators to the teachers' needs.

Suggestions for improvement

There were some suggestions for improvement. The administration was noted as an area of concern because a few teachers spoke about late release of assessaiment marks. One teacher mentioned that the mark he received was initially a pass and then the next day it was reported as a fail. The teachers were also frustrated about the pacing of the examinations especially when they had more than one examination scheduled on one day.

Concluding remarks

In this paper we looked at the throughput rate of an inservice qualification programme (designed for rural Senior Phase Mathematics, Science and Technology teachers) administered by the University of Zululand. We then analysed students' responses to a questionnaire and a focus group interview to identify possible factors which contributed to the high throughput rate. This data was supplemented by an interview with the programme coordinator. One of the aims of the study is to contribute to knowledge about successful practices in designing inservice programmes for teachers. The study identified three main factors which accounted for the teachers' satisfaction with the programme —the residential contact sessions, the student support and the freedom given to the programme coordinators to meet the teachers ' needs.

Although accommodation for in-service teachers is an expensive programme investment, the teachers' responses indicate that this facet of the programme helped them to get more work done without being distracted by their family duties. The evening discussions with their peers also contributed positively to their learning experiences. The teachers and coordinator also commented that the financial support via the bursaries, the welcoming attitudes of the institution, the tutorial support as well as the classroom visits were all beneficial to them.

However the success of the whole programme cannot just be judged on the basis of the successful delivery and high throughput rate. A crucial indicator of the success is the degree to which the teachers are able to improve the teaching and learning of mathematics in their classrooms. It is important to investigate whether the programme translates to sustained improvement in classroom practice. This is one of the focuses of the current research project which aims to investigate the impact of various ACE programmes on the teaching and learning in KZN.

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Mathematics through Language

Allen Lambert, MEd
Middle Years Program Teacher, Wellspring Learning Community, Beirut, Lebanon
allenl@wellspring.edu.lb

Abstract
This workshop explored the importance of mathematics in written and oral language development. Mathematics has its own unique words and expressions, but is learned through a student's natural language. Writing in mathematics can help students grasp concepts better, explain and assess their thinking, and bring more enjoyment to the subject. When a student writes about math, he/she gains a deeper understanding of what he/she knows and is able to do. In the workshop we explored the following questions: How does writing in mathematics develop language? When should students write in mathematics? What does good mathematical writing look like? How can mathematical writing be used to assess students? Participants analyzed an article, investigated lesson ideas, reflected on student work, and generated ideas for their own practice.

Introduction: How is language developed through the use of writing in mathematics?
Traditionally mathematics has been taught through repetition and memorization. Classroom practices must be transformed by teaching concepts and strategies through meaningful learning experiences. The use of language and specifically written language in mathematics is a valuable paradigm switch in teaching and learning. Instead of assessing the right answer to a problem, teachers should assess the ideas and the strategies behind finding the solution. To solve a problem in mathematics students need to use their language skills to comprehend the problem, reason the best strategy for solving it and to communicate how they found the solution to others. Integrating writing into this process allows the student to be reflective of their understanding and abilities.

Studies show that effective teaching in mathematics includes a focus on the development of conceptual knowledge and language that requires the teacher to use clear and understandable dialogue with the students. This supports them in learning new ways of expressing their thinking and models the appropriate uses of the mathematics register; the use of the special vocabulary that is particularly to math as well as natural language. Even natural language takes on special meaning within the mathematics register. Everyday words may be transformed or broadened to include new meaning peculiar to the concepts of mathematics. At times, students may struggle with understanding concepts described within the mathematical register. Integration of writing allows for more exposure, repetition and reflection on the use of the appropriate register while students are developing their natural language.

Benefits and challenges of using writing in mathematics

Most educators were taught mathematics through traditional means, yet recognize the need to be more progressive in their own teaching practices. Conceptual based teaching and the integration of disciplines benefits the learner. This requires action research, innovation, sharing of ideas, the battling of misconceptions and collaboration. Transforming teaching practices in the classroom requires taking risks and spending more time in preparation and planning. In addition, when students write in mathematics, more time is needed to read and reflect on their work.

There are many ways that writing and mathematics can be integrated. Students can write short reflections or explanations of their problem solving, which can make over a more traditional approach to teaching math. They can be incorporated into an interdisciplinary unit project. Mathematics can be used as a theme for creative writing. Some of the more practical math writings can be journals or logs, writing about the strategies used during the problem solving process, writing about understanding of concepts, and writing about feelings or attitudes towards math. Each form of writing has its purpose and value. Writing can be used to bring enjoyment to the subject or to reveal the knowledge and misconceptions a student may have.

Understanding the Approach

Workshop attendees read article selections and shared their reflections with the group. Half of the attendees read an article by Gilberto J. Cuevas. Cuevas asserts that the student's ability in language not only determines performance in mathematics but also in the acquiring of conceptual learning and therefore is an integral part of the teaching process. Although there is not a clear understanding of the relationship between language factors and mathematical achievement, research has shown that there is a correlation between mathematical achievement and reading ability. This speaks of the need for language development. Teachers must support students in their development of written and oral language and be mindful of the students in the class that need additional support in this area. Mathematical language has its own functional register. It is important to help students develop the language so that they can adequately describe the problem solving process and explain their thinking. They must use a combination of mathematical terms and expressions along with their natural language to communicate effectively. Their ability to understand and share their thinking helps them in processing their reasoning. Naturally, language has a role in assessment.

The other half read an article by Dr. Marcia Frank. Frank concluded that integration of writing with mathematics is integral for teaching concepts. Although there are limitations to how writing can be used, it is one of the best ways to assess whether students understand the concepts and the process of the mathematics learned in class. It is a means to conference with each individual student when time is limited. It is through their writing that students can demonstrate what they know and what they can do. By asking students to respond to simple writing prompts, much of their understanding can be revealed.

Following the analysis of the articles, we focused on how different forms of writing can contribute to helping students process their own reasoning, demonstrate their learning and develop their language skills. The use of journals can help students reflect on their learning and monitor their growth. In class writing prompts can assist in working through a process, communicating to the teacher individual needs and demonstrating understanding of a newly learned skill. Portfolios are also great for monitoring growth and to give the students an opportunity to highlight work that they are proud of. Graphic organizers can be useful in developing vocabulary and working through a problem solving process. In everyday class work, homework, test and quizzes, writing can be incorporated to develop language and make the task more enjoyable for the students.

Attendees also looked at some sample student work from an international school in Beirut, Lebanon. The artifacts represented various types of mathematical writing. Participants were given a rubric to assess the work. This activity was followed by a discussion of how to assess mathematical writing and the useful data that can be gained from it. Finally, attendees were asked to reflect on their teaching practices or the practices of their teaching community and set goals for the future.

Conclusion

Students are expected to use language appropriately when communicating mathematical ideas, reasoning and findings – both orally and in writing. Through mathematics students have access to a unique and powerful universal language while developing their primary academic language. Students should be able to communicate a coherent mathematical line of reasoning using different forms of representation when investigating complex problems. Writing in mathematics is integral to working through a process, communicating to the teacher individual needs and demonstrating understanding of newly learned skill.

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An action research study of the growth and development of teacher proficiency in mathematics in the intermediate phase – an enactivist perspective. Work-in-progress

Mandy Lee MEd (Higher Education)
Embury Institute for Teacher Education, Durban, KZN, South Africa, mandylee@telkomsa.net
Prof. M. Schäfer,
Rhodes University Grahamstown, South Africa, m.schafer@ru.ac.za

Abstract
This paper reports on a current research project that focuses on an action research study of a pre-service mathematics education module to grow and develop proficient teaching in mathematics in the intermediate/middle school phase. The aim of this research study is to ascertain if a fourth year mathematics education module whose teaching pedagogy is informed by the underlying themes of enactivism will develop and grow pre-service teachers' proficiency in teaching mathematics in the intermediate phase. The focus of this presentation will be to report on the outcome and my analysis of the first research cycle and to highlight the way forward in the second cycle of this action research project.

Introduction
The intention of this research project is to determine if a mathematics education module informed by an enactivist orientation and teaching pedagogy will enable pre-service teachers participating in the module to unpack the reality of their teaching practice in terms of proficient teaching through active participation in lectures and practical tutorial sessions. My theoretical framework is informed firstly by enactivism and secondly by the use of the underlying themes of enactivism, as introduced by Di Paolo, Rohde and De Jaegher (2007), as a vehicle to develop teaching proficiency.
The investigation is an action research study that encompasses a fourth year mathematics module for BEd. Degree pre-service teachers training to teach in the foundation phase. The module has been designed to equip the foundation phase pre-service teachers with the necessary skills and content to teach mathematics in the intermediate phase. The aim of this mathematics module is to develop and grow pre-service teachers who will one day be role models for their learners and instil in their learners a love and interest in mathematics. Thus the module has been designed to augment the pre-service teachers' basic skills in mathematics and provide them with opportunities to become competent in basic mathematics and pedagogic skills for the Intermediate Phase. The intention of this module is to develop both their confidence and proficiency in mathematics and their proficiency in teaching mathematics in the Intermediate Phase.

Enactivism
Enactivism is a philosophy that views the body, mind and world as inseparable and was originally developed by Merleau – Ponty. It was then further developed by Maturana and Varela; Thompson and Rosch (Davis, 1996), as an alternative to the bipolar and divisive nature traditionally associated with western thinking. As a philosophy enactivism recognises the structure of an individual to be a combination of both their biological composition and their personal history of interacting with the world. Enactivism also "focuses on the importance of embodiment and action to cognition" (Li, Clark & Winchester, 2010:403). Consequently the structure of the individual is considered to be adaptable as it changes to make sense of new experiences and challenges.
Since enactivism considers cognition to be a complex co-evolving process of systems interacting and affecting each other, cognition is deemed to be a producer of meaning and not a processor of information. Therefore an individual's interactions with the world and their past experiences will shape and influence the meaning that they make of their world (Lozano, 2005). Thus any perturbation will present an opportunity for an individual to act according to his/her structure and it is the structural make up of the individual that determines what and the extent of the change that occurs. Enactivism encourages learning and the construction of knowledge by means of a collaborative process, thus any given learning situation must encompass the lecturer, the student, the content and the context in order for some form of interaction to take place.
From an enactivist perspective the role of the lecturer is that of a "perturbator" in order to encourage and provoke learners into "thinking differently" about mathematical concepts that may not form part of their personal

construct. With this in mind, I have chosen to use five underlying themes of enactivism (Di Paolo, Rohde and De Jaegher, 2007; McGann, 2008) namely autonomy, sense-making, emergence, embodiment and experience to underpin my pedagogic practice. A number of tasks and practice will be developed or drawn from the literature where they have been applied in other mathematics contexts and adapted with the intention of developing teaching for mathematical proficiency skills.

Underlying themes

I have used the five underlying themes to assist pre-service teachers to determine what embodied views of cognition reveal about their personal proficiency in mathematics, their mathematical identity and their self development. Furthermore the themes will help to ascertain how the pre-service teachers' embodied perceptions of their mathematical proficiency support their own teaching for mathematical proficiency during the practical tutorial sessions.

Di Paolo, Rohde and De Jaegher (2007) indicate that living organisms are autonomous due to their ability to create their own identity that characterises them as a unique entity. In this research study the pre-service teachers' mathematical identity is representative of autonomy since enactivism argues that one's identity is enacted and therefore determined by the interplay between biology and human culture and the individual's manner of dealing with life's experiences.

With regard to the second theme, namely sense-making, according to Di Paolo, Rohde and De Jaegher (2007) this refers to the way an individual actively participates in generating meaning of their world with changes occurring as a result of dialogue between the individual and the environment. Davis (1996) introduces the notion of listening as an embodied action that enables one to understand human communication and collective action. Thus sense making will manifest in my pedagogy through encouraging and creating opportunities for the pre-service teachers to play an active and participatory role in emerging conversations and to engage in different types of listening.

The third theme, emergence, is explained by Davis (2004) as the notion that understanding and interpretations are generated through shared activities over a period of time, as opposed to predetermined learning objectives. In addition, Davis (2004) raises the point that learning is not the site of the individual, be it learner centred or teacher centred, but rather the collective generation of knowledge and understanding. Therefore this research study will use various reflective techniques to encourage the students to reflect on what they have learnt during a particular lecture or tutorial and to acknowledge the role that the community of practice, autonomy and sense making have played in their understanding of and culmination of the learning process.

According to Li, Clark and Winchester (2010) the embodied experience that an individual undergoes is due to the bringing together of the mind and body by means of reflection. Di Paolo, Rohde and De Jaegher (2007) augment this notion indicating that cognition is an "embodied action" (p.11) in that the mind and the body form part on a living system composed of various "autonomous layers of self-coordination and self-organization" (p.12) which allow it to interact with the world in "sense-making" activities, with the understanding that the body is not controlled by the brain. Davis (2004) suggests that embodiment refers to the idea that individuals are all part of a larger collective system and cites Kauffman's description of the relationship between the collective and the individual as one in which "the collective is enfolded in and unfolds from the individual" (p. 213). With this in mind an aspect of the pedagogical practice is to encourage the formation of a mathematical community of practice amongst the pre-service teachers to research its influence on the lesson preparation and sense making processes.

Finally the pre-service teachers will be given the opportunity to experience developing lessons for the intermediate phase and teaching for proficiency. Furthermore, it is anticipated that these experiences will change through embodied action and the emergent process. In the initial stages of the module,

activities are incorporated to encourage pre-service teachers to discuss their experiences and critical incidents of mathematics teaching. Further the students will need to determine the influence that these experiences have had in determining what they understand by proficiency and their role as a teacher in trying to teach for mathematical proficiency.
In order to analyse the growth in proficiency, Kilpatrick, Swafford & Findell's (2001) framework for mathematical proficiency is the core analytical framework for the research project, since the authors are cognisant of the importance of noticing and analysing learners' interpretations and of the role that different contexts, especially environmental and situational elements, play in the teaching and learning process. Kilpatrick et al (2001) identify mathematical proficiency as encompassing five interwoven and interdependent strands, namely, conceptual understanding; procedural fluency; strategic competence; adaptive reasoning and a productive disposition.

Action Research Cycle

In this study the pre-service teachers' involvement in the decision process entails providing feedback on the tutorial sessions in the form of a class group interview at the conclusion of each tutorial session and through their reflective journals. Enactivism views cognition as a complex co-evolving process of systems interacting and affecting each other, not as processors of information but as producers of meaning. Lozano (2005) explains this as the manner in which an individual's interactions with the world and their past experiences shape and influence the meaning that they make of their world. The pre-service teachers will reflect on their experiences at each tutorial session and identify what aspects or critical incidents they believe have influenced their proficiency. As the researcher, I will analyse this data using Kilpatrick et al's (2001) five strands of mathematical proficiency to determine the growth and development of proficiency. This information in turn will be used to inform the development of the next lecture so that different activities and tasks can be developed to encourage and grow proficiency.

The module comprises a double lecture weekly in which enactivist pedagogy is role-modelled and mathematical theory and content discussed. In addition to the lectures, weekly practical tutorials are held during which the pre-service teachers are given the opportunity to develop their mathematics teaching skills. Since there are fifty pre-service teachers, the class has been divided into two groups with each group attending fortnightly tutorials.
Each of the tutorial groups is further subdivided into five groups with 5 pre-service teachers in each, three teaching groups, one learner group and one observation group (see figure 1). Therefore at each tutorial session the three teaching groups are given the opportunity to teach a mathematical concept, the learner group takes on the role of the learners and the observation group monitors and analyses the teaching process identifying critical incidents and completing an observation sheet underpinned by the 5 strands of proficiency. Over the duration of the year the groups in each tutorial session will rotate so that each pre-service teacher will have the opportunity to be either in a teaching, learner or observation group. The pre-service teacher will remain in the same group of 5, both for tutorial sessions and lectures, thus forming their own mathematical community to assist with the sense making process.

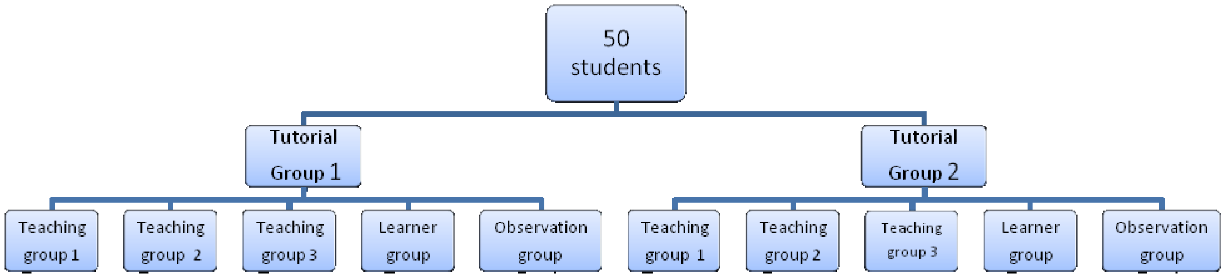

Figure 1

The three teaching groups will be given prior warning of the content that they will be required to teach with the expectation that they will then develop a 15 minute micro lesson that encompasses some of the strands of mathematical proficiency. All the members of the teaching groups contribute to the preparation of the lesson and resources, although only one member from each teaching group is given the opportunity to teach the lesson at the tutorial session. During the course of the year a number of pre-service teachers will be given an opportunity to teach a lesson. At the conclusion of the first micro lesson, the pre-service teachers will be given a few minutes entering into conversation as to the outcome of the lesson with regard to teaching for proficiency. The teaching group and the learner group will discuss what strands of proficiency they believed had been addressed, while the observation group will identify critical incidents that they deem to have impacted on and affected teaching for proficiency. The role of the observation group is to notice and identify which strands of proficiency have been addressed, what enactivist pedagogic strategies and themes have had a positive effect and where and what difficulties have arisen in the lesson delivery and activities. This same process will then be repeated for the second and third teaching group. Following the final lesson, the tutorial session concludes with a class group interview with all the pre-service teachers, relating to the lessons observed and the perceived growth and development of teaching for mathematical proficiency and the contributing factors.

Conclusion

In presenting this paper it is my intention to discuss and reflect on my experiences pertaining to the first cycle of the research. Since enactivism is a theory that has not been researched widely in South Africa, I hope to engage the audience in a conversation as to the merits of my research to date and my objectives for the second cycle.

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This work is based upon research supported by the FirstRand Foundation Mathematics Education Chairs Initiative of the FirstRand Foundation, Rand Merchant Bank and the Department of Science and Technology.
Any opinions, findings, conclusions or recommendations expressed in this material are those of the authors and therefore the FirstRand Foundation, Rand Merchant Bank and the Department of Science and Technology do not accept any liability with regard thereto.

MATHEMATICAL COMPETENCE ASSESMENT OF A LARGE GROUPS OF STUDENTS IN A DISTANCE EDUCATION SYSTEM

Genoveva Leví
Departamento de Didáctica, Educación Escolar y Didácticas Especiales,
Facultad de Educación, UNED (Spain).
genovevalevi@edu.uned.es

Eduardo Ramos
Departamento de Estadística, Investigación Operativa y Cálculo Numérico,
Facultad de Ciencias, UNED (Spain).
eramos@ccia.uned.es

Abstract

We present a model for the assessment of mathematical competence of a large group of students in a distance education system. The model can be applied automatically and allows not only a summative evaluation type, but it has also educational character. We present some results of applying the model to a real situation for the Course *Applied Mathematics to Social Sciences* taught at UNED of Spain as a part of the Foundation Course for access to University for people 25 years old and over.

Mathematical Competence

The use of the term competence has set out as the cornerstone for the design of teaching and learning process in diverse areas and educational levels. Currently, in Spanish universities, all degrees must be designed in accordance with the teaching and learning model aimed to developing learning skills. It is not possible here to make a profound discussion on the various meanings of the term. For purposes of this paper we understand that "*competence is values, attitudes and motivations, as well as knowledge, skills and abilities, all as part of a person who is insert in a given context, considering also that he/she learns continuously and progressively throughout life*" (Sevillano, 2009).

According to the above idea, to define the mathematical competence it is necessary to identify the components that comprise it. That is, mathematical competence is developed when successfully integrated mathematical knowledge and skills, so they emerged in different areas of personal performance directed by certain values and attitudes. Ramos, 2009, includes an extensive discussion on how to achieve the mathematical competence. The determination of the components leads to the following specifications, Ramos *et al*., 2010:

- *Mathematical knowledge* formed around certain key ideas that have historically set the scope of mathematics and the core of mathematical thinking. These ideas include: i) *Mathematical language*, ii) *Quantity*, iii) *Space and shape*, iv) *Change and relationships*, v) *Uncertainty*. Each of these sections includes a set of concepts that formalize the different experiences of men in real situations that present aspects of mathematical nature. It is clear that among those sections there are many common points and any attempt to strictly separate them leads only to artificial situations. However, the above classification is useful not only for its traditional character, but also for its important methodological implications. The description and contents of the five sections can be found in the text by Hernández, Ramos, Velez and Yáñez, 2008.

- *Mathematical capabilities* that are required at different stages of the mathematical process, putting into action the mathematical knowledge in a given context. These capabilities, according to PISA, 2003, 2006, 2009, can be stated as: i) *Thinking and reasoning,* ii) *Argumentation,* iii) *Communication* iv) *Modelling,* v) *Problem posing and solving,* vi) *Representation*. vii) *Using symbolic, formal and technical language and operations,* viii) *Use of aids and tools*. The cognitive activities that include these capabilities are three levels of development or skill clusters: *reproduction* cluster, the *connections* cluster and the *reflection* cluster, each of which represents, respectively, a higher level of development of mathematical competence. It is also possible to consider the following contexts*: personal, educational/occupational, public and scientific*. The meaning and scope of the above expressions is conveniently detailed in PISA 2003, 2006, 2009, (*vid.* Ramos 2009, Ramos *et al.* 2010).
- *Attitudes* that show: i) *Security and confidence with information and situations that contain mathematical elements,* ii) *Respect and enthusiasm for certainty through the correct reasoning*.

Evaluation of mathematical competence

One of the key issues of a teaching and learning model designed to develop skills is how to truly evaluate the achievement of the desired competences. Traditional models aimed at acquiring knowledge include an evaluation system designed to observe mainly whether or not a student has certain knowledge. In a model of competences we must go further and try to assess whether or not students have acquired the expected competences. In the assessment of competences we should keep in mind that these, in addition to knowledge, include skills, abilities, attitudes and values that the student has to develop. We then face the problem of assessing the level a student show in certain qualities, that can only be seen with a degree of subjectivity and are more difficult to detect by the usual assessment methods.

Generally, the competence models use assessment systems such as oral presentations, analysis of data or texts, skills practice while the individual is under observation, professional folder (portfolio), reports field work, written thesis, or similar. However, these activities encounter practical problems that can make them inapplicable in certain situations. In particular, we note that they are usually only practicable in groups of few students, they represent a major increase in workload for both students and faculty and are difficult to apply in some contexts such as distance education model. If we do not want to refuse to use a competence model, the alternative is to investigate the possibility of adapting the traditional assessment schemes so that they can be used as an acceptable method for assessing competence.

Ramos *et al*., 2010, present the theoretical ideas for the development of a competence evaluation system inspired by traditional systems, assessing not only knowledge but also skills and attitudes. The idea is to design assessment activities provided with a number of appropriate characteristics so that it is possible to assess the degree of acquisition of the various components and subcomponents of the competences.

The practical application of this theoretical model of evaluation to the case of the mathematical competence, in a large group of students and in a distance education system, leads to designing evaluation activities with certain attributes that allow the combination of simplicity of the traditional evaluation with the objectives of the competency assessment. We use automatic evaluation tests in order to obtain an objective qualification within a reasonable period of time. Moreover, the design of these tests include a quantification of all aspects that make up the components and subcomponents of the competences, that is, the acquisition of knowledge, skills and attitudes within a given context. Finally, it is necessary to consider an indicator for the accuracy or adequacy of the response to a particular reference. The above considerations lead us to define a test or evaluation activity as an object identified by the attributes that are listed in Table 1. Specific examples of such tests can be seen in Ramos *et. al*., 2010

TEXT	ASSESSMENT			CORRECTION INDICATOR
	Knowledge	Capabilities	Attitudes	
Statement (TST). Correct alternative (TCA) Distractor 1 (TD1) Distractor 2 (TD2)	Mathematical Language (VKL) Quantity (VKQ). Space and Shape (VKS) Change and Relationships (VKC) Uncertainty (VKU)	Thinking and reasoning (VCT) Argumentation (VCA) Communication (VCC) Modelling (VCM) Problem posing and solving (VCP) Representation (VCR) Using symbolic, formal and technical language and operations (VCST) Use of aids and tools (VCU)	Security and Confidence (VAS) Enthusiasm for certainty and correct reasoning (VAE)	Success. (S) Failure (F) No answer (W)

Table 1

Thus, mathematical competence can be assessed by applying an evaluation form consisting of a number *N* of activities with the characteristics in Table 1. To do this, we must have a function of the attributes that, based on indicators of correctness of each of the *N* tests, provide an overall score, either quantitative or qualitative, in each of the components and subcomponents of the competences.

The quality of an evaluation form can be assessed by considering the overall characteristics of the attributes of the activities that it comprises, like the mean, minimum, maximum, range, etc. of the valuation of knowledge and skills. Thus, it is possible to compare different forms and even prepare forms that meet certain requirements desired by the teacher.

Case study

We present in this section the results of applying in practice the model described above. The framework is the subject *Mathematics Applied to Social Sciences* in the exam of June 2010. The examination form consists of ten evaluation questions. We use a numerical scale from 0 to 4 to assess the intensity with which a question measures the level of acquisition of each of the subcomponents of the competences. The attributes of each question in the form are summarized in Table 2. The indicator correction is the same for all question: S = 1, F =- 0.25 and W = 0.

Question	Knowledge					Capabilities								Attitudes	
	VKL	VKQ	VKS	VKC	VKU	VCT	VCA	VCC	VCM	VCP	VCR	VCS	VCU	VAS	VAE
1	1	2	1	4	0	2	1	1	1	1	2	3	1	1	0
2	1	4	0	0	0	1	0	2	1	2	2	3	3	1	0
3	4	2	0	0	0	4	1	2	3	2	2	2	2	1	3
4	0	0	0	0	4	4	2	2	4	2	1	1	0	4	1
5	2	2	4	1	0	2	2	2	3	2	3	2	2	1	1
6	2	1	1	4	0	2	2	2	3	3	2	3	2	1	1
7	1	0	4	0	0	3	3	3	3	2	3	2	0	3	0
8	4	0	0	0	0	3	4	3	1	1	1	2	0	1	4
9	0	3	0	0	4	3	2	3	4	3	4	2	4	4	1
10	0	4	0	0	0	1	1	1	2	2	2	2	4	1	1

Table 2

The student responses are collected by automatic reading sheets and form a vector $R = (R_1, R_2, \ldots, R_{10})$, where, R_i, *i=1,...,10*, takes one of the values *TCA(i), TD1(i), TD2(i)* or it is a blank or null answer. If we denote

$$E_i = \begin{cases} 1 & \text{if } R_i = TCA(i) \\ -0.25 & \text{if } R_i = TD1(i) \text{ or } R_i = TD2(i) \\ 0 & \text{otherwise} \end{cases}$$

then the traditional numerical grade, on a scale of 0 to 10, is calculated by the formula $E = \text{Máx}\left\{0, \sum_{i=1}^{10} E_i\right\}$.

For an independent assessment of each of the components of the competences, the above expression can be extended in several ways. For instance, it is possible to assign a different weight to each response, according to different attributes of the question. However, the current assessment model considered only assesses the different subcomponents of competence, using the data included in Table 2. Thus, the score obtained in knowledge, K_j, *j=1,...5,* capability C_k, *k=1,...,8,* and attitude A_l, *l=1,2* can be expressed as

$$K_j = \text{Max}\left\{0, \frac{10}{k_{.j}} \sum_{i=1}^{10} k_{ij} E_i\right\} \quad C_k = \text{Max}\left\{0, \frac{10}{c_{.k}} \sum_{i=1}^{10} c_{ik} E_i\right\} \quad A_l = \text{Max}\left\{0, \frac{10}{a_{.l}} \sum_{i=1}^{10} a_{il} E_i\right\}$$

where k_{ij} is the valuation of question *i* in knowledge *j*, c_{ik} is the valuation of question *i* in capability *k* and a_{il} is the valuation of question *i* in attitude *l*, given by Table 2; moreover, $k_{.j} = \sum_{i=1}^{10} k_{ij}$, $c_{.k} = \sum_{i=1}^{10} c_{ij}$ and $a_{.l} = \sum_{i=1}^{10} a_{il}$ are, respectively, the total of the values of the knowledge (*j=1,...,5*), capabilities (*k=1,...,8*), and attitudes (*l=1,2*) in the form. Finally, the average values can be calculated:

$$K = \sum_{j=1}^{5} \frac{K_j}{5}; \qquad C = \sum_{k=1}^{8} \frac{C_j}{8}; \qquad A = \sum_{l=1}^{2} \frac{A_l}{2}; \qquad EC = \frac{K + C + A}{3}$$

providing a measure of the level of each component of competence, (*Knowledge K, Capabilities C, Attitudes A*) and overall assessment of the mathematical competence *EC*.

Table 3 shows some records in the database of results. Each row corresponds to a particular student, while the columns include identification (ID), the valuation of each of the five subcomponents of knowledge, the eight subcomponents of capabilities and the two subcomponents of attitudes, the average valuation on knowledge *(K)*, capability (*C*) and attitude *(A)*, as well as traditional grade *(E)*.

ID	VKL	VKQ	VKS	VKC	VKU	VCT	VCA	VCC	VCM	VCP	VCR	VCS	VCU	VAS	VAE	K	C	A	EC	E
1	8,53	10,0	10,0	5,45	10,0	8,71	8,30	8,91	9,11	9,43	8,21	8,44	8,96	9,34	9,22	8,80	8,76	9,28	8,95	8,75
2	7,06	3,75	2,05	3,18	3,75	5,69	5,45	4,57	3,75	4,32	3,75	5,31	4,27	4,08	6,09	3,96	4,64	5,09	4,56	5,00
3	6,67	4,44	4,00	1,11	,00	4,40	4,44	3,81	3,60	3,50	3,64	3,64	4,44	2,22	7,50	3,24	3,93	4,86	4,01	4,00
4	10,0	10,0	10,0	10,0	10,0	10,0	10,0	10,0	10,00	10,0	10,0	10,0	10,0	10,0	10,0	10,0	10,0	10,0	10,0	10,0
5	3,83	6,39	,00	4,44	5,00	3,30	4,44	4,64	3,80	5,00	4,43	5,00	6,94	3,89	5,21	3,93	4,70	4,55	4,39	4,50
6	4,17	3,75	3,75	3,06	3,75	5,00	5,83	5,24	4,50	5,00	3,75	4,89	3,75	5,14	4,79	3,69	4,74	4,97	4,47	5,00

Table 3: Some records in the database of results

Table 3 illustrates the possibilities of the evaluation model. Instead of a single data about each student, the standard grade *E*, we have a complete information on each competence level. For example, in the student with ID #5 has a grade of 4.50, typically assessed as insufficient. However, we can see that has an acceptable development of knowledge *Quantity* (VKQ=6.39) and *Uncertainty* (VKU=5.00). Similarly, he/she has a satisfactory level on *Problem posing and solving* (VCP = 5.00) and *Use of aids and tools* (VCU=6.94). On the other hand, the student with ID #6 has a classical grade satisfactory (E=5.00); however we see a lack in many components of the mathematical competence, since most of her scores are less than five. In both cases, we may recommend conducting some activities which had the objective of strengthening the shortfalls.

Conclusions

The assessment model we present has several advantages. We can highlight that it can be easily applied in educational contexts where it is difficult to make a more personalized assessment of the student, as may occur when the group is too big, or in a distance learning system. It is also possible to obtain not only a summative assessment, but also a formative evaluation designed to assess the individual student's level in each of the competences, setting minimum standards in each one of them to meet the objectives of the curriculum, identify gaps, to be able to recommend learning activities needed to achieve the desired level in each component of competence: knowledge, ability or others that we may consider.

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THE INFLUENCE OF GEOGRAPHICAL, SOCIAL AND CULTURAL FACTORS IN THE MATHEMATICAL COMPETENCE LEVEL

Genoveva Leví
Departamento de Didáctica, Educación Escolar y Didácticas Especiales,
Facultad de Educación, UNED (Spain).
genovevalevi@edu.uned.es

Eduardo Ramos
Departamento de Estadística, Investigación Operativa y Cálculo Numérico,
Facultad de Ciencias, UNED (Spain).
eramos@ccia.uned.es

Abstract

In this paper we present some results on the influence of different geographical, cultural and social conditions in the observed level of mathematical competence in a group of students. The experimental situation refers to the Course *Mathematics Applied to Social Sciences* taught at UNED of Spain as a part of the Foundation Course for access to the University of people 25 years old and over. As a tool, we used an evaluation model designed by the authors for the assessment of competences.

Introduction

The implementation of teaching models aimed at developing competences has been generalized to all levels of education in diverse fields. One of the keys to the effectiveness of these models is to find a solution to the problem of evaluating competence, i.e., it is necessary to have useful tools to adequately assess whether students have acquired the competences designed by the curriculum.

Typical evaluation systems of competence models, such as oral presentations, skills practice while the individual is under observation, or others, are difficult to apply in many real situations because the circumstances of the teaching-learning process make it complicated to follow the student on an individual basis. A particular case is a distance education system with large groups of students distributed around the world. In this context, if we do not want to give up the use of a competences model, it is necessary to resort to adapting traditional evaluation systems by adding certain features to measure the level that the student has in each of the competences set in the curriculum.

In previous work, Ramos *et al*. 2009, 2010, Leví and Ramos 2011, we presented various ideas for developing a competence assessment model capable of being applied in situations such as the noted above. We present an application of this model to a real situation, consisting of evaluating the mathematical competence shown by students in a course that is part of *Direct Access Course for over 25* years taught at the National University of Distance Education (UNED) from Spain. The main results focus on a quantitative study of mathematical competence of students grouped by various characteristics of geographical, social and cultural factors. In general, we obtained satisfactory levels of competence, although there are minimal differences between groups.

Objectives and Hypothesis

Mathematical competence is developed when mathematical knowledge and skills are integrated successfully so that they emerge in different areas of personal performance directed by certain values and attitudes, Leví and Ramos 2011. The components and subcomponents of mathematical competence that are considered are the following:

- *Knowledge*: i) Mathematical language, ii) Quantity, iii) Space and shape, iv) Change and relationships, v) Uncertainty.
- *Capabilities*: i) Thinking and reasoning, ii) Argumentation, iii) Communication iv) Modelling, v) Problem posing and solving, vi) Representation. vii) Using symbolic, formal and technical language and operations, viii) Use of aids and tools.
- *Attitudes*: i) Security and confidence with information and situations that contain mathematical elements, ii) Respect and enthusiasm for certainty through the correct reasoning.

The main objective of this paper is to analyze the level of mathematical competence, expressed in terms of its components and subcomponents, which reaches a group of students from the *Direct Access Course for over 25* years taught at the UNED of Spain. This course serves as preparation for the entrance exam to the university for students who have not finished the usual way of the "*Selectividad"* to complete secondary education. The analyzed matter is *Mathematics Applied to Social Sciences;* this matter is mandatory for students following options for Social Sciences and Health Sciences, while it is optional for students of Arts and Humanities. Students in the field of Science, Architecture and Engineering were not included in the study.

The course is taught according to the methodology of distance education distinctive of UNED. Basically, it uses a set of studying materials developed by faculty and designed for individual study, while monitoring of learning takes place through a series of technological means of all kinds, which include the so-called virtual course through internet. In addition, most students have in person tutorial support in study centers spread throughout the country and even internationally, although in some cases there are several factors that hinder or even preclude the effective use of such support. In any case, students can always address the teaching staff directly to solve any difficulties they may encounter in their study.

Although most students are Spanish and live in the country, there is a significant participation of foreign students spread mainly around various countries in Europe and America. Special mention may be made for a group of students from Equatorial Guinea (Africa). UNED has maintained since its inception a historic commitment to the training of Guinean higher education, with significant contribution of Access Course. Another peculiar group is the group of students boarders in a prison. UNED has a program of collaboration with the *Directorate General of Prisons of Spain* to facilitate access to higher education of students in prison. The Access Course is also a way used by many inmates to start university studies.

In our study we intend to analyzed mathematical competence of the different groups of students mentioned above. The objective of the analysis is to compare the influence of different social, cultural and geographical factors in the acquisition of mathematical competence.

Since Mathematics is a basic science and uses universal methods for the rational man, we want to corroborate the hypotheses that there should be no difference in math competence levels among different groups due to the influence of social, cultural or geographical factors.

Methods

The target population was the group of students who take the examination in June 2010. Considering the social, cultural and geographical factors outlined above, the students were classified into four groups: *Foreing, Guinea, Prison, Spain*.

The assessment of mathematical competence was carried out by considering a form of exam consisting of ten questions of objective-type with one foot and three alternatives of which only one is correct. The correction answer indicator for the question *i = 1,..., 10,* is $E_i = 1$ if the right choice is selected, $E_i = -0.25$ when a wrong alternative is selected and $E_i = 0$ otherwise. Traditional grade is $E = \text{Máx}\left\{0, \sum_{i=1}^{10} E_i\right\}$. Moreover, each question of the form has an assessment value in each of the subcomponents of the competence, given on a scale of 0 to 4 and prepared by the faculty. From these, it is possible to calculate the score that the student achieves in each subcomponent of the mathematical competence by using the expressions:

$$K_j = \text{Max}\left\{0, \frac{10}{k_{.j}} \sum_{i=1}^{10} k_{ij} E_i\right\} \quad C_k = \text{Max}\left\{0, \frac{10}{c_{.k}} \sum_{i=1}^{10} c_{ik} E_i\right\} \quad A_l = \text{Max}\left\{0, \frac{10}{a_{.l}} \sum_{i=1}^{10} a_{il}\, E_i\right\}$$

where k_{ij} is the valuation of question *i* in knowledge *j*, c_{ik} is the valuation of question *i* in capability *k* and a_{il} is the valuation of question *i* in attitude *l*; moreover, $k_{.j} = \sum_{i=1}^{10} k_{ij}$, $c_{.k} = \sum_{i=1}^{10} c_{ij}$ and $a_{.l} = \sum_{i=1}^{10} a_{il}$ are, respectively, the total of the values of the knowledge *j=1,...,5,* capabilities *k=1,...,8,* and attitudes *l=1,2* in the form. Finally, the average values can be calculated:

$$K = \sum_{j=1}^{5} \frac{K_j}{5}; \qquad C = \sum_{k=1}^{8} \frac{C_j}{8}; \qquad A = \sum_{l=1}^{2} \frac{A_l}{2}; \qquad EC = \frac{K + C + A}{3}$$

providing a measure of the level of each component of competence, (*Knowledge K, Capabilities C, Attitudes A*) and overall assessment of the mathematical competence *EC*.

Because multiple meetings for the examination with different locations and times are required, it is necessary to use various forms of examination, identified by a code called *Type*. To eliminate possible biases that may result from this fact, the analysis will be presented later in place of direct scores we used the standardized scores, i.e., those obtained by subtracting the mean and dividing by the standard deviation of the results of each type of form. Thus, all assessments of the components and subcomponents of the competence have a zero average and standard deviation equal to one.

Results

Table 1 and Figure 1 show the distribution of the group of students. As we can see, the largest group is the Spanish one (94,7%), although there is a considerable number, in absolute terms, of members of the other groups.

GROUP	Frequency	Percentage
FOREING	71	0,9%
GUINEA	190	2,3%
PRISON	169	2,1%
SPAIN	7719	94,7%
Total	8149	100,0%

Table 1: Frequency Distribution of Groups

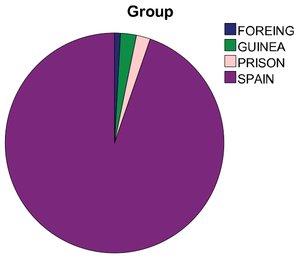


Figure 1: Sector Diagram of Groups

Figure 2 shows the distribution of standardized value of the different components of mathematical competence and global assessment of the mathematical competence for each of the groups considered.

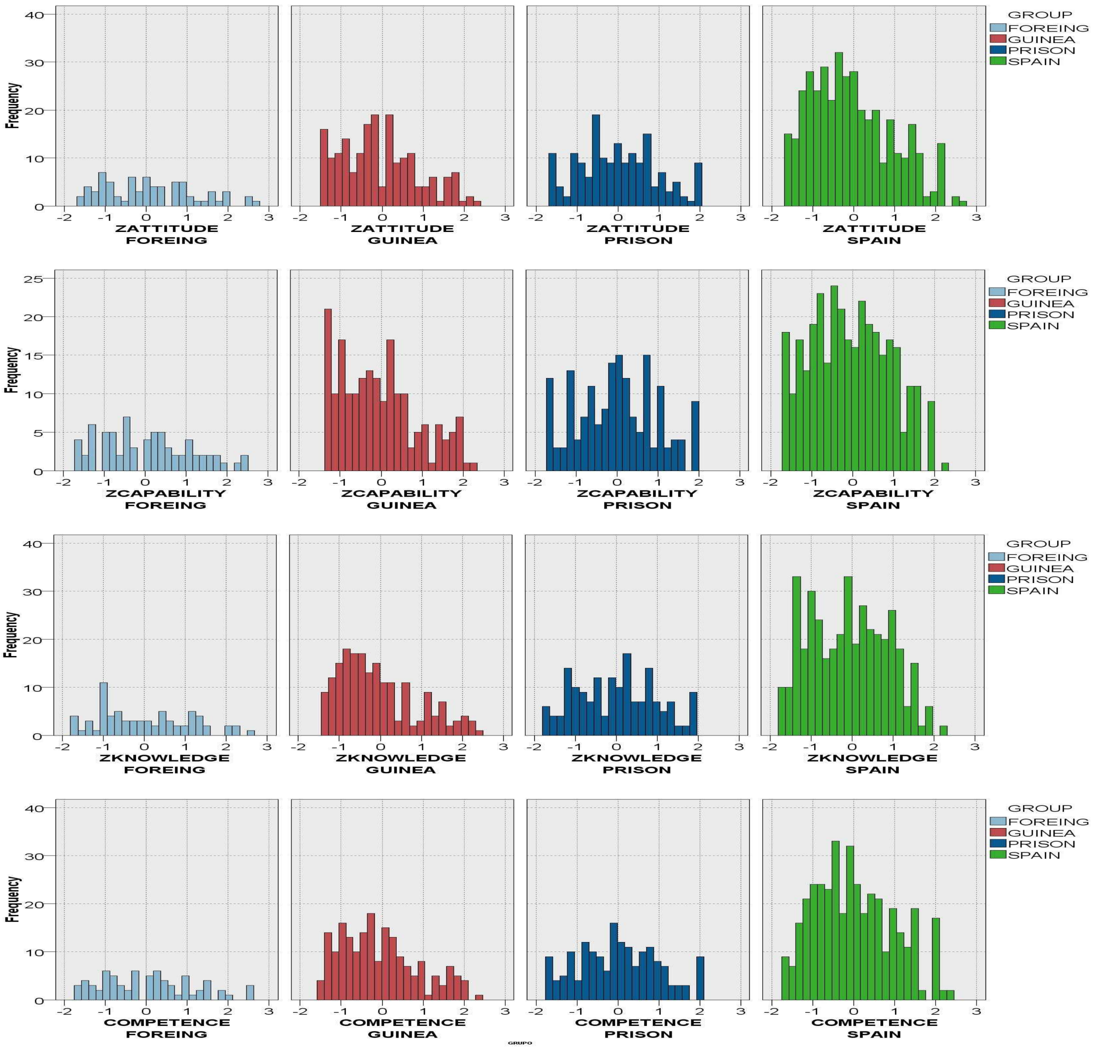
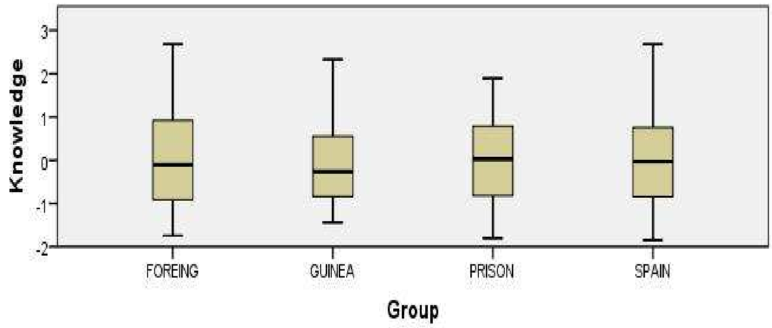
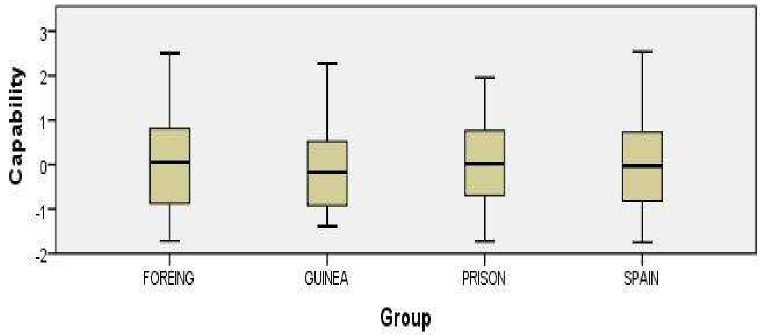


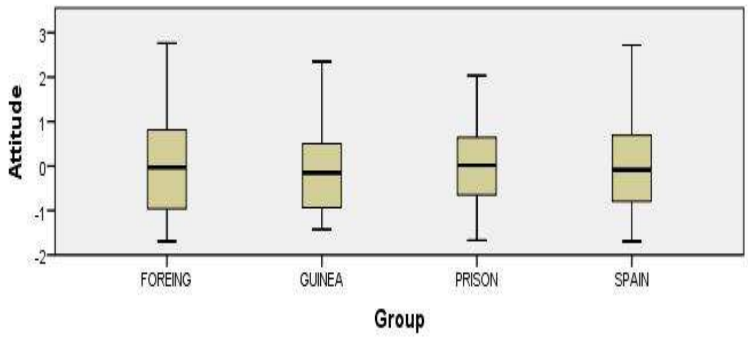
Figure 2: Distribution of the components of mathematical competence by group.



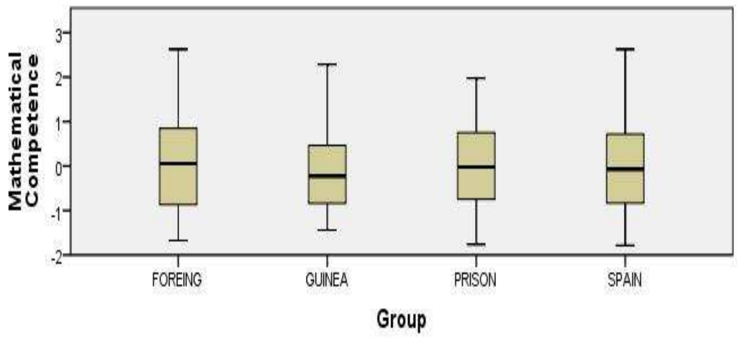
Statistic adjusted for ties	p-value
1.006	0.800



Statistic adjusted for ties	p-value
1.030	0.792



Statistic adjusted for ties	p-value
1.085	0.781



Statistic adjusted for ties	p-value
1.097	0.790

Figure 3: Kruskal Wallis test for comparison of levels of the components of the mathematical competence.

Figure 3 shows the results of Kruskal-Wallis test for comparison of the four groups, including statistical significance levels of the test. As it can be seen, all the comparisons are not significant, so we can accept the hypothesis of equal distributions of the components of mathematical competence in the differente groups.

Comparing the subcomponents of mathematical competence there were differences only in case of *Uncertainty*, see Figure 4. In this case, the statistics adjusted for ties is 55.496 and the p-value is 0.000.

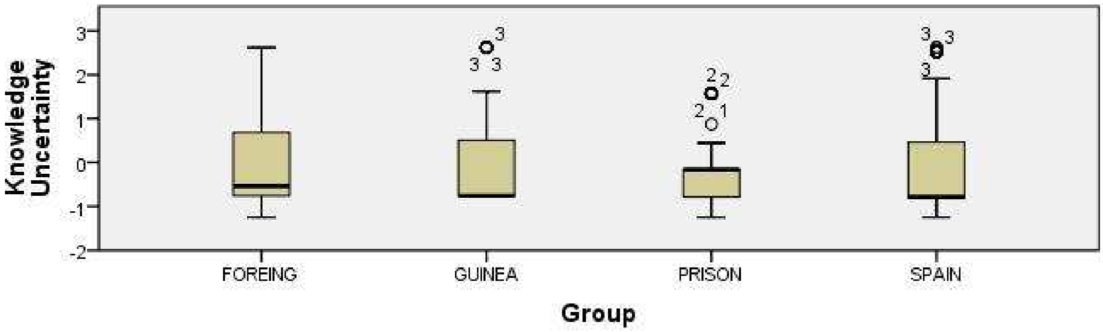


Figure 4 Kruskal Wallis test for comparison of levels of the subcomponent *Uncertainty*

Discussion

Assessment of the mathematical competence can be accomplished by adapting traditional assessment systems, such as the use of objective type tests, introducing aspects that allow them to quantify the level of each competence's component and subcomponent. Using this evaluation method is possible to obtain information about the student competence level in a more complete way than that obtained by traditional methods.

When comparing the level of mathematical competence among different groups of students attending to social, cultural and geographical factors, such as the origin of different countries, belonging to a developing country or the circumstances of these criminal serving time in a center prison, there are no significant differences in the level shown in the various components of mathematical competence. However, there are significant differences in the level of *Uncertainty* subcomponent. This subcomponent relates to knowledge concerning the study of phenomena involving chance and probability, as well as mathematics relating to statistical thinking. We have no clear explanation for this, although it may perhaps be due to cultural differences that may exist in the understanding of chance and statistics.

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Ramos, E; R. Vélez; V. Hernández; J. Navarro; E. Carmena and J.A. Carrillo (2009): *Sistemas inteligentes para el diseño de procedimientos equilibrados para la evaluación de competencias*, In M. Santamaría y A. Sánchez-Elvira (coord.): *La UNED ante el EESS. Redes de investigación en innovación docente 2006/2007*, Colección Estudios de la UNED, UNED, Madrid, pp.597-610.

Ramos, E; R. Vélez; V. Hernández; J. Navarro; E. Carmena and J.A. Carrillo (2010): C*ompetencias en Matemáticas Aplicadas a las Ciencias Sociales y su Evaluación Inteligente*, In M. Santamaría y A. Sánchez-Elvira (coord.): *La UNED ante el EESS. Redes de investigación en innovación docente 2007/2008*, Colección Estudios de la UNED, UNED, to be published.

PHANTOM GRAPHS.

Philip Lloyd. Epsom Girls Grammar School, Auckland, New Zealand. philiplloyd1@gmail.com

Abstract. While teaching "solutions of quadratics" and emphasising the idea that, in general, the solutions of $ax^2 + bx + c = 0$ are obviously where the graph of $y = ax^2 + bx + c$ crosses the x axis, I started to be troubled by the special case of parabolas that do not even cross the x axis. We say these equations have "complex solutions" but **physically, where are these solutions**?
With a little bit of lateral thinking, I realised that **we can physically find the actual positions of the complex solutions of any polynomial equation** and indeed many other common functions!
The theory also shows clearly and pictorially, why the complex solutions of polynomial equations with real coefficients occur in conjugate pairs.

Fig 1: The big breakthrough is
to change from an **x AXIS**…..

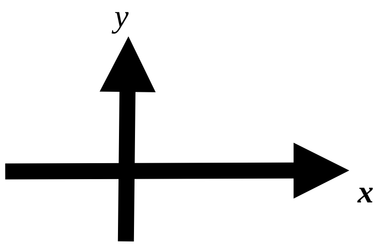
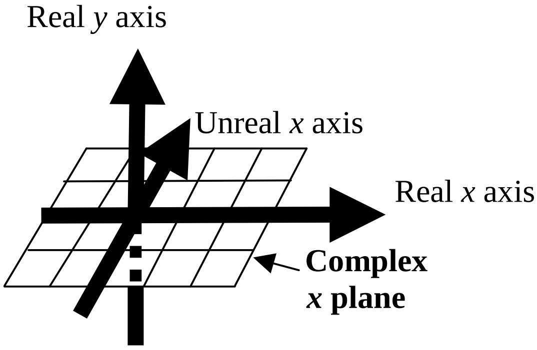


Fig 2
……..to a **complex *x* plane**!

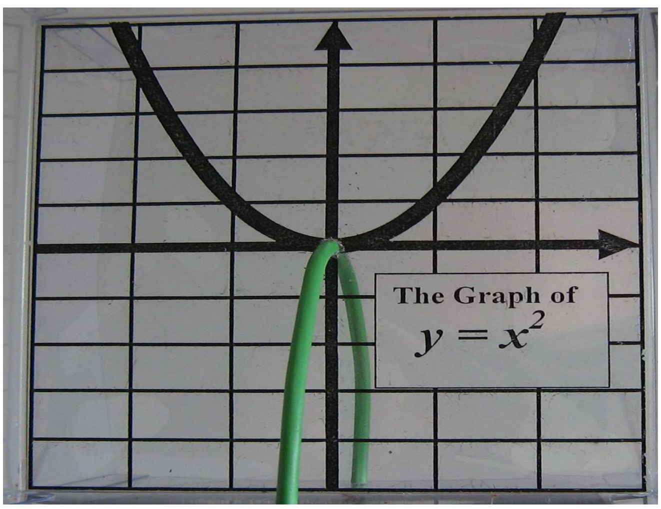


Introduction. Consider the graph $y = x^2$.
We normally just find the positive y values such as: (±1, 1) , (±2, 4), (±3, 9) but we can also find **negative y values** even though the graph does not seem to exist under the x axis:

If $y = -1$ then $x^2 = -1$ and $x = \pm i$.
If $y = -4$ then $x^2 = -4$ and $x = \pm 2i$.
If $y = -9$ then $x^2 = -9$ and $x = \pm 3i$

Thinking very laterally, I thought that instead of just having a ***y axis*** and an ***x AXIS*** (as shown in **Fig 1**) we should have a ***y axis*** but a **complex *x* PLANE!** (as shown on **Fig 2**)

Fig 3

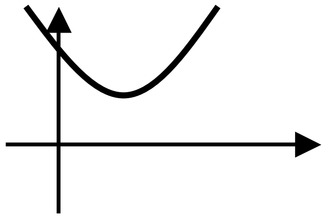


This means that the usual form of the parabola $y = x^2$ exists in the normal ***x, y*** plane **but another part of the parabola exists at right angles to the usual graph.**

Fig 3 is a Perspex model of $y = x^2$ and its "phantom" hanging at right angles to it.

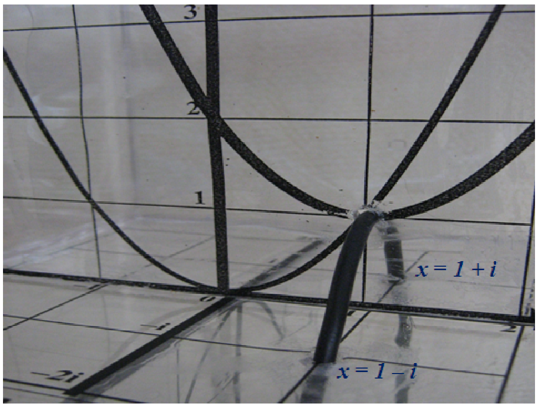
<u>"PHANTOM GRAPHS".</u> Now let us consider the graph $y = (x-1)^2 + 1 = x^2 - 2x + 2$

The minimum real y value is normally thought to be $y = 1$
but now we can have any real y values!



If $y = 0$ then $(x-1)^2 + 1 = 0$
so that $(x-1)^2 = -1$
producing $x - 1 = \pm i$
therefore $x = 1 + i$ *and* $x = 1 - i$

Fig 4

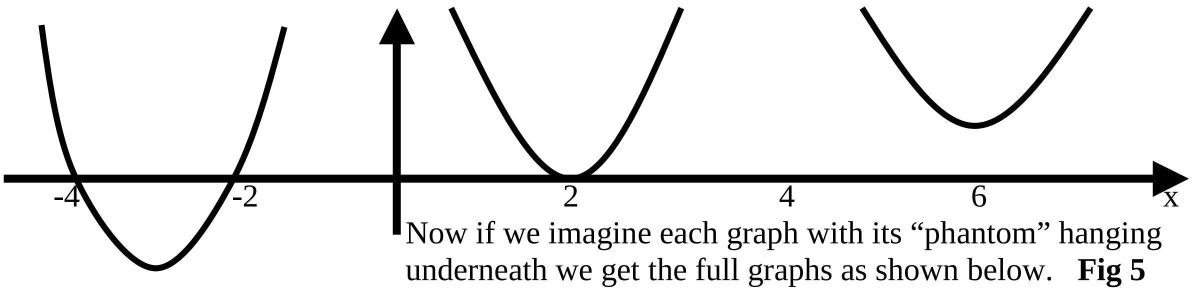


If $y = -3$ then $(x-1)^2 + 1 = -3$
so that $(x-1)^2 = -4$
therefore $x = 1 + 2i$ *and* $x = 1 - 2i$

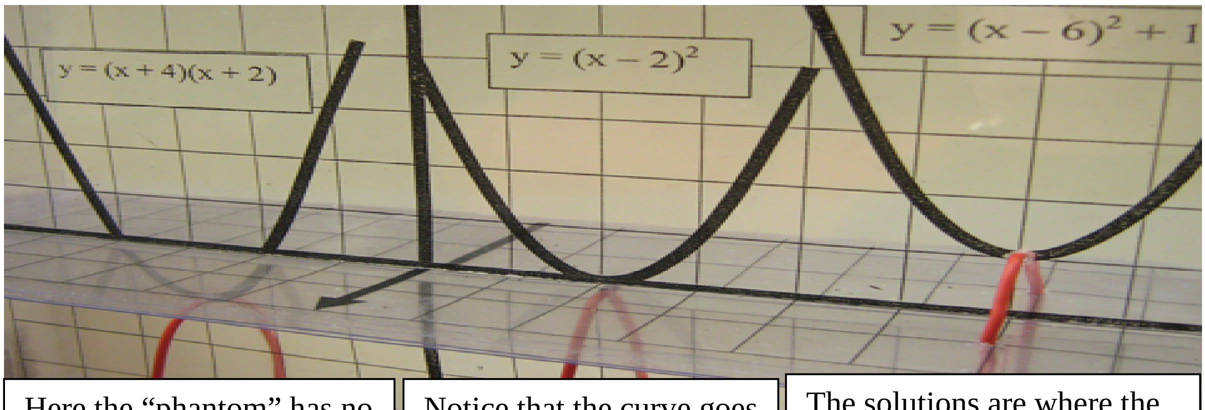
Similarly if $y = -8$ then $(x-1)^2 + 1 = -8$
so that $(x-1)^2 = -9$
therefore $x = 1 + 3i$ *and* $x = 1 - 3i$

The result is another "phantom" parabola which is "hanging" from the normal graph $y = x^2 - 2x + 2$ and the exciting and fascinating part is that the **solutions of** $x^2 - 2x + 2 = 0$ are $1 + i$ and $1 - i$ which are **where the graph crosses the x plane**! See **Fig 4**

In fact ALL parabolas have these "phantom" parts hanging from their lowest points and at right angles to the normal x, y plane.
It is interesting to consider the 3 types of solutions of quadratics.
Consider these cases: $y = (x+4)(x+2)$; $y = (x-2)^2$; $y = (x-6)^2 + 1$

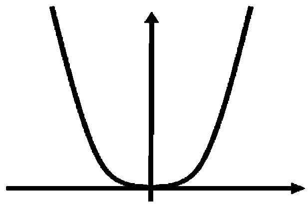


Now if we imagine each graph with its "phantom" hanging underneath we get the full graphs as shown below. **Fig 5**



Here the "phantom" has no effect on the solutions $x = -4$ *and* $x = -2$.	Notice that the curve goes through the point $x = 2$ twice! (a double solution)	The solutions are where the graph crosses the ***x plane*** at $x = 6 \pm i$ (**a conjugate pair**)

Now consider the graph of $y = x^4$



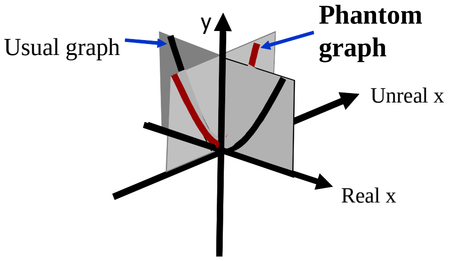
We normally think of this as just a U shaped curve as shown.
This consists of points (0, 0), (±1, 1), (±2, 16), (±3, 81) etc
The fundamental theorem of algebra tells us that equations of the form $\boldsymbol{x^4 = c}$ should have 4 solutions not just 2 solutions.

If $\boldsymbol{y = 1}$, $\boldsymbol{x^4 = 1}$ so using De Moivre's Theorem: $\boldsymbol{r^4 cis\ 4\theta = 1cis\ (360n)}$
$\boldsymbol{r = 1}$ ***and*** $\boldsymbol{4\theta = 360n}$ therefore $\boldsymbol{\theta = 0, 90, 180, 270}$ producing the 4 solutions :
$\boldsymbol{x_1 = 1\ cis\ 0 = 1,\ x_2 = 1\ cis\ 90 = i,\ x_3 = 1\ cis\ 180 = -1}$ ***and*** $\boldsymbol{x_4 = 1\ cis\ 270 = -i}$

If $\boldsymbol{y = 16}$, $\boldsymbol{x^4 = 16}$ so using De Moivre's Theorem: $\boldsymbol{r^4 cis\ 4\theta = 16cis\ (360n)}$
$\boldsymbol{r = 2}$ ***and*** $\boldsymbol{4\theta = 360n}$ therefore $\boldsymbol{\theta = 0, 90, 180, 270}$ producing the 4 solutions :
$\boldsymbol{x_1 = 2\ cis\ 0 = 2,\ x_2 = 2\ cis\ 90 = 2i,\ x_3 = 2\ cis\ 180 = -2,\ x_4 = 2\ cis\ 270 = -2i}$

Fig 6

This means $\boldsymbol{y = x^4}$ has another **phantom** part at right angles to the usual graph.



The points (1, 1), (–1, 1), (2, 16), (–2, 16) will produce the ordinary graph but the points (i, 1), (–i, 1), (2i, 16), (–2i, 16) will produce a similar curve at right angles to the ordinary graph. **Fig 6**

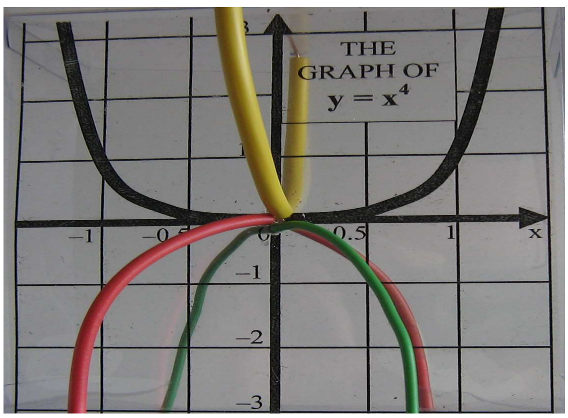
But this is not all!

We now consider **negative** real y values!

Consider $\boldsymbol{y = -1}$ ***so*** $\boldsymbol{x^4 = -1}$
Using De Moivre's Theorem:
$\boldsymbol{r^4 cis\ 4\theta = 1cis\ (180 + 360n)}$
$\boldsymbol{r = 1}$ and $\boldsymbol{4\theta = 180 + 360n}$ so $\boldsymbol{\theta = 45 + 90n}$
$\boldsymbol{x_1 = 1\ cis\ 45,\quad x_2 = 1\ cis\ 135,}$
$\boldsymbol{x_3 = 1\ cis\ 225,\ x_4 = 1\ cis\ 315}$

Similarly, if $\boldsymbol{y = -16}$, $\boldsymbol{x^4 = -16}$
Using De Moivre's Theorem:
$\boldsymbol{r^4 cis\ 4\theta = 16cis\ (180 + 360n)}$
$\boldsymbol{r = 2}$ ***and*** $\boldsymbol{4\theta = 180 + 360n}$ ***so*** $\boldsymbol{\theta = 45 + 90n}$
$\boldsymbol{x_1 = 2\ cis\ 45,\ x_2 = 2\ cis\ 135,}$
$\boldsymbol{x_3 = 2\ cis\ 225,\ x_4 = 2\ cis\ 315}$

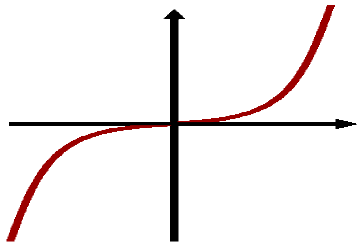
Fig 7 (photo of Perspex model)



The points corresponding to negative y values produce two curves identical in shape to the two curves for positive y values but they are rotated 45 degrees as shown on **Fig 7.**

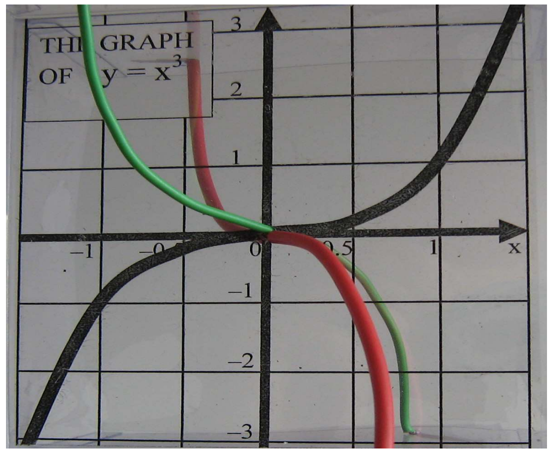
NOTE: Any horizontal plane crosses the curve in 4 places because all equations of the form $\boldsymbol{x^4 = \pm c}$ have **4** solutions and it is clear from the photo that the solutions are conjugate pairs!

<u>Consider the basic cubic curve $y = x^3$.</u>



Equations with x^3 have 3 solutions.
If $y = 1$ then $x^3 = 1$
so $r^3 cis\ 3\theta = 1 cis\ (360n)$
$r = 1$ and $\theta = 120n = 0, 120, 240$
$x_1 = 1\ cis\ 0,\ \ x_2 = 1\ cis\ 120,\ \ x_3 = 1\ cis\ 240$

Fig 8 (photo of Perspex model)

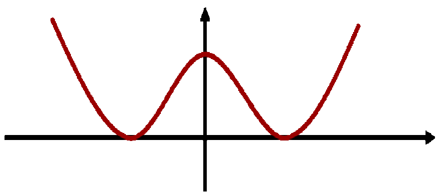


Similarly if $y = 8$ then $x^3 = 8$
so $r^3 cis\ 3\theta = 8 cis\ (360n)$
$r = 2$ and $\theta = 120n = 0, 120, 240$
$x_1 = 2\ cis\ 0,\ \ x_2 = 2\ cis\ 120,\ \ x_3 = 2\ cis\ 240$

Also y can be negative. If $y = -1,\ \ x^3 = -1$
so $r^3 cis\ 3\theta = 1 cis\ (180 + 360n)$
$r = 1$ and $3\theta = 180 + 360n$ so $\theta = 60 + 120n$
$x_1 = 1\ cis\ 60,\ \ x_2 = 1\ cis\ 180,\ x_3 = 1\ cis\ 300$

The result is THREE identical curves situated at 120 degrees to each other! (See Fig 8)

<u>Now consider the graph $y = (x + 1)^2(x - 1)^2 = (x^2 - 1)(x^2 - 1) = x^4 - 2x^2 + 1$</u>



Any horizontal line (or plane) should cross this graph at 4 places because any equation of the form $x^4 - 2x^2 + 1 = C$ (where C is a constant) has 4 solutions.

If $x = \pm 2$ then $y = 9$
so solving $x^4 - 2x^2 + 1 = 9$
we get : $x^4 - 2x^2 - 8 = 0$
so $(x + 2)(x - 2)(x^2 + 2) = 0$
giving **x = ±2 and ±√2 *i***

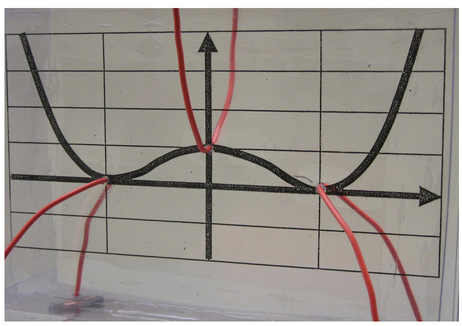
Similarly if $x = \pm 3$ then $y = 64$
so solving $x^4 - 2x^2 + 1 = 64$
we get $x^4 - 2x^2 - 63 = 0$
so $(x + 3)(x - 3)(x^2 + 7) = 0$
giving **x = ±3 and ±√7 *i***

The complex solutions are all of the form **0** ± n***i***. This means that a **phantom curve**, **<u>at right angles</u>** to the basic curve, stretches upwards from the maximum point.

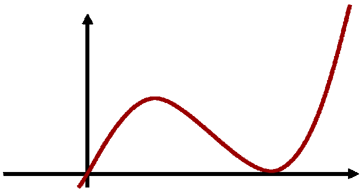
If $y = -1,\ \ x = -1.1 \pm 0.46i,\ 1.1 \pm 0.46i$
If $y = -2,\ \ x = -1.2 \pm 0.6i,\ \ 1.2 \pm 0.6i$
If $y = -4,\ \ x = -1.3 \pm 0.78i,\ 1.3 \pm 0.78i$

Notice that the real parts of the *x* values vary. This means that the **phantom** curves hanging off from the two minimum points are not in a vertical plane as they were for the parabola. See **Fig 9. Clearly all complex solutions to $x^4 - 2x^2 + 1 = C$ are conjugate pairs.**

Fig 9 (photo of Perspex model)

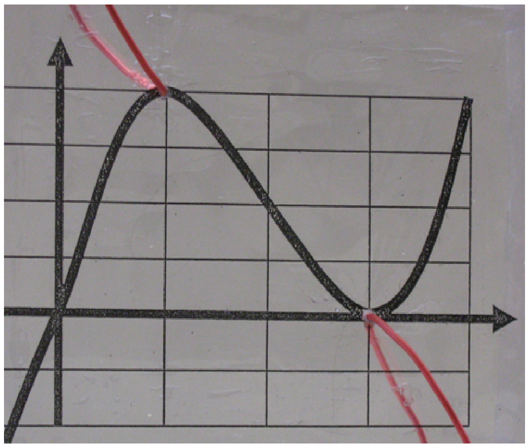


Consider the cubic curve $y = x(x - 3)^2$



As before, any horizontal line (or plane) should cross this graph at 3 places because any equation of the form: $x^3 - 6x^2 + 9x =$ ***"a constant"***, has 3 solutions.

Fig 10 (photo of Perspex model)



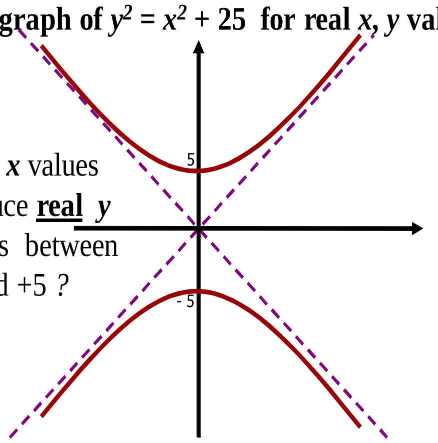
If $x^3 - 6x^2 + 9x = 5$ *then* $x = 4.1$ and $\underline{0.95 \pm 0.6i}$
If $x^3 - 6x^2 + 9x = 6$ *then* $x = 4.2$ and $\underline{0.90 \pm 0.8i}$
If $x^3 - 6x^2 + 9x = 7$ *then* $x = 4.3$ and $\underline{0.86 \pm 0.9i}$
So the left hand phantom is leaning to the left from the maximum point (1, 4).

If $x^3 - 6x^2 + 9x = -1$ *then* $x = -0.1$ and $\underline{3.05 \pm 0.6i}$
If $x^3 - 6x^2 + 9x = -2$ *then* $x = -0.2$ and $\underline{3.1 \pm 0.8i}$
If $x^3 - 6x^2 + 9x = -3$ *then* $x = -0.3$ and $\underline{3.14 \pm 0.9i}$
So the right hand phantom is leaning to the right from the minimum point (3, 0). **See Fig 10**

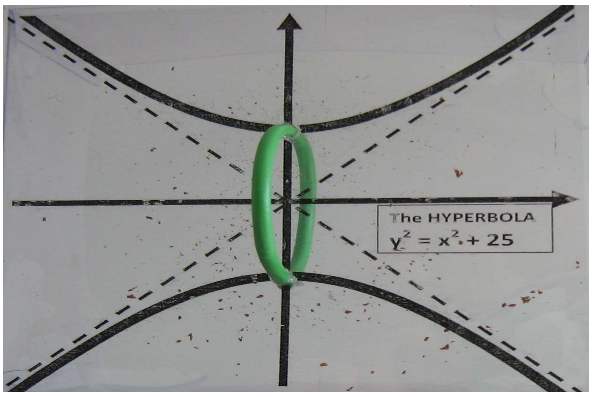
The HYPERBOLA $\underline{y^2 = x^2 + 25.}$ This was the most surprising and absolutely delightful Phantom Graph that I found whilst researching this concept.

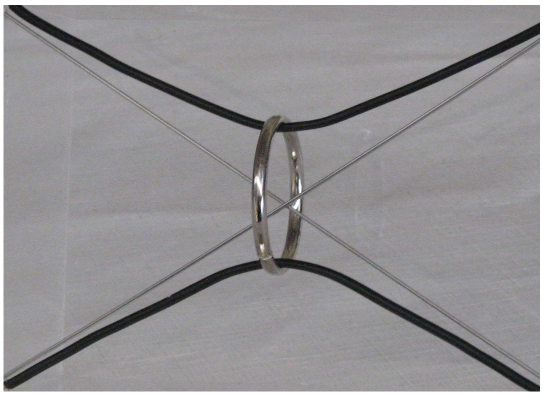
The graph of $y^2 = x^2 + 25$ for real x, y values.

What x values produce **real** y values between -5 and +5 ?



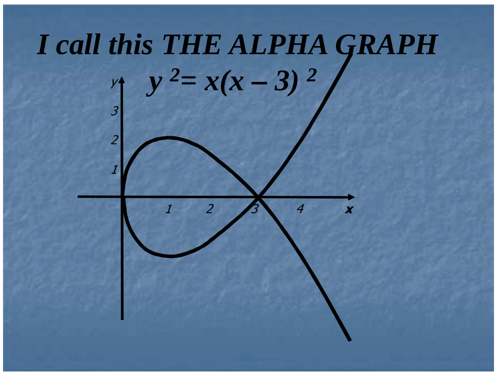
If $y = 4$ then $16 = x^2 + 25$
and $-9 = x^2$
so $x = \pm 3i$
Similarly if $y = 3$ then $9 = x^2 + 25$
so $x = \pm 4i$
And if $y = 0$ then $0 = x^2 + 25$
so $x = \pm 5i$
These are points on a circle of radius 5 units.
$(0, 5)$ $(\pm 3i, 4)$ $(\pm 4i, 3)$ $(\pm 5i, 0)$
The circle has complex x values but real y values.
This circle is in the plane at right angles to the hyperbola and joining its two halves!
See photos below of the Perspex models.

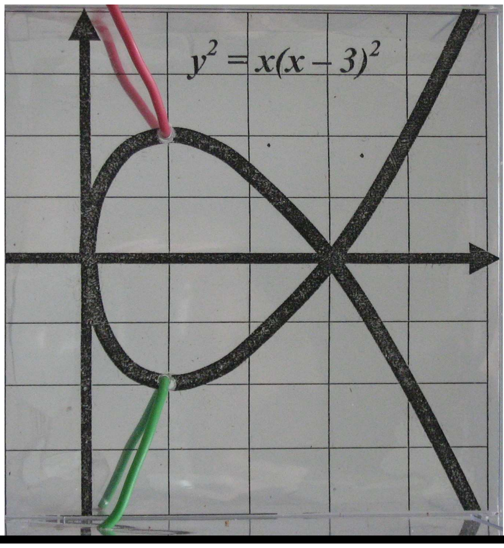




AFTERMATH!!!

I recently started to think about other curves and thought it worthwhile to include them.





Using a technique from previous graphs:

I choose an x value such as $x = 5$,

calculate the y^2 value, ie $y^2 = 20$ and $y \approx 4.5$

then solve the equation $x(x-3)^2 = 20$

already knowing one factor is $(x-5)$

$$\text{ie} \quad x(x-3)^2 = 20$$
$$x^3 - 6x^2 + 9x - 20 = 0$$
$$(x-5)(x^2 - x + 4) = 0$$
$$x = 5 \text{ or } \tfrac{1}{2} \pm 1.9i$$

This means (5, 4.5) is an "ordinary" point on the graph but two "phantom" points are (½ ± 1.9i, 4.5)

Similarly:

If $x = 6$, $y^2 = 54$ and $y = \pm 7.3$ so $x(x-3)^2 = 54$

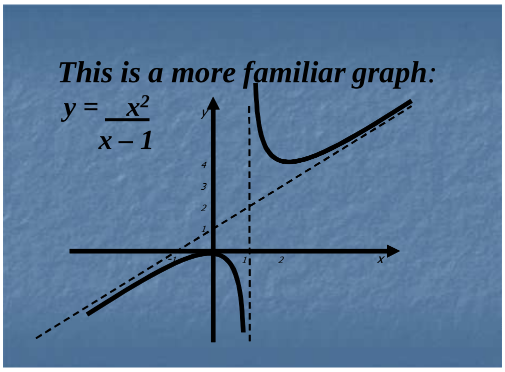
$$x^3 - 6x^2 + 9x - 54 = 0$$
$$(x-6)(x^2 + 9) = 0$$
$$x = 6 \text{ or } \pm 3i$$

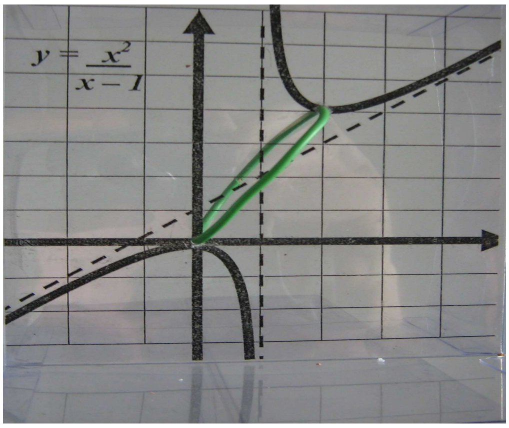
And

If $x = 7$, $y^2 = 112$ and $y = \pm 10.6$ so $x(x-3)^2 = 112$

$$x^3 - 6x^2 + 9x - 112 = 0$$
$$(x-7)(x^2 + x + 16) = 0$$
$$x = 7 \text{ or } -\tfrac{1}{2} \pm 4i$$

Hence we get the two phantom graphs as shown.





$$y = \frac{x^2}{x-1}$$

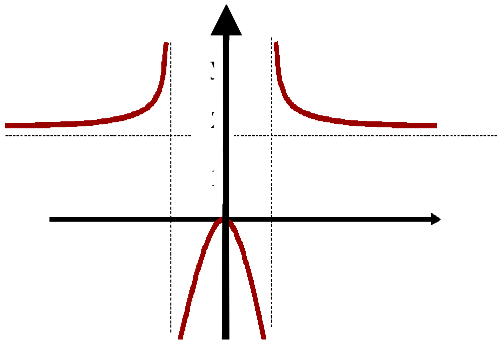
Here we need to find complex x values which produce real y values from 0 to 4.

$$\left.\begin{array}{ll} \text{If } y = 0 & x = 0 \\[2ex] \text{If } y = 1 & x = \dfrac{1}{2} \pm \dfrac{\sqrt{3}i}{2} \\[2ex] \text{If } y = 2 & x = 1 \pm i \\[2ex] \text{If } y = 3 & x = \dfrac{3}{2} \pm \dfrac{\sqrt{3}i}{2} \\[2ex] \text{If } y = 4 & x = 2 \end{array}\right\}$$

These points produce the phantom "oval" shape as shown in the picture on the left.

Consider the graph $y = \dfrac{2x^2}{x^2 - 1} = 2 + \dfrac{2}{x^2 - 1}$

This has a horizontal asymptote $y = 2$
and two vertical asymptotes $x = \pm 1$



If $y = 1$ ***then*** $\dfrac{2x^2}{x^2 - 1} = 1$

so $2x^2 = x^2 - 1$

and $x^2 = -1$

producing $x = \pm i$

If $y = 1.999$ ***then*** $\dfrac{2x^2}{x^2 - 1} = 1.999$

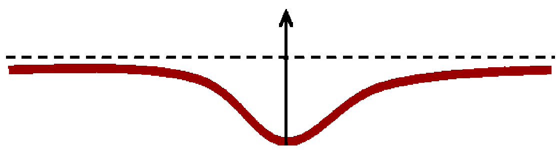
so $2x^2 = 1.999x^2 - 1.999$

and $0.001x^2 = -1.999$

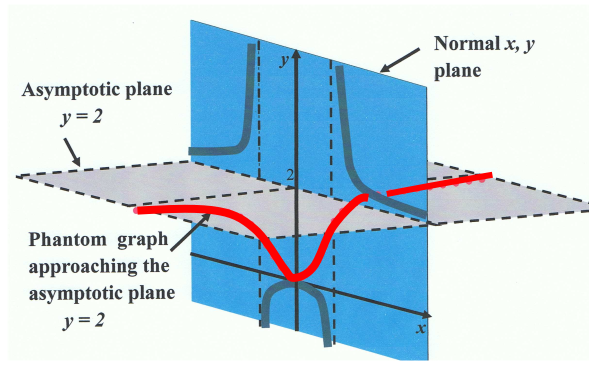
Producing $x^2 = -1999$

$x \approx \pm 45\,i$

This implies there is a "phantom graph" which approaches the horizontal asymptotic plane y = 2 and is at right angles to the x, y plane, resembling an upside down normal distribution curve.

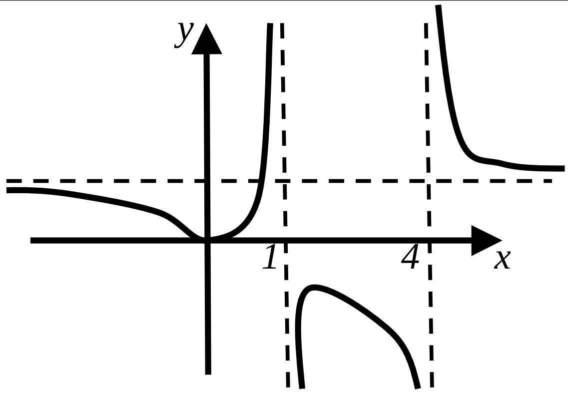


Side view of "phantom" approaching y = 2



Consider an apparently "similar" equation but with a completely different "Phantom".

$y = \dfrac{x^2}{(x-1)(x-4)} = \dfrac{x^2}{x^2 - 5x + 4}$



The minimum point is (0, 0)
The maximum point is (1.6, –1.8)

If $y = -0.1$, $x = 0.2 \pm 0.56i$

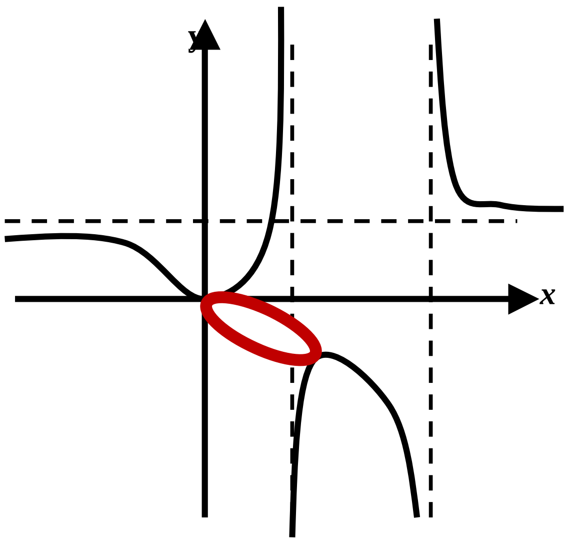
If $y = -0.2$, $x = 0.4 \pm 0.7i$

If $y = -0.5$, $x = 0.8 \pm 0.8i$

If $y = -1$, $x = 1.25 \pm 0.66i$

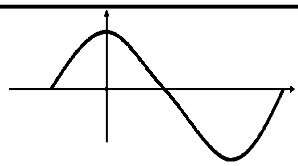
If $y = -1.5$, $x = 1.5 \pm 0.4i$

If $y = -1.7$, $x = 1.6 \pm 0.2i$



These results imply that a "phantom" ***oval*** *shape joins the minimum point (0, 0) to the maximum point (1.6, –1.78).*

The final two graphs I have included in this paper involve some theory too advanced for secondary students but I found them absolutely fascinating!



If $y = \cos(x)$ what about y values > 1 and < -1 ?

Using $\cos(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} - \ldots\ldots$

Let's find $\cos(\pm\, i) = 1 + \dfrac{1}{2!} + \dfrac{1}{4!} + \dfrac{1}{6!} + \dfrac{1}{8!} \ldots\ldots$

$\approx$ **1.54 (ie > 1)**

Similarly $\cos(\pm 2i) = 1 + \dfrac{4}{2!} + \dfrac{16}{4!} + \dfrac{64}{6!} + \ldots$

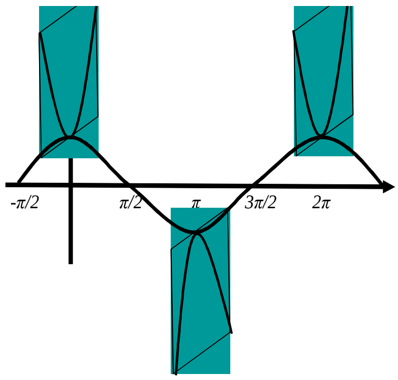
$\approx$ **3.8**

Also find $\cos(\pi + i) = \cos(\pi)\cos(i) - \sin(\pi)\sin(i)$

$= -1 \times \cos(i) - 0$

$\approx$ **– 1.54 (ie < –1)**

These results imply that the cosine graph also has its own "**phantoms**" in vertical planes at right angles to the usual ***x, y*** graph, emanating from each max/min point.



Finally consider the exponential function $y = e^x$. How can we find x if $e^x = -1$?

Using the expansion for $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \ldots\ldots$

We can find $e^{xi} = 1 + xi + \dfrac{(xi)^2}{2!} + \dfrac{(xi)^3}{3!} + \dfrac{(xi)^4}{4!} + \dfrac{(xi)^5}{5!} + \ldots\ldots$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$

$$= \qquad (\cos x) \qquad + \qquad i\,(\sin x)$$

If we are to get **REAL y values** then using $e^{xi} = \cos x + i \sin x$, we see that **sin *x*** must be zero.

This only occurs when $x = 0, \pi, 2\pi, 3\pi, \ldots$ (or generally $n\pi$)

$e^{\pi i} = \cos\pi + i\sin\pi = {}^{-}1 + 0i$, $e^{2\pi i} = \cos 2\pi + i\sin 2\pi = +1 + 0i$

$e^{3\pi i} = \cos 3\pi + i\sin 3\pi = {}^{-}1 + 0i$, $e^{4\pi i} = \cos 4\pi + i\sin 4\pi = +1 + 0i$

Now consider $y = e^X$ where $X = x + 2n\pi i$ (ie even numbers of π)

ie $y = e^{x + 2n\pi i} = e^x \times e^{2n\pi i} = e^x \times 1 = e^x$

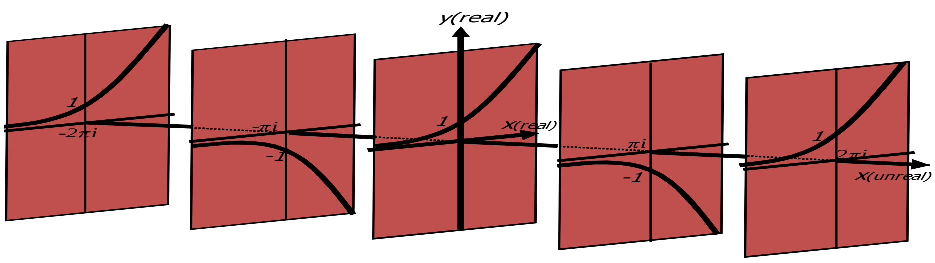
Also consider $y = e^X$ where $X = x + (2n+1)\pi i$ (ie odd numbers of π)

ie $y = e^{x + (2n+1)\pi i} = e^x \times e^{(2n+1)\pi i} = e^x \times {}^{-}1 = -e^x$

This means that the graph of $y = e^X$ consists of **parallel identical curves** if $X = x + 2n\pi i$

$= x$ + even N[os] of πi

and, **upside down parallel identical curves** occurring at $X = x + (2n+1)\pi i = x$ + odd N[os] of πi



Graph of $y = e^X$ where $X = x + n\pi i$

Workshop: Error Analysis of Mathematics Test Items

Rencia Lourens; Nico Molefe and Karin Brodie

Florencia.Lourens@wits.ac.za; Nicholas.Molefe@wits.ac.za; Karin.Brodie@wits.ac.za

School of Education, University of the Witwatersrand, Johannesburg

The workshop will help mathematics teachers develop an understanding of learner errors and misconceptions and how these errors and misconceptions can be embraced in the teaching and learning of mathematics. The workshop will draw on an activity developed by the Data Informed Practice Improvement Project (DIPIP). Some examples of learner errors from an international test will be given. Participants will be given the opportunity to discuss and identify the mathematical content needed to answer the question, analyse the correct answer and then analyse the incorrect answers, identifying the reasons for learner errors and the misconceptions that produce these errors.

Background:

The Data Informed Practice Improvement Project (DIPIP) is an innovative professional development project aiming to develop sustainable professional learning communities amongst mathematics teachers. Teachers engage with data from their classrooms and work towards understanding learners' errors better. This is done by trying to understand why learners are making errors and how teachers should respond to and work with the errors.

We have developed a number of activities in the project to engage teachers with their learners' thinking. One of these activities is error analysis, where we look at test items and have conversations about the errors that learners make, and the reasons behind these errors. We work with teachers to support them in understanding and engaging with learner errors in the classroom, and to view errors as reasonable (Nesher, 1987; Smith, diSessa & Roschelle, 1993; Drews, 2005).

Purpose of the workshop:

To share ideas on how teachers can be assisted in working with errors in the teaching and learning of mathematics by

- Engaging with learner errors,
- Making meaning of the learners' thinking behind the errors
- Recognising that errors can show valid learner thinking.

Theoretical ideas on error analysis:

Error analysis forms part of the key activities in the DIPIP project. Errors are a result of a "consistent conceptual framework based on earlier acquired knowledge", called misconceptions (Nesher, 1987: 33) and make sense to learners in terms of their current thinking. Learners do not just make errors - these errors make sense to them as a result of the conceptual links that they make to knowledge they acquired previously. It should be noted that misconceptions may lead to correct answers, even when the mathematical thinking producing these answers is partly or entirely incorrect (Nesher, 1987). A classic example of this is the case of Benny as discussed in Erlwanger (1973).

Borasi (1987) views errors as “springboards for enquiry”. In other words, errors can raise important issues for further exploration of the mathematics. Because errors make sense to those who make them, it’s important that errors be embraced in the teaching and learning of mathematics, and not be ignored, or merely corrected. It is also important to know how errors and misconceptions can inform our instructional practices. Errors “provide evidence that the expected result has not been reached, and that something else has to be done” (Borasi, 1987: 4) and errors can therefore be used to “investigate the nature of fundamental mathematical notions” (Borasi, 1987: 5). While we agree with Borasi (1987) that errors can be springboards for enquiry we need to acknowledge that errors can be an indication to teachers of misconceptions that exist and need to be addressed. Teachers would normally not teach errors or misconceptions. For learners to develop misconceptions is a normal part of learning. All learners develop misconceptions at some point, even those who have good teaching. When teachers teach in ways that do not embrace errors, there is a likelihood that misconceptions might develop or be perpetuated. It is for this reason that teachers need to understand the type of errors that learners make and also understand why they make those errors. Understanding learners’ errors can help teachers develop teaching strategies that can engage with learners’ misconceptions. Teachers can never entirely prevent errors, so it is important that they can deal with them as they come up.

Assessment tasks on which the workshop is based:

The workshop activity is based on results of a standardised, international test that learners of participating schools have written, with focus on Algebra. This test was developed as part of the research programme “Concepts in Secondary Mathematics and Science” (CSMS) (Hart, 1981), and has been extended by “Increasing Competence and Confidence in Algebra and Multiplicative Structures” (ICCAMS). According to Hart (1981), there are six different ways of interpreting and using letters: letter evaluated, letter not used, letter used as an object, letter used as a specific unknown, letter used as a generalised number and letter used as a variable (p.4). Hart (1981) discusses these categories as follows:

Letter Evaluated – when the response suggests that the letter is given a numerical value instead of being treated as an unknown or generalised number.

Letter Not Used – letter is acknowledged without being given a meaning or it is simply ignored

Letter as Object – letter used to denote an object

Letter as Specific Unknown – letter thought of as a particular but unknown number

Letter as Generalised Number – letter seen as being able to take several values

Letter as Variable – letter seen as representing a range of values, which is more of a dynamic view

The above categories are used as guidelines for analysis of errors. When teachers use these categories to analyse learner responses, they will identify the type of errors that learners have made and discuss these errors in relation to the teaching and learning of mathematics. Templates that we have developed through DIPIP are used to analyse errors that learners make.

Items chosen:

For this workshop we have chosen six items to look at and will use the answers of different learners to work on the error analysis activity with participants. We will identify the errors that learners made and workshop participants will discuss why we think learners made those errors. The items were chosen because of the richness in the errors that the learners made.

The workshop will provide participants with opportunities to look at classroom data in different ways. Data from classroom can help teachers develop methods of looking deeply into learner needs and see how these learner needs can inform teachers' learning needs (Earl & Katz, 2006)

Boudett, City, & Murnane (2006) argue that teachers can deepen their understanding of learners' strengths and misconceptions by looking collaboratively at their learners' work and talking about it in meaningful ways. In looking at their learners' work teachers can get opportunities to understand their learners' thinking, especially when learners make errors.

Analyses:

In analysing the items participants in the workshop will use a template that will be given to them to look at the following:

- Test Item

  Here we will look at the question and provide participants the opportunity to answer the question in as many different ways as possible.

- Content Description

  Participants will need to identify the mathematical content needed to answer the question.

- Analyses of Correct Response

  Analysing what knowledge, skills and procedures are required to get the correct response for this question

- Analysis of Errors

  This is the stage in the analysis that normally brings about the most discussion. Looking at the answers of the learners we look at the actual errors made by learners. Not all errors are discussed in depth, but the ones which are common. The aim is to understand the thinking processes of the learners and to identify any misconceptions that could have contributed to this particular error. All possible reasons for the error should be discussed.

- Critical Concept(s)

  The final stage, having done the error analysis, is to identify the critical concept(s) needed by the learner. The questions asked are what is missing between the error and the correct response. This can be seen as the link between the error and the correct response. These critical concepts become the learning needs of the learners and in the DIPIP project lessons would be developed around this concept.

Conclusion:

It is important for teachers to understand the role that errors play in the teaching and learning of mathematics. The workshop will try and raise an awareness of how errors can be embraced in mathematics classrooms instead of being ignored, or merely corrected. The error analysis activity can provide participants with skills to handle errors that learners make in class as well as the errors they make when they write their homework.

We will provide participants with an opportunity at the end of the workshop to give an evaluation of how the workshop went; what they gained; what works well for them and what does not; and also make input on any matter they deem necessary in working with learner errors and misconceptions.

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Isomorphic Visualization and Understanding of the Commutativity of Multiplication: from multiplication of whole numbers to multiplication of fractions

George Malaty, University of Eastern Finland, george.malaty@uef.fi

In 2010, I wrote a paper on the developments of visualization, functional materials and actions in teaching mathematics. This paper was a historical survey, but in the recent paper we put emphases on practical issues, taking from the case of the commutativity of multiplication a model for the isomorphic type of visualization we have developed.

From historical view to practical issues on the development of using visualization

In a previous paper of 2010, a discussion was given on the type of visualization used in Finnish mathematics textbooks in the 19th and 20th centuries (Malaty 2010). In the recent work, we are moving to put emphasis on practical issues and present an example, through which we can explore some of the main roles visualization can have in teaching mathematics. Developing mathematics teaching approaches has been always one of my main interests, and working in teacher education and mathematical clubs, in Finland, has given me a chance to develop and test my teaching approaches. One of the elements used in these approaches is visualization (Malaty 2006a, Malaty 2006b, Malaty 2006c, Malaty 1996). Visualization has been for us a facilitator to give students a chance to understand and discover mathematical concepts and relations, and as well a tool to demonstrate, solve and pose mathematical problems. The example we are going to present is related to the teaching of the commutativity of multiplication.

Common mistakes in the visualization of the commutativity of multiplication

Fig.1 represents one of our textbooks' visualization (Haapaniemi et al. 2002, 126). At first, we can notice that the figure contains unneeded elements, and this makes it far from the simplicity of iconic visualization. Such mistake is common in textbooks, but the main problem, we see, is the use of visualization to make ungrounded generalizations. This is as well the case in our example. The textbook uses the given figure (Fig.1) to present two statements $5 \cdot 2 = 10$ and $2 \cdot 5 = 10$. But these statements are not justified by the figures presented, but by the multiplication tables. Visualization role here is a modest one, and near to be only decorative. This is still the case, when textbooks provide more iconic visualization. In Fig.2, of the same textbook, and in the same page, two statements, $3 \cdot 2 = 6$ and $2 \cdot 3 = 6$, are introduced and justified by multiplication tables and not upon visualization.

Fig. 1. Complicated visualization

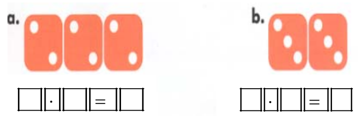
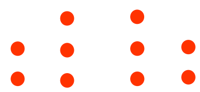


Fig. 2. Iconic visualization

Textbooks' effect is rematkable on what happanes in the classroom, including the use of textbooks' visualization. But, other types of mistakes have been observed in classroom teaching. For the commutativity of multiplication, not rare to see on the boards a type of mechanical performance like the one of Fig.3. While the numbers included in the statement $2 \cdot 3 = 3 \cdot 2$ are visualized (Fig.3), no attention is given to the meaning of the multiplication operation. Numbers are visualized, as if for summands of a sum and not for multiplier and multiplicand of a product.



$$2 \cdot 3 = 3 \cdot 2$$

Fig.3. Mechanical performance

Iconic, dynamic 2-dimensional visualization

In building visualization for mathematical relations, we have developed iconic dynamic types of visualization to get isomorphism with the relation we visualize. For the commutativity of multiplication, our types of visualization are 2-dimentional ones. At the beginning, and for young children, we use perpendicular dimensions, but at relevant stage we ask students to investigate the possibility of using oblique dimensions.

Visualization of the commutativity of multiplication of whole numbers

This I start by writing on the board the expression $2\cdot 3$. After that, I draw only one row of three circles, as in Fig. 4. Then I ask children:

Fig.4. One row of three circles

How many times three red circles I have drawn? – Only one time. After getting this answer, I draw an arrow as in Fig. 5, and then continue: *Right. But, here above is written* $2\cdot 3$ *, what I have to do?* – Draw another row. *Here?* I add a new arrow as in Fig.6, and then I draw the new row of three red circles as in Fig.7

Fig.5. Adding an arrow to mean one time of three circles

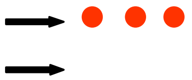


Fig.6. Adding another arrow

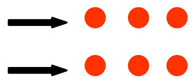


Fig.7. Drawing the other row of three circles

The 2-dimentional iconic visualization (Fig.7) has been built in a dynamic way to get a figure isomorphic to the expression $2\cdot 3$ **.** After that, we can continue our work with children in different ways, taking in mind the level of the group. A challenging and interesting one is to continue as follows: *But, this figure* (Fig.7) *is also representing the expression* $3\cdot 2$ (where I write numeral 2 in red and numeral 3 in other color like green), *who can tell me how this is possible?* If my last question does not help, I can use another approach like the one of the next dialog. *How many red circles you can see in one column?* (I draw in green vertical arrow as in Fig. 8, and then also in green a vertical segment as in Fig.9) – Two. *How many times such 'two circles' we have?* - Three. Yes, *who can draw more arrows to show that we have'3 times 2 circles'?* When a student gets a figure like that of Fig.10,

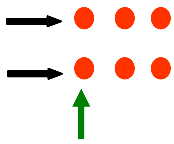


Fig.8. Adding in green vertical arrow

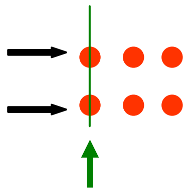


Fig.9. Adding in green vertical segment

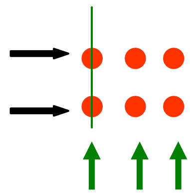


Fig.10. Drawing more vertical arrows

we continue: Great. *'3 times 2',* this expression I want to write here; *which color is relevant for writing numeral 3?* – Green. Right. *What about the color relevant for numeral 2?* – Red. *Why not green or other color?* – Because, here, 2 is the number of red circles. At this moment, I write the expression $3\cdot 2$ in the same row of the expression $2\cdot 3$, which has been already written on the top of the board, and leave a space between the two expressions to draw a box in a form of a rectangle, as in Fig.11. In addition, I put three large cards over the open sentence, two for inequality relations and the third for the relation of equality, as in Fig.12. Then, I ask children to have a time to think about the relevant card to change the open sentence into a true statement? *Take your time, and raise your hand when you are ready to tell me, which card is relevant and where to put it?*

$$2\cdot 3 \ \boxed{} \ 3\cdot 2$$

Fig.11. Open sentence

$$\boxed{} \quad \boxed{} \quad \boxed{} \quad ?$$

$$2\cdot 3 \ \boxed{} \ 3\cdot 2$$

Fig.12. Adding three large cards of relations

Two meanings of dynamic visualization, and isomorphism

Getting Fig.10 for the commutativity of multiplication was an outcome of a process and not given as such, at once. For this reason we call such

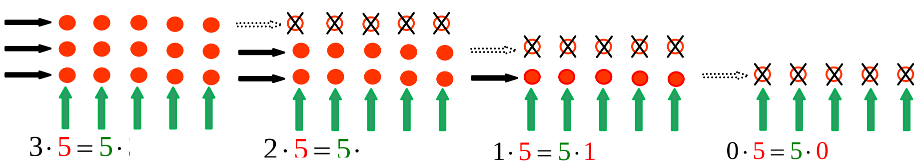


Fig.13. From $3\cdot 5=5\cdot 3$, **to** $2\cdot 5=5\cdot 2$ **then** $1\cdot 5=5\cdot 1$ **and Finally** $0\cdot 5=5\cdot 0$

visualization a dynamic visualization. Nevertheless, the main reason of giving this name is the possibility of modifying such figure to present other cases, including the general case. For instance, Fig.10 can be used by students to show that the commutativity of multiplication does exist for any two whole numbers. Among others, students can modify Fig.10 to show that, $2\cdot 5 = 5\cdot 2$, $3\cdot 5 = 5\cdot 3$, $3\cdot 7 = 7\cdot 3$, $4\cdot 7 = 7\cdot 4$, $5\cdot 7 = 7\cdot 5$, $5\cdot 10 = 10\cdot 5$, $5\cdot 100 = 100\cdot 5$, $5\cdot 104 = 104\cdot 5$ and $7\cdot 104 = 104\cdot 7$. Not only such cases can be discussed, but students can visualize a statement like $17\cdot 104 = 104\cdot 17$ before they learn how to perform in any of this statement's sides. This is an evidence of the isomorphism of our type of visualization with the relation it visualizes. In addition, we can modify this 2-dimensional iconic dynamic visualization to allow us to visualize special cases, in particular $0\cdot 5 = 5\cdot 0$ and $0\cdot 0 = 0\cdot 0$, as Fig.13 and Fig. 14 show.

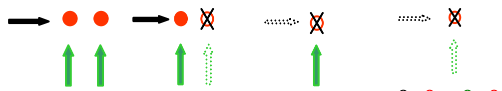
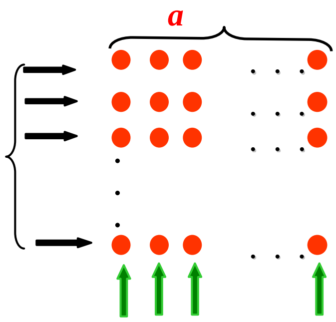


Fig.14. From $1\cdot 2 = 2\cdot 1$**, to** $0\cdot 0 = 0\cdot 0$

Isomorphic visualization and generalization

The most important is that, our visualization can be used for the general case of the commutativity of multiplication for whole numbers. The general case can be presented by the statement; $n\,a = a\,n$, where n and a are whole numbers. In Fig.15, because the number of red circles in a row is a and we have n rows, the total number of circles is na. As we have n rows, in a column we must have n circles. But, the number of such columns is a, because in a row we have a circles. Thus, the total number of circles is an. Now, as Fig.15 gives us the chance to visualize the general case, this figure can be modified to visualize the special cases $1\cdot 5 = 5\cdot 1$ and $0\cdot 5 = 5\cdot 0$, presented above (Fig.13), in the general form; $1\cdot a = a\cdot 1$, $0\cdot a = a\cdot 0$.



Visualization's isomorphism with the algebraic proof and the need for visualization in writing proofs

$$
\begin{aligned}
n\,a &= n\,(1+1+1+\ldots+1 \quad [a\ terms])\\
&= \; n+n+n+\ldots+n \quad [a\ terms]\\
&= a\,n
\end{aligned}
$$

Fig.16. A proof of distributivity

In Fig.16 we give a simple proof for the commutativity of multiplication for whole numbers. This proof is built on the distributivity of multiplication over addition for whole numbers. And, this distributivity can be proved easily, upon the associativity of addition for whole numbers. For this chain of proofs, we need to have this associativity as an axiom. In addition, both associativity and distributivity properties can be easily visualized. Regards the isomorphism of algebraic proof (Fig.16) with the visualization of the general case (Fig.15), we can notice that ones in the first line of the proof (Fig.16) match the red circles in the first row of Fig.15. In addition, n in the first line of the proof (Fig.16) corresponds the rows' number in Fig. 15. In the second line of the proof (Fig.16), n corresponds the number of circles in a column (Fig.15), while in the algebraic proof it is the result of multiplying 1's by n according to the distributivity of multiplication over addition. Without using distributivity, we can rely on the concept of multiplication to prove that:

$$
\begin{aligned}
&n\,(1+1+1+\ldots+1 \quad [a\ terms])\\
&= \quad 1+1+1+\ldots+1 \quad [a\ terms]+\\
&\qquad 1+1+1+\ldots+1 \quad [a\ terms]+\\
&\qquad 1+1+1+\ldots+1 \quad [a\ terms]+\\
&\qquad \vdots\\
&\qquad 1+1+1+\ldots+1 \quad [a\ terms]\\
&= \; n+n+n+\ldots+n \quad [a\ terms].
\end{aligned}
$$

Fig.17. Proof based on the concept of multiplication

$$n\,(1+1+1+\ldots+1 \quad [a\ terms]) = \; n+n+n+\ldots+n\,[a\ terms] \text{ (Fig.17).}$$

This last proof (Fig.17) shows how our visualization for the general case of the commutativity of multiplication for whole numbers (Fig.15) is not different than this proof. The n series in the proof (Fig.17) correspond the n rows in our visualization (Fig.15), and the ones in the proof (Fig.17) correspond the red circles (Fig.15).

This shows the need to develop visualization to become as much as possible isomorphic to the mathematical entity visualized. For young children, this gives them a chance to think mathematically and be ready to move to more abstract level, like the algebraic level. When we look at the last proof, we can notice that, bringing n's at the last line (Fig.17) is done visually from the ones of the columns over these n's, and this what we are doing in an isomorphic way in our visualization (Fig.15). Getting these n's of the proof (Fig.17) algebraically is possible, and this can make the algebraic proof rather closed to our visualization of the general case (Fig.15). For space reason, we leave this proof outside this paper. But, we have to mention that such proof can show how visualization is still needed in writing this proof. This means that visualization is not only needed for young children, but as well in studying mathematics and making mathematics.

Isomorphic visualization, multiplication by a fraction and unit fraction

The isomorphism between our 2-dimensional dynamic iconic visualization and the commutativity of multiplication for whole numbers is clearly strong, and therefore we can use it in a modified form to visualize the commutativity of multiplication of fractions. Fractions here are proper fractions, those less than one, and improper fractions, those equal or greater than one. Thus, whole numbers are also fractions. When we start to work with multiplication by fractions, different than whole numbers, we need to develop our reading of a product. For instance reading $\frac{1}{3}\cdot 6$ as "one third times 6" is difficult to understand, and therefore we need to modify our reading style as we in fact making enlargement for the concept of multiplication. Thus, instead of saying three times six, two times six, one times six and "one third 'times' six"; it is now a time to say three sixes, or three of sixes; two sixes, or two of sixes; one six; one third six or "one third 'of' six". Reading "one third of six" brings to light the relation between multiplication by a fraction and division by the reciprocal of this fraction. With this reading style, we can easy get the value of the product. For instance, one third of six is two as it is the quotient we get in dividing 6 by 3. Here to remember that in algebra we do not say 2 times a, but $2a$. One thing more, related to fractions and fractions multiplication, we have to here mention. This is the concept of unit fraction, which is a fraction with 1 as nominator. This concept wasn't known in Finnish schools, but our work has made a positive change towards understanding and using of it. For instance, in multiplying $\frac{3}{8}$ by 2 we underline the fact that we need to multiply 3 by 2, exactly as multiplying 3 ones by 2. The only difference is the unit of the product. In multiplying three eighths by two the product is not 6 ones, but 6 eighths, as the unit of the multiplicand is not of ones, but of eighths. The idea of "unit" has a wider role in our work. Regards arithmetic and decimal system, place value has a special meaning, as each digit's value depends on the unit of the place. Our experience shows that understanding of the unit idea brings meaning to children for their performance of skills and makes this performance easier. And, both achievements bring motivation for the study of mathematics.

Commutativity and visualization of the multiplication of a whole number by a unit fraction

Multiplying a whole number by $\frac{1}{1}$, as improper unit fraction, is possible to discuss. But, this is an easy and trivial case, and therefore we move to discuss a more substantial case, and take the expression $\frac{1}{5}\cdot {\color{red}3}$, where 3 is multiplied by a unit fraction $\frac{1}{5}$.

Fig.18. Drawing a rectangle

To get isomorphic visualization to the expression $\frac{1}{5}\cdot {\color{red}3}$, we start by writing this expression on the board, where I use a color, like red here, in writing the multiplicand.

Then I ask students: *What is the multiplicand we have in this expression?* – Three. *We now shall try to represent the multiplicand 3 by rectangles, are you ready?* –Yes. Which color, I have to use in drawing the rectangles? – Red. Then I draw a rectangle as in Fig.18, and ask students: *How many rectangles I have drawn?* – One. Yes, *how many of such rectangles I need to draw more?* – Two. Right, then I draw two more rectangles as in Fig.19. *Who can read the expression, written on the top of the board?* – "One fifth of three". Great, *who can use blue color and draw a straight line to find and present to us one fifth of three?* Such approach or a modified one has to lead to get a figure like the one of Fig.20. Then we again give students a challenge by saying: but, this figure in fact shows that $\frac{1}{5}\cdot 3 = 3\cdot\frac{1}{5}$. *Who can show us that this statement is true?* This discussion has to lead us to a figure with three arrows as in Fig.21.

Fig.19. Visualization of 3

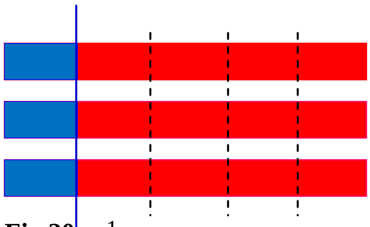


Fig.20. $\frac{1}{5}\cdot 3$

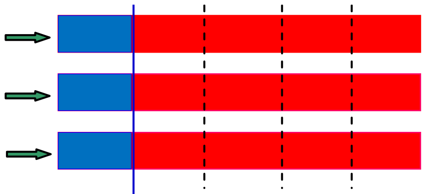


Fig.21. $\frac{1}{5}\cdot 3 = 3\cdot\frac{1}{5}$

The color used in writing the fraction $\frac{1}{5}$ is blue, and this is because we have agreed with students to color the part of the three rectangles, which represents one fifth of three, by the same color we give to the line we use in cutting the rectangles. We can make a story about the effect of the color of the line used. Speaking about using of stories in teaching mathematics, in multiplying 3 by one fifth we can use a story like 5 children in a party sharing equally three cakes of different taste. We can use also a type of functional materials to help those in need of working on hands, where they can cut by themselves strips of red colored papers.

Commutativity and visualization of the multiplication of a whole number by a fraction different than unit fraction

Let us here start with a proper fraction $\frac{2}{5}$ as multiplier and use the same multiplicand of the last case, i.e. 3. After dealing with the case of $\frac{1}{5}\cdot 3 = 3\cdot\frac{1}{5}$, it is not difficult to visualize $\frac{2}{5}\cdot 3 = 3\cdot\frac{2}{5}$, and this can be given to students as a problem to solve. Fig.22 presents such visualization. Here, we do not use colors in writing the statement. As we need to free students from using such colors. About such coloring use, on one hand, we can go back to use such type of coloring, when we find it necessary to recall ideas, and on the other hand colors can work in an imaginary form for both students and teachers in their discussions. In our case, when we do not use colors in writing statements, we can visualize the expression 'two fifth of three' using two colors, instead of one. In Fig.22 we have colored "one fifth of three" in blue and the other "one fifth of three" in yellow. The other choice is to color both in one color, like blue. Here to notice that, Fig.22 is modified from Fig.21, and this show that we can continue this process, to go from Fig.21 to any case of the multiplication of a whole number by a fraction, proper or improper. In addition, we can modify Fig.21 to visualize the general case of multiplying a whole number by a fraction, i.e. we can use a visualization to show that $\frac{m}{n}\cdot a = a\cdot\frac{m}{n}$, where *m, n* and a are whole numbers and $n \neq 0$. We can also investigate the isomorphism of the algebraic proof of this statement and the figure visualizes it.

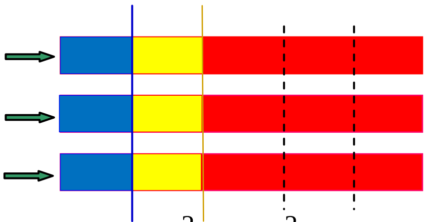


Fig.22. $\frac{2}{5}\cdot 3 = 3\cdot\frac{2}{5}$

Commutativity and visualization of the multiplication of a fraction by a fraction

Let us start with the visualization of the commutativity of multiplication of a unit fraction by a unit fraction, like $\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{3}\cdot\frac{1}{2}$. At first we visualize a whole by a rectangle (Fig.23), and then draw two segments to divide the rectangle into 3 congruent rectangles and color one in blue to visualize $\frac{1}{3}$ (Fig.24). To get 'half of one third' we draw a horizontal segment, in red to divide the blue rectangle into two congruent rectangles and color one in red, here the lowest. This 'half of one third' (Fig.25) is obviously 'one third of a half', as we can imagine, or here draw in a similar way a horizontal segment in the whole rectangle to divide it into two congruent rectangles and color the lowest in red to visualize half (Fig.26), then draw a vertical segment in blue to get 'one third of this half' (Fig.27).

Fig.23. Visualization of a whole

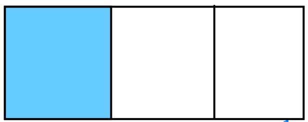


Fig.24. Visualization of $\frac{1}{3}$

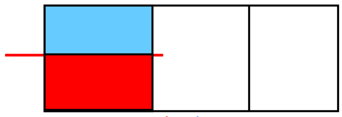


Fig.25. $\frac{1}{2}\cdot\frac{1}{3}$

Fig.26. Visualization of a $\frac{1}{2}$

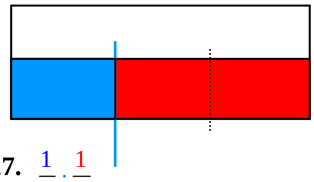


Fig.27. $\frac{1}{3}\cdot\frac{1}{2}$

Reflections

Last discussed visualization is for the commutativity of the multiplication of two unit fractions, but we can easy modify it for use for any two fractions and for the general case, where improper fractions are included. Algebraic proof can be provided to the statement of general case, and this proof can show how our iconic dynamic visualization is isomorphic to mathematics. Visualization can bring understanding of mathematics, but on the other hand making visualization needs understanding of mathematics. All our figures in this paper are 2-dimensional ones. Using of rectangles is possible to the case of whole numbers, and this use has a closed relation with the proof of rectangle's area statement. In our visualization above, parallelograms can replace rectangles. This is similar to the case of the 2-dimensional visualization for the multiplication of a whole number by another whole number, where we can use oblique dimensions.

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Assessing the teaching efficacy beliefs of teacher trainees

Dr Sheila N Matoti, Senior Lecturer, School of Teacher Education, Faculty of Humanities, Central University of Technology, Free State, Bloemfontein
smatoti@cut.ac.za

Dr Karen E Junqueira, Lecturer, School of Mathematics, Natural Sciences and Technology Education, Faculty of Education, University of the Free State, Bloemfontein
junquierake@ufs.ac.za

Abstract

Research shows that self-efficacy is an important concept which influences a teacher's ability to teach and the effectiveness with which the teaching is done. Each teacher trainee has a sense of efficacy with regards to teaching which is influenced by many factors. This study aimed to determine the teaching self-efficacy of third-year teacher education students in three categories: student engagement, instructional strategies and classroom management. A questionnaire was administered as the survey instrument and provided data which the researchers analysed and interpreted. It was found that at this stage of the student teachers' careers, that is, at the end of their third year of study, the student teachers responded with overwhelming positive self-efficacy beliefs with regard to their future occupation.

Introduction

A growing number of educational researchers are interested in relationships between teacher efficacy and other educational variables. For example, teachers' efficacy judgements have been correlated with decreased burnout (Brouwers & Tomic 2000), increased job satisfaction (Caprara, Barbaranelli, Borgogni, & Steca 2003), and commitment to teaching (Coladarci 1992). Ross (1998) reviewed 88 teacher efficacy studies and suggested that teachers with higher levels of efficacy are more likely to (1) learn and use new approaches and strategies for teaching, (2) use management techniques that enhance student autonomy and diminish student control, (3) provide special assistance to low achieving students, (4) build students' self-perceptions of their academic skills, (5) set attainable goals, and (6) persist in the face of student failure. This shows that there is a relationship between teaching efficacy and student academic performance.

Self-efficacy refers to the beliefs about one's capabilities to learn or perform behaviours at designated levels (Bandura 1997) and is said to have a measure of control over individual's thoughts, feelings and actions. The beliefs that individuals hold about their abilities and outcome of their efforts influence in great ways how they will behave. It is the realization of this relationship between individual beliefs and subsequent behaviours that prompted researchers' interest in self-efficacy. Self-efficacy has been applied in educational settings. The influence that self-efficacy has on motivation, learning and academic achievement has been investigated and reported (Pajares 1996; Schunk 1995). Self-efficacy has also been reported for individual subjects such as mathematics (Pajares & Miller 1994).

Furthermore, researchers have shown increasing interest in the teaching efficacy of prospective teachers. Student teaching or teaching practice is generally considered the most beneficial component of preparation by prospective and practising teachers as well as teacher educators (Borko & Mayfield 1995). It is during teaching practice that students develop a positive or a negative attitude towards teaching as a career, indicating that teaching practice can have both positive and negative influences. For example, poorly chosen placements result in feelings of inadequacy, low teacher efficacy and an unfavourable attitude towards teaching (Fallin & Royse 2000) whereas extensive and well-planned field experiences can help prospective teachers develop confidence, self-esteem and an enhanced awareness of the profession.

Theoretical framework

Self-efficacy is explained in the theoretical framework of social cognitive theory espoused by Bandura (1986, 1997) which states that human achievement depends on interactions between one's behaviours, personal factors and environmental conditions. The behaviour of an individual depends largely on early experiences at home. The home environment that stimulates curiosity will help build self-efficacy by displaying more of that curiosity, and exploring activities would invite active and positive reciprocity. This stimulation enhances the cognitive and affective structures of the individual which include his ability to sympathise, learn from others, plan alternative strategies and regulate his own behavior and engage in self-reflection (self-efficacy) (Mahyuddin, Elias, Cheong, Muhamad, Noordin & Abdullah 2006).

The self-regulating system referred to, affords an individual the capacity to alter his environment, and influences his subsequent performance. Therefore, the beliefs he has of himself are the key elements in exercising control and personal efficacy. This affects behaviour in two ways: either he engages in tasks he feels competent and confident in or he simply avoids those that he feels incompetent in. Self-efficacy helps to determine how much effort, perseverance and resilience are being put into a task. The higher an individual's sense of efficacy, the greater the effort, persistence and resilience that will be put into a task. Efficacy beliefs also trigger emotional reactions. Individuals with low self-efficacy and who believe that a task is tough can develop stress, depression and a narrow vision on how to solve the problems. On the other hand, those with high efficacy would be more relaxed in solving difficult tasks. Therefore, these influences are strong determinants of the individual's level of achievement.

The development of self-efficacy of prospective teachers is influenced by many factors such as mastery learning and vicarious experience. The positive and negative influences of self-efficacy on prospective teachers have been found to be context specific. It is against this background that this study sought to determine the self-efficacy beliefs of prospective teachers studying in the School of Teacher Education at a University of Technology with the aim of also determining the predictors of their teaching efficacy.

Method

This study used a descriptive survey research design. Efficacy beliefs of pre-service teachers were examined through a survey instrument administered at the end of the first semester of the third year of the programme. The participants in this study were third year B.Ed (FET) students in the School of Teacher Education at a University of Technology in South Africa. Students in all five B.Ed (FET) programmes offered by the School of Teacher Education participated in the study.

A questionnaire was used as the instrument to collect data from the respondents. The TSES included 24 items on a 5-point scale yielding three subscales: Efficacy for Classroom Management, Efficacy for Instructional Strategies, and Efficacy for Student Engagement. 136 students answered and returned the questionnaire. The questionnaires were issued to the students during class time to optimize participation. It was emphasized, however, that participation was not compulsory. The data obtained from the questionnaires was analysed by the researchers themselves. The information was then presented in tables from which written interpretations were made.

Results

Descriptive statistics provided a sample profile and summarized variables.

Table 1: Gender of respondents (N=136)

Gender	Frequency	%
Male	70	51.47
Female	66	48.53
Total	136	100

A good distribution of male and female respondents was obtained with close to 50% making up each group.

Table 2: Distribution of respondents per programme

Programme	Male	Female	Total	%
Natural Sciences (NS)	36	25	61	45
Economic and Management Sciences (EMS)	15	24	39	29
Technology	13	1	14	10
Languages	1	11	12	9
Computer Science	5	5	10	7
Total	70	66	136	100

The distribution of respondents per programme was not even with nearly half of the respondents in the Natural Sciences programme and nearly a third of the respondents in the Economic and Management Sciences programme, while the Technology, Languages and Computer Sciences programmes together made up the other 26% of respondents. This distribution is due to the number of students that registered and which can be accommodated in the different programmes in the School of Teacher Education. Fortunately, it does not influence the validity of the data as no comparison between the respondents in the different programmes was made.

Table 3: Teaching Efficacy Beliefs of the respondents (N=136)

Questions	Mean	SD
Q1	3.881	0.993
Q2	4.326	0.854
Q3	4.378	0.854
Q4	4.474	0.905
Q5	4.170	0.966
Q6	4.459	0.689
Q7	3.970	0.810
Q8	4.000	0.914
Q9	4.341	0.774
Q10	4.230	0.897
Q11	4.133	0.991
Q12	3.793	1.037
Q13	4.296	0.856
Q14	4.459	0.655
Q15	4.170	0.877
Q16	4.022	0.902

Q17	4.015	1.007
Q18	4.096	0.800
Q19	4.193	0.877
Q20	4.400	0.794
Q21	3.911	0.966
Q22	3.815	1.038
Q23	4.126	0.814
Q24	4.222	0.870
Average	4.162	0.881

Respondents were requested to indicate their opinion about what they would do to deal with teaching situations presented to the respondents as questionnaire statements and indicated in Table 3 as Q1 to Q24. A scale from 1 to 5 was presented with 1 representing “Nothing”, 2 representing “Very Little”, 3 representing “Some Influence”, 4 representing “Quite a Bit” and 5 representing “A Great Deal”.

On the whole it seems as if the teacher trainees felt that they could have a very big influence on the learners’ learning as the average of the means of all the questions is 4.162 out of a possible 5. They are therefore 83.24% percent certain that they can have a positive influence on their learners. The question in which the respondents scored highest was Q4: How much can you do to motivate learners who show low interest in school work? A mean of 4.474 was achieved in this question. The teacher trainees are therefore 89.48% certain that they will be able to motivate their learners to work harder once they start to teach full-time. The question in which the respondents scored lowest was Q12: How much can you do to foster learner creativity? A mean of 3.793 was achieved in this question. The teacher trainees are therefore 75.86% certain that they can develop their learners’ creativity.

The lowest standard deviation of 0.655 was obtained in Q14: How much can you do to improve the content understanding of a learner who is failing, with a mean of 4.459, while the greatest standard deviation of 1.038 was obtained in Q22: How much can you assist families in helping their children do well in school, with a mean of 3.815. The high mean of 4.459 and relatively low standard deviation of 0.655 in Q14, mean that more student trainees feel that they can improve the content understanding of a learner than the number of trainees who feel that they can assist families in helping their children do well in school, as Q22 has a lower mean of 3.815 and a relatively higher standard deviation of 1.038.

Table 4: Teaching Efficacy Beliefs regarding student engagement

Questions	**Mean**	**SD**
Q1	3.881	0.993
Q2	4.326	0.854
Q4	4.474	0.905
Q6	4.459	0.689
Q9	4.341	0.774
Q12	3.793	1.037
Q14	4.459	0.655
Q22	3.815	1.038
Average	4.194	0.868

The average of the means and standard deviations determined with regard to questions on student engagement produced scores of 4.194 and 0.868 respectively.

Table 5: Teaching Efficacy beliefs regarding instructional strategies

Questions	Mean	SD
Q7	3.970	0.810
Q10	4.230	0.897
Q11	4.133	0.991
Q17	4.015	1.007
Q18	4.096	0.800
Q20	4.400	0.794
Q23	4.126	0.814
Q24	4.222	0.870
Average	4.149	0.873

The average of the means and standard deviations determined with regard to questions on instructional strategies produced scores of 4.149 and 0.873 respectively.

Table 6: Teaching Efficacy beliefs regarding classroom management

Questions	Mean	SD
Q3	4.378	0.854
Q5	4.170	0.966
Q8	4.000	0.914
Q13	4.296	0.856
Q15	4.170	0.877
Q16	4.022	0.902
Q19	4.193	0.877
Q21	3.911	0.966
Average	4.143	0.901

The average of the means and standard deviations determined with regard to questions on classroom management produced scores of 4.143 and 0.901 respectively.

Herewith a summary of the information provided in Tables 4, 5 and 6.

Table 7: Summary Table of Teaching Efficacy Beliefs

Category	Mean	SD
Student engagement	4.194	0.868
Instructional strategies	4.149	0.873
Classroom management	4.143	0.901
Overall Teaching Efficacy	4.162	0.881

Clearly, the teacher trainees' efficacy beliefs with regard to the three sub-scales do not differ much. The difference in mean scores between Instructional strategies and Classroom management is a mere 0.006, the difference in mean scores between Instructional strategies and Student engagement is 0.045 and the difference in mean scores between Student engagement and Classroom management is 0.051. The greatest difference in means between the sub-scales is therefore 1.216%, while the smallest difference in means between the sub-scales is a mere 0.145%. The standard deviations of the three sub-scales show a similar pattern.

Conclusion

This paper reported on the first half of a study into the self-efficacy beliefs of third-year teacher trainees studying in the School of Teacher Education at the Central University of Technology, in the Free State Province of South Africa. A questionnaire was administered to the respondents to determine their self-efficacy beliefs with regard to teaching at this stage of their careers. The students responded with overwhelming positive self-efficacy beliefs with regard to their future occupation. A follow-up questionnaire will be administered to these same students after they have completed a six-month work-integrated learning experience during the first six months of 2011. Their self-efficacy beliefs will then once again be determined and changes in their beliefs are predicted.

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On Economic Interpretation of Lagrange Multipliers

Ivan Mezník

Professor of Mathematics, Faculty of Business and Management, Brno University of Technology, Brno, Czech Republic, meznik@fbm.vutbr.cz

Abstract

Lagrange multipliers play a standard role in constraint extrema problems of functions of more variables. In teaching of engineering mathematics they are readily presented as quantities of formal type in the algorithm for finding of constraint extrema. The paper points to important interpretations Lagrange multipliers in optimization tasks in economics

Introduction

The *Lagrange multipliers method* is one of methods for solving constrained extrema problems. Instead of rigorous presentation we point to the rationale of this method. Recall that for a function f of n variables the necessary condition for local extrema is that at the point of extrema all partial derivatives (supposing they exist) must be zero. There are therefore n equations in n unknowns (the $x's$), that may be solved to find the potential extrema point (called *critical* point). When the $x's$ are constrained, there is (at least one) additional equation (constraimt) but no additional variables, so that the set of equations is overdetermined. Hence the method introduces an additional variable (the Lagrange multiplier), that enables to solve the problem. More specifically (we may restrict to finding of maxima), suppose we wish to find values $x_1,...,x_n$ maximizing

$$y = f(x_1,...,x_n)$$

subject to a constraint that permits only some values of the $x's$. That constraint is expressed in the form

$$g(x_1,...,x_n).$$

The Lagrange multipliers method is based on setting up the new function (the *Lagrange function*)

$$L(x_1,...,x_n,\lambda) = f(x_1,...,x_n) + \lambda g(x_1,...,x_n), \tag{1}$$

where λ is an additional variable called the *Lagrange multiplier*. From (1) the conditions for a critical point are

$$\begin{aligned} L'_{x_1} &= f'_{x_1} + \lambda g'_{x_1} \\ &\vdots \\ L'_{x_n} &= f'_{x_n} + \lambda g'_{x_n} \\ L'_{\lambda} &= g(x_1,...,x_n), \end{aligned} \tag{2}$$

where the symbols L', g' are to denote partial derivatives with respect to the variables listed in the indices. Of course, equations (2) are only necessary conditions for a local maximum. To confirm that the calculated result is indeed a local maximum second order conditions must be verified. Practically, in all current economic problems there is on economic grounds only a single local maximum.

In a standard course of engineering mathematics the Lagrange multiplier is usually presented as a clever mathematical tool („trick“) to reach the wanted solution. There is no large spectrum of sensible examples (mostly a limited number of simple “well-tried” school examples) to show convincingly the power of the method. Economic interpretation of the Lagrangian multiplier provides a strong stimulus to strengthen its importance. This will be central to our next considerations.

Economic interpretations

In the sequel we will examine two useful interpretations $1^0, 2^0$ of the Lagrange multipliers.

1^0 Rearrange the first n equations in (2) as

$$\frac{f'_{x_1}}{-g'_{x_1}} = ... = \frac{f'_{x_n}}{-g_{x_n}} = \lambda. \tag{3}$$

Equations(3) say that at maximum point the ratio of f'_{x_i} to g'_{x_i} is the same for every x_i and moreover it equals λ. The numerators f'_{x_i} give the **marginal contribution** (or **benefit**) of each x_i to the function f to be maximized, in other words they give the approximate change in f due to a one unit change in x_i. Similarly, the denominators have a marginal cost interpretation, namely, $-g'_{x_i}$ gives the **marginal cost** of using x_i (or marginal "taking" from g), in other words the approximate change in g due to a unit change in x_i. In the light of this we may summarize, that λ is the *common benefit-cost ratio* for all the $x's$, ie.

$$\lambda = \frac{marginal}{marginal}\ \frac{benefit}{cost}\ \frac{of}{of}\ \frac{x_i}{y_i} = \frac{f'_{xi}}{-g'_{x_i}}. \tag{4}$$

<u>Example</u> A farmer has a given length of fence, F, and wishes to enclose the largest possible rectangular area. The question is about the shape of this area. To solve it, let x, y be lengths of sides of the rectangle. The problem is to find x and y maximizing the area $S(x, y) = xy$ of the field, subject to the condition (constraint) that the perimeter is fixed at $F = 2x + 2y$. This is obviously a problem in constraint maximization. We put $f(x, y) = S(x, y)$, $g(x, y) = F - 2x - 2y$ and set up the Lagrange function (1)

$$L(x, y, \lambda) = xy + \lambda(F - 2x - 2y). \tag{5}$$

Conditions (2) are $L'_x = y - 2\lambda, L'_y = x - 2\lambda, L'_\lambda = F - 2x - 2y$. These three equations must be solved. The first two equations give $x = y = 2\lambda$, i.e. x must be equal to y and due to (5) they should be chosen so that the ratio of marginal benefits to marginal cost is the same for both variables. The benefit (in terms of area) of one more unit of x is due to (4) given by $S'_x = y$, (area is increased by y), and the marginal cost (in terms of perimeter) is $-g'_x = 2$ (from the available perimeter is taken 2 for both variables are increased by the same length 1). As mentioned above, the conditions (4) state that this ratio must be equal for each of the variables. Completing the solution we get $x = y = \frac{F}{4}, \lambda = \frac{F}{8}$. Now let us discuss the interpretation of λ. If the farmer wants to know, how much more field could be enclosed by adding an extra unit of the length of fence, the Lagrange multiplier provides the answer $\frac{F}{8}$ (approximately), i.e. the present perimeter should be divided by 8. For instance, let 400 be a current perimeter of the fence. With a view to our solution, the optimal field will be a square with sides of lengths $\frac{F}{4} = 100$ and the enclosed area will be 10 000 square units. Now if perimeter were enlarged by one unit, the value $\lambda = \frac{F}{8} = \frac{400}{8} = 50$ estimates the increase of the total area. Calculating the "exact" increase of the total area, we get: the perimeter is now 401, each side of the square will be $\frac{401}{4}$, the total area of the field is $(\frac{401}{4})^2 = 10050{,}06$

square units. Hence, the prediction of 50 square units given by the Lagrange multiplier proves to be sufficiently close.

2^0 Rewrite the condition $g(x,y)=0$ in (1) as $c(x,y)=k,$ where k is a parameter. Then for the partial derivative of the Lagrange function with respect to k we get $L'_k=-\lambda.$ From the interpretation of a partial derivative we conclude, that the value $-\lambda$ states the approximate change in L(and also f) due to a unit change of k. Hence the value $-\lambda$ of the multiplier showss the approximate change that occurs in f at the point of its maxima in response to the change of k by one in the condition $c(x,y)=k.$ Since usually $c(x,y)=k$ means economic restrictions imposed (budget, cost, production limitation), the value of multiplier indicates so called the **oportunity cost** (of this constraint). If we could reduce the restriction (i.e. increase k) then the extra cost is $-\lambda$. If we are able to realize an extra unit of output under the cost less than $-\lambda$, then it represents the benefit due to the increase of the value at the point of maxima. Clearly to the economic decision maker such information on opportunity costs is of considerable importance.

<u>Example</u> The profit for some firm is given by $PR(x,y)=-100+80x-0{,}1x^2+100y-0{,}2y^2,$ where x,y represent the levels of output of two products produced by the firm. Let us further assume that the firm knows its maximum combined feasible production to be 325. It represents the constraint $x+y=325.$ Putting $g(x,y)=x+y-325=0$ we set up the Lagrange function $L(x,y,\lambda)=-100+80x-0{,}1x^2+100y-0{,}2y^2+\lambda(x+y-325).$ Applying Lagrange multipliers method we get the solution $x=183{,}335, y=141{,}667, \lambda=-43{,}333$ with the corresponding value of the profit $PR(183{,}335;141{,}667)=21358{,}420.$ Now we reduce the restriction altering the constraint equation to $x+y=326$. Finding the new solution as before we have $x=184, y=142, PR(184{,}142)=21401{,}6.$ We see that the increase in profit brought about by increasing the constraint restriction by 1 unit has been $43{,}18$ - approximately the same as the value $-\lambda$ that we derived in the original formulation. It indicates that the additional increase of labour and capital in order to increase the production has the opportunity cost approximately $43\ ,3$.

Conclusion

In instructing of engineering mathematics the Lagrange multipliers method is mostly applied in cases when the constraint condition $g(x,y)=0$ can not be uniquely expressed explicitly as the function $y=f(x)$ or $x=h(y)$. When solving constraint extrema problems in economics the bulk of constraint conditions may be expressed explicitly, so the reason to use the Lagrange multipliers method would seem to be too sophisticated regardless of its theoretical aspects. With a view to the crucial importance of the economic interpretations of Lagrange multipliers is the use of the method primarily preferred. Concrete applications of the presented interpretation principle may be developed in many economic processes. More deeper study on the role of the Langrange multipliers in optimization tasks may be found in Rockafellar (1993).

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Dreams, Paradigm Shifts and Reforms in Mathematics Education: Classification and Plan of Action

Fayez M. Mina, MA PhD C. Math FIMA, Emeritus Professor, Faculty of Education
Ain Shams University, Roxy, Heliopolis, Cairo, Egypt, fmmina@link.com.eg

The present paper is concerned with the classification of possible areas of change in mathematics education into dreams, paradigm shifts and reform in order to set up an appropriate plan of action to deal with each. Dreams include integration and non-formal teaching. Paradigm shifts include: developing creativity, interests of students, self education, changing methods of teaching and evaluation, considering complexity and the role of "teacher", while reforms include paying more attention to application of mathematics, the use of technology, and educational activities, and concentrating on mathematical concepts and the "points of departure" in mathematics. A special section is devoted for the suggested plan of action in each of the classified areas of change. However, changing the mentality of the concerned people - especially teachers - is needed in all cases as well as consistency among all components of the educational system and curricula. The studied changes are applicable in almost all countries, may be in different manner and different time.

Introduction

The present paper is concerned with the classification of possible areas of change in mathematics education into dreams, paradigm shifts and reforms [1] in order to set up an appropriate plan of action to deal with each. The major criterion of classification is the degree of applicability of these changes. Dreams seems to be in one stream - not applicable at the moment, while paradigm shifts are applicable under certain conditions and reforms - in the other stream- are ready to be applied after taking necessary decisions and procedures. However, the position of those changes is changeable according to many factors, with time and availability of encouraging environment to come first as major factors.

Classification of possible areas of change [2]

Dreams

1- Teaching mathematics in integrated contexts to the extend to which it could be said that there is no mathematics education as such [3].

2- School materials will be presented in the form of integrated activities through knowledgeable projects.

3- There will be no "traditional" formal teaching of mathematics. Students will "theorize" for themselves.

4- Higher education, especially the study for the first degree, will follow the same pattern, with more direction to the field of study and research assignments (instead of projects).

Paradigm shifts

1- The major aims of teaching mathematics are to develop creativity, to make the study enjoyable and to prepare students to deal with future changes (both in knowledge and jobs).

2- The whole school system will be based upon multiple intelligences theory [4].

3- Adopting problem-solving and "research problems" as dominating methods of teaching mathematics.

4- Complexity is considered in all educational activities[5]. Emphasis must be given to "commons" among different systems to state assumptions underlying different formulas and the existence of different possible solutions.

5- Evaluation of students is mainly based on continuous and non-formal evaluation. Great attention will be given to self evaluation and discussion of student's reports and "research work".

6- The major job of a teacher is as facilitator.

Reforms

1- Studying applications of mathematics in other disciplines and in life, as an essential part of school mathematics.

2- Concentrating on conceptual bases with very little attention to computations, with the use of calculators and computers.

3- Intensive use of technology, with emphasis on data collection, building knowledge and self-learning.

4- Studying the history of mathematics, with particular emphasis on the "departure points" (the cultural historical approach).

5- Paying more attention to school activities relevant to mathematics education.

Suggested plan of action

1- Starting with reforms and attempting some "paradigm shifts".

2- As for reforms:

   a) Plan to implement reforms, e.g. prepare necessary equipments, conducting in-service teacher education programmes, changing current pre-service teacher education programmes- when needed … etc.

   b) Studying past experiences and considering lessons from them. For instance, some good applications of mathematics are included in some text books[6], but ignored by teachers, may be because they are not included in examinations.

   c) Participation of teachers in all procedures leading to reform.

3- As for adopting paradigm shifts, the following procedures can be taken:

   a) Convincing teachers and the public opinion - in general, with the value of these paradigm shifts and explaining the way to implement them.

   b) Consistency of all components of the whole educational system as well as those of mathematics curricula including aims, content, methods of teaching, using technology, educational activities and evaluation.

   c) Priorities for change are for: text-books, teaching methods, means and tools of evaluation and school activities.

4- As regards dreams, many steps can be taken, such as:

   a) Paving the way to change the mentality of planners, text-book writers, administrators, teachers, parents and students for specific changes, particularly integration.

b) Changing the whole system of both pre-service and in-service teacher education.

c) Encouraging attempts to integrate branches within a subject and among some subjects.

A Final Word

The author would like to confirm the following:

a) The previous classification is flexible and some of its items are interacted and interrelated.

b) Many of these previous elements- especially reforms- are applied in some countries.

c) The mentioned areas of change are applicable in almost all countries, may be in different manner and different time.

Notes

(1) Although the author can not deal with all the terms involved within the available space, he would like to explain that he means by paradigm shift "a revolution due to a fundamental change in our world view which changes even the way reality is perceived and understand".

See: Kuhn, Thomas S. (1972). **The Structure of Scientific Revolutions** London: Phoenix.

Quoted from: Rugerson, Alan (2010). "The DQME Project as Part of a World-Wide Paradigm Shift In Mathematics Education", **A Background Paper Presented to the DQME3 Meeting**, June 29 - July 2, 2010, Ciechocinek (Poland).

(2) For sources of these areas of change, the writer reviewed his papers which were presented to ME21 Conferences (1999, 2000, 2003, 2004 and 2005), in addition to the following paper:

Mina, Fayez M. (2010). "Some Suggested Alternatives to Activate Some New Trends in Mathematics Education", **A Paper Presented to the Conference of the Egyptian Society of Mathematics Education**, Cairo, 3rd August, 2010. (In Arabic).

(3) It seems that the only attempt to integrate mathematics with life is done by MISP in rather primary education. See:

Gamble, Andy and Rogerson, Alan (1997). **Making Links**. New Zealand: User Friendly Resource Enterprises.

Rogerson, Alan (1995). **Playing Cards**, and **Building Bridges**. (The same publisher).

Rogerson, Alan (1998). **Divine Designs;** and **Patterns & Tiling** (The same publisher).

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(4) See:

Gardner, Howard (1983). **Frames of Mind: The Theory of Multiple Intelligences**. New York: Basic Books.

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(5) Complexity can be described in terms of the following terms:

1- There is no more simple and absolute laws controlling motion and globe.
2- Unity of human knowledge.
3- Research is no more neutral.
4- Thought is no more controlled by logic, and knowledge is no more certain.
5- It is suggested that the main goal of science is to understand realty with the intention to influence and change it.
6- Cohesion of knowledge and its technological applications.
7- The development in technologies of communication, measurement and its units and scientific calculations.

See:

Mina, Fayez M. (2003). **Issues in Curricula of Education**. Cairo the Anglo-Egyptian Bookshop, pp. 23 - 26. (In Arabic).

(6) For Example: See the series of mathematics text-books MATH POWER™ from 7th to 12th Grades, published by McGraw-Hill Ryerson Limited in Canada.

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An Initial Examination of Effect Sizes for Virtual Manipulatives and Other Instructional Treatments

Patricia S. Moyer-Packenham, PhD
Professor, Mathematics Education
Utah State University
patricia.moyer-packenham@usu.edu
Arla Westenskow
Doctoral Candidate, Mathematics Education
Utah State University
Arlawestenskow@gmail.com

Abstract
This paper is a meta-analysis comparing the use of virtual manipulatives with other instructional treatments. Comparisons were made using Cohen *d* effect size scores, scores which report treatment effect magnitude but are independent of sample size. Findings from 29 research reports yielded 79 effect size scores. Effect size scores were grouped and averaged to determine overall effects comparing use of virtual manipulatives alone, and in combination with physical manipulatives, to other instructional treatments. Results yielded moderate effects when virtual manipulatives were compared to all other instructional methods combined, large effects when compared to traditional instruction with textbooks only, and small effects when compared to instruction using physical manipulatives only. Combining physical and virtual manipulatives and comparing this treatment with other instructional methods resulted in moderate effect sizes for all comparisons.

Virtual Manipulatives and Mathematics Learning

Virtual manipulatives are "computer-based renditions of common mathematics manipulatives and tools" (Dorward, 2002, p.330). Moyer, Bolyard and Spikell (2002) define them as "an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (p. 373). Virtual objects such as pattern blocks, base -10 blocks, tangrams, and geoboards can be found as internet applets or as small stand-alone application programs. Many virtual manipulatives include features designed to focus the attention of learners by highlighting and enforcing mathematical concepts which support children's integrated-concrete knowledge (Dorward & Heal, 1999; Sarama & Clements, 2009). In the past two decades since the emergence of virtual manipulatives, there have been a number of research studies documenting the effects of virtual manipulatives as a mathematics instructional treatment. This meta-analysis synthesizes the research on this treatment by calculating averaged effects of virtual manipulatives on student achievement when compared with other instructional treatments.

Learning Mathematics with Virtual Manipulatives

Several constructs support the use of virtual manipulatives for learning mathematics concepts. *Representational fluency* is defined as a student's ability to transfer ideas easily from one representation to another, a skill which some researchers suggest can be strengthened through the use of technology by providing students with greater access to multiple and dynamic mathematical representations (Zbiek, Heid, Blume, & Dick, 2007). Virtual manipulatives develop representational fluency by linking symbolic, pictorial and concrete representations (e.g., placing a "90^{o}" beside a picture of a right angle); and by linking different types of representational models (e.g., a number line model showing ½ and a region model showing ½). By interacting with dynamic objects, students using virtual manipulatives learn to define, solve and prove mathematical problems by observing connections between their actions and the virtual objects (Durmus & Karakirik, 2006). When virtual manipulatives focus on linking representations, this can influence students' selection of problem solving methods. Manches, O'Malley and Benford (2010) observed that, during partitioning activities, students using virtual manipulatives used more compensation strategies while students using physical manipulatives used more commutative strategies.

Another construct which plays an important role in student learning is the *fidelity* of the technology tools (Zbiek et al., 2007). The degree of alignment between a tool and the mathematical properties is a measure of a tools' *mathematical fidelity*. The degree to which the tools reflect the users' thought processes is defined as *cognitive fidelity*. Seeing visually the consequences of their actions on virtual objects provides students with visual feedback as they test and prove new understandings. When using technology, cognitive fidelity can be even further enhanced as the user's actions are both represented and constrained, making the mathematical properties and relationships even more explicit for learners (Durmus & Karakirik, 2006). There are large collections of virtual manipulatives on the internet with resources linked to national mathematics standards (e. g., National Library of Virtual Manipulatives, http://nlvm.usu.edu; National Council of Teachers of Mathematics Illuminations http://illuminations.nctm.org; and Shodor Curriculum Materials http://shodor.com/curriculum/).

Research Questions

The purpose of this meta-analysis was to conduct an initial examination of effect sizes for virtual manipulatives when compared with other instructional treatments. Two research questions guided the analysis: 1) What are the effects of virtual manipulatives as an instructional treatment in mathematics on gains in student achievement? 2) What are the effects of virtual manipulatives as an instructional treatment in studies of differing durations?

Methods

The study used quantitative methods for a meta-analysis examining the effect sizes of multiple studies. Effect size scores were calculated and used to answer the research questions.

Data Sources

Following the search procedures and standard criteria outlined by Boote and Beile (2005), we conducted a comprehensive search of databases. These included electronic and manual library searches in educational and international databases such as ERIC, PsycInfo, Dissertation Abstracts, Web of Science, Google Scholar, and Social Sciences Index using search terms such as: virtual manipulatives, dynamic manipulatives, computer manipulatives, virtual tools, mathematics manipulatives, mathematics tools, technology tools, computer tools, mathematics applets, and computer applets. In addition to uncovering research on virtual manipulatives, the search located a large body of research focusing specifically on commercially developed dynamic geometry software (e.g., Geometer's Sketchpad, Cabri, and GeoGebra), which is a separate line of inquiry and beyond the scope of this analysis.

Criteria for Inclusion in the Meta-Analysis

From a collection of 135 publications discussing the use of virtual manipulatives, 74 articles and dissertations were identified as empirical research studies, 66 of which had been peer reviewed. The other 61 articles were papers expressing opinions, developing theories, or suggestions for instruction. To build a comprehensive base of studies, only three criteria were used to remove studies from the empirical pool originally identified. Sixteen studies were excluded because of study design, type of applet, and threats to validity such as history, mortality, instrumentation, testing, selection, regression, and maturation (Gall, Gall & Borg, 2003). A final total of 58 studies met our criteria for further review, and 29 of these studies contained effect sizes.

Analysis

Within the 29 studies, there were 79 effect score cases comparing virtual manipulatives with other instructional treatments. Effect size scores are used to report the magnitude of treatment effects and are independent of sample sizes, thus making the comparison of studies across multiple settings possible. Effect size scores were computed using gain scores, differences between post test scores, and *F* values used to calculate Cohen's *d* scores. The 79 effect score cases were grouped to obtain averaged effect size scores.

Results

The following results report comparisons between virtual manipulatives as an instructional treatment, a) with all other instructional treatments, b) with instruction using physical manipulatives only, and c) with traditional instruction using textbooks only. The effect sizes reported for each of the comparisons of the analysis are averages of 79 case effect sizes yielded from the 29 studies. Descriptions of effect sizes are based on the suggestions of Urdan (2010) that an effect size of less than 0.20 be considered small, effect sizes in the range of .25 to .75 are considered moderate, and those over .80 are considered large.

Effects of Virtual Manipulatives as an Instructional Treatment

The first research question focused on the effects of virtual manipulatives as an instructional treatment in mathematics on gains in student achievement. The following comparisons are presented in Table 1: a) all instruction using virtual manipulatives compared with all other methods of instruction; b) instruction in which only virtual manipulatives were used compared with all other methods of instruction, with instruction using physical manipulatives, and with traditional instruction using textbooks; and, c) instruction in which virtual manipulatives and physical manipulatives were combined as a treatment compared with all other methods of instruction, with virtual manipulatives used alone, with physical manipulatives used alone, and with traditional instruction using textbooks.

Table 1
Effect Size Scores for Virtual Manipulatives Compared with Other Treatments

Comparisons	Number of Comparisons	Effect Size
Virtual Manipulatives Used		
& Other Instructional Treatments	70	0.37 (0.44)*
Virtual Manipulatives Used Alone		
& Other Instructional Treatments (combined)	53	0.37 (0.46)*
Physical Manipulatives	35	0.18 (0.32)*
Traditional Instruction (textbook)	18	0.73
Virtual and Physical Manipulatives Used Together		
& Other Instructional Treatments (combined)	26	0.33
Virtual Manipulatives	9	0.26
Physical Manipulatives	11	0.20
Traditional Instruction (textbook)	6	0.69

Note: *Effect size with one outlier.

The comparison of all studies using virtual manipulatives for instruction with all other instructional treatments yielded a moderate averaged effect score (0.37; 0.44, with one outlier). The analysis of the 53 cases comparing instruction using only virtual manipulatives with other instructional treatments also yielded a moderate effect (0.37/0.46 with one outlier). An analysis of the 35 cases comparing instruction using only virtual manipulatives to instruction using physical manipulatives yielded a small/moderate effect (0.18/0.32 with one outlier); and the analysis of 18 cases comparing instruction using only virtual manipulatives to classroom instruction using textbooks yielded a moderate effect (0.73).

In the analysis of studies where virtual manipulatives were combined with physical manipulatives (VM/PM combined) for instruction and compared with other instructional methods, 26 cases yielded a moderate effect (0.33). In the comparison of VM/PM combined with the use of virtual manipulatives alone, nine scores yielded a moderate effect (0.26). VM/PM combined compared with physical manipulatives alone in 11 scores produced a small effect (0.20). Finally when VM/PM combined was compared with classroom instruction using textbooks this produced a moderate effect (0.69). In summary, the largest averaged effect scores for the virtual manipulatives were produced when comparisons were made between virtual manipulatives and classroom instruction using textbooks. Other comparisons produced moderate or small averaged effects. Overall the effect size results demonstrated that virtual manipulatives produced positive averaged effects on student achievement when they were used as an instructional treatment for mathematics teaching.

Effects of Virtual Manipulatives Based on Treatment Duration

The second research question focused on the effects of virtual manipulatives as an instructional treatment in studies of differing durations. This analysis examined the length of the instructional treatments when virtual manipulatives were used for instruction. Length of treatment categories were aggregated by days and number of effect size scores per category. The five categories for the analysis were 1 day, 2 days, 3-5 days, 6-10 days, and more than 10 days. These results are reported in Table 2.

Table 2

Effect Size Scores for Virtual Manipulatives by Length of Treatment

Length of Treatment	Number of Comparisons	Effect Sizes
1 day	10	0.13
2 days	5	0.36
3-5 days	12	0.21
6-10 days	10	0.48
More than 10 days	31	0.47 (0.62)*

Note: *Effect size with one outlier.

In approximately half of the comparisons students participated in instruction involving virtual manipulatives for durations longer than ten days. The shortest length of treatment (1 day) yielded the smallest averaged effect size score (0.13) when comparing instruction using virtual manipulatives to other methods of instruction. Treatments of 2, 6-10, and more than 10 days of virtual manipulative treatment, all yielded moderate average effect size scores (0.36, 0.48, and 0.47/0.62, respectively). The results of the comparisons indicate that studies of longer durations tend to report larger effect sizes while studies in which virtual manipulatives are used for shorter durations tend to report smaller effect sizes.

Discussion & Conclusions

The purpose of this study was to use a meta-analysis to synthesize the quantitative results from research on virtual manipulatives. From the meta-analysis, there are several patterns that emerged. Overall, the virtual manipulatives are an effective instructional treatment for teaching mathematics when compared with other instructional methods. It is also interesting to note that the average effect size scores for virtual manipulatives compared with traditional instruction using textbooks are larger than when comparing virtual manipulatives with physical manipulatives. However, combining virtual and physical manipulatives as a treatment compared with traditional instruction using textbooks resulted in some of the largest effects produced in this study. These results suggest that the virtual manipulatives have unique affordances that have a positive impact on student achievement in the learning of mathematics. The results also suggest that combining virtual and physical manipulatives for instruction provides students with representations available in each manipulative type that are a visual support for students and promote students' representational fluency.

Results of the meta-analysis also suggest that the length of treatment for virtual manipulatives influences the average effect size scores. This result is similar to other studies on instructional treatments showing that longer treatment durations provide more opportunity for the effects of a treatment to be determined through research. This result makes sense, particularly since there are various factors in the virtual manipulative environment which may be new to students, such as finding webpages or manipulating dynamic objects, and these activities take time for students to learn so that they can interact effectively with the virtual manipulatives.

Although averaged effect sizes indicate that virtual manipulatives are as effective, and may even be more effective tools of instruction than other methods, little is known about how learner characteristics, applet features or instructional methods affect student learning while using virtual manipulatives. Additional research is needed to determine if the use of virtual manipulatives as an instructional tool is more effective for some students than others. Although it has been suggested that virtual manipulatives could be successfully used in both gifted and intervention instruction, to date, there are limited research studies investigating differences in virtual manipulative use as related to student abilities. There is also great variability in applet features, structures and the amount of guidance they provide. To further enhance the use of virtual manipulative applets, research is needed which compares the effects of different applet characteristics on student learning and compares which applets are most effective for teaching which specific concepts within each mathematical domain. For example, Haistings (2009) indentified variations in learning when students used the same virtual manipulative applet with and without symbolic linking; and, Bolyard (2006) compared the effects of two different virtual manipulative applets for integer instruction. These types of investigations may help researchers to identify relationships between applet features and impacts on student learning and achievement.

This meta-analysis found that virtual manipulatives have a moderate average effect on student achievement when compared with other methods of instruction, and that larger effect size scores are produced when studies have longer treatment durations. While these results confirm the effectiveness of virtual manipulatives for mathematics instruction, they do not reveal *why* virtual manipulatives are effective. Further research on specific affordances that promote learning, effects for different mathematical domains, and implications of virtual manipulative use for different students will significantly contribute to our understanding of the features that make virtual manipulatives effective and will answer the question of why virtual manipulatives impact student achievement during mathematics instruction.

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New and Emerging Applications of Tablet Computers such as iPad in Mathematics and Science Education.

MEHRYAR NOORIAFSHAR
mehryar@usq.edu.au

University of Southern Queensland, Toowoomba, Australia

Abstract

This research paves the way for work and development in adopting the latest technologies in tablet computing in learning and teaching mathematics and science related topics. In particular, the more human-like interface features, offered by Apple's iPad and other touch devices is being investigated for educational development. Preliminary studies by the author have demonstrated that students have a preference for using a device such as iPad in helping them with their studies. They have reported the unique touch interface, portability and easy eBook reading abilities as some of the significant features of iPad. This study has also identified the teaching capabilities of iPad from the teacher's point of view. As part of this component of the research, the iPad was used to actively involve students in discussing and undertaking a series of specially developed case studies in classroom.

A number of useful and relevant apps for learning and teaching mathematics and science have also been identified as part of this research project.

Key words: Technology, interface, Education

Computers' Application in Education – An Evolution

When back in the 1980s, Commodore 64 (C64) entered the homers of several hundred thousands of people in different parts of the world; it revolutionized how one should work with and use computers. That is whenever and to a certain extent wherever one wishes. In other words, computers were not restricted to only computer labs of learning and industrial organizations. The other significant contribution to computer aided learning was the multimedia features of these relatively inexpensive home computers.

Numerous software packages for educational purposes were designed and developed for personal computers such as Commodore 64. They included programs with abilities to teach with text, sound, images and relevant graphs. A significant approach to reinforcing learning was the use of multimedia quizzes with sound and colour for learning enhancement. See Schembri, T., & Boisseau, O. (2001) for details on Commodore 64.

The progress in the technology, its capabilities and educational applications have continued and enhanced exponentially over the recent years. The latest developments focus on the web based learning systems for the purposes of better understanding. For further information, see Chau (2007). Although the Internet based learning plays a major role in education and its delivery, the latest hardware features promise exciting developments. These features have enabled the application developers create interesting and educationally effective apps.

The iPad's Potential Use in Education

When Apple introduced iPad in 2010, its potential and applications in education were realized and considered by many academics around the world. In one university alone, with which the author is familiar, there are around 400 of these devices. Academics and others involved in education are eager to find ways of using these sleek devices. With the release of iPad in 2010, the Touch technology has had even more serious implications for education. Just before the formal launch of iPad in the US, Fry S (2010) had the following comment after interviewing Steve Jobs (Apple's CEO) and reviewing the product for Time Magazine:

"When I eventually got my hands on one, I discovered that one doesn't relate to it as "tool"; the experience is closer to one's relationship with a person or an animal."

According to Fry (2010), Tracy Futhey, of Duke University, was quite optimistic about iPad's potential in education and commented that:

"The iPad is going to herald a revolution in mashing up text, video, course materials, students input ...We are very excited."

Putting the speculations aside, the burning question, at the moment, is: What makes iPad superior to conventional notebook type computers?

iPad offers a different and more natural way of interface. For instance Apple's tap, pinch and draw capabilities using fingers on iPad, iPhone or even iPod are good examples.

The experience through the Apple's Touch technology does certainly create a more natural interface between the user and the machine. To demonstrate this capability, applications which utilize the touch features intensively may be referenced here. For instance, the painting and drawing apps for iPhone enable a user to experiment with painting in a totally innovative fashion. The painter uses iPhone screen as a canvas and the fingers as brushes. The colours are selected by tapping and touching a colour wheel. The chosen colour is placed on the user's palette and the index finger then starts drawing and painting on the screen. iPhone is extremely responsive to strokes and the tiniest detail as desired by the painter are depicted on the canvas. The pinch and zoom feature is used to draw and paint the fine details. The following image (Figure 1) was painted by the author using Brushes app on the iPhone. The painting experience does certainly create a much closer relationship between the painter and the subject. Hence, this experiment demonstrates the special technological advantages offered by iPad/iPhone.

Figure 1 – Diamond Head and a Waterfall Scene Painted by Finger on iPhone

Let us do not forget that progress in notebooks has also been taking place. Many modern notebooks are quite technologically advanced and have interesting and useful features. Hence, the hardware features of iPad cannot be the main reasons and basis for its popularity of use in fields such as education.

One of the main contributory factors towards the popularity of iPad and the desire to explore its potential in learning and teaching is the availability of the apps. These apps are basically pieces of software programs (in traditional terms) which run on devices such as iPad and iPhone. They cover numerous fields such as languages, arts, music, science, mathematics and statistics. The list is continually growing. These apps are readily available at reasonably modest prices on the app store and are accessible via Apple's iTunes. According to the iTune's app store prices, the majority of the educational apps cost under the $10 mark. A very large proportion of these sell for an amount less than five dollars (AUD and USD are almost the same at the time of writing). The apps have several distinct features and advantages over the conventional programs. Firstly, they are inexpensive. For a fraction of the price of a traditional PC software package, one can purchase an app. In terms of features, they are not too far behind their older cousins either. For instance, the author has recently investigated the suitability of an app for teaching the fundamentals of planning, execution and control in a Project Management. The outcome of this investigation was a pleasant surprise compared with the large-scale packages such as MS Project. This app (Project Planner) priced only $3.99 satisfies the needs of teaching the basics of Plan, Execute, Control and Report quite adequately.

Another innovative technology which certainly has a place in the modern approaches to learning is the Amazon Kindle. Kindle is a specially developed hardware and software packaged into a very compact and attractive tablet. Kindle has free international electronic book, magazine or document download capabilities via 3G and wireless connection. Kindle with its whispenet synchronization between the user's different devices, is a very good example of seamless technology for learning. For further information on seamless learning, see Looi (2010). Hence the user can download numerous items of interest from the Amazon's Kindle Store. In addition to its very useful features such as an active dictionary and free 3G access to Wikipedia and the Internet, it is equipped with an experimental text to speech function. When switched on, this function allows the reader to listen to the text on the page. The author has experimented with this feature for the purposes of speed reading training. This experiment was carried out by setting the speech pace to fast and the text on the screen was scanned at the same speed by the author. It was observed that the need for sub-vocalization was removed from the process. Although sub-vocalization is an important factor in comprehension, it is also an inhibitor in achieving higher speeds. The author has comfortably achieved speeds above 250 words per minute with a close to full comprehension outcome.

One of the main features of the Amazon Kindle is its ability to access several hundreds of thousands of books from the Kindle Store. There are also several major international magazines, periodicals and newspapers available on the site. The book collection includes a comprehensive coverage of topics in mathematics and science. The list of these books, in size and coverage, is growing all the time. It must be mentioned that Amazon has also developed and provided a Kindle eReader app for iPad and iPhone which is free to download and use. The app contains most of the features and functionalities of the very successful Kindle eReader. The eReader capabilities of iPad, in general, are certainly a preferred feature amongst the learners. According to the feedback collected from the author′s students in three different classes, this feature is definitely amongst the top three. Although text is a traditional learning mode, it continues to remain one of the most effective forms of media. With the technological advancement in the tablet computers such as iPad, text can become even more powerful in terms of learning and also an appealing way of fully immersing in the topic. The features such as quick access to books, reasonable prices of eBooks, portability of a large collection and text to speech capabilities will certainly help this popularity.

These and similar technologies are very likely to become readily available on other tablet computers. They have a great potential for education in many fields. They can even build on the immersive and real-time engagement as in Virtual Reality in online courses. For challenges of using virtual reality in online courses, see Stewart et al (2010). In order to test the technology′s acceptance and perceptions about its suitability and effectiveness, a series of surveys were conducted by the author in the past two years. As a challenge to determine these technologies' serious uses in education, the author set himself the task of undertaking the research and writing this paper utilizing several apps on an iPhone. Some examples included apps on communication (text and voice mail), data collection and Statistics, MS Word, document scanning and PDF converter and image cropping. The statistical analysis was also carried out using an iPhone app. The next sections presents the methodology and results for this ongoing work.

Learners’ Perceptions and Preferences

In 2010, the author conducted an investigation into the educational applications of the latest developments in modern computing. The main purpose was to determine the learners’ needs and preferences in terms of the latest developments. The participants of this investigation were people who were either directly involved in some form of learning for themselves or closely related to others such as their children or spouses. Adults of both genders from totally different walks of life and backgrounds were selected and contacted for the survey and data collection in this study. These people included college and university students, professionals such as nurses, dentists, technicians and teachers. The study included respondents with varying cultural, linguistic and geographical characteristics too. An aspect of this investigation was to study and compare the levels of interaction-enjoyment for both computer and human teacher. The respondents were asked to rate their perception of the level of enjoyment on a 1 to 5 scale. An initial analysis of the responses determined that the interaction in terms of enjoyment for human teacher has a much larger mean (4.1) than computer teacher (2.8). As Normal Distribution curves in Figure 2 show, the standard deviation for human teacher is also smaller than computer teacher and the respondents appear to have preferences very close to 4 (3 and 5). A t-test even at 1% level of significance indicated that indeed the null hypothesis of identical population means (for computer and human teachers) ought to be rejected. Therefore, it can be concluded that learners, in general, perceive that the learning process with an actual (human) teacher is more enjoyable than a virtual (computer) teacher.

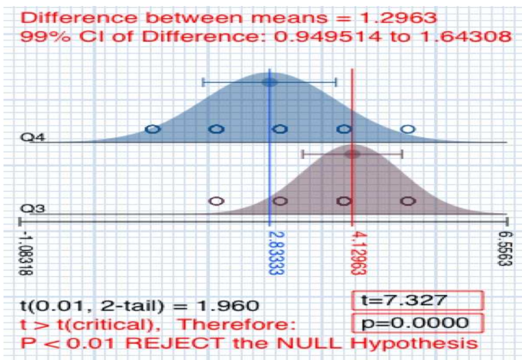

Figure 2 – Difference between the Means of the Responses – Computer and Human Teachers

This finding is rather interesting because the respondents would perceive a computer teacher to have a place in the future education. Hence, the innovative approaches offered by iPad and the available apps is, without a doubt, embraced by the users but education, in its traditional format, is definitely preferred. The respondents' very positive response (Figure 3) to the following question (Q7) is demonstrative of their belief in future technologies for learning and teaching.

Please rate the effectiveness of the following scenario which may take place in the future:

You buy/borrow a book on a topic of your choice, take it home and open it. You then ask the book in your language of choice some questions. The book starts talking and explaining to you by showing you 3 dimensional images. It then invites you to physically (but virtually) interact with them. So, it helps you to learn your topic (e.g. a craft or a skill) by letting you experiment; and it gives you feedback all the time! 1 (Low) 2 3 4 5 (High)

	Q7
Mean	3.7037
Std Dev	1.22289
Sxbar	0.166414
95% CI	3.3742 to 4.0332
Sum	200.0
Median	4.000
Min.	1.000
Max.	5.000
Range	4.000
n	54

Figure 3 – Future Technologies

Conclusions

The main purpose of this paper was to establishment the main reasons and motives for iPad's popularity. The availability of numerous apps related to mathematics and science learning is also a very positive feature. App developers are continuously producing new applications and updating the existing ones. It was concluded that iPad's sleek and contemporary design should not be the sole contributing factor to a desire to explore the educational applications for it. There are other reasons which should be considered. For instance, due to its special features such as portability, software (app) cost effectiveness and easy access, iPad does offer a great deal to learning and teaching in general.

Although there seems to be a positive belief in the future of iPad-like devices in learning and teaching, the learners still have a preference for having a human teacher. The outcome of the investigation conducted by the author, regarding the effectiveness and level of enjoyment with an actual teacher support this finding. The respondents, however, are certainly in favour of an advanced and intelligent system which can respond to learners' needs. Hence, tablet computers such as iPad will have a place in mathematics and science education.

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Science, Technology, Engineering, and Mathematics (STEM) Development: Pathways for Universities to Promote Success

Eric D. Packenham
Utah State University
eric.packenham@usu.edu

Abstract
The focus of this paper is on polcies used in university communities to promote Science, Technology, Engineering, and Mathematics (STEM) development. The paper outlines how STEM can be expanded to engage stakeholders including colleges and universities, students and faculty, and academic and industry leaders. The paper emphasizes proven practices that engage and sustain educators and students in STEM experiences. Most needed to facilitate STEM efforts is a strong commitment from institutional leadership and clear measures to scale-up efforts throughout the campus. Key partners in this effort include university department heads, university deans, community leaders, business and industry leaders (e.g., Chamber of Commerce), departments of education, higher education committees, and the governor's office. Successful STEM efforts attract multiple partners who collectively share risks and rewards of the STEM efforts.

Introduction
In recent years, there has been an increased focus on the critical importance of educating individuals in the fields of science, technology, engineering, and mathematics (STEM). This focus has resulted in wide-spread support and funding to develop and encourage numerous initiatives that place an emphasis on STEM. A variety of reports, including *Rising Above the Gathering Storm* (National Academy of Sciences, 2007) and *Before It's Too Late* (National Commission on Mathematics and Science Teaching for the 21st Century, 2000), echo the purpose of a high quality mathematics and science education and its importance in preparing citizens for competition in a global society with STEM-based careers and professions. These reports emphasize the need for better preparation of K-20 teachers and students in all areas of STEM. To address this need, many STEM faculties in disciplinary departments at universities are working in collaboration with faculty in colleges of education. These partnerships are often formed in an effort to support mathematics and science teaching and learning in partnerships with school systems. Many of these partnerships are funded by outside agencies as a way to encourage the partnership and collaboration (for example, the National Science Foundation has funded these types of collaborations in the Math and Science Partnership Program throughout the United States). This paper outlines three proven strategies for engaging and sustaining STEM disciplinary faculty, College of Education faculty, and K-20 students in STEM experiences. These strategies will be outlined throughout the remainder of the paper and include: 1) Expanding the STEM pathway for student success, 2) Professional development and learning for partners in the STEM effort, and 3) Using technology to facilitate STEM efforts.

Expanding the STEM Pathway for Student Success

A STEM Pathway includes all of the experiences that students have from K-12 school to a career in a STEM-based field (for example, experiences in elementary school, middle school, high school, undergraduate education, internships, and graduate education). One of the first proven strategies for engaging and sustaining STEM disciplinary faculty, College of Education faculty, and K-20 students in STEM experiences in the STEM Pathway is to develop structures on the university campus to increase student interest, participation, and achievement in STEM fields, especially among underrepresented students. Student diversity and a commitment to preparing all students must be a fundamental component of the work of our universities. It is imperative to engage faculty members who are experts in the science, mathematics, engineering and technology disciplines with students to provide experiences for the students that reveal the types of activities in which STEM faculty are engaged as part of their work. These scientists can also expose students to employers who work in the STEM fields to demonstrate how the college degree in STEM transfers into the world of work. Leadership for these experiences and structures that facilitate this type of interaction are provided by college and university presidents, provosts, and chancellors who have a vision for STEM education on their campuses. In order for the STEM pathways to be transformative and seamless, leadership should include federal agencies, STEM disciplinary societies, faculty and administrators employed in a wide variety of academic institutions, and employers in private industry.

One essential goal in expanding the STEM pathway should be on encouraging, recruiting, and mentoring students as they enroll in more rigorous mathematics, science, technology and engineering courses, both as high school students and when they matriculate to college as undergraduate students. However, attaining this goal is no small feat. First, university faculty who work with STEM students should agree that quality matters substantially more than quantity, and that a focal point should be on how STEM faculty teach rather than how much they teach. This is particularly true in the introductory science, mathematics, engineering and technology courses at the university where STEM students experience their first interaction with the activities of their future careers. These STEM courses must be well-connected and aligned to current issues and societal challenges so that students can easily recognize the application of the STEM content they are studying to real world problems. Indicators from STEM industries can be used to assist university faculty in this alignment and to emphasize the connections between bodies of knowledge and STEM work. It is also important to connect and integrate research and education and use research findings to drive and inform education. The best practices must be assembled and become tools and the knowledge basis for learning. This practice will improve the quality and productivity of undergraduate intellectual experiences. However, there seems to be de-emphasis on the importance of the STEM Pathway in some countries. For example, in the United States, federal support of basic research in engineering and physical sciences has experienced modest to no growth over the last thirty years. In fact, as a percentage of GDP, funding for physical science research has been in a thirty year decline. In contrast, at no time in history has the possession of knowledge been so strong an indicator of economic wealth.

In the midst of these economic constraints, one approach to strengthening the STEM Pathway depends on broad visions for collaboration among various stakeholders. These broad initiatives include the development of learning opportunities to improve undergraduate education and engage the university in wide reform approaches that analyze teaching and learning. Universities need to understand exemplary teaching and provide ample opportunities for colleagues to witness these practices so that faculty can transfer innovation from one department to another and one university to another. To do this will require the university to invest in building intellectual communities that share knowledge and synthesize shared data, thereby forging partnerships that stimulate research and teaching innovation on the university campus. Universities can then expand their partnerships to include business, industry, chambers of commerce, professional societies, National Science Foundation and other funding sources. Ultimately, and probably the most difficult undertaking, is for universities to inform and encourage the public to understand how critical the STEM pathway enterprise is to their welfare.

Professional Development and Learning for Partners in the STEM Effort

A second strategy, that is absolutely necessary for engaging and sustaining STEM disciplinary faculty, College of Education faculty, and K-20 students in STEM experiences, is professional development and learning opportunities for each of the partners in the effort. For these opportunities to be institutionalized, the leadership of the institution must facilitate the development structure, put into place mechanisms for scaling up the number of STEM faculty participants, and sustaining the interdisciplinary STEM learning environments. One example of a leadership question to ask might be: How can STEM faculty be supported and rewarded for their work in STEM education? There are currently many more science, technology, engineering, and mathematics (STEM) faculty and other professionals working with school systems in partnerships to support mathematics and science teaching and learning as a result of funding incentives. One unique feature of some of these partnerships is that K-12 school teachers work together with mathematicians and scientists in the field. This type of interaction leads to collaborations between teachers and scientists doing field work and can result in improvements in teachers' classroom instruction. Researchers report that there are benefits for everyone involved, including teachers, scientists, and school students, in these types of STEM partnerships (Siegel, Mlynarczyk-Evans, Brenner, & Nielsen, 2005). For example, as a result of a teacher and scientist collaboration in a STEM partnership, there is teacher learning and scientist learning which then impacts the teacher's school classroom teaching and the scientist's university teaching (Canton, Brewer, & Brown, 2000; Dresner, 2002; Dresner & Starvel, 2004). Another outcome of these types of interactions is that research scientists and mathematicians who work with schools and teachers begin to better understand mathematics and science teaching in school systems and how their own work and expertise connects with school teaching and learning for students (McCombs, Ufnar, & Shepherd, 2007).

The poor rankings for students in the United States on measures of mathematics and science, when compared with their international peers, are one indicator that the need for STEM professional development at the K-20 levels is a growing concern. In *Rising above the Gathering Storm* (2007), the National Academies concluded that "the scientific and technological building blocks critical to our economic leadership are eroding at a time when many other nations are gathering strength." At the core of effective science teaching is "student-centered and inquired based opportunities for students to actively be engaged in learning." One of the most important ways to initiate institution-wide reform is to begin with an examination of the pedagogy being used to teach STEM courses at the K-20 levels. Institutions should ask important questions such as: Does this pedagogy meet the needs of 21st Century learners and of the STEM discipline and its careers? These critical examinations of pedagogy can spur the need for training and support as faculty adopt new cutting-edge pedagogies that meet the needs of learners and the discipline. Some changes will focus on intellectual communities and situated cognition that integrate new social interaction theories and new technologies that facilitate technologically advanced collaborations. The universities' colleges of education are likely experts in pedagogy and can assist in modeling some of these instructional innovations. Faculty from academic units across the university campus can also learn about diverse learning styles, ways to apply instructional technology to sustain learning, and particularly how social media plays an important role in the current generation's models for learning.

The preparation and development of teachers has come under fire and some leaders question the value and necessity of teacher preparation programs, implying that they do not take teacher education seriously. There are some very good teacher education models, yet many are expensive. Recently the Association of Public Land Grant Universities (APLU) designed an Analytical Framework for STEM Teacher Education to be used in the development and assessment of STEM teacher preparation programs. This framework describes an effective teacher preparation program for science and mathematics teachers, and provides some very specific examples of effective criteria and processes that should be used in teacher preparation programs. These criteria serve as a model and a guide for improving the preparation of STEM teachers. Once teachers are in the classroom it is important to continue their development as lifelong learners throughout their profession. But as reports show, career development for teachers in the United States is not systematic and cohesive, as it is in some other countries (Stigler & Hiebert, 1999). The National Staff Development Council's (Wei, Darling-Hammond, Andree, Richardson, & Orphanos, 2009) report about professional development trends in education reported that teachers experience much less in-depth professional development and spend less time in professional development than they did four years ago. The decline in professional development, in a system where teachers are already lacking professional development opportunities in STEM, will have significant impacts in teachers' instruction and student learning in STEM for years to come.

Utilizing Technology to Facilitate STEM Efforts

A third proven strategies for engaging and sustaining STEM disciplinary faculty, College of Education faculty, and K-20 students in STEM experiences is to use current technologies to enhance learning and interaction. Advancing technologies have minimized academic institutions as a depository of knowledge. Using knowledge networks is a much more vital and instrumental way of shaping new knowledge today, rather than having students recite what is already known. It is important for universities to identify the kinds of technology that are available and to remain current on those technologies to facilitate and engage more learner-centered teaching in STEM courses. Connecting with students in the K-12 STEM Pathway means that institutions should find ways to effectively use social and interactive asynchronous and synchronous media such as twitter, wikis, podcasts, e-books, blogs, social networking, gaming, videos, internet2, simulations, Second Life, or other interactive programs. These tools can be used effectively to promote student interest and achievement in STEM, in general education, and in STEM majors. One example could be the use of robots to teach students engineering principles and design. In fact, one Korean Education Policy is planning to use robots to teach thousands of kindergarteners by 2013.

Many universities are beginning to offer coursework and entire programs of study through the use of technology. While digital education is designed currently for adult learners, in time it is likely that the formats of on-line instruction will adapt to provide technological options for all levels of education and learners. These advances have the potential to make learning more individualized, interactive, and self-directed and could be used in a variety of STEM learning environments with students. For example, animal dissection using computer models has already replaced the dissection of real animals in many biology classrooms.

Both universities and public schools will need to keep current with the technologies that are already infused in students' daily lives. Unfortunately rapid shifts toward the use of social media in society have not been followed by equally rapid shifts to these technologies by schools and universities. Institutions must become more agile in responding to technology changes so that the technologies students have learned to use in their daily lives can be applied to problem solving in their school and university experiences, particularly as those technologies apply to the solution of STEM problems. Unfortunately in many places, it is easy for students to see more technology at the gas station in their public school or university classrooms. An important way to change this trend it to ensure that instructors have opportunities to experience these technologies as part of their daily work so that they can integrate the technologies into their classrooms and interactions with students. Some college campuses were early adopters of podcast technologies, which provide their incoming students with tools to acclimate and navigate around campus. For these campuses it is common to see students walking around the campus with ear buds listening to campus map information, campus event information, or a STEM lecture for one of their classes. There are also public schools where students are using their IPod technology to listen to podcasts from their teach or another source on topics such as the process of photosynthesis or the difference between equilateral and isosceles triangles. This technology individualizes learning and allows students to have anytime access to the STEM content they are studying in their courses.

Closing Thoughts

The purpose of this paper was to outline three proven strategies for engaging and sustaining STEM disciplinary faculty, College of Education faculty, and K-20 students in STEM experiences. These strategies focused on the importance of enhancing the STEM pathway for student success, the critical need for professional development and learning for partners in STEM efforts, and impact of new technologies on facilitating STEM efforts. Without the advances in knowledge and understanding that are brought about by study in STEM fields, our future generations will experience a great deficiency of understanding. In the book, *The Demon Haunted World*, Carl Sagan implies that this deficiency of understanding foreshadows an alarming future. He writes: "Finding the occasional straw of truth awash in a great ocean of confusion and bamboozle requires vigilance, dedication, and courage. But if we don't practice these tough habits of thought ... we risk becoming a nation of suckers, a world of suckers, up for grabs by the next charlatan who saunters along." Learning in the sciences, technology, engineering and mathematics are essential to the type of understanding that Sagan describes. Yet at many institutions of higher education only a small portion of the student undergraduate population completes a mathematics or science course after their freshman or sophomore year of college. This unfortunate pattern signals a lack of understanding about the importance and applicability of STEM to their daily lives and future careers and contributes to a citizenry that is less and less knowledgeable about science, technology, engineering and mathematics. The challenges of renewable energy, clean water, and global climate change will be the problems faced by the current generation of students studying STEM in K-20 education. Will this generation be less informed about science, technology, engineering and mathematics and make their state and national decisions based on what their Facebook friends value? Or can we count on this generation to use science, technology, engineering and mathematics to solve the world's problems and advance STEM for the entire planet? Only an ongoing commitment to STEM pathways, instructional development, and integrated technologies will produce a favorable result for all.

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The basics of set theory – some new possibilities with ClassPad

Ludwig Paditz, University of Applied Sciences Dresden, Germany
paditz@informatik.htw-dresden.de

Abstract:

The basics of set theory consists in sets, elements, lists, set-builder notation, subsets, equal sets, the empty set, union, intersection, difference, symmetric difference and Cartesian product.
Sets are one of the most fundamental concepts in mathematics but we can not calculate with set operations and set relations on the ClassPad. On the other hand we can write in the text mode with special symbols of the set theory in the ClassPad, e.g. $\in$, $\notin$, $\cup$, $\cap$, $\setminus$, $\subset$, $\subseteq$, $\neq$, …

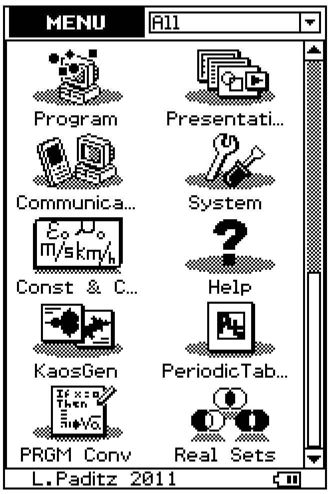
Thus some students of informatics tried to introduce the set theory in the operating system version 3.05 (published 2010). They followed two ways of solution:
1. Create a so called Add-In for ClassPad to calculate with sets of real numbers.
2. Create a Basic-program for ClassPad to calculate with finite sets of numbers or words.

In the first case we use the set-builder notation {x|…}, e.g. $\{x|a<x<b\}$ or $\{x|x\geqq c\}$ or {x|2,3,5}. In the other case we use the notation “{2,3,5}” or “{x1,x2,x3}” or “{(a,b),(a,c),(2,c),(d,4)}” or“{Anna, Alan, Max, Marc, Tanja}” to fulfil set theory operations. Here we work with strings. Thus we get no problems with system variables x1, x2, x3 or Max.

During the workshop the participants could download the new programs to their own calculators or notebooks and check the new possibilities or they follow a demonstration with the ClassPad Manager software.

1. The Add-In “Real Sets”

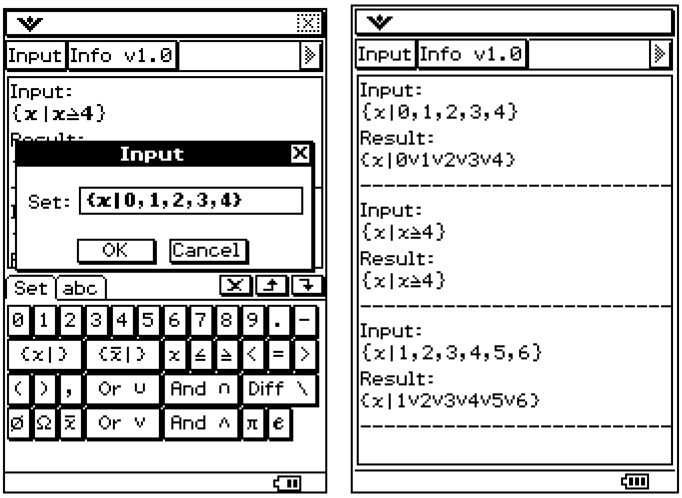
The students wrote a program in C^{++} and used than the CASIO-SDK (software development kid) to compile the source program into a ClassPad Add-In. The Add-In can be installed in a handheld calculator but not in the PC-manager-software. Thus the students compiled their program additional in an exe-file, which runs on a Windows-PC (**Real_Sets.exe**). Here we need additional a special library, the **ClassPadDLLgcc.dll**, in the exe-file folder.



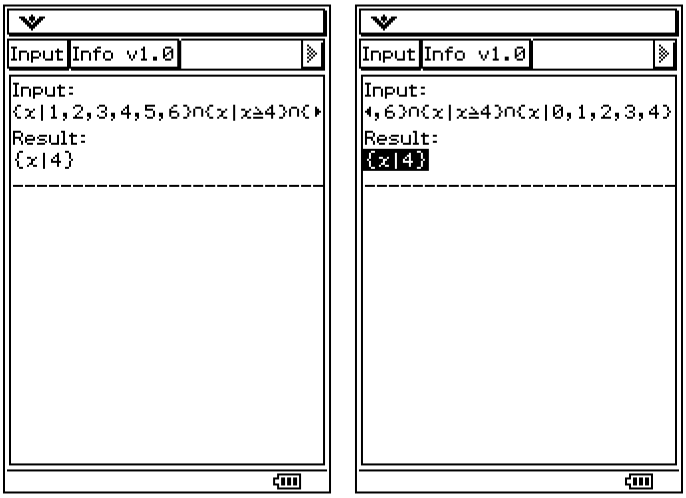
The new **Real Sets**-Icon in the menu

The students created a special keyboard for the set theory!

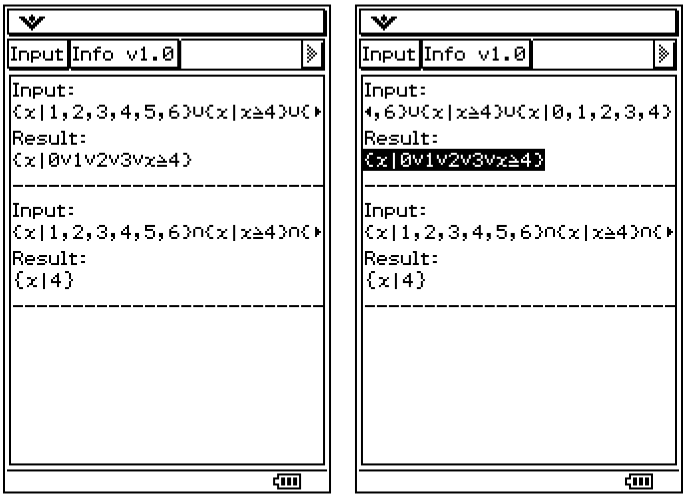
Now let us calculate some examples:
Let be A = {x| 1,2,3,4,5,6}, B = {x| x≥4} and C = {x| 0,1,2,3,4}



The output-window shows 3 results

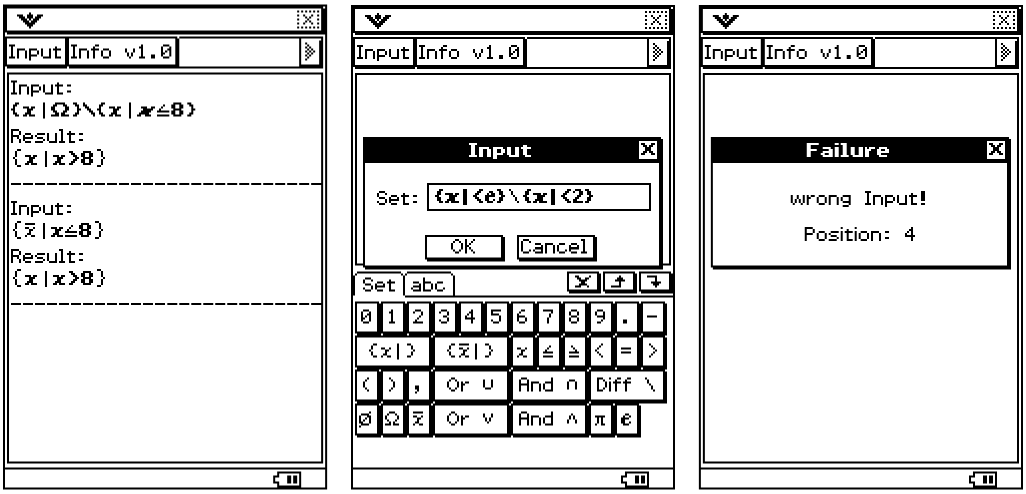


A∩B∩C

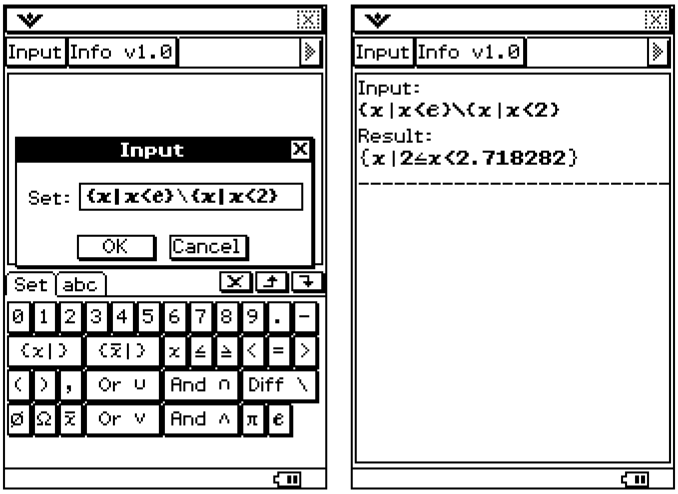


A∪B∪C

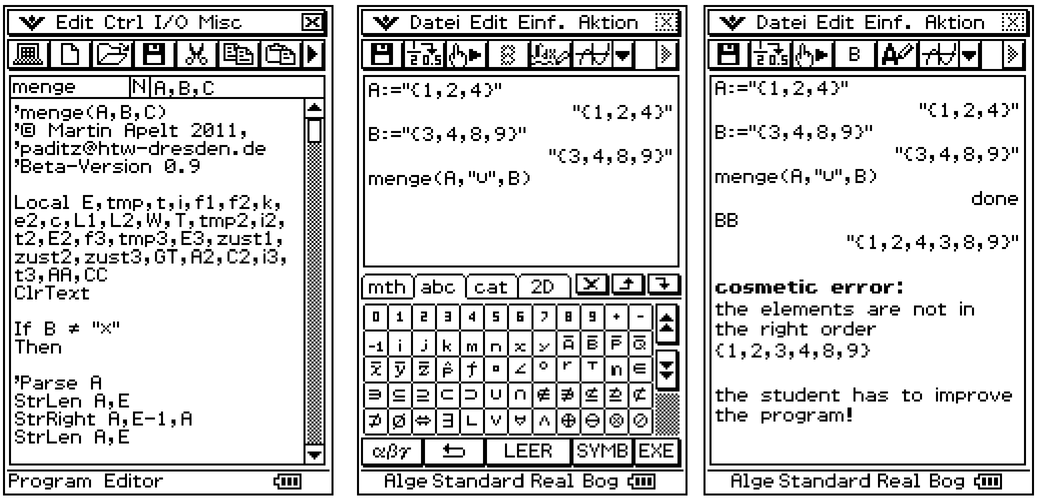
For the empty set here we have the symbol Ø and the full set (all real numbers) is Ω.
Additionally we can calculate the complementary set in Ω by the help of the x-bar notation.
We can use brackets (…) for multiple operations, e.g. A∩(B∪C).

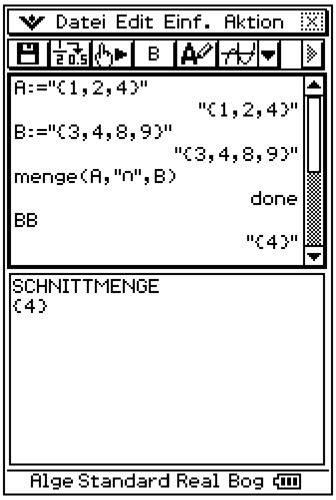


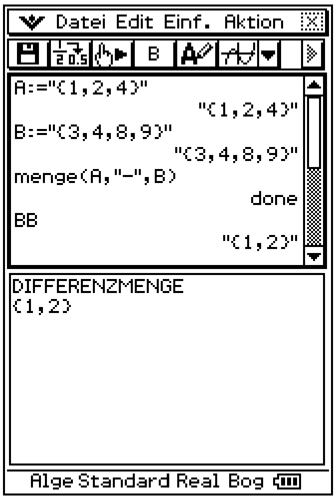
Sometimes we get an error message (we forgot x after |).

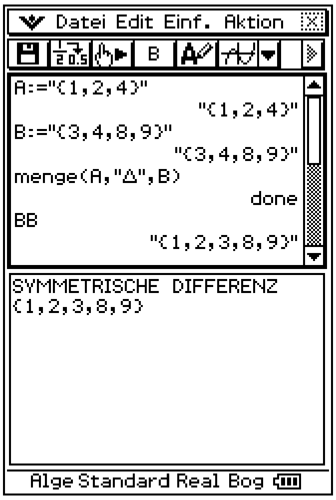


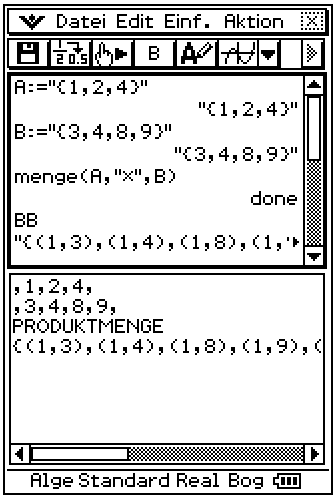
2. The program "menge" (see SA2011-workshop-Paditz.vcp)
After download the program in the library-folder of the ClassPad we can use it e.g. in the eActivity-menu. The function menge has the syntax **menge("A","operation","B")** or **menge("A","relation","B")**. The result is stored in the variable BB and appears in a separate window. At first have a look in the source text (Beta version 0.9).



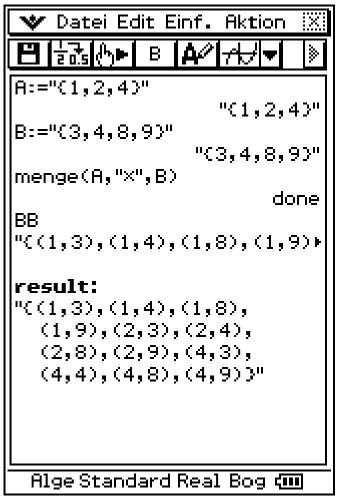


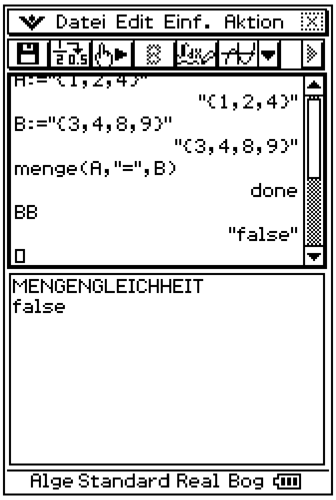


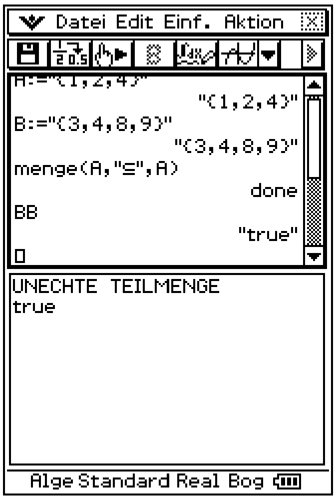


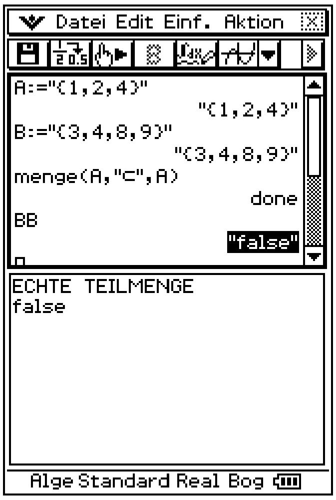


the result window is not optimal



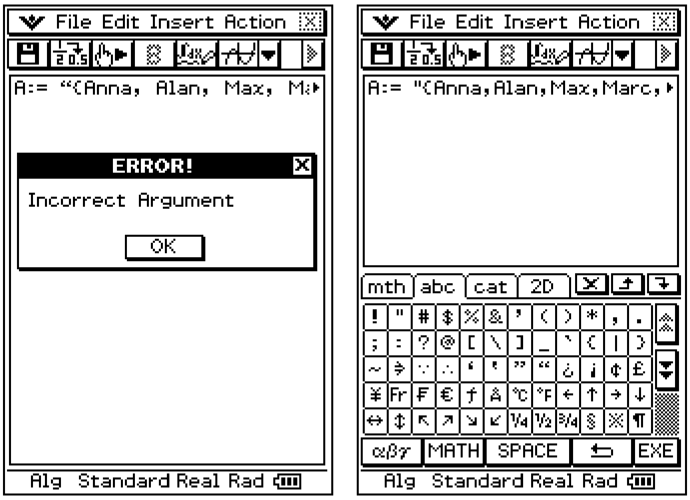




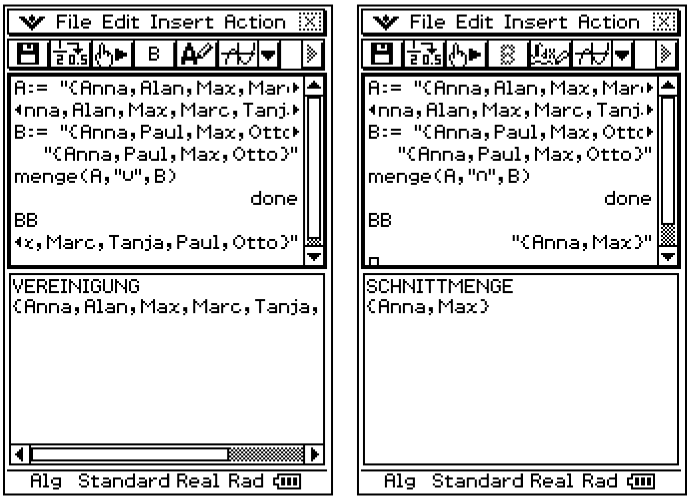


Now we can work with names:

Let be A:= “{Anna, Alan, Max, Marc, Tanja}” and B:= “{Anna, Paul, Max, Otto}”

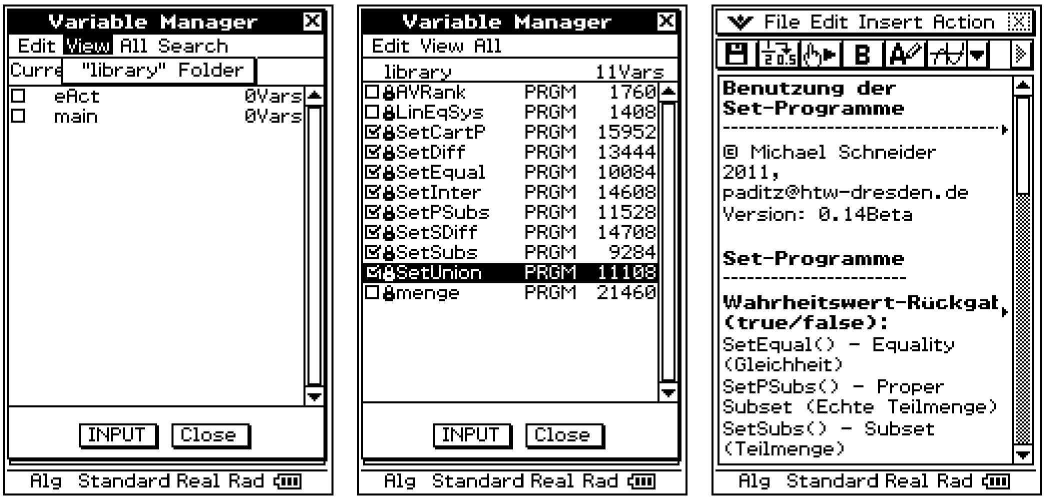


use the correct strings "…"



and so on.

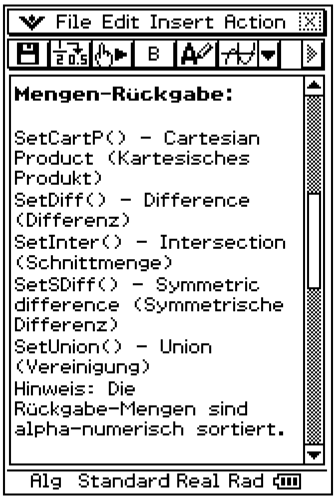
3. The individual programs for one task (see SA2011-workshop-Paditz.vcp)
Another student has created 8 several single programs for one operation or relation.

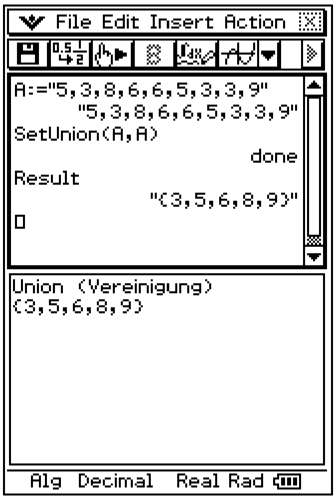


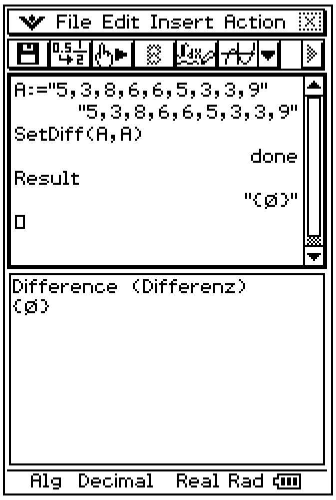
The programs have English names: SetUnion (for $\cup$), SetInter (for $\cap$), SetDiff (for \ ), SetSDiff (for Δ), SetCartP (for $\times$), SetSubs (for $\subseteq$), SetPSubs (for $\subset$), SetEqual(for =).

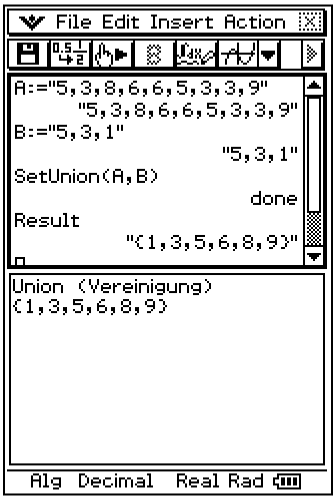
Let us check these programs. We idn't need any operation or relation symbol. The syntax is as follows: Set…(A,B), where the sets are given in strings "   " but without brackets { and } .

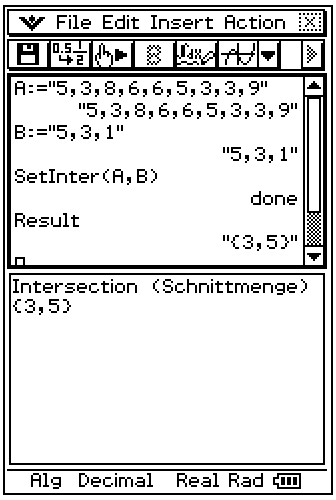
The result is stored in the variable **Result**, the elements are in alpha-numerical order (sorted by their ASCII-codes)

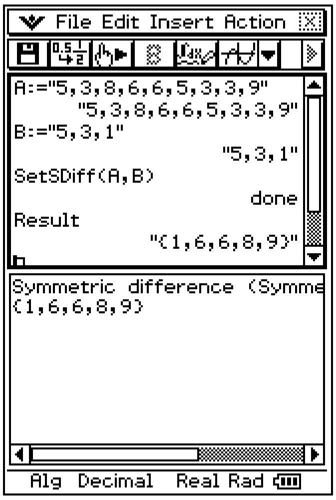




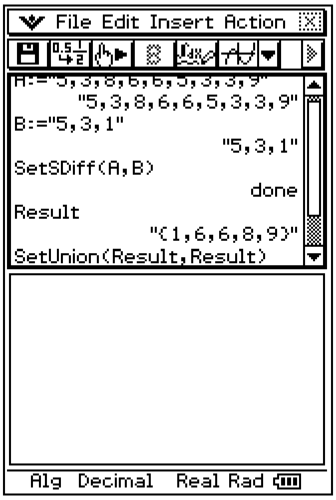


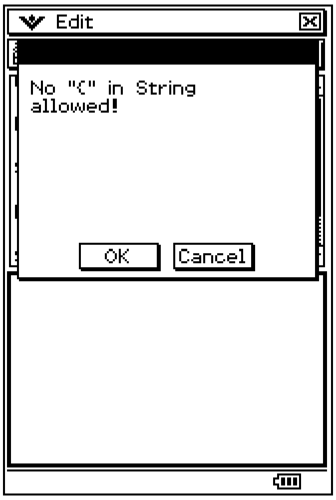


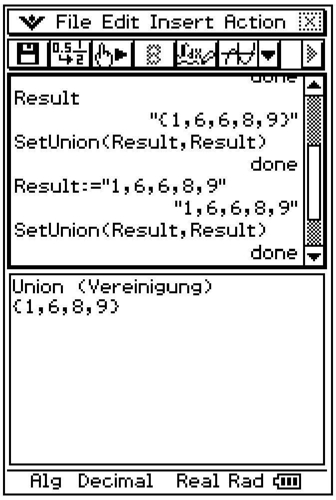




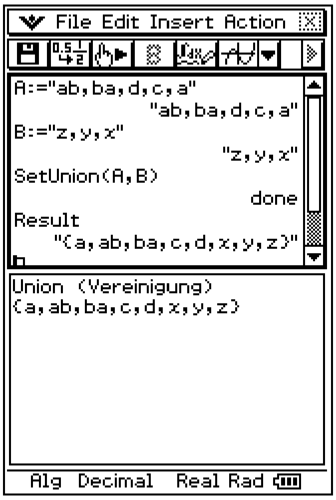
In the last result is a cosmetic error: double 6, better {1,6,8,9}. The student has to improve his program. To simplify the result we use SetUnion(Result,Result) = Result. At first we get an error message.

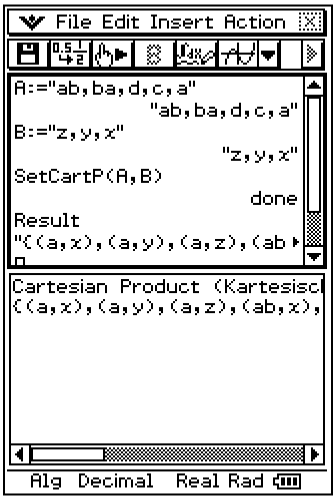






Here we remark, that the input is without brackets { and }, but the result variable has brackets!





4. Final remarks

During the learning process our students have good ideas to solve several problems in development new programs. The final step is to check the new programs and improve the errors, which appear during the first check of the new created programs. Here we have the Add-In in the final version 1.0 but the other programs in a Beta version 0.9 and 0.14 respectively. Later we will publish some updates.

For the HP 50g calculator you can find set programs here (by Clemens Heuson)
http://www.heuson-software.de/heusoneng.htm

In the internet we have a nice symbolic and numeric calculator:
http://www.tusanga.com/
Here we can do set theory calculations too.

A next step could be to create a program for drawing Venn diagrams, cp.
http://en.wikipedia.org/wiki/Set_%28mathematics%29

Download:
http://www.informatik.htw-dresden.de/~paditz/Set-Theory-SA2011.zip

The Set-Theory-SA2011.zip contains following parts:
- the **Real_Sets.exe** together with **ClassPadDLLgcc.dll** for a Windows-PC
- the CASIO ClassPad Add-in application: **Real Sets.cpa**
- the CASIO ClassPad Manager Virtual ClassPad File: **SA2011-workshop-Paditz.vcp**

CHALLENGES AND POSSIBILITIES IN EMERGENCY EDUCATION: INSIGHTS FOR MATHS TEACHING AND LEARNING AT A JOHANNESBURG REFUGEE SCHOOL.

Pausigere Peter, PhD Fellow, Numeracy Chair,
Rhodes University, Grahamstown, South Africa.
peterpausigere@yahoo.com

Abstract
Zimbabwean refugees and economic migrants at the Central Methodist Church (CMC) Refugee House, in central Johannesburg have successfully established a combined school-St Albert Street Refugee School. This paper comes out of research carried over a period of five months which employed the ethnographic approach and gathered data through classroom non-participant observation, interviewing and document collection. Using the framework of the Direct Instructional Model (DIM), an approach recommended in emergency education, the paper highlights the challenges facing teaching and learning at the Refugee School and explores possible alternatives to some of the challenges. The paper then reflects on how these challenges affect the effective teaching and learning of maths and looks for feasible pathways for maths classroom practices in refugee situations. The possibilities discussed for the refugee maths classrooms are informed by the literature on emergency education and acceptable maths practices.

Introduction: Context of the Study

Zimbabweans have migrated to South Africa mainly because of the country's economic crisis which started slowly in the late 1990s. Political violence and intimidation also led many Zimbabweans to flee their country. It is estimated that about half of Zimbabwe's adult population have either migrated or fled to South Africa.

Some Zimbabwean economic migrants and political refugees have been given refuge and provided with shelter at the Central Methodist Church (CMC), in central Johannesburg. Between 2004 and 2005 a few Zimbabweans trickled to the church to seek accommodation, basic provisions and financial assistance from the generous Bishop Paul Verryn.

In 2007 more than 2 500 refugees were staying at the Refugee House, sleeping on the bare floors, corridors, steps and halls in the five-storey building. At the peak of the Zimbabwe crisis in 2008 the church housed close to 4 500 political refugees and economic migrants.

It is within this context that the need for education was identified. Thus the refugees at the church house started and established a primary and a secondary school (St Albert Street Refugee School) with a combined enrolment of about 500 learners.

This paper comes out of my Master of Education research report and draws from the research report, extracts and instances of maths teaching and learning observed in and across primary and secondary classes. The paper highlights challenges facing teaching and learning in an emergency education context at the Refugee School and provides possible ways forward to some of these challenges. It then reflects on how these challenges affect maths teaching and learning and looks for feasible pathways for maths classroom practices in refugee situations.

Investigation of the challenges facing emergency education, and how these generally affect maths teaching and learning is done through the theoretical framework of the United Nations' High Commission of Refugees' recommended teaching and learning approach called the

"Direct Instructional Model" (UNHCR, 2003).

St Albert Street Refugee School and its curriculum

The St Albert Street Refugee School is about a kilometre away from CMC Refugee Centre and was opened in July 2008 by four volunteer refugee teachers, after they found out that there was an increase in the number of children at the centre who were not attending school. When I left the research site on November 15 2009 the school had a total of 21 teachers, 534 learners of which 421 were accompanied students and 113 unaccompanied students. The latter term denotes learners who came from their country of origin without parents or guardians and stay at the CMC Refugee House, most of these are Zimbabwean children. The combined school provides tuition from Grade 0 (R) to Form 6 (equivalent to Grade 12). The school follows the Cambridge Curriculum. The decision to follow this British originating curriculum was necessitated by the fact that the refugee learners could not register for the South African Matric as they did not have identity documents. The local Cambridge examination centre run by the British Council did not require the refugee exam-writing candidates to have identification cards, birth certificates or refugee papers. It is on these grounds that the refugees collectively decided to adopt the Cambridge curriculum.

Research Methodology

In carrying out my research I used the ethnographic methodology. I employed three strategies for gathering data that is non-participant observation, interviewing and document collection, over a period of five months from mid-June up to mid-November 2009. Daily observations of the teaching and learning interactions were done for one week in Grade 1, 7, Form 2 (equivalent of Grade 9) and Form 3 (equivalent of Grade 10) classes at the St Albert Street Refugee School. During these non-participant observations of classrooms teaching and learning interactions, field notes were compiled. I used standardised open-ended interview schedules to solicit for information from key knowledgeable members of the refugee community. I also managed to collect school principal reports, the school timetable, curriculum guides and school pamphlets.

Challenges facing the St Albert Street Refugee School and their implications for the teaching and learning of mathematics.

The CMC education system, like any other refugee education initiatives, was regularly in endless financial crises. Lack of funding is the main reason cited for poor quality refugee education (Sinclair 2001). Inadequate finance implies limited supplies in teaching and learning materials, textbooks and furniture and this inhibits learners from concentrating on learning (Williams 2001). There is need for make shift writing pads for the primary school learners who use two large open church halls which have long fixed benches (pews), used by the church congregants. The school lacks relevant current textbooks, teaching and learning materials, stationery for learners and this hinders teaching and learning. In the observed Grade 7, Form 2 and 3 maths lessons only the teachers had Mathematics textbooks, from which they copied examples that were worked in the classes (Fieldwork observation notes, 2 & 3 October, 2009). The non-availability of textbooks in maths classes at the Refugee School inhibits effective and independent maths learning and closes opportunities for students to experience and encounter authentic maths practices.

The deteriorating physical conditions at the CMC Refugee School were at par or even worse than those reported by Bird (2003, p. 62), in Rwandan and Burundian refugees classes at

Goma and Ngara refugee camps in Tanzania where children were reported to have learnt in "cramped, underequipped, poorly lighted classes with fixed benches". The subdivided Form 2 classroom was cramped with learners, dirty, dusty, too small and poorly aerated. The Grade 4 to 7 primary school classes used the 'Chapel' and the 'Main Sanctuary' halls at the CMC church. The Main Sanctuary accommodated three primary classes whilst the Chapel Hall had two classes. These halls were poorly ventilated and unpartitioned thus teaching in one class would interfere with other classes. The Grade 1 class at the St Albert Street School had 43 learners yet the UNHCR (1995) recommends a class of not more than 35 pupils in crisis situations. Therefore the greatest challenge to teaching and learning as well as effective maths classroom practices at the CMC Refugee School is inadequate space and poor physical learning conditions.

Like any other refugee programmes, the St Albert School faced high staff turnover as teachers complained of poor incentives, the teachers received a R3 000 monthly stipend. Poor staff retention can disrupt learning as was the case in one Bhutanese refugee class where it was reported that there had been five different teachers for one class in a single year (Brown, 2001). At the Refugee school the most affected by an absence of qualified teachers were Maths and Science classes as Maths and Science teachers are most sought after in South Africa. Most Maths and Science teachers lasted for only one term before seeking for higher paying jobs in government schools. Teaching staff attrition severely affected the teaching of maths and science at the refugee school. The progressive and cumulative nature of maths teaching make high teacher turnover in this subject particularly problematic.

In addition the choice of language of instruction is a problematic issue in the refugee curriculum. Sinclair (2001) argues that the issue of language instruction is a human rights issue and advocates for the use of the mother tongue medium of instruction amongst refugee children. On this issue the Refugee school's official and agreed medium of instruction and communication is English. Teachers were not allowed to teach in vernacular languages, in the three maths classes observed in Grade 7, in Form 2 and in Form 3, the different Maths subject teachers always used English. However it emerged during my fieldwork observation that the Grade 1 teacher at times used Shona in a class of 43 learners where 15 children spoke Ndebele and 3 students were Congolese with the remainder being Shona speaking learners. The issue of the language of teaching and learning was problematic at the refugee centre and is a persistent problem even in South African multi-lingual Maths classes (Setati, 2005).

Teaching and Learning approaches and Maths classroom practices at the CMC Refugee School.

Refugee education is also embedded in contested pedagogical issues that have characterised the education terrain in the 20th and 21st centuries. The question has been; should refugee teaching and learning methods be child-centred or teacher centred? (Williams 2001). Kagawa (2005) argues for learner-centred progressive pedagogies, which are closely linked to the democratisation of society and also allows for the learner's critical thinking and exploration. However, democratic teaching methods are difficult to carry out in overcrowded classrooms, are seen as time consuming in a congested curriculum and do contradict with the drive for examination passing (Williams 2001). While many curricula are moving towards participatory methods of teaching and learning, limited space and resources, the nature of the curriculum and assessment methods also determine the pedagogical approach to be

employed. Whilst the school principal in the School Council meeting dated 16 November 2008 had recommended for the enhancement of teacher-pupils interaction in classrooms, in practice the opposite seemed to prevail. At the Refugee School, the maths classrooms were characterised by authoritative teacher centred teaching practices that inhibited learner participation thus depriving the refugee learners of maths discussion and engagement.
In the observed Grade 1 class Numeracy lesson the teacher encouraged rote learning and employed closed questions. It was characteristic of the, Grade 7 and Form 2 maths teachers to pose "funnelling questions" thereby inhibiting extended learners' participation in the lessons. In the observed Form 3 maths class lesson, on Matrices, teaching was teacher dominated with students' participation being limited to closed questions which were chorused and chanted by the learners (Fieldwork observation notes, 1 October 2009). Such maths teaching and learning approaches are problematic and a challenge to democratic learning that aims to empower learners and ensure that students are given opportunities to explain and make sense of mathematical ideas and procedures. According to Kilpatrick, Swafford and Findell (2001) traditional methods of instruction in maths classes generally lead students to develop procedural fluency at the expense of conceptual understanding, strategic competence, adaptive reasoning and productive disposition. The observed Form 3 "Matrices" maths lesson mostly involved simple addition rather than relating to why, when and what to do next, thus limiting student access and engagement even with procedural knowledge of matrices. The usage of traditional methods of teaching and learning maths at the refugee school deprived learners of a holistic understanding of mathematics.

Direct Instructional/ Active Teaching/Mastery Learning Model

Whilst recommending more participatory approaches to maths teaching and learning at the refugee school, it is important to note that the Cambridge curriculum is a "collection code" that primarily focuses on disseminating the structure of the discipline to the students and has strong pacing and sequencing (Bernstein, 2000). Democratic learning approaches could be difficult to carry out at the Refugee school which is characterised by overcrowding and lack of space. The best teaching and learning approach for use in refugee contexts and most suited for the Refugee community school and for effective maths classroom practices is what the UNHCR (2003, p. 43) calls the "Direct Instructional Model" (DIM).
The DIM had been piloted and introduced in Afghan Refugee Camp schools in the Peshawar region in Pakistani (UNHCR, 2003). This model is teacher directed but in a positive manner that ensures student engagement and is mostly used to teach in difficult circumstances with limited space and resources (UNHCR, 2003). The teacher under this approach is expected to carefully structure every skill and concept, yet ensuring student engagement through the use of task-orientated approaches (UNHCR, 2003). Such an approach to teaching and learning suits the adopted Cambridge Refugee school curriculum which is knowledge focused and would also ensure that learners are engaged in extended subject practices. The Direct Instructional model augurs well with highly regarded maths practices which value learners' mathematical contributions in classes and at the same time "keeping an eye" on maths conceptual knowledge. The refugee school needs to consider such a teaching and learning possibility both for their school classrooms and maths lessons practices.

Possibilities in Refugee education: Implications for maths teaching and learning.

This part of the paper provides possible pathways for the challenges facing the Refugee schools. The problems facing the refugee school also spilled into maths lessons thereby affecting the effective teaching and learning of maths. To overcome the problem of textbooks at the CMC Refugee School, I borrowed textbooks from the university libraries which the School administrators photocopied for teaching purposes. A well-resourced independent school that used the Cambridge curriculum, St Johns College had expressed interest in helping the refugee school with textbooks. Maths textbooks enable learners to work independently and at different paces. The supply of teaching and learning materials and stationery had improved substantially at the Refugee School, thanks to the numerous donors which were beginning to support the refugees' education initiatives.

The greatest challenge to teaching and learning at the CMC Refugee School is inadequate space and poor physical learning conditions. The "double-shift system", could be one solution to this problem as was the case in Rwandan refugee schools in Tanzania where such a practice was introduced to overcome overcrowding (Bird, 2003, p. 65). However even if this system is introduced there would still be need for more teachers.

One possibility of overcoming the problem of multi-lingual classes has been suggested by Gutierrez, Baquedano-Lopez & Tejeda, (1999). Gutierrez et al, (1999) argue for the utilisation of 'hybrid languages' under which educators use different languages from the learners' community when teaching basic concepts, this increases the possibility of classroom dialogue. In the same vein Setati (2005) calls for "code-switching" in South African multi-lingual maths classes. This ability to draw on learners' languages as a resource for learning and teaching mathematics has positive implication for learners' access to Mathematical knowledge (Setati, 2005). Applying this theory into practice in Refugee maths classes implies that Shona, Ndebele, Zulu and Xhosa could be used to teach maths concepts to learners. I think such a measure would be very helpful to support learner engagement in mathematical discussions at the Refugee School.

To overcome the challenge of teaching staff turnover the Refugee school administrators had resorted to recruiting untrained teachers who had passed 'Advanced level' from the refugee community. Four such volunteer teachers are teaching at the St Albert Street School. The move of recruiting temporary teachers from the emergency population, according to the UNHCR (1995), ensures sustainability of the education system. The recruitment of untrained Maths teachers from the refugee could possibly lead to the normalisation and stabilisation in maths classes. There is however tension between avoiding teacher attrition by employing temporary untrained teachers at the expense of qualified experienced teachers.

A move towards progressive teaching approaches (similar to the Direct Instructional model) was being spearheaded by a group of lecturers from the Wits School of Education which was holding fortnightly workshops on teaching and learning (Slo, personal communication, 8 October 2009). Such staff development initiatives are highly appreciated and valuable in schools and are more than helpful in fragile education contexts. On this note the UNHCR (1995) recommends that there be in-service training for teachers in emergency situations. We hope that the refugee staff development initiatives offers one viable avenue that would change classroom practices and also impact positively on maths teaching and learning.

Conclusion

The emergency education field is new, full of possibilities and one area of study that is under researched in South Africa. Refugee education borrows theories from other fields. Just as refugee education borrows from other fields, classroom and maths teaching and learning practices can also be viewed within the frameworks of emergency education approaches such as the Direct Instructional Model. Such approaches offer possible ways and new horizons for emergency education. Of key importance is the need for methodologies to be employed which enable active democratic participation of learners since it is often the undemocratic circumstances in refugee home countries which have led to the influx of refugees. We hope that the possible alternatives discussed in this paper flow into as well as influence the refugee classrooms and maths teaching and learning practices resulting in quality effective education that leads to mathematically proficient and competent learners.

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Mathematics Connections to Current Events

Esther M. Pearson, M.S., Ed.D.
Assistant Professor, Mathematics
Lasell College
1844 Commonwealth Avenue
Newton, Massachusetts 02466
Phone: 617-243-2455 office; 978-257-5725 cell
Email: epearson@lasell.edu

Abstract

The "Mathematics Connections to Current Events" provides pedagogy for introducing current events into Mathematics courses thus providing humanistic mathematics examples and discussions in an instructional environment. The outcome is instructional approaches that result in student's gaining not only an understanding of mathematical concepts but how to apply the concepts to current events thus engaging students in and beyond the classroom. This allows students to apply their knowledge to new mathematical situations providing a sense of empowerment and active participation in social environments with mathematics as their basis of putting current events into proper perspective.

Introduction

Mathematics is occurring all around us. It is veiled under the covering of everyday life and current events. Reaching students with mathematics occurs by lifting that veil to uncover mathematics as a humanistic endeavor. Mathematics curriculum must be broadened in its connections to relevant global societal events. This is accomplished using global events that occur in selected subject areas of society. Mathematics is presented as a relevant and useful tool that is used to identify and assist in resolving global problems, concerns, and crises.

Mathematics in proper perspective is mathematics in personification. It is put into perspective as a tool for viewing our daily environment and making decisions that personal lives and society in general. A paradigm shift occurs as mathematics is portrayed in a humanistic view, rather than as depersonalized, asocial, and without much human context or relevance.[1] Mathematics is learned and performed within the context of human purpose and meaningful human enterprise.

Richard Felder, North Carolina State University and Rebecca Brent, EDI - Education Designs Inc. developed a list of pedagogical mistakes that are made in the presentation of Mathematics. Mistake #5 on the list is the failure to establish relevance.[2] Establishing relevance takes into consideration both the student and the mathematics curriculum content. The curriculum is built around the student not the student molded and shaped around the curriculum.

Students learn best when they clearly perceive the relevance of course content to their lives. Connecting theory and practice must be achieved by bringing classroom instruction to bear on real world events. To foster motivation you must begin the course by describing how the content

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[1] Brown, S. I., "Towards Humanistic Mathematics Education", Mathematics Ulterior Motives, http://mumnet.easyquestion.net/sibrown/sib003.htm

[2] Felder, R.M., and R. Brent, "The 10 Worst Teaching Mistakes. Mistakes 5–10," *Chem. Engr. Education*, 42(4) 201 (2008) <*http://www. ncsu.edu/felder-public/Columns/BadIdeas1.pdf*>

relates to important societal events to include whatever you know of the students' experience, interests, and career goals.[3]

To find current events related to mathematics there are several techniques that can be used. These techniques locate the events but do not provide the mathematical context for them in a curricular format. It is up to the professor, teacher, instructor to place the current event into a humanistic context in which mathematics can be drawn out. The steps of doing this include:

- Determine categories of Mathematics of Interest
- Provide search categories for Media outlets
- Develop Unique search criteria

The humanistic categories upon which the current events can be placed are taken from mathematics topics to include:

- Mathematics of Social Choice
- Mathematics of Finance and Economics
- Mathematics of Shape and Form
- Mathematics of Change and Growth

This is not an all inclusive list of categories. Others may be added based on curriculum and context of instruction.

Descriptions of each of these topics are as follows:

Mathematics of Social Choice – mathematics represented in government, voting, sharing, and apportionment

Mathematics of Finance and Economics – mathematics represented in money and commerce

Mathematics of Symmetry – mathematics represented in nature, art, human body, and architecture to include fractals

Mathematics of Change – mathematics of growth and measurement

These categories link current events with the recognitions of mathematics in daily life. Mathematics is involved in governance, finance, nature, art, music, growth, data collection and interpretations. Each of these daily recognitions interacts with our lives and enriches them with awareness of humanity.

To find current events the common and advanced Internet search engines can be used. The search engines containing "Alert" systems which provide notification when a topical subject of your interest is posted to a public database of information and accumulated knowledge are used. The alerts are triggered by "Key Words" based on meta-tags. So, the choice of key words is critical in notification of the alerted current events, as well as, syntactical limiters. The

[3] Dewar J., Loyola Marymount University, CA. –AWM Newsletter Nov- Dec 2009

syntactical limiters include braces, brackets, wildcards, quotes, and excluders. Each of these narrows the focus of the search.

Along with search engines, there are software applications known as “web crawlers” that locate and make available current events based on selection criteria. These software applications require more programming expertise and are not without cost and thus upgrades, and revisions. These applications are utilities that extract URL, meta-tags, plain text, page size, last modified date value from Web sites, Web directories, search results and lists of URLs from file. Thus, they are heavily dependent upon the depositor of the information’s creativity in labeling or timeliness in depositing information.

Each of the humanistic categories must be matched with current events. In other words, the current events that are found using Internet searches and/or web crawler applications are matched with the humanistic categories. Three current examples are as follows:

HUMANISTIC MATHEMATICS CATEGORY: Mathematics of Finance and Economics
CURRENT EVENT: Anthropogenic Oil Spill in the Gulf of Mexico
MATHEMATICS EDUCATION: Algebra
CONCLUSION: Students discover costs of real life anthropogenic societal events

HUMANISTIC MATHEMATICS CATEGORY: Mathematics of Change (and Growth)
CURRENT EVENT: Haiti Earthquake
MATHEMATICS EDUCATION: Pre-calculus
CONCLUSION: Students discover the devastation of an earthquake in relationship to seismic releases of energy/power.

HUMANISTIC MATHEMATICS CATEGORY: Mathematics of Symmetry (Shape and Form)
CURRENT EVENT: State of Glaciers in High Asia
MATHEMATICS EDUCATION: Geometry as Fractals
CONCLUSION: Students discover mathematics relationship to nature.

These three societal events are examples of how mathematics can be taught humanistically. Thus, the curricular and pedagogical environment is alive with events that have both affect, influences on emotion and culture and effect as it demonstrates changes that occur in human situations and circumstances that are of global importance. Thus, students become engaged in and aware of mathematics’ relevance, its humanity, and its global ability to make current events a mathematics classroom common experience.

Exploring the challenges of teachers' and learners' understanding of solution strategies using whole numbers

Tom Penlington
Rhodes University Mathematics Education Project
Rhodes University, Grahamstown, South Africa
t.penlington@ru.ac.za

Abstract
This paper is a qualitative study and reports on work with both teachers and learners in nine schools in the Eastern Cape Province of South Africa on mathematical reasoning and problem solving, specifically related to developing computational fluency with whole numbers. As a facilitator who has been involved with the upgrading of mathematics in-service teachers, the study involved both teachers and learners, being able to calculate in multiple ways by using a variety of solution strategies in order to solve problems. Fluency require of both teachers and learners more than just memorizing a single procedure. It rests on an understanding of the meaning of the operations and their relationship with one another.
The work with learners and teachers involved them working individually on solving contextual word problems. The study showed that in most tasks, learners relied heavily on procedural understanding at the expense of conceptual understanding. Traditional standard algorithms appeared to have been learned in isolation from concepts, failing to relate them to understanding.

Introduction
The rationale for choosing whole number computation and studying the solution strategies of teachers and learners was triggered by the poor TIMSS results and my interest in the problem-centred approach that I had used in my own class.
The Third International Mathematics and Science Study (TIMSS, 1996) and the (TIMSS –R, 1998) had found that the performance of South African students was placed at the bottom of a list of over 41 countries that participated in the study. Having worked with many in-service teachers and learners over a number of years, trying to developing their computational fluency and reasoning skills, involving the four basic operations, I have come across a plethora of diverse solution strategies that both learners and teachers use in solving these problems.

As both studies from TIMSS showed no real difference in the performance of learners, a study on whole numbers was developed to ascertain whether learners and teachers were able to use different solution strategies when solving these problems. Encouraging learners to develop and use their own solution strategies is regarded as consistent with a move from being 'teacher-centred' to a more process driven problem solving 'learner-centred approach (Southwood & Spanneberg 1996). The Revised National Curriculum Statement (2002:1) reaffirms this when it states, "the outcomes encourage a learner-centred and activity-based approach to education." With the learner at the centre of the learning process, much more emphasis is placed on learners developing conceptual understanding and learning computational skills (Bransford, Brown & Cocking 1999).

Context
The Rhodes University Mathematics Education Project (RUMEP) is an independently funded non-governmental organisation (NGO), linked to the university, with the specific aim of improving the quality of teaching and learning mathematics in schools. The Project had its beginnings, on an informal, regional level, in 1983 and grew so rapidly in stature and effect

that it became a teacher development institute of the university in 1993, as a formal numeracy project.
A major aspect of the project is the upgrading of both primary and secondary mathematics teachers. These accredited courses represent a direct response to the challenge of reaching the many teachers in deep rural areas who have not had access to in-service training, in the past. The ACE programme is offered through two blocks of university based tuition over two years of part-time study. During these contact sessions, teachers are immersed into all five learning outcomes in Mathematics developing both their mathematics content knowledge and pedagogy.

Theoretical perspective

This study is underpinned by the social constructivist model (Vygotsky 1978) and the problem-centred approach. The social constructivist philosophy emphasises language, culture and the social milieu. In the social constructivist model of the teaching-learning process, four key elements interact and affect each other – the learner, the teacher, the task and the context. As knowledge is socially constructed, the classroom is seen as an extension of the learners' environment. That knowledge which the learner knows is built on the existing knowledge gained through social interactions other than those found in the formal classroom.
Problem solving is consistent with the constructivist philosophy and my submission that learners are encouraged to invent their own procedures as advocated by (McClain & Cobb 2001) so that learners build their own meaning for themselves in order to better understand the concepts and skills of mathematics is pertinent here.

Methodology

The study is qualitative in nature and lies in the interpretive paradigm. It deals with individuals and is interested in describing processes rather than just an outcome or end result (Cohen & Manion 1994; Mwira & Wamahui 1995). Grounded theory was the underpinning methodology selected for the study. It was postulated by Glaser & Strauss (1967) and appropriate for the study as it was a small scale investigation into the solution strategies of learners. The initial justification for this research method was that I intended to develop categories of children's solution strategies. However, when examining the strategies, I found that the level of each task only allowed me to go as far as the first type of coding, namely open coding.
To ascertain the problem solving ability of learners and teachers and their ability to use different solution strategies, nine schools (both teachers and learners) in the Northern region of the Eastern Cape Province took part in the study.

The research instruments consisted of a test, individual clinical interviews with each of the nine learners and teachers and a structured teacher interview schedule.

The majority of the learners were isiXhosa speaking. However for transparency sake, all the questions were translated from English into Afrikaans and isiXhosa. The test consisted of 15 multiple choice type questions and 12 problem type word problems. Grade 7 learners were chosen because it was felt that learners at this age are able to articulate their thought processes and their communication skills are also sufficiently developed at this level.

My interest in the test was to find out how the learners and teachers confronted the problems and what strategies they had used which made sense to them. Further I was interested to see whether the solution strategies chosen, were the same or different from those their teachers had used.

The individual interview schedule was semi-structured in nature (Bogdan & Biklen 1992) and contained a checklist of suggested questions. However, not all questions were predetermined. This kind of interview allowed me to have more flexibility and freedom to explore the solution strategies adopted by the participants and whether they could explain carefully what they had done. The goal and emphasis of using the semi-structured interview was to probe for understanding.

A short, additional structured interview schedule was drawn up asking participant teachers to comment on aspects of the test. The intention behind doing this was to gauge whether the tasks were too difficult for their classes and whether translating the tasks into their mother tongue had any effect on how learners approached them.

Interpretation and discussion of solution strategies

The TIMSS Curriculum Framework was used in this study to place each task into categories and into performance expectations. The performance expectation component refers to the cognitive dimension and describes the kinds of performance or behaviours that might be expected of learners. I shall discuss three whole number problems from the test.

Task 1

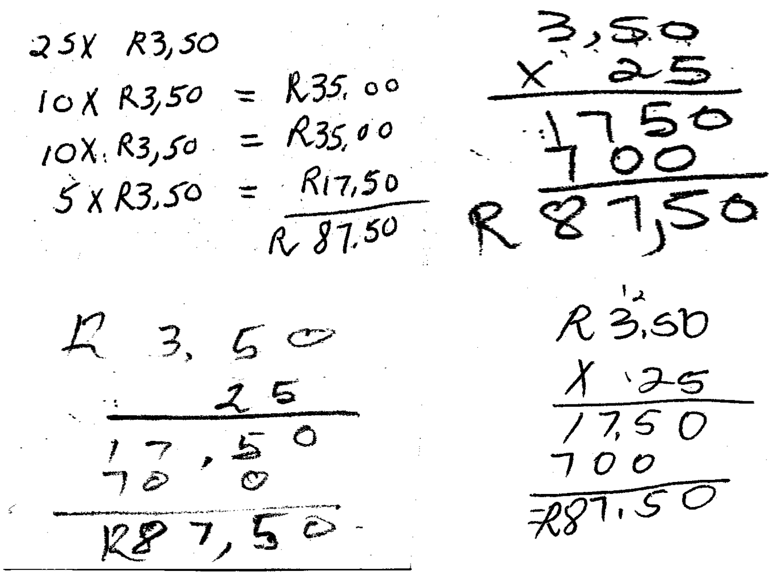
25 learners go on an outing to the beach. They each buy an ice-cream which costs R3, 50.
How much must they pay altogether?
Open coding was used and the following strategies were developed.

Learner	**CS**	**IS**	**Strategies**	**Teacher**	**CS**	**IS**	**Strategies**
1	√		Vertical algorithm	1	√		Vertical algorithm
2		×	Counting strategy	2	√		Decomposition of the multiplier
3		×	Vertical algorithm	3	√		Vertical algorithm
4	√		Mathematical model	4	√		Decomposition of the multiplier
5	√		Vertical algorithm	5	√		Vertical algorithm
6		×	Vertical algorithm	6	√		Fraction multiplication
7		×	Fraction multiplication	7	√		Decomposes multiplier
8		×	Counting strategy	8	√		Vertical algorithm
9	√		Vertical algorithm	9	√		Short method of X

Content area category	Number
Subcategory	Decimal Fractions
Subordinate subcategory	Properties of operations
Performance expectation	Using routine procedures
Subcategory	Performing routine procedures

Only 44% of the learners managed to solve this problem with the majority using the traditional vertical algorithm strategy. Only a few were able to explain the process involved. Many got lost along the way because they lacked the mastery of the multiplications tables. Strategies identified were: decomposing the multiplier, using the vertical algorithm, use of a mathematical model, a counting strategy and fraction multiplication. This example was correctly answered by all nine teachers (100%).

Teacher and learner strategies



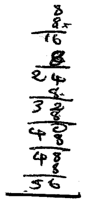
Task 2

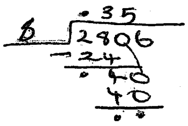
Farmer Zodwa and his workers pick apples. They pick 2 806 apples and place them in packets with 8 apples to a packet. How many packets do they fill? Are there any apples left?

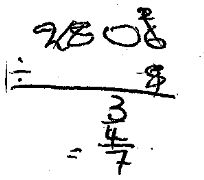
Learner	CS	IS	Strategies	Teacher	CS	IS	Strategies
1		×	Vertical algorithm	1		×	Vertical algorithm
2		×	Unclear strategy	2		×	Partitioned dividend
3		×	Unclear strategy	3	√		Partitioned dividend
4		×	Factors of 2 and 4	4	√		Partitioned dividend
5		×	Partitioned dividend	5		×	Vertical algorithm
6		×	Vertical algorithm	6		×	Vertical algorithm
7		×	Partitioned dividend	7		×	Partitioned dividend
8		×	Guess work	8		×	Partitioned dividend
9	√		Vertical algorithm	9	√		Vertical algorithm

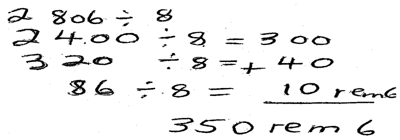
Only 11% of the learners solved this problem while only 33% of the teachers accurately answered it. Most learners relied on memorisation but got stuck halfway as they tried to solve the problem. Teachers still teach division as a ‘goes into’ model which we know is an ineffective model as digits are treated separately (Fosnot & Dolk 2001). Teachers tend not to emphasis the reciprocal relationship between multiplication and division or what the meaning of a remainder is.

Learner and teacher strategies







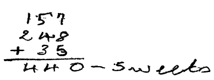


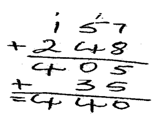
Task 3
Mrs Khumalo has a bag of sweets to give to her Grade 7 classes. She gives the first class 167 sweets and the second class 248 sweets. She then has 35 sweets left in her bag. How many sweets were in her bag at the start?

Learner	**CS**	**IS**	**Strategies**	**Teacher**	**CS**	**IS**	**Strategies**
1	√		Vertical algorithm	1	√		Vertical algorithm
2	√		Decomposition of numbers	2	√		Vertical algorithm
3	√		Vertical algorithm	3	√		Vertical algorithm
4	√		Vertical algorithm	4	√		Vertical algorithm
5	√		Vertical algorithm	5	√		Vertical algorithm
6	√		Vertical algorithm	6	√		Horizontal and vertical algorithm
7	√		Vertical algorithm	7	√		As above
8	√		Vertical algorithm	8	√		Vertical algorithm
9	√		Vertical algorithm	9	√		Vertical algorithm

There was a 100% success rate with this problem. This shows that some teachers spend more time teaching this operation at the expense of the other operations.

Learners' strategies





Findings

Most of the solution strategies that the learners used were straight forward procedures that they had learnt. They relied mostly on procedural understanding at the expense of conceptual understanding. The solution strategies of whole numbers adopted by the learners in the study were similar to the whole number solution strategies used by their teachers. A few teachers and learners did employ their own constructed solution strategies. They were able to make sense of the problems and to 'mathematize.' Language played a role in that learners sometimes struggled to communicate their thought processes in a coherent manner, even though the problems had been translated into mother tongue as well.

Challenges

Teachers should ensure that learners be given sufficient opportunities to solve problem-solving type word problems. Besides addition, the other whole number operations must also be given the recognition they deserve. In each lesson planned, time should be given to allow learners to master basic skills to enable them to use this information when planning their solution strategies. The need to take cognisance of language problems is a further aspect that teachers need to take into consideration. They should note that if word problems are written in English and mother tongue, reading, comprehension and encoding errors would be lessened. To make sense of the mathematics, it is important for English second or third language speaking learners to be allowed to rely on their own language, since background knowledge is the basis of any learning process in mathematics. The implication is that teachers need to spend time allowing learners to look at the processes involved in arriving at a solution, rather than just focusing on the solution. By allowing learners to develop their own solution strategies, teachers will be sensitized to the thinking and reasoning of learners as they strive to make sense of the mathematics. Teaching algorithms as fixed procedures restricts the thinking ability of learners to reason, communicate and consequently, their ability to do mathematics. Teaching for understanding should be emphasized at the expense of 'teacher taught procedures'. The idea of being computationally fluent will result in learners being able to explain and analyse their methods and this will result in their developing efficient, accurate and flexible strategies. As their knowledge of different strategies grows, so does their computational fluency.

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Stepping into Statistics: Providing a Head Start for students

Anne Porter and Norhayati Baharun

School of Mathematics and Applied Statistics, University of Wollongong, alp@uow.edu.au

Abstract

The major aim of this paper is to discuss the learning design of a head start introductory statistics module made available to students on-line. It explores the combination of different kinds of resources in particular genres of video to support learning and the learning of discipline content and processes. Different mechanisms for facilitating communication between students, and production issues and issues in merging the head start program with the introductory statistics subject.

Introduction

For the past 15 years the development of an introductory first year level university statistics subject, STAT has been guided by a quality cycle of planning, acting, reviewing and improving. Each year students valuing of resources has been evaluated in terms of their usefulness in helping students learn and understand statistics. Resources that have not been considered valuable terms of helping students learn have been reworked or replaced. For some years students have placed a high value on their laboratory manual, lectures, assessment, provision of marking criteria and fully worked solutions. Innovations have included the collection and use of real data and working topics of social significance. Students have since the 1990s have been provided with access to all resources and communication with other students on line through an E-Learning system.

The subject fits a traditional subject teaching profile in that it provides three weekly lectures (numbers 80-300 students) and two hours in a laboratory class (numbers 20-30 students). In this subject lectures are highly interactive even with large class sizes. New topics are started with an activity to elicit student ideas and these are then refined and formalized in terms of the appropriate statistical language and theory. Students work with a mix of real data gathered by them selves and data supplied by others. Strategies for learning and learning issues that arise are also explored. In recent years there has been an increasing numbers of students using online resources rather than face-to-face. In 2010 students with no previous failures were allowed to complete their laboratory work at home or in their own time. Only the lecturer of the subject is avaible outside of class time to assist students.

In the search for the best approach a variety of assessment tasks have been employed over the years. These have included assignments involving the collection and analysis of data, portfolios, summaries, tests and presentations to class and final examinations. Different weightings have been applied. The approach that has been adopted involves a final examination worth 50 per cent, a presentation where the student in a pair collects their own data and makes a presentation to class worth 10 per cent and four competency "tests" commencing in even weeks from the fourth week of a thirteen week session. Typically only the first of these tests is held within the class and the remainder completed by the students in their own time. The "tests" involve analysing data and working with theory, like an assignment but the allowed time is a few days. The first tests have a rapid turn around in terms of marking and students are provided with fully worked solutions. In odd weeks the students who do not gain 70 per cent are required to undertake a similar test, and those not passing the second test are identified as "at risk" students. The "at risk" students are given more support through directing them to resources, interviewing the student, or direct comments on the work. In the final week, make-up tests are available to all students.

In 2009 the online resources were expanded to include a variety of video clips relating to different topics. Early approaches to providing resources to students included grouping all similar resources for example lectures, video clips and data.

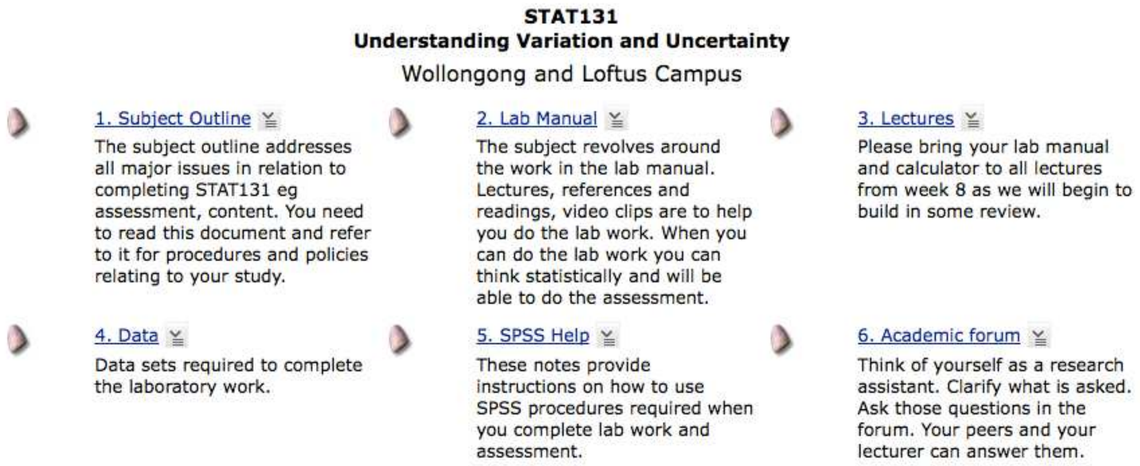


Figure 1. Organisation of resources for students by common category

An alternative approach has been to Group resources by week as in Figure 2.

Figure 2. Organisation of resources by week.

A third option has been to group both by category of resources and by week.

Student change evaluations revealed that when the students used the video clips they were considered extremely useful in helping students learn and understand statistics. However not all students *found* the video clips. As a consequence in 2010 the innovation focus was on the learning design itself. In this introductory subject students were provided with a *learning design visual sequence* illustrating the “chronology of tasks, resources and supports” (Aghostinho et al, 2009). Originally conceived of as a learning design from the perspective of the teacher, in this subject it was used to communicate with the students, the tasks they had to complete, the primary resources and support materials if they needed additional resources, see for example Figure 3.

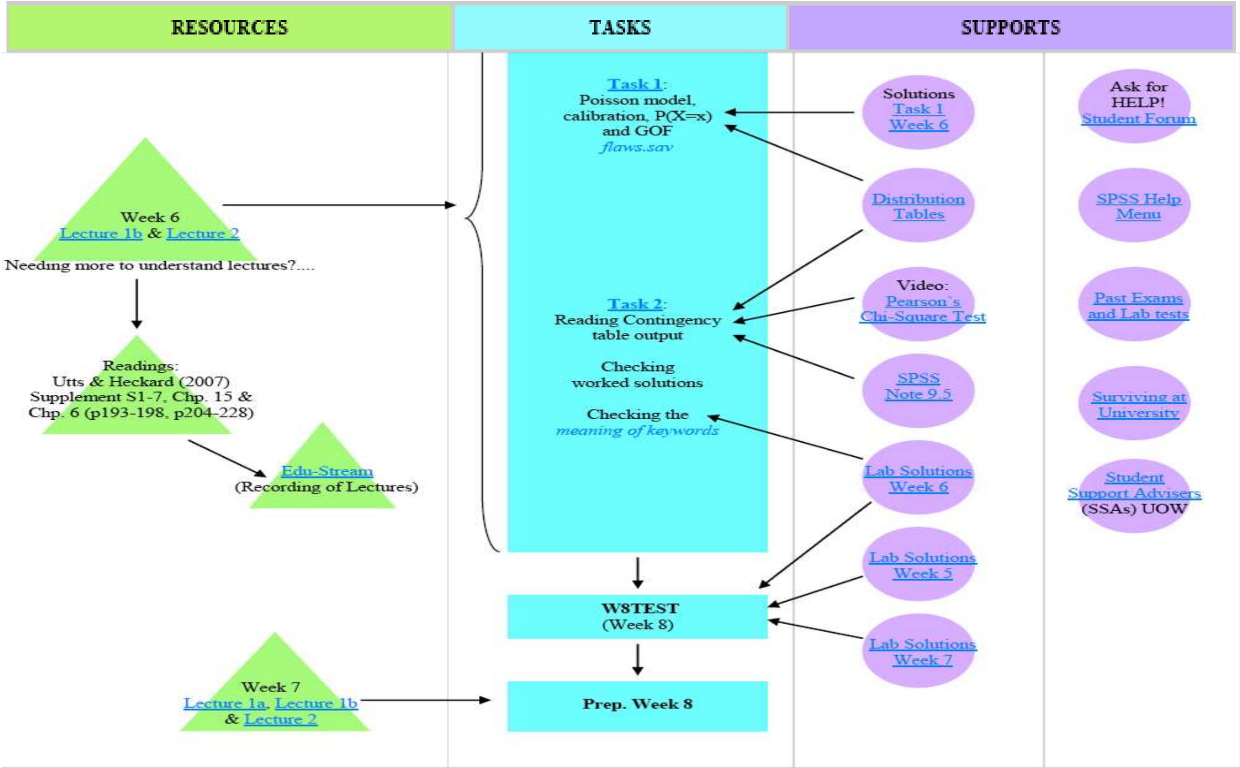


Figure 3. An extract of a learning design visual sequence for a week of STAT (Porter & Baharun, 2010)

The visual learning design maps linked all student supports, resources and tasks. The competency based assessment system allowed early identification of students at risk, that is those who did not satisfactorily complete any test/re-test sequence. Video clips were used to provide students with discipline support and this was supplemented by direct communication, both email and in-person, between the lecturer and at risk students. The map provided a clear picture of what was required, when and what help was available.

One outcome of trying new approaches is that as a teacher one can see what was previously not evident. It makes no sense that students would enroll in a subject, pay fees and walk away with a zero, low mark and a fail grade. There are students who are inactive, do not come to class and do not submit assignments that feasibly deserve such a grade. For some years there has been an increase in zero/low mark fails typically interpreted as students who cease to be active in the subject but who do not withdraw. How to engage these students has inspired innovations in the statistics subject for the past fifteen years. As teachers we have worked with the mantra “If we can get students to class we can assist them through the learning.” At a graduate level the innovations with on-campus, distance and international students have been highly successful with many learning outcomes including a low failure rate (Baharun & Porter, 2009). At the undergraduate level the learning design map teamed with competency-based assessment highlighted another group of students, seemingly with learning difficulties, namely students who attend lectures and laboratory classes, are active in class answering questions, but who have difficulties submitting assessment even after guidance and with the possibility of re-submission.

Non-submission of assessment is the key indicator of pending student failure. For this group of students being able to communicate what they know through the assessment system appeared problematic. However, at the end of session students with the exception of one, unanimously endorsed the assessment system. Contrasting with this, approximately two-thirds of completing students indicated that they would like access to a head start program. Students wanted more time to cover the curricula. Hence, the development of a Head Start module.

Head-Start Module

The Head-Start module implements a draft-final assignment model rather than the test-retest model of STAT, potentially a more positive experience than “failure” feedback repeat test. This should assist more fragile learners. The module is accessed on-line after students have enrolled in the subject. For students electing to complete the module, they will have approximately three weeks head start and will be able to complete a first piece of assessment which can substitute for the first in-class test in the subject proper. The first two weeks of session then become consolidation of the Head Start program (See Figure 4.)

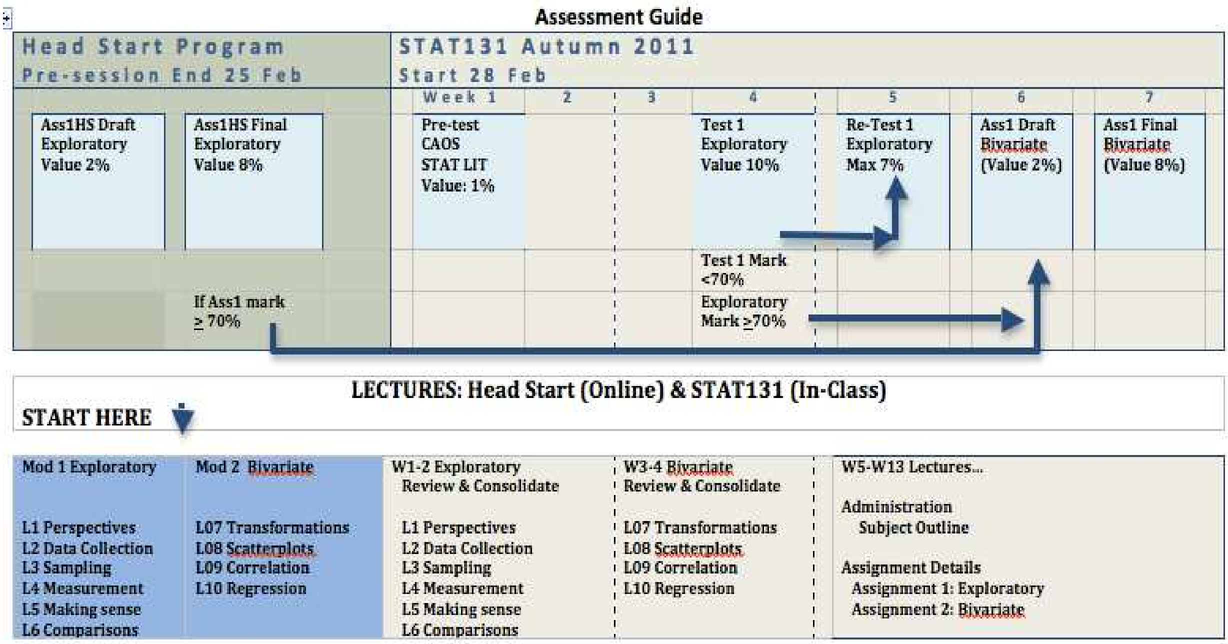


Figure 4. Merging the head-start module and STAT subject

From the outset a pattern is established for the Head Start on-line lectures.

1) Orientate the learner with text and video

> ***Orientation:*** *In 2010 approximately two thirds of the students completing STAT131 indicated that they would like to be able to commence STAT131 before the official start to classes. They felt too rushed to take on board the new skills. The Head Start Program Stepping into Statistics allows students to …*

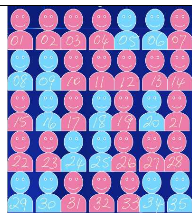
2) Provide an activity to get students communicating with each other

Task 1 Meeting colleagues and identifying your pre-conceptions about Statistics Introduce yourself to other students in the academic forum. Without looking at any text or what other students say write down: What you think statistics involves? Do you have any concerns about studying statistics? What you are looking forward to in studying statistics? See what others have to say. Review your answers at the end of session.

3) Provide an activity to address a relevant learning issue

Task 2 Experience the rush Many students find complex, multistep math problems particularly difficult. The problem set on the next page is designed to evoke in you the feelings a student might feel working out a problem that requires combining and using several mathematics skills in a short period of time. Give yourself **two minutes** to solve all three problems. **Adapted from** **http://www.pbs.org/wgbh/misunderstoodminds/math.html**	Follow all four instructions below to solve each of the three problems. Record your answers. A. Multiply the third number in the first row by the seventh number in the third row. B. Add this result to the fifth number in the second row. C. Add to this total ten times the fourth number in the third row. D. Subtract the eighth number in the first row from the result. Problem 1: 6 5 8 7 4 5 6 8 4 3 2 1 9 5 6 4 2 1 6 5 1 5 1 3 2 3 5 etc Debrief

4) Provide a sequence of tasks for the students to complete that cause them to interact with the discipline issues, content and processes.

Task 3: Not comfortable accepting statistical results Many people are uncomfortable when research outcomes are based on a **sample** or portion of the entire **population** for which data on some **variable** are collected. The video clip has an exercise where you conduct a census. Do this and answer the following: Why it is often acceptable and maybe even better to collect data using a sample rather than a census?	
Task 4: Simple Random Sample For a sample to represent the population you must use a probability sampling plan. Here we have a population of 35 men and women, they represent your sampling frame. Your task is to take a simple random sample of size 7 using a random numbers table. What observations are in your sample? How did you get them?	 Population of 35 Random Numbers Table

Conclusions

At this stage evaluation remains at the design level. From the perspective of Boud and Prosser (2001) to have the potential for high quality learning the design must address engagement of learners, consider the implementation within the broader learning context, challenge learners seeking active participation supporting students ampliative skills and provide practice. The Head Start module relies heavily on a variety of resources combined with different types of activities to connect students with their discipline and everyday life decision-making. The video resources include orientation chats, worked examples to demonstrate calculations, theory snippets, stimulus materials, procedural matters such as taking a sample (Task 4) or using statistical packages and interpreting output . The opening forum communication, Task 1, sets the scene for students to begin to explore their perspective as to what statistics involves and this is followed by activities that challenge narrow definitions of what statistics is about. Activities such as those in Task 3 where students conduct a census and discover that they have probably not "counted" correctly and that maybe sampling is a good approach to collecting data are to engage students, through providing an activity that challenges their perceptions. Throughout the module students are provided with practice, working with data and theory and debriefing through video clips.

Students have only just commenced the head start module and further evaluation will be required to explore whether the uptake of the program and the benefits to students has warranted its development.

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Transforming Mathematical Tastes: a Twist of Lemon – or a Pretzel?

Shirley Porter, B.A., Trained Teachers Certificate,
Certificate in Adult Learning and Teaching, Senior Academic Staff Member
Kahurangi Student Services, Bay of Plenty Polytechnic, Tauranga, New Zealand
shirley.porter@boppoly.ac.nz

Abstract

Since the early 1990s, the nature of students enrolling in universities around the world has been changing [(1)]. In New Zealand, the initial government-led focus was "bums on seats" with a drop in prerequisites, open entry and a rise in the average age of students. A subsequent focus, underway now, is "success and retention" – in order to attract the maximum funding, tertiary institutions need to ensure the students achieve. Meanwhile secondary school mathematics has been changing its attention from abstract to more utilitarian content [(2)]. When students arrive at tertiary institutions, many have insufficient mathematics understanding and skills to cope with the demands of such courses as economics and engineering. "Service mathematics" courses have been designed to counteract this lack of mathematical ability. This presentation considers two such courses offered at Bay of Plenty Polytechnic in Tauranga, New Zealand, taught in a manner mostly traditional for secondary but not tertiary level, but with a "twist", in an effort to bring students up to speed. A survey of the students over two years provides evaluations to complement this reflective overview.

Introduction

Bay of Plenty Polytechnic is a small institution, situated in Tauranga in the North Island of New Zealand, with approximately 8000 full and part time students. Tauranga (population 140000) is not a university city but the Polytechnic has degree agreements with several institutions across New Zealand. I am in my 12th year as a mathematics tutor and advisor attached to the Kahurangi Student Services department of the Polytechnic. During this time I have received two peer-nominated sustained teacher excellence awards (2009 and 2005) and one student-nominated exceptional adult-educator award in 2002. I work with students from all programmes where mathematics is involved. The range is broad; from learning multiplication tables and understanding division to calculus and complex numbers and often in context. I have a passion for mathematics; many of the students have a "maths phobia" and I have been particularly interested in analysing how best they learn. In researching this, I became aware of the "mathematics problem" which appears to be world-wide. Students have been arriving at tertiary institutions, enrolled in courses which require a reasonable level of mathematics competency, but without the necessary skills and often with negative attitudes. I have investigated within our Polytechnic with my two classes that I tutor to research how to best teach to alleviate this problem. In my presentation I will discuss my experiences, the papers I teach, and the results of surveys I have administered.

Transforming Mathematical Tastes: a Twist of Lemon – or a Pretzel?

"I hate maths. I can't do it. I'm only taking this paper because it's compulsory. So long as I get a C, I'll be happy", announced a student (I'll call her Marie) as she entered my management mathematics class for the first time. As I learned later, Marie was an A+ student in all her business, management and accounting subjects. What a tragedy that such an obviously bright person had such an attitude towards mathematics!

This attitude can often be traced back to one particular mathematics teacher at secondary or even primary level. Duncan remembered a teacher in year nine "torturing" him with his sarcasm when he gave a wrong answer; Lauren recalls her teacher telling her at the age of six to "go home and play with your dolls. You will never be good at maths". Yet both of these people now have high qualifications, but certainly not in mathematics. Of course, not all students dislike mathematics. In many cases the students arriving at tertiary institutions "like" mathematics but have either never advanced to any great level for a variety of reasons or they last attended school many years ago and have long since forgotten most of the mathematics they learned. Other students, more recently at secondary school, have become so dependent on graphics calculators (poorly taught and poorly used) that mathematics has become like the "Yellow Pages" – their fingers do the walking but their thinking brain is not involved. When they can no longer reach for their crutch, they topple. Added to this is a unit-based assessment system at secondary level which partitions mathematics and fails to produce a coherent overview. Secondary students can achieve sufficient credits, yet still be inadequately prepared for tertiary demands.

The "Mathematics Problem"

Students are enrolling in programmes such as business and engineering which require mathematics content well above their level of expertise. There have always been some students in this category but over the past 20 years the problem has grown exponentially. This "mathematics problem" is not confined to my institution, or to just New Zealand; it is of global proportions. In the UK in 1995, the Engineering Council [3] concluded that "students are now accepted on engineering courses with relatively low mathematics qualifications". In the U.S., also in 1995, a Survey on Remedial Education in Higher Education Institutions [4] found that 78% of higher institutions had found the need to offer remedial courses in reading, writing or mathematics. In 2004, also in the US, the Committee on the Undergraduate Programme (CUPM), reporting on four years of findings, discovered that twice as many students were enrolled in "remedial" or "introductory" mathematics courses compared with those in calculus or statistics courses [5]. They noted that the remedial courses were particularly challenging to teach because of the diversity of mathematical backgrounds of the students, who also brought with them negative attitudes created by past experiences with mathematics.

From a diversity of countries, e.g. Ireland, Australia and Canada, there is a body of literature that documents the "mathematics problem". Neither is it confined to English speaking countries. At the International Congress on Mathematical Education (ICME-9) held in Japan in 2000, a group which was very broadly international, identified as a serious issue the challenge posed by non-specialist mathematics students requiring service mathematics [6]. Here in South Africa, Engelbrecht and Harding in 2008 acknowledged that university lecturers could no longer assume that students have certain mathematics skills [7]. Also in South Africa, in 2010, Winnips, Brouwer and Mwambakana confirm there is a "mathematics problem" and have held workshops on using e-learning to improve the situation [8].

In New Zealand the problem has become even more pronounced because of a change in government policy. Originally the attitude was "get them off the dole and into education", the so-called "bums-on-seats" policy. The current government has refocused and the policy almost underway is "funding according to success and retention". In fact, the government now requires us to have a "good idea" that students have the capability of succeeding before they can be enrolled. The government has spent millions of dollars investing in the development of "The Tool" which diagnoses students' literacy and numeracy at an elementary level. The Tool is designed to (a) help prevent students from enrolling in courses for which they have little hope of success; and (b) indicating to both tutors and students the personal specific mathematics skills requiring extra attention.

Institutional Intervention

What steps have we been taking at Bay of Plenty Polytechnic to alleviate this "mathematics problem"?

Unfortunately, due to IT constraints, The Tool is not administered to students until after they are enrolled. However, difficulties and knowledge gaps are highlighted, tutors can become more informed about their students and assistance is offered by our Student Learning Centre (Kahurangi). The drawback of The Tool is that it is limited to an elementary level and assesses whether students have sufficient mathematics to succeed at levels 1 - 3 programmes such as carpentry and electrical level 3. It does not assess whether students have the skills to achieve at higher level Diploma or degree papers.

The current policy within Kahurangi is to focus on providing assistance within the classroom or in workshops specific to an identified problem area. There are fewer one-to-one appointments available due to resource limitations. The main focus is on levels 1 - 3 as per a government directive. This makes the two papers that I teach even more important in helping remedy the mathematics problem at higher levels.

Four years ago, as a mathematics advisor in Kahurangi, I supported students enrolled in a Management Mathematics paper which is taught under an articulation agreement with Waikato University in Hamilton, NZ. It became obvious that some of the students were inadequately prepared to meet the demands of the university paper, which itself was a "service mathematics" paper and had been designed specifically to support those students who needed a higher level of mathematics to cope with their business, economics or management degrees. Subsequently, I wrote a "Mathematics Bridging" paper, tailor-made for the Management Mathematics students, to be taught over three weeks at Summer School. Interestingly, although the paper targeted Management Mathematics students, it has proved to be suitable in content for Engineering students as well. This paper, worth ten credits, is a "short course" for which government has cancelled all funding but I am still able to run it as I am paid to support mathematics and it is classified as "support". It is free for the students and they receive a certificate upon successful completion, although the credits do not count towards any other programme. Ten topics are covered in ten days, and there is a test each day on the previous topic with a chance at a resit in each topic halfway through and at the end. It is an intensive pressure paper and serves as a great refresher for those students who have studied successfully at year 12 level or equivalent somewhere in their past (year 13 is the last year of our secondary schooling).. For those who had never reached that standard, the paper is formidable indeed. With absolute dedication, some students with little background do achieve a pass but it has become obvious the fast pace is not suiting the needs of all the students.

In response to this, I am now also running the Mathematics Bridging paper across a whole semester in the evening and the students have a two-week time span in which to digest the concepts and content of each topic. This mode of delivery will be evaluated at the end of the year. It may be that there is a need for a simpler paper to be provided for those who struggle even at the slower pace.

Over the past three years, while I have been teaching the Management Mathematics, all students who completed the paper passed except for one student who rarely attended lectures and was dependent on his (poorly taught) graphics calculator. The paper is assessed using ten assignments (one per week), two short tests and an examination of three hours. Of the ten assignments, the two lowest are discarded, a feature which the students felt was very fair, allowing for their pressure times or personally difficult topics. (As a Polytechnic, a good proportion of our students are more mature, returning to education to either upskill or begin a new career, but often with family commitments making a demand on their time). Alternatively, the pass mark can rely entirely on the examination, a helpful situation if there has been a crisis in the student's life. The teaching programme was also planned to allow for in-class practice time before tests and the examination. The classes were small (9 to 14 students) and this lent itself to a tutorial style classroom where questions could be raised and answered.

These two papers are a response to the mathematics problem. As well, in a dual project with a colleague, I am helping develop an online diagnostic system specific to each paper that has a mathematical content. Students will be provided with an analysis of their readiness to cope with the mathematics involved in their prospective courses of study. They will then be supported by eMathematics workshops; this should be a successful intervention method for some students. There will still remain a body of students who need an oral explanation and Kahurangi will provide the back-up if the students have enrolled.

The Survey and Key Results

I surveyed the students at the end of the semester using an online survey. Not all students returned the surveys, with non-returns mainly in 2010 when I was delayed from sending out the surveys. In the survey I explored their mathematics background including their feelings towards mathematics at the start of the paper and what changes had occurred by the end of the paper. I investigated their learning needs and styles, their thoughts on the teaching style, whether they used the extra support from Kahurangi, their opinion on the assessment structure and any external factors that had impacted on their progress. The Bay of Plenty Polytechnic also undertakes an evaluation by the students regarding the paper and the teaching. This is completely anonymous and provided not just further data but a check on the reliability of the survey responses.

The mathematics backgrounds of the participants were varied, ranging from being successful at year 12 in secondary school to never being successful in mathematics. Accordingly, feelings towards mathematics ranged from "I love it" to "I am terrified of it". By the end of the paper, everybody had positive feelings towards mathematics and felt confident approaching it. "I feel far more confident in all areas" was a typical comment. Students were quite clear that their learning styles were a mixture of, particularly, visual and aural but often kinesthetic as well. They were happy that all of these had been catered for in the teaching. They particularly liked the build up on the whiteboard of the processes for solving problems, then having the time to practice in class, even discuss it with another student. No one wanted PowerPoint presentations, (and none were given), which is probably a reaction to many hours spent in other lectures where they were subjected to a misuse of PowerPoint. (It can be a great tool when used imaginatively). They were not aware that I analysed every step of the process and identified the basic skills embedded. It takes just a moment to run over something like $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$ before proving the completion of the square to find maximum/minimum values. If they lose steps like these, they lose the whole proof while they are worrying over basic algebra.

The students really appreciated the relaxed atmosphere in the class, feeling comfortable enough to ask any question. In the first class, I always run an Ice-breaker so everybody knows a little about everyone else. The teaching language used was important and I made a conscious effort to identify new vocabulary, especially in a topic like calculus. Time spent in class on revision was much appreciated; it helped “pull it all together”. Students having difficulty grasping concepts were able to book extra time with me through Kahurangi on a one-to-one basis early on before the problem became insurmountable. According to their survey responses, this was essential. The students recognised the sound organisation, the provision of extra resources for practice and the prompt return (by the following lecture) of their assignments. Of course, this last factor was made possible by the size of the class.

Conclusion

Although the classes surveyed were small, extrapolating the findings to larger groups underscores the usefulness of studies such as this. In many ways, the lectures were very similar to the “chalk and talk” often seen in secondary school. The differences are subtle, such as ensuring everyone feels they can achieve despite their backgrounds. The students learn early on that their progress is important to both them and me and are willing to be helped, even if they have a negative attitude towards mathematics itself. The content is thoroughly analysed and considered and taught well. There is time to be assisted, whether in class from peer or tutor, or through Kahurangi.

It is not just one magic factor that has made this class successful, it is a conglomeration of factors interwoven like a pretzel. In my experience, it eventually depends on the attitude of the students. Those who enjoy mathematics and have had success in the past, given that there is “quality teaching”, will achieve a good result. For those who arrive with negative backgrounds, learning that their progress is valued and that there is help available can be sufficient for students to overcome their personal barriers – and perhaps that is the dash of lemon that makes the difference. The last words must go to my student, Marie, who passed with her usual A+, not the C she was aiming for. She wrote “I now realise I can do anything I put my mind to”. That is a true transformation!

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Tangram-base Problem Solving in Radical Constructivist Paradigm: High School Student-Teachers Conjectures

Medhat H. Rahim, mhrahim@lakeheadu.ca, Radcliffe Siddo, rsiddo@lakeheadu.ca, Moushira Issa, missa@lakeheadu.ca

955 Oliver Road, Lakehead University, Faculty of Education, Thunder Bay, Ontario, Canada P7B 5E1

Abstract

A series of tangram-based problem solving tasks, focusing at visual geometric construction and justification by high school teacher candidates, are reported. Sociocultural and psychological components of von Glasersfeld Theory of *radical constructivism* have been utilized. The purpose was to describe and analyze how students' cognitive constructions have been initiated, modified, and re-modified as they were proceeding in their attempts to solve and justify spatial tangrams-based problems.

Background

The 'tangram' has been originally referred to as the 7-pieces dissection (or tangram problem) consisting of seven flat shapes forming together a square shape (five triangles: two identical large, two identical small and one medium triangles; a small square and a parallelogram). It was originally invented in China at some unknown year in history, and then carried over to the world by trading ships in the early 19th century to become well-known since then (Wang & Hsiung, 1942; Read, 1965). In particular, assuming that the small square has an area of one unit square then each large triangle has an area of two unit square and each small triangle of an area of half unit square and medium triangle of an area of one unit square and a parallelogram have an area of one unit square too. The objective of the tangram problem, often called tangram puzzle, is to form a specific shape, given only its outline or description, using all seven pieces with no overlapping (Figure 1).

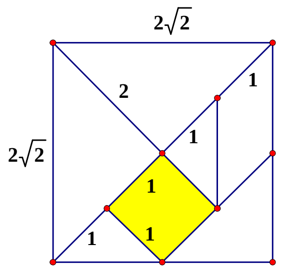


Figure 1: Seven pieces tangram set resembled into a square of side $2\sqrt{2}$ units

Clearly a given tangram set is not confined to a particular rigid and fixed shape formation such as the case of a "jigsaw" puzzle set of pieces, rather the pieces can be rearranged into several geometric shapes each of which is area equivalent to the original tangram square. Meanwhile the new constructed shapes may be convex: a shape is convex if the line joining any two points within the shape falls entirely in the shape; otherwise it is a non-convex shape. This tangram shapes formation represents a rich environment through which shape-to-shape, shape-to- parts and part-to-part interrelationships would be explored by the learners and possibly at all levels of schooling, from pre-kindergarten to university teacher training classes.
The literature provides a variety of examples using the seven pieces tangram set. The pieces may be rearranged into other distinct shapes each of which contains also seven pieces yet are of equal

area of the original seven pieces tangram set whether a resultant shape is convex or not. Below are the *only* thirteen convex shapes that are the resultant of the shape formation using the seven piece tangram (see Wang & Hsiung, 1942, p. 596; Scott, 2006, p. 5).

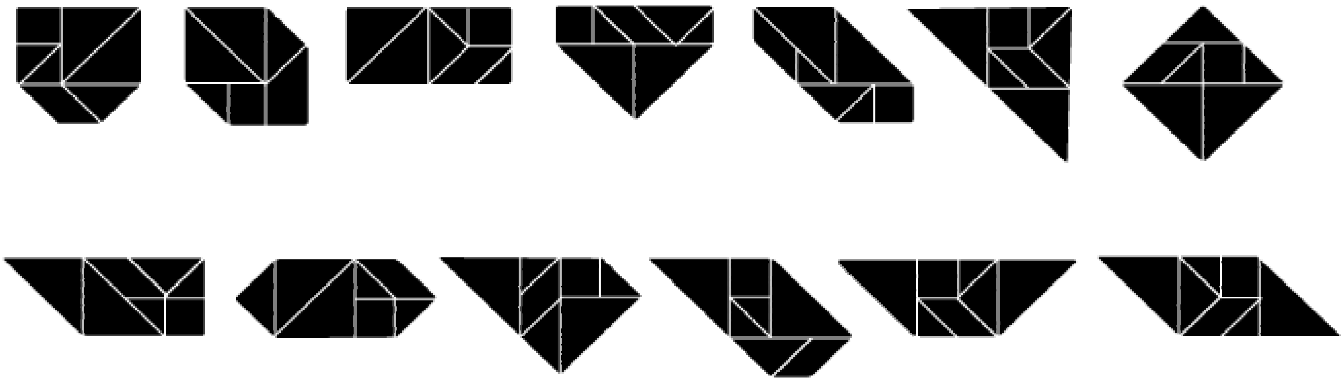


Figure 2: The thirteen convex shapes made out of the seven pieces tangram set

Also, tangram shapes formation may be carried out by using only two, three, four, five or six pieces in addition to the whole seven pieces.

Radical Constructivism: Sociocultural and Psychological Mechanisms in Justification

It has been suggested by von Glasersfeld (1995) that, based on his radical constructivism theory known as "*Radical Constructivism: A Way of Knowing & Learning*", when individuals deal with the physical world, their minds construct, through certain mental mechanisms and collections of cognitive structures, their conceptualization, reason, and coordination of their engagements (von Glasersfeld, 1995; 1984; 1974). Battista (1999) has described the notion of *abstraction* as the process through which the mind selects, coordinates, unifies, and registers in memory a collection of mental acts that appear in the attentional field (p. 418). Further, Battista (1999) has referred to von Glasersfeld's (1995, p. 69) ideas of *abstraction* and added that abstraction has several levels: At its *perceptual level* (most basic), abstraction isolates an item in the stream of an experience and seizes it as a unit. Battista added that material or entity is said to have reached the *internalized level* whenever it has been sufficiently abstracted so that it can be re-presented (re-created) in the absence of its perceptual input. Material or entity is said to have reached *interiorized level* whenever it has been disembodied from its original perceptual context and it can be freely operated on in imagination, including being "projected" into other perceptual material and utilized in novel situations (Battista, 1999, p. 418). Earlier, Steffe & Cobb (1988) asserted that *interiorization* is "the most general form of abstraction; it leads to the isolation of structure (form), pattern (coordination), and operations (actions) from experiential things and activities" (p. 337). von Glasersfeld, in presenting his radical constructivism theory, stated that *understanding* requires more than abstraction; it requires *reflection* which is the conscious process of re-presenting experiences, actions, or mental processes and considering their results or how they are composed. *Reflective abstraction* takes mental operations performed on previously abstracted items as elements and coordinates them into new forms or structures that, in turn, can become the content -what is acted upon- in future acts of abstraction (von Glasersfeld, 1995, p. 69). Battista (1999), in reporting his 3D cube arrays' study, suggested that besides von Glasersfeld's (1995) list of mental mechanisms that includes abstraction and reflection mechanisms there are three additional mechanisms that are fundamental to understanding students’ reasoning. They are *spatial structuring, mental models, and schemes* (Battista, 1999; Battista & Clements 1996). *Spatial structuring* is the mental act of constructing an organization or form for an object or set of objects. It determines an object’s nature or shape by identifying its spatial components, combining components into spatial composites, and establishing interrelationships between and among components and composites. *Mental models* are nonverbal recall-of-experience-like mental versions of situations; they have structures isomorphic to the perceived structures of situations they represent (Battista, 1999, p. 418; 1994). Mental models consist of integrated sets of abstractions that are activated to interpret and reason about situations that one is dealing with in action or thought. A *scheme* is an organized sequence of actions or operations that has been abstracted from experience and can be applied in response to similar circumstances. It consists of a mechanism for recognizing a situation, a mental model that is activated to interpret actions within the situation, and a set of expectations (usually embedded in the behaviour of the model) about the possible results of those actions (Battista, 1999). Meaningful learning occurs as students make adoptions to their current cognitive structures as a result of their reflection on an experience (Steffe, 1988; Battista, 1999). An accommodation is triggered by a *perturbation* which is described as a disturbance in mental equilibrium caused by an unexpected result or a realization that something is missed or does not work (von Glasersfeld, 1995, p. 67). Perturbation arises when students interact with other individuals or with the physical world (Battista, 1999). von Glasersfeld (1995) made a clear distinction between teaching and training stating that, “From a educator point of view one of the most important features of radical constructivism is the sharp distinction it draws between teaching and training. The first aims at generating understanding, the second at competent performance” (p. xvi). Further, von Glasersfeld in referring to learning mathematics stated that “To know mathematics is to know how and why one operates in specific ways and not in others, how and why the results one obtains are derived from the operations one carries out” (p.xvi).

The Classroom Sessions

First Session**:** Through the classroom problem solving sessions using the seven pieces tangram set, the set was introduced to the students by providing each of them with a colored plastic made model of the seven pieces tangram. The students were instructed that these seven pieces together can be manipulated through motions of translation, rotation and/or reflection to formulate other geometric shapes without overlapping, where the area of each of the resultant shapes made by the seven pieces, is invariant. The students were encouraged to use these seven pieces collectively to form new shapes of equal area. Each of which must be a simply polygon. The instructors explained the concepts of a simple polygon as opposed to a non simple polygon. The students were curious to know the difference and for that the instructors elaborated in saying that a simply polygon is any polygon in which each vertex is created by only two external sides. That is a vertex can have no more than two sides passing through it. This emphasis on the concept of simple polygon is essential since the concept of non simple polygon is rarely included in the middle/high school curriculum. Further, the concept of convex and non convex was also introduced.

Second Session*:* The students started forming as many shapes as they can within the instructions given above using the plastic tangram pieces. They were encouraged to ask questions and were advised to trace and make copies of their constructions. As the students being familiar with the Geometer’s Sketchpad (GSP) they were encouraged to use GSP to construct digital images for their shape formations based on the seven pieces tangram.

Third Session: This session was a continuation on the second session but through the computer lab applying GSP. During the computer lab session, the students were challenged with the following question: ***"Given a simple polygonal region and assume it is dissected into a finite number of sub-regions, then what is the maximum number of sides for a simple polygonal region that can be constructed using all sub-regions? Make a conjecture."*** The students then were allowed to take the tangram pieces home for further shape construction, refinement, and show-and-tell opportunity at the following class session. Further, the students were told that they were about to introduce a *new mathematical proposition* should their answer be verifiable. At the tail of the session, one student has asked: "so how far we can go with the number of sides of the new tangram?" The instructor replied: "you may go as far as you see it possible; as a hint, the maximum number of sides possible for a simple polygon is more than twenty!"

Students' Uses of Radical Constructivist Paradigm

Due to the space restriction, two case studies are presented.

Case Study 1: Student, Lee, using the tangram pieces, she has come up with an answer to the question: "*What is the maximum number of sides for a simple polygonal region that can be constructed using all the sub-regions? Make a conjecture as an answer to this question.*" Through a series of constructions started with a square, rectangle, right triangle, parallelogram, trapezoid, pentagon, hexagon, heptagon, octagon, nonagon, decagon, hendecagon, dodecagon, but then Lee emerged with an answer to the question above by stating: "shape with maximum number of sides that can be produced with tangram set is 23 sided polygon." Lee also stated: "the sum of the number of sides of all 7 pieces tangram is equal to the number of sides of the shape that contains the maximum number of sides that can be possibly created using these pieces". She further stated: "Conjecture: **Several shapes can be combined to form a simple, closed shape with maximum number of sides n, where n is the sum of all the sides contained by the sub-shapes altogether**" (Lee's bolding). Lee then offered the figures shown below in Figure 3 a & b.

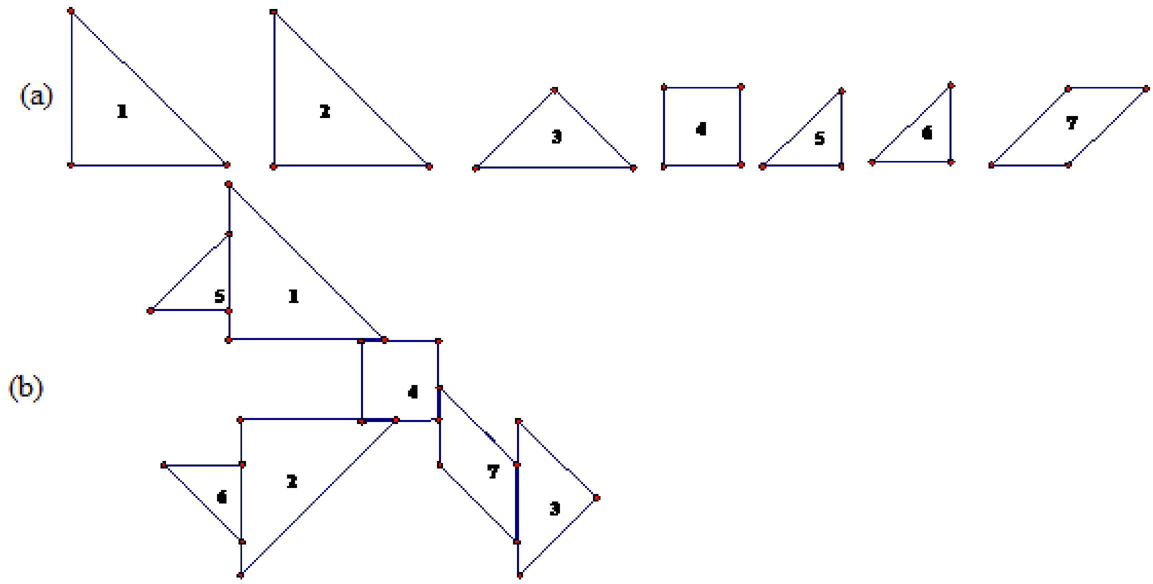
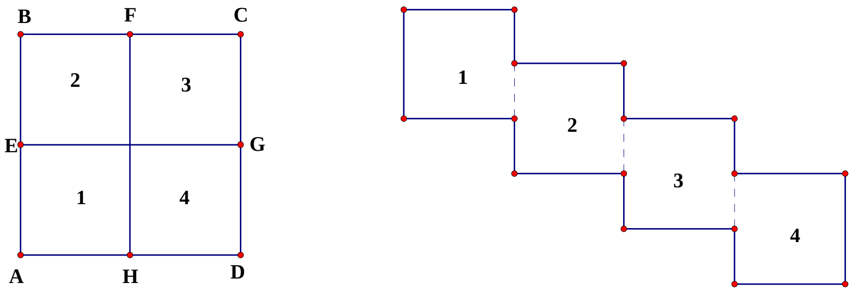


Figure 3: Lee's conjecture

Case Study 2

Student, Brooks, came up with an interesting method for his conjecture in stating that, “for a given square ABCD when dissected into four congruent squares symbolized as 1, 2, 3 and 4 as shown in Figure 4a, the four squares can be rearranged into the shape shown in Figure 4b, and hence, in his words: “*The highest sided polygon is equal to the cumulative number of sides of all of the individual shapes.*” Clearly, Brooks meant by his statement that the ultimate created shape using the resultant pieces by dissecting the given square has to have its number of sides to be equal to the total number of the sides of all resultant pieces due to the dissection process. Brooks did not further elaborate on the case when the square ABCD contains the 7 pieces tangram set. Nonetheless, his spatial structuring presents an elegant answer to the question presented above.



(a) **(b)**

Figure 4: Brooks Conjecture

Epilogue

As a reflection on the two cases, out of several cases, within the experimental observations reported, it would seem clear that the von Glasersfeld’s radical constructivism theory has been present throughout the students’ work on the tangram 7-pieces set activities. Our interpretation of the student, Lee, is that: Lee’s processes described in Case Study 1 above (Figure 3 a & b) was directly exemplifying Battista (1999) and Battista & Clements’ (1996) interpretation of *spatial structuring* concepts. Evidently, her mental acts of constructing an organization or a form for an object (or set of objects) has been by determining the object’s nature or shape, identifying its spatial components, combining components into spatial composites, and establishing interrelationships between and among components and composites. Further, Battista (1999) indicated that material or entity is said to have reached *interiorized level* whenever it has been disembodied from its original perceptual context and it can be freely operated on in imagination, including being "projected" into other perceptual material and utilized in novel situations (p. 418) – Lee was virtually acting along this path. The use of *reflective abstraction* through her mental operations, performed on the previously abstracted items (the 7-pieces shown in Figure 3a), seems to be her building elements that she coordinated into new forms or structures (von Glasersfeld, 1995, pp. 69-70). Then, she coordinated these abstracted pieces or elements to come to make the new form or resultant shape, the 23 sided figure, depicted in Figure 3b. Her acts on the 7-pieces were clearly and directly performed exemplifying Battista’s (1995) notion of *interiorized level of abstraction* (p. 418). For the question: “*What is the maximum number of sides you can make for your constructed polygons? Make a conjecture as an answer to this question.*” Brooks has also used the notion of *spatial structuring* applied on his pieces, 1, 2, 3, and 4 shown in Figure 4a; Brooks used these perceptual pieces to form the newly created shape

shown in Figure 4b. Brooks, in dealing with his 4 squares, seems to have reached "*interiorized level*" of abstraction in disembodied them from "their original perceptual context". Brooks then has freely operated on the 4 squares "in imagination, including being "projected" into other perceptual material and utilized in novel situations" (Battista, 1999, p. 418). It would seem evident that Brooks has been using *reflective abstraction* through his mental operations, performed on the previously abstracted items along the von Glasersfeld theory (1995, p. 69). Finally, the conjecture that is the crux of these mathematical activities is stated below:
For a given polygonal region, when dissected into a 'k' number of sub-regions, 1, 2, 3, ….., k, with the corresponding number of sides $n_1, n_2, n_3, \ldots\ldots, n_k$, the simple polygonal region with the maximum number of sides that can possibly be constructed by all these sub-regions is
= $n_1 + n_2 + n_3 + \ldots\ldots + n_k$.

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VITALmaths – Transforming learning experiences through mathematical video clips

Duncan Samson
FRF Mathematics Education Chair, Rhodes University, Grahamstown, South Africa
d.samson@ru.ac.za

Helmut Linneweber-Lammerskitten
School of Teacher Education, University of Applied Sciences Northwestern Switzerland, helmut.linneweber@fhnw.ch

Marc Schäfer
FRF Mathematics Education Chair, Rhodes University, Grahamstown, South Africa
m.schafer@ru.ac.za

Abstract

This paper provides an overview of the VITALmaths project (Visual Technology for the Autonomous Learning of Mathematics), a collaborative initiative between the School of Teacher Education of the University of Applied Sciences Northwestern Switzerland (PH FHNW) and the FRF Mathematics Education Chair hosted by Rhodes University in South Africa. This project seeks to produce, disseminate and research the efficacy and use of a growing databank of short video clips designed specifically for the autonomous learning of Mathematics. A dedicated website has been established to house this growing databank of video clips (www.ru.ac.za/VITALmaths) from which the video files can either be freely downloaded or streamed. Specific to the South African context is our interest in capitalising on the ubiquity of cellphone technology and the autonomous affordances offered by mobile learning. This paper engages with a number of theoretical and pedagogical issues relating to the design, production and use of these video clips, a number of which will be shown during the presentation.

Introduction

Among the implications that can be expected from the implementation of the National Educational Standards in Switzerland, as well as a number of other European countries, the following three appear to be most important for Mathematics education (Linneweber-Lammerskitten & Wälti, 2008): (a) It will be necessary to find better ways to deal with heterogeneity, particularly with regard to supporting weaker pupils; (b) It will be necessary to give more attention to the non-cognitive dimensions of Mathematical competency such as motivation, sustaining interest and the ability to work in a team; (c) It will be necessary to deal with aspects of Mathematical competence such as the ability and readiness to explore mathematical states of affairs, to formulate conjectures, and to establish ideas for testing conjectures.

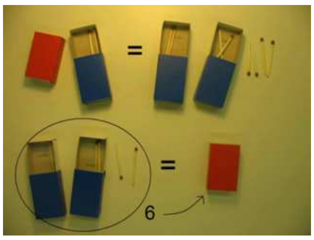
These necessities find resonance with similar implications which have arisen from the implementation of the Revised National Curriculum Statement in South Africa. This is true not only in terms of subject-specific outcomes and assessment standards, but it is also echoed by the social transformation imperatives of the curriculum and the desired attributes of the kind of learner envisaged within the South African education system (Department of Education, 2002, 2003).

Bearing in mind that these educational standards and curriculum statements are representative of the minimum levels of knowledge and skills achievable at each grade, it follows that the establishment of such aspects of competence as minimal standards *for all pupils* can only be successful if appropriate measures are taken to (a) fully integrate weaker pupils in the learning experience and (b) create an environment that is stimulating, motivating, interesting and encourages social competence. The VITALmaths project began as a response to this challenge of developing auxiliary means that could not only release teachers from the frontal introduction to mathematical themes, but also provide an opportunity for learners, particularly weaker learners, to experience genuine and challenging mathematical activities and explorations.

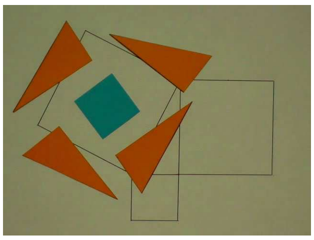
Autonomous learning through video clips

Although the use of video clips in the pedagogical context of the classroom is nothing new, the majority of these videos tend to be instructional in nature and consequently are underpinned by specific pre-determined outcomes and pedagogical imperatives. The VITALmaths project was born out of the identification of a need for short, succinct video material that could be swiftly and easily accessed and used autonomously by both teachers and pupils alike. Common design principles of these video clips are that they are short, aesthetically delightful, and are self-explanatory in the sense that they require minimal instruction. An important aspect of these video clips is that they encourage genuine mathematical exploration that transcends the mathematical content of the film by encouraging a desire to experiment, use trial-and-error, formulate conjectures, and generalise results.

VITALmaths video clips are silent, short in duration (typically 1 to 3 minutes), and are produced using a stop-go animation technique incorporating natural materials as opposed to high-tech graphics. The video clips explore and develop mathematical themes in a progressive manner that supports and encourages genuine mathematical exploration. These themes include, amongst others, striking visual approaches to proving the Theorem of Pythagoras; patterns and symmetry generated through tiling activities; elegant visual support for various results from elementary number theory; interior angles of polygons; equivalence of different area formulae; visual insight into numerical operations. A dedicated website has been established to house this growing databank of video clips (http://www.ru.ac.za/VITALmaths) from which the video files can either be freely downloaded or streamed. A selection of screenshots is shown in Figure 1.



This video clip models the solving of simultaneous equations through a process of logical reasoning without the introduction of algebra.



A proposal is made for a visual proof of the Theorem of Pythagoras. The question is raised as to whether or not this constitutes a general proof.

A model of a rectangle is used to visualise the sum of two fractions, a quarter and a third.

Figure 1. A selection of screenshots from three VITALmaths video clips.

One of the guiding tenets behind the project is that of autonomy. Autonomy represents an inner endorsement of one's actions – a sense that one's actions emanate from within and are one's own (Deci & Ryan, 1987 as cited in Reeve & Jang, 2006:209). As Mousley, Lambdin and Koc (2003) succinctly comment, “Autonomy is not a function of rich and innovative materials themselves, but relates to genuine freedoms and support given to students” (p. 425). Thus, teachers cannot directly provide learners with an experience of autonomy (Reeve & Jang, 2006), but rather they need to provide genuine opportunities that encourage, nurture and support autonomous learning. Being sensitive to this, critical elements of the design principles of the video clips take into account both cognitive and non-cognitive dimensions.

Design principles & design process

Of fundamental importance to the VITALmaths project are the design principles on which the video clips are modelled, since this plays a critical role in terms of how students are likely to interact with the technological medium. At the heart of this design process is the notion that designers design the *experience*, not simply the *product*. The basic design cycle is shown in Figure 2.

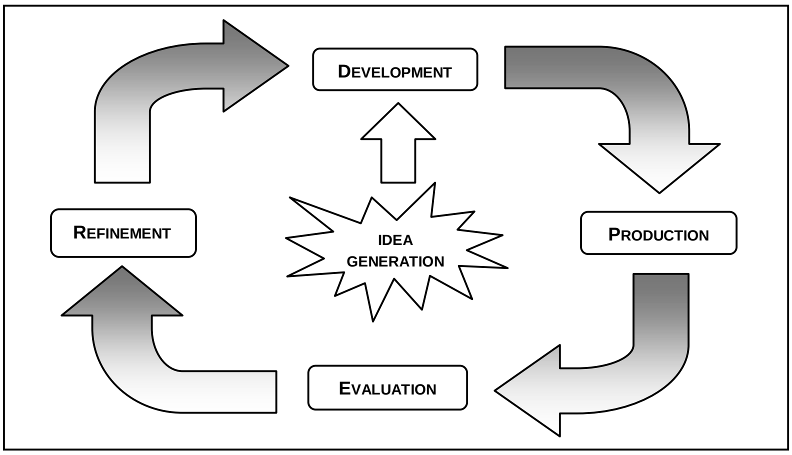


Figure 2. The design cycle.

The idea generation process is multi-faceted. One aspect of the VITALmaths project relates to the production of video clips specifically aligned with certain textbooks used in Switzerland. The idea here is that teachers will be able to use these clips as an auxiliary means to the introduction of new mathematical themes thereby allowing more time to focus on weaker pupils. However, another important aspect of the project is the production of video clips that are purposefully *not* aligned with the mathematical content of school curricula. It is envisaged that these video clips will be used in the preparation of exploratory lessons, for personal conceptualisation of mathematical concepts, and as motivational and explanatory tools, with the emphasis lying on teachers and learners using them as autonomously and independently as they wish. Ideas of appropriate topics are sourced from teachers, pupils, and experts in the field. In addition, we are exploring the possibility of a group of pupils conceptualising and creating their own video clips as part of a school Design & Technology project.

Once an idea has been generated it is developed into a workable video clip. In terms of the design principles that relate specifically to the video clips themselves, a purposeful decision was taken to eschew high-tech graphics animations in favour of using natural materials. This design consideration supports autonomous learning on two levels. Firstly, in terms of cognitive access, the use of natural materials should allow for a more direct and personally meaningful engagement with the content of the video clips when compared with the additional abstract dimension associated with high-tech graphics animations. Secondly, learners will be able to personally source all the required material to explore identical or similar scenarios, thus encouraging hands-on mathematical exploration that will have personal meaning for each learner. The production process presently used incorporates a stop-go animation technique using the free software *VideoPad Video Editor*. Once the video files are created they are then converted from AVI format to both MP4 and 3G2 formats. The MP4 format is appropriate for PCs, laptops, iPhones and most cellphones. The 3G2 format is appropriate for older cellphones that aren't MP4-compatible.

The evaluation process relies on feedback from teachers, learners and experts in the field, and is continuously used to reflect on and refine both the production process itself as well as the design principles that inform the conceptualisation of the video clips. This feedback forms a critical component of the refinement stage of the design cycle in which video clips are either modified or reconceptualised.

Mobile technology

Specific to the South African context is our interest in the use of cellphone technology as the primary distribution platform for these video clips. Our interest lies not only in the use of cellphone technology as a means of viewing the video clips, but ultimately as their primary distribution platform. Not only will cellphone technology enhance and support the autonomous learning objective of the enterprise, but it will greatly facilitate access to these video materials. In addition, it is anticipated that this innovation will have a significant positive impact for teachers in deep rural settings where access to mathematics resources is very limited.

There are a variety of mobile devices that have found application within the education arena - Personal Digital Assistants (PDAs), tablet PCs, iPods, and some games devices. However, fuelled by the development of powerful telecommunication networks which support an ever increasing range of data access services, coupled with technological advances and steadily declining costs of cellphones themselves, cellphones have emerged as a viable option for mobile learning. The VITALmaths project aims to capitalise on the flexible and versatile potential of cellphones for mobile learning.

The educational potential for mobile learning afforded by cellphone technology is diverse (Kolb, 2008; Prensky, 2005). Within South Africa a number of projects have already harnessed the ubiquity of cellphone technology to support the learning of Mathematics (e.g. ImfundoYami / ImfundoYethu, *M4Girls*, *MOBI*™ *maths*, and *Dr Math*). Selanikio (2008) makes the pertinent comment that "for the majority of the world's population, and for the foreseeable future, the cell phone is the computer". This sentiment is echoed by Ford (2009) in her pronouncement that "the cellphone is poised to become the 'PC of Africa'". The challenge for educators is thus "to capitalize on the pervasive use of cell phones by younger students for educational purposes" (Pursell, 2009:1219). The VITALmaths project aims to take up this challenge and to capitalise on the flexible and versatile potential of cellphones for mobile learning.

Conclusion

The VITALmaths project began as a response to the challenge of developing auxiliary means that could not only release teachers from the frontal introduction to mathematical themes, but also provide an opportunity for learners, particularly weaker learners, to experience genuine and challenging mathematical activities. The desire for teachers and learners to make autonomous use of the video clips is supported by the broad and open philosophy embraced by the design principles on which the video clips are conceived. The video clips are short, succinct, visually and intellectually appealing, relevant and mathematically inspirational, and a growing databank of video clips has been established from which they can be freely downloaded (http://www.ru.ac.za/VITALmaths).

The broad and open philosophy embraced by the design principles on which the video clips are conceived aims to encourage teachers and learners to use the video clips as autonomously as they desire. Continued research into the use and impact of these video clips seeks to develop a base for sustained growth and development, while at the same time contributing and participating in the academic discourse surrounding the use and development of visual technologies in the Mathematics education arena.

Acknowledgements

This work is based upon research supported by the FirstRand Foundation Mathematics Education Chairs Initiative of the FirstRand Foundation, Rand Merchant Bank and the Department of Science and Technology, as well as the Swiss South African Joint Research Programme (SSAJRP).

Any opinion, findings, conclusions or recommendations expressed in this paper are those of the authors and therefore the FirstRand Foundation, Rand Merchant Bank and the Department of Science and Technology do not accept any liability with regard thereto.

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Figural pattern generalisation – the role of rhythm

Duncan Samson
FRF Mathematics Education Chair, Rhodes University, Grahamstown, South Africa
d.samson@ru.ac.za

Marc Schäfer
FRF Mathematics Education Chair, Rhodes University, Grahamstown, South Africa
m.schafer@ru.ac.za

Abstract

This paper uses a theoretical backdrop of *enactivism* and *knowledge objectification* to explore the role of rhythm in the generalisation of pictorial patterns. Through an analysis of two vignettes, rhythm is shown not only to be an indicator of the conscious or unconscious perception of structure, but it is also shown that rhythm can be an artefact born out of specific counting processes, an artefact which in turn can *lead* to structural awareness.

Introduction

The use of number patterns, specifically *pictorial* or *figural* number patterns, has been advocated by numerous mathematics educators as a didactic approach to the introduction of algebra and as a means of promoting algebraic reasoning. From a pedagogic standpoint, French (2002) comments that the introduction of algebra through what is potentially a wide range of pattern generalisation activities may be effective in assisting pupils to see algebra as both meaningful and purposeful right from the earliest stages. Although this route to the introduction of algebra is not without its problems (see e.g. Warren, 2005), pattern generalisation activities nonetheless present a meaningful way of arriving at algebraically equivalent expressions of generality. This lends itself well to exploring the notion of algebraic equivalence in a practical context where pupils would experience the process of negotiation towards meaning (Mason et al., 1985), a process which in itself has the potential to develop and support pupils' *productive disposition* (Kilpatrick, Swafford & Findell, 2001). Vogel (2005) also suggests that patterning tasks have the potential to develop important metacognitive abilities. This paper focuses specifically on the role of *rhythm* in the process of figural pattern generalisation.

Enactivism

Enactivism is a theory of cognition that draws on ideas from ecology, complexity theory, phenomenology, neural biology, and post-Darwinian evolutionary thought. As Nickson (2000:8) discusses, the basic tenet of enactivism is that there is no division between mind and body, and thus no separation between cognition and any other kind of activity. Enactivist theory brings together action, knowledge and identity so that there is a conflation of *doing*, *knowing*, and *being* (Davis, Sumara, & Kieren, 1996).

Within an enactivist framework, there is a purposeful blurring of the line between thought and behaviour, with the focus on the dynamic interdependence of thought and action, knowledge and knower, self and other, individual and collective (Davis, 1997:370). Cognition is not seen as a representation of an external world, but rather as "an ongoing bringing forth of a world through the process of living itself" (Maturana

& Varela, 1998:11). Thus, cognition is viewed as an embodied and co-emergent interactive process where the emphasis is on *knowing* as opposed to *knowledge*.

From an enactivist stance, we need to consider not only the formal mathematical ideas that emerge from action, but to give close scrutiny to those preceding actions – "the unformulated exploration, the undirected movement, the unstructured interaction, wherein the body is wholly engaged in mathematical play" (Davis et al.,1996:156). Within an enactivist framework, language and action (e.g. rhythm) are seen not merely as outward manifestations of internal workings, but rather as "visible aspects of … embodied (enacted) understandings" (Davis, 1995:4).

Knowledge Objectification

To *perceive* something means "to endow it with meaning, to subsume it in a general frame that makes the object of perception *recognizable*" (Sabena, Radford & Bardini, 2005:129). Radford (2008:87) refers to the process of making the objects of knowledge apparent as *objectification*, a multi-systemic, semiotic-mediated activity during which the perceptual act of noticing progressively unfolds and through which a stable form of awareness is achieved. Use of the word "objectification" in this context needs to be interpreted in a phenomenological sense, a process whereby something is brought to one's attention or view (Radford, 2002:14). Knowledge objectification is premised on the notion that semiotic means such as gestures, rhythm and speech are not simply epiphenomena, but are seen to play a fundamental role in the formation of knowledge (Radford, 2005:142).

Rhythm, the "ordered characteristic of succession" (Fraisse, 1982:150), is an important element in the process of knowledge objectification. Rhythm, whether in speech or gesture, is not merely the conscious or unconscious perception of order, but crucially it creates a sense of expectation or the "…anticipation for something to come" (You, 1994:363). There is thus an inherent sense of expectancy associated with rhythm, and it is seen as a crucial semiotic device in the process of generalisation (Radford, Bardini & Sabena, 2006). Rhythm is a subtle yet powerful semiotic device since it is able to operate on multiple levels – verbal, aural, kinaesthetic and visual. In the process of generalisation, rhythm aids and supports the move from the particular to the general by enabling pupils to project and make apparent a regularity or perception of order that transcends the specific cases under scrutiny (Radford et al., 2006).

Discussion

Two vignettes are presented to highlight the different roles that rhythm can play in the process of figural pattern generalisation. In the first vignette, rhythm is identified as an indicator of unconscious structural awareness. In the second vignette, a specific counting procedure leads to a rhythmic pulse which in turn leads to the development of structural awareness.

Vignette 1

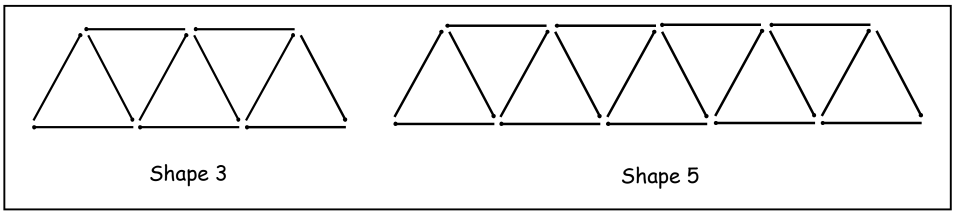

Figure 1 Visual stimulus (two terms of a pictorial pattern) presented to Grant.

Grant was presented with the two terms shown in Figure 1. Grant began by counting the forward-leaning parallel matches of Shape 5 from left to right (see Figure 2). After a brief pause he then worked his way back from right to left counting the backward-leaning parallel matches. He then counted the remaining top and bottom matches in pairs, rhythmically alternating between top and bottom as shown in Figure 2: 1,2…3,4…5,6…7,8…9. This counting procedure was central in alerting Grant, whether consciously or unconsciously, to the non-paired match in the bottom row. Based on this counting procedure, Grant was able to arrive at the following general expression for the nth term of the sequence: $n+n+2n-1$.

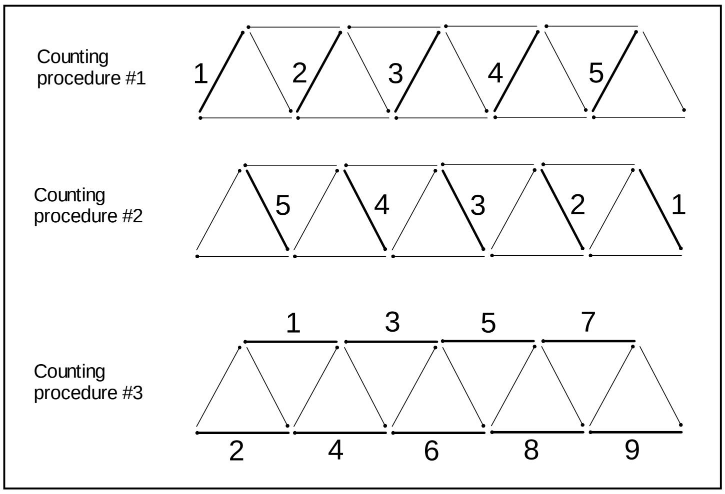

Figure 2 Grant's different counting procedures.

Grant was able to justify his general expression ($n+n+2n-1$) by relating the $n+n$ portion to two sets of "parallel central matches", while the $2n-1$ he associated with what he referred to as the "outside matches". Just prior to writing the $2n-1$ part of the expression, Grant made use of indexical gesturing - he first gestured a horizontal line across the top of Term 5 and then a second horizontal line across the bottom of Term 5. As Grant wrote down the $2n-1$expression he commented that he was just simplifying $n+[n-1]$. When asked to articulate how he was "seeing" it, he was insistent that he saw the structure as $n+[n-1]$, i.e., in terms of n matches along the bottom and $n-1$ matches along the top, and that the $2n-1$ portion of his expression was in fact an algebraic simplification of $n+[n-1]$. Grant then re-wrote his expression for the nth term as $n+n+n+[n-1]$ which he explained as being a truer representation of his visual apprehension of the pictorial pattern.

Grant's initial formula ($n+n+2n-1$) suggests that the "outside" matches have been split into pairs - one match of each pair forming part of the upper horizontal row with its paired match positioned below it in the bottom row. The $2n-1$ "outside" matches in Grant's initial formula seem to represent n pairs of matches, making $2n$ matches in total, the "-1" being an adjustment required due to the right-most pair missing a match in the upper row. Although his final expression ($n+n+n+[n-1]$) suggests that the "outside" matches were in fact sub-divided into two distinct horizontal rows, with n matches along the bottom and $n-1$ matches along the top, the $2n-1$ portion of his original expression is likely to have been inspired by his counting procedure shown in Figure 2, in which the rhythmical pairing of the top and bottom matches was central to alerting him to the non-paired match in the bottom row. In this instance the rhythmic counting procedure seems to be an indicator of an unconscious perception of structure which subsequently was manifested in the general algebraic expression for the nth term.

Vignette 2

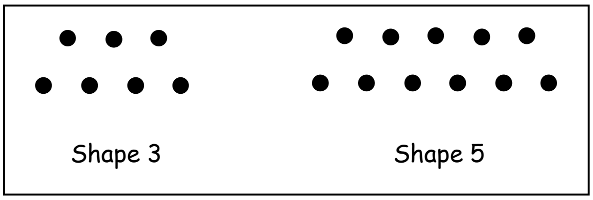

Figure 3 Visual stimulus (two terms of a pictorial pattern) presented to Anthea.

Anthea was presented with the two terms shown in Figure 3. Upon initial presentation of her pictorial pattern, Anthea counted the dots in Shape 3 and Shape 5 in an economical zigzag manner as shown in Figure 4 (a) and (b). She then double-checked her two answers. When she re-counted the dots in Shape 3 she used the same counting method as she used in the initial count. However, when she re-counted the dots in Shape 5 she did so in a slightly modified manner. This new counting procedure is shown in Figure 4(c). After a bit of silent thinking she counted the dots in Shape 3 one final time, using her *initial* counting method, before writing down the formula $n+(n+1)$. She then tested her formula mentally and was satisfied that the formula worked for these two cases.

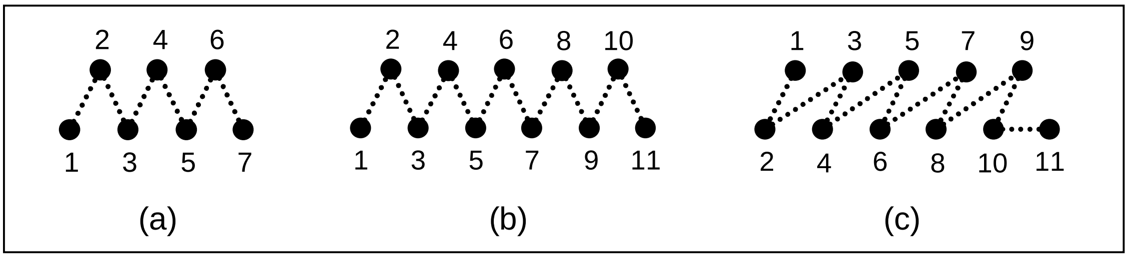

Figure 4 Anthea's different counting procedures.

We now move onto the development of Anthea's second algebraic expression. After sitting silently for about 30 seconds she suddenly wrote down $2n+1$. Anthea explained that her formula $2n+1$ was arrived at through numerical considerations only. However, what is interesting is that the formula $2n+1$ seems to resonate with the counting method shown in Figure 4(c). Although Anthea arrived at her general expression $T_n = 2n+1$ through a process of numeric rather than visual reasoning, it is likely that the second of her two earlier counting processes unconsciously inspired this algebraic expression. With the first counting method the starting point is the dot furthest to the left on the *bottom*. For the second counting method the starting point is the dot furthest to the left on the *top*. Both methods result in an overall zigzag movement from left to right, but the first method is more economical in terms of total distance traversed during the counting process. The second method is uneconomical since each movement from top to bottom has a small lateral component to the left (e.g. the move from dot 1 to dot 2 in Figure 4(c)) with the result that a longer distance needs to be covered when moving from bottom to top (e.g. the move from dot 2 to dot 3 in Figure 4(c)). The distance traversed in the second counting method is in fact just over 20% longer than that traversed in the first method. However, what is crucial to appreciate is that this alternating top-to-bottom and bottom-to-top movement, where the top-to-bottom movement is accomplished slightly faster than the bottom-to-top movement as a result of the two different path lengths, creates a critical sense of *rhythm*: 1,2…3,4…5,6…7,8…9,10..11. The critical distinction here is that instead of the rhythm being an artefact of a counting method inspired by a perceived structural regularity, the rhythm is actually an artefact born out of the counting process itself, an artefact which in turn may *lead* to perceived, albeit possibly unconscious, structural regularity and thus to the development of a new apprehension and associated general algebraic expression.

Concluding comments

The cognitive significance of the body has become one of the major topics in current psychology (Radford et al., 2005:113). Knowledge objectification, premised on the notion that semiotic means such as gestures, rhythm and speech are not simply epiphenomena but play a fundamental role in the formation of knowledge (Radford, 2005:142), proved to be a useful theoretical construct in the exploration of rhythmic aspects of embodied knowing. Rhythm was found not only to be an indicator of the unconscious perception of structure but also an artefact of the manner of engagement with the visual stimulus, an artefact which in turn *led* to structural awareness.

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Probability in Mathematics: Facing Probability in Everyday Life

Malka Sheffet, PhD   Bassan-Cincinatus Ronit. M.A
Kibbuzim College of Education, Math department,
Tel-Aviv, Israel

Abstract
Understanding chance is essential to understand probability. The aim of this study was to find out how teenagers understand this idiom. We interview students from 6^{th}, 8^{th} and 11^{th} grades. The results from the interviews pointed out that the idiom 'chance' is not clear enough. They understand better theoretical situations like throwing a dice. They understand the random and the certain in this situation. On the other hand, when they connect chance to everyday situations they relate to them as yes/no situations. They have difficulties in defining the random and the certain.

Introduction
Let's look at any typical "text book" activity devised for 8^{th} grad students to understand what 'probability' is. The students are doing an experiment with tossing a coin (or a few coins together). They find that the ratio between the number of heads they got all together, and the number of coin tosses, is stabilized on 1/2. So they define the probability to get head on a random coin toss as this number. Do students, after this activity, understand what probability is? Are they able to apply this notion to everyday life? There is a need to find how teenagers understand probability. More specifically, how they understand 'chance' and what meaning do they apply when they use it in everyday life. These are the goals of this research.

Background
Piaget and Inhelder (Piaget and Inhelder, 1954) said that understanding probability means the ability to produce some quantification assessment (for the probability) that a certain event will occur. In order to give such assessment, one should understand both 'chance' and the usage of the combinatory operations. Piaget and Inhelder also said that understanding 'chance' is to understand total possibilities and randomness. Total possibilities events are events that will never occur (like throwing two dices and getting 14) or, events that one can be sure that they will occur (like throwing two dices and getting a result bigger then 0 and smaller then 13). They defined 3 stages for developing understanding of 'chance' and probability. Only at the third stage (from the age 11/12) one understands 'chance' and is able to use the combinatory operations.
According to the theory of Piaget and Inhelder, understanding 'chance' is essential for understanding probability. Understanding ‘chance’ is a combination of the discrimination between certain and uncertain events, and the understanding the distribution and the variety of the results.
Researchers in the last decade focused on understanding the variety of the results of an experiment that is repeated many times (like: Watson and Kelly, 2003; Watson et. al. 2003; Watson and Moritz, 2000, 2003; Sanches and Matinez, 2006). These researchers claim that understanding the variation of the results reflects actually understanding of the 'chance'. This understanding is combined of 3 aspects: understanding the randomness, understanding the structure of probability, and connecting this structure to empirical results.
The main aim of our research is to find what meaning teenagers, ranging from 11 years-old up to 17 y”o, give to the word 'chance' in everyday situations, including probabilistic situations.

Methodology

56 students (19 from 6th grade, 18 from 8th grade and 19 from 11th grade) were interviewed. In addition to free conversation, each interview included the following assignments:

1) Explain what is 'chance'.
2) Compose a sentence with the word 'chance'.
3) Explain what is 'almost zero chance'?
4) Read the story of the weather man. Did the weather man was wrong when he said that there is a 'almost zero chance' for the next day to be rainy?[1]
5) Playing "Ladders and Ropes" game. It is the same usual game with one additional rule: before throwing the regular dice, each player has to throw a special dice, which has 3 faces marked with a circle and 3 faces marked with a X sign. If the player gets the circle sign, he may proceed as usual and throw the regular dices. But if the player gets the X sign, the turn is passed over to the next player. Before the game, the students were asked what was the chance to get a 'circle' and what was the chance to get 'X'. When one player (or more) got many more Xs then circles (or vice-versa), the interviewer stopped the game and talked with the students about those chances again.

We asked about 'almost zero chance' in addition to 'chance', because we wanted to delve deeper into Piage's "understanding of the random and the certain". 'Almost zero chance' can easily be taken as the equation "chance = zero". While 'almost zero chance' **is fitted to** describe probability of random events, the notion of "chance = zero" is fitted to the probability of certain events. In the case of 'almost zero chance' there is still some probability that the event can occur, though it likely to be very poor.

Findings

Explanations to 'chance':

Most of the subjects said that chance means uncertainness. Like: "something that may happen", or: "a possibility that I'll get one out of two options". Very few subjects added some quantification. Like: "the percentages for something that will occur". (See table 1).

One note that almost 25% of the subjects could not explain what 'chance' is. They said that they understand what 'chance' is but they cannot explain it.

	6th grade		8th grade		11th grade	
	N	%	N	%	N	%
uncertainty	13	68	11	61	10	53
quantification	1	5	3	17	3	16
Could not explain	5	26	4	22	6	31
total	19	99	18	100	19	100

Table 1: Distribution explanation to 'chance'

Composing a sentence with 'chance':

All of the subjects composed a sentence with it the word 'chance', although there were some subjects that could not explain it. All the sentences were connected to everyday situations. In all the sentences, the meaning of 'chance' was uncertainty that an event will occur.

[1] Last night the weatherman said that there is almost zero chance of rain today. Today it did rain. Was the weatherman wrong?

One can classify the sentences into two categories. The first category - sentences with yes/no situations. Will the event happen or not. Like: “there is a chance that tomorrow we will win the football game”, or: “there’s no chance that I’ll eat tomatoes”. The other category - sentences with some quantification for 'chance'. Like: “there’s 20% chance that I’ll win the lottery game”, or: “low chances of having a rainy day tomorrow”. (See table 2.)

	6th grade		8th grade		11th grade	
	N	%	N	%	N	%
Yes/no situation	12	63	11	61	11	58
quantification	7	37	7	39	8	42
total	19	100	18	100	19	100

Table 2: Distribution of composing a sentence with 'chance'

Note that almost 60% of the subjects use the word 'chance' for dichotomy situation (will this-and-that happen or not). There is a slight decline with age. The rest of the subjects (about 40%) added some quantification.

Explanations to 'almost zero chance':

All the subjects explained what is 'almost zero chance'. Most of the 6th grade subjects (54%) said that it means “no chance at all”. Most of the 8th grade subjects (56%) explained it well. The number of subjects from the 11th grade who explained it well exceeded 68%.

The weatherman story:

More than 50% of the subjects from each grade said that the weatherman was mistaken. The rest said that he gave the right information, because their still was some probability that there will be rain the next day.

Consistency:

About 40% from the subjects at each grade showed right consistency. They said that 'almost zero chance' means a very small probability, and the weatherman was right. Most of the subjects from the 6th grade (53%) showed wrong consistency. They said that 'almost zero chance' means no chance at all and the weatherman was wrong. Only 44% from the 8th grade subjects showed the wrong consistency. The number decreased (32%) at the 11th grade. There was also a third group - those who had no consistency. They said that 'almost zero chance' means no chance al all and that the weatherman was right, or the other way around - that 'almost zero chance' means a very small chance and the weatherman was mistaken. This group percentage increased from 11% al the 6th and 8th grades to 26% at the 11th grade.
All the above details are gathered in table 3.

We can conclude that about 40% from all subjects understand correctly 'almost zero chance'. They understand that it means very poor chance, but there is still some chance that the event will occur. They understand it correctly when it is related to everyday situations. This rate does not change with age. However, about 40% from the subjects understand 'almost zero chance' as “no chance at all”, or the equation “probability = 0”. This rate decreases with age. (53% at the 6th grate, 44% at the 8th grade, 32% at the 11th grade). An average of 16% from the subjects is inconsistent. This rate increases with age (from 10% at 6th grade up to 26% at the 11th grade).

		6th grade (N=19)		8th grade (N=18)		11th grade (N=19)		Total (N=56)	
		N	%	N	%	N	%	N	%
Explanation	chance=0	12	63	8	44	6	31	26	47
	very poor	7	37	10	56	13	68	30	54
The weather-man story	Wrong	10	53	9	50	11	58	30	54
	Right	8	42	9	50	8	42	26	47
consistency	wrong.	10	53	8	44	6	32	24	43
	right con.	7	37	8	44	8	42	23	41
	No con.	2	10	2	11	5	26	9	16

Table 3: Distribution of 'almost zero chance'

The 'Ladders and Ropes' game:

Before starting the game all member of each group of players were asked what is the chance to get a circle, and what is the chance to get an X. In each group there was at least one player who counted the number of circled and the number of X-s on the faces of the dice. All the subjects agreed that the chances are 50:50. This means they have the same chance to appear. The subjects then started playing the game, and the interviewer stopped the game when one symbol appeared many times more then the other. At this point the interviewer asked one or two players (those who got one symbol more then the other one) what symbol they gut most of the time? Then he asked them what are the chances to get a circle and to get an X, and if it is still 50/50?

Most of the subjects said that the chances to get a circle or an X did not change and they are still 50:50. Only few subjects (3 from the 6th grade, 2 from the 8th grade and 2 from the 11th grade) changed the probability to set a circle or an X according to the results they achieved. In one of the groups, one of the subjects said that, the chances are still the same 50:50 but he had “a special ability to throw the dice and get a circle”, which explained why he got so many circles.

At this point the interviewer asked them to explain how come the dice-tosses are so unbalanced if the chances are the same. The subjects from the 6th grade said that the differences between the numbers of the symbols are a matter of luck. The subjects from the 11th grade said that the numbers of all circles appeared at the game is very close to the number of the all X-s. Another answer was that in the short run one can get one symbol more times then the other, but in the long run the numbers will be almost equal.

Discussion:

From the findings we can see that most of the subjects understand chance as a word that is connected to uncertain events in the future. Following this one may think that they understand correctly what 'chance ' means. **But the key question is: How they understand uncertain events in the future**. That's why the subjects were asked to combine a sentence with 'chance'. We found that the majority of the subjects described yes/no events, and we also note that about a 1/4 of the subjects could not explain what 'chance' means (yet they could compose a sentence with the word). Had we found that the majority of subject who couldn’t explain it was of the youngest (6th) grade, we could have justified it by the limited ability of young students to express themselves. Yet, almost a 1/3 of the subjects from the 11th grade said that they know what 'chance' is and they cannot explain it. We believe this fact alone gives evidence to the fact that many students do not understand 'chance' at all, even though they often use it in everyday life.

Some connected ‘chance’ with yes/no situations, and if so they connect it to equal probability. For example:

Interviewer: Everything that may happen is “50% chance”?
Student: Yes.
Interviewer: Your friend told me that team so-and-so has no chance to win the next game. Do you agree with him?
Student: Yes, there is a great chance.
Interviewer: So there is a great chance. Is it still 50/50?
Student: Yes, it is 50/50.
Interviewer: Why?
Student: Because, if there is a chance - then it is 50/50.
Interviewer: What is the chance that tomorrow will be a nice day?
Student: Very high.
Interviewer: Very high, do you mean higher then 50?
Student: 50/50.
Interviewer: Still 50/50?
Student: Yes, because it may vary.

We got the same picture when we asked about 'almost zero chance'. From the subjects explanations one might conclude that they understand it well. But their reaction to the weatherman story showed that more then 50% of the subjects said that the weather prediction was wrong.
Almost everyone who explained 'almost zero chance' as ‘no chance’ also said that the weatherman was wrong. This was more common among the young subjects. Alternatively, most of the subjects who explain ‘almost zero chance’ as ‘a very poor probability’, said that the weatherman was right. This was more common among the older subjects. These two groups were almost the same in size – about 40% each.
We can see the same phenomena in 'chance' and in 'almost zero chance'. Theoretically, teenagers understand these notions well. But when students connect them to everyday life, they might change their meanings – these notions became synonyms to yes/no situations (will happen or will not happen).

The "Ladders and ropes" game demonstrated that most of the subjects showed a correct understanding of 'chance'. For example:

Interviewer: What did you get?
Student: Majority of X-s.
Interviewer: So what are the chances to get X?
Student: Still 50/50.
Interviewer: So how come you got so many X-s?
Student: Because when you throw the dice 3 times, and it does not tell anything (he threw the dice more then 3 times).
Interviewer: So how many times we need to throw the dice?
Student: about 100 times.
Interviewer: If I throw the dice 100 times then what?
Student: Most of the chances that you will get 50/50.

From this we can see that they understand that in the short run one might get one symbol more times then the other, but in the long run the numbers will be equals.

Earlier we claimed that understanding 'chance' in everyday situations is connected to yes/no situations. Here, the game reflects a well profound understanding. There is a contradiction. One way to settle the contradiction is to claim that only few of the subjects understood these ideas (the subjects were gathered in groups during the game). Even though the interviewer posed the question to one player, the other

answered, causing the first players, who were confused by the interviewer's questions, to adopt the answer and agree the student who answered. This is what Way and Ayres (2002) called the fragile probability knowledge and added that subjects do not feel the need to reason consistently.
Another way to explain the contradiction is that throwing a dice in a game like this presents a situation with many outcomes. Experiencing situations like this can build-up a correct understanding of probability. We have no doubt in our minds about the importance and contribution of a game like this to the understanding of probability. However, we believe that because of the many repetitions, and because students learned probability based on such games, students classify chance and probability as theoretic notions, and do not connect them to everyday life.

In this research, we found that there are two types of understanding 'chance'. One type is to understand 'chance' as a theoretical notion, and the other type is to understand 'chance' in connection to everyday life. Students display better understanding of probability when it is related to theoretical situations, especially older ones. When they are asked about everyday situations they connected them to yes/no situation and usually they estimate them as 50/50 chance. Amit and Jan (2006) pointed out that students are developing intuition to differ between numerical probability situations and empirical probability situations. Numerical probability situations are we here “theoretical situations”, and empirical probability situations are everyday situations.

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Teaching Derivations of Area and Measurement Concepts of the Circle: A Conceptual-Based Learning Approach through Dissection Motion Operations

Tracy Shields — tracyshields@shaw.ca
Medhat H. Rahim — mhrahim@lakeheadu.ca

955 Oliver Road, Lakehead University, Faculty of Education, Thunder Bay, Ontario, Canada P7B 5E1

Abstract

In this article a variety of Dissection Motion Operations (DMO) are presented; they are primarily focused at the derivation of the area formula of the circle through hands-on manipulation for the classroom practices in a *conceptually-based learning fashion*. The National Council of Teachers of Mathematics (NCTM, 2000), within its dedication to the improvement of mathematics pedagogy and assessment, has established the standards for mathematics teachers and policymakers in the United States and beyond. Adopted by the Ontario Ministry of Education (2005), the NCTM's principles have infiltrated the Ontario mathematics curriculum. As such, *conceptually-based learning of mathematics versus procedural-based learning* has had an opportunity to flourish in the classrooms of the province of Ontario, Canada, for the past six years. The movement toward reform based mathematics classrooms finds its philosophy grounded in the concerns about procedural based learning that is taught without conceptual understanding. Several conceptual-based approaches to the origin of the formulas such as the area of the circle will be presented in this article.

Background

In reference to the pitfalls of the procedural-based learning methodology in teaching, Freire (2003) has raised a red flag about this methodology and has long written about teachers' responsibility to do more than "fill" students with the contents of lectures. Freire has directly talked about the status of teacher-directed education where "banking" concepts through which students were seen to have the responsibilities in regards to the storing, receiving and filing of deposits. Freire (2003) further argued that in the process of such rote learning procedures, it is the students themselves that are being filed away (p. 55). Freire's position has echoed the National Council of Teachers of Mathematics Standards (NCTM, 2000) calling for conceptual meaningful understanding versus rote learning and memorization. This article will investigate the literature as it pertains to the numeracy level of secondary school graduates in Canada. Research indicates that both students and teachers in North America have difficulty with measurements involving area and that this difficulty can be traced to area attribute problems (Baturo and Nason, 1996; Battista, 2003; Kouba, Brown, Carpenter, Lindquist, Silver and Swafford, 1988). Battista (2003) further discussed the lack of "spatial structuring" which is the underpinning of area and volume concepts and which result in the inability of students to apply what they have learned (p. 132). Baturo and Naon (1996) have made a specific emphasis at the use of manipulatives as a medium for hands-on manipulation in addition to the specific *dynamic* aspect of the area concept stating that:

> In those schools where teachers do provide their students with concrete experiences in developing the notion of area and its measurement, it is often done in a cursory and disconnected fashion or, if done conscientiously, tends to focus almost exclusively on the static perspective of the notion of area to the exclusion of the dynamic of area (Baturo and Nason, 1996, p. 239).

Conceptual understanding of the concepts and processes involved in area measurement includes "knowing that the area of a shape remains the same when it has been changed by 'Cutting and Pasting' to form different shapes" for concrete knowledge (Baturo and Nason, 1996, p. 239). Further, conceptual understanding includes "knowing that congruence and 'cut and paste' transformations conserve the area of a shape; area is a continuous attribute that can be divided into discrete subunits" (Baturo and Nason, 1996, p. 236).
Baturo and Nason (1996) further discussed the importance of students understanding of how pi was derived and the shapes it is associated with along with how the formulae associated with each shape are developed and connected. In this connection, Rahim (2010) has introduced a treatment for teaching the derivation of area formulas for polygonal regions through Dissection-Motion-Operations (DMO) as a systematic refined process for "Cutting and Pasting" acts cited above. Baturo and Nason (1996) further stated that students who combined shapes when attempting solving problems were more stimulated than if they were simply finding the areas of the shapes. Alarmingly, their study revealed that only two of their 13 students understood the relationship between the area of a rectangle and the area of a triangle. As a result, their use of the formula for area held no meaning for them. The students admitted that they did not remember engaging in activities that were meant to enhance their understanding of this relationship (p. 256). Stephan and Clements (2003) recognized that maintaining area is an important concept that it is often not included in measurement instruction. The authors stated that students have difficulty to understand that when a shape is cut up into a finite number of pieces and when the pieces are rearranged into a different shape with no overlapping, its area remains the same. Further, Stephan and Clements (2003) found that research indicated that children use differing strategies to conceptually comprehend the ideas behind the concept of area.

Historically, Ma (1999) stated that the approximation of the area of a circle using the area of a parallelogram has been known since the 17^{th} century (p. 116), (see Smith & Mikami, 1914, p. 131). Ma has indicated how Chinese teachers have made a full lesson by inspiring students to subdivide a circle into sectors and rearrange the pieces into a parallelogram like shape, then, by imagining the number of the sectors increases, a closer resemblance of a parallelogram shape will be reached. And as the approximation process continues the ultimate shape ends up into a parallelogram (and hence into a rectangle by further cutting and moving) implying the presence of the area formula of the circle (p. 116).

Teaching the Derivation of the Area of a Circle: A Dissection-Motion-Operation Approach

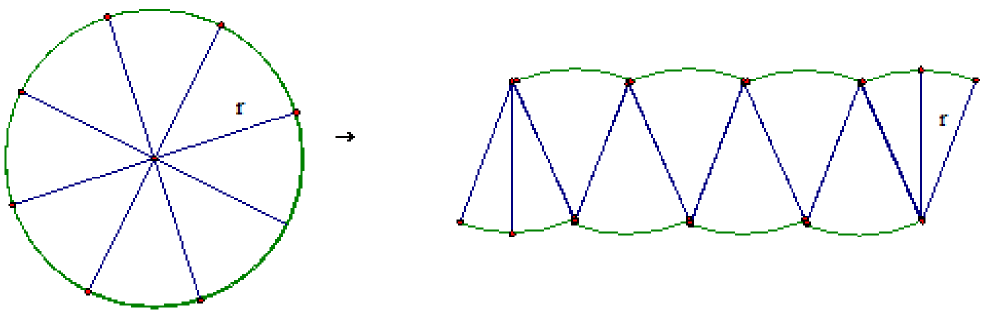
Clearly, the number of equivalent sectors in which a circle can be dissected would be either odd or even. As such, there will be three possibilities of dissecting a circle:
Case 1. The number of the equivalent sectors, n, in which a circle can be divided, is even;
Case 2. The number of the equivalent sectors, n, in which a circle can be divided, is odd; and,
Case 3. The number of equivalent sectors, n (even or odd), is a perfect square.

As Case 1 (will be shown below) deals with the relationship between the circle and the parallelogram, it should be noted that this relationship has been known and used for centuries. Most recent elementary mathematics texts have used this relationship to introduce a visual conceptual understanding of the area of a circle in a dynamic and meaningful fashion.

Incidentally, Rahim (2010) has introduced a decomposition-composition (Dissection-Motion) treatment of showing the origins of all area formulas of polygons of all types. Rahim was focusing at the dynamic teaching approach for the derivation of area formulas for polygonal regions through what is called Dissection-Motion-Operations (p. 195). The following are derivations of the formula for the area of the circle through *shape transforms* for each of the three cases listed below.

Case 1: The number of the equivalent sectors, n, in which a circle can be divided, is ***even***. In this case, the circle is transformed into a ***parallelogram*** like shape of equal area through the following steps:

1. Dissection Step: Consider the dissection of the circle given in Figure 1. Note that, in this case, the circle can be dissected into ***n*** *number of equivalent sectors where n can be any* ***even*** *positive integer* **≥ 4**. However, for simplicity, assume it is dissected into *eight* equivalent sectors as shown in the left part of Figure1 below.
2. Motion Step: Each of the eight individual sectors is appropriately moved under a transformation operation of translation, rotation, and or reflection to form the parallelogram like shape shown at the right part of Figure 1.



|------ ½(Circumference= 2πr) = πr -----|

Figure 1. Shape transform of a circle into a parallelogram resemblance through DMO

Then, from Figure 1, it is clear that

$$\text{Area of the circle} = \text{Area of the parallelogram like shape}$$
$$\approx \tfrac{1}{2}(2\pi r)\times r = \pi r^2.$$

This result holds true when the number of equivalent sectors increases to approach infinity (∞). Accordingly, the shape at the right side of Figure 1 will take the shape of the parallelogram.

Case 2: The number of the equivalent sectors, n, in which a circle can be divided, is *odd*. In this case, the circle is transformed into a ***trapezoid*** like shape of equal area through the following steps:

1. Dissection Step: Consider the dissection of the circle given in Figure 2. Note that, in this case, the circle can be dissected into ***n*** *number of equivalent sectors where n can be any* ***odd*** *positive integer* **≥ 3**. For simplicity, assume it is dissected into five equivalent sectors as shown in the left part of Figure2 below.

2. **Motion Step:** Each of the five individual sectors is appropriately moved under a transformation operation of translation, rotation, and or reflection to form the trapezoid like shape shown at the right part of Figure 2.

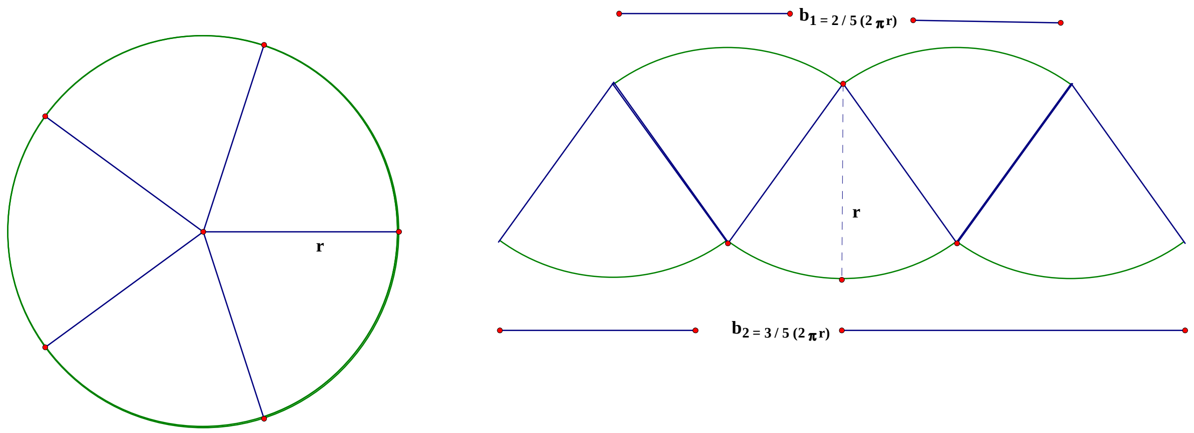


Figure 2. Shape transform of a circle into a trapezoid resemblance through DMO

Then, from Figure 2, it follows that

Area of the circle = Area of the trapezoid like shape

$\approx$ ½ (h)($b_1 + b_2$)

= ½(r)[(2/5)(**2πr**) + (3/5)(**2πr**) = $\boldsymbol{\pi r^2}$**.**

This result holds true when the number of equivalent sectors increases approaching infinity (∞). As a result, the shape at the right of Figure 2 should take the shape of the isosceles trapezoid.

Case 3. The number of equivalent sectors, n (even or odd), is a perfect square. In this case, the circle is transformed into an isosceles triangle like shape of equal area through the following:

1. Dissection Step: Consider the dissection of the circle given in Figure 3. Note that, in this case, the circle can be dissected into ***n*** *number of equivalent sectors where n can be any* ***perfect square*** *positive integer* **≥ 4**. Assume the circle is dissected into nine equivalent sectors as shown in the left part of Figure3 below.
2. Motion Step: Each of the nine individual sectors is appropriately moved under a transformation operation of translation, rotation, and or reflection to form the isosceles triangle like shape shown at the right part of Figure 3.

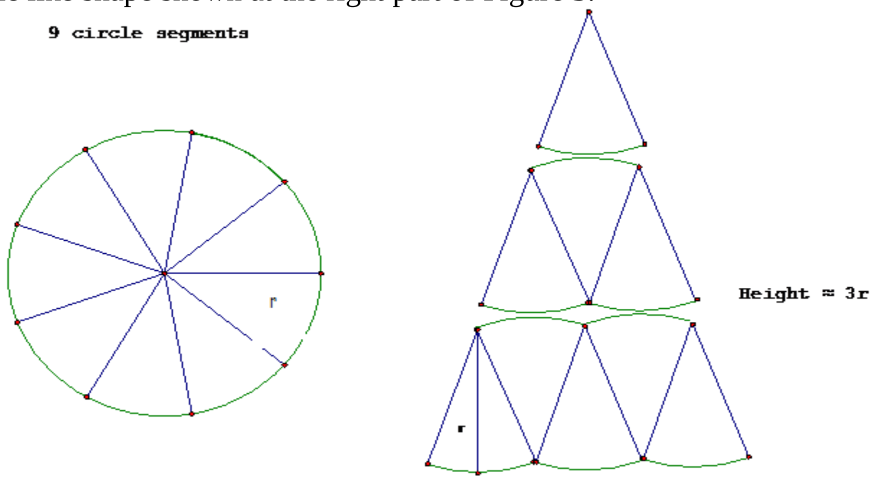


Figure 3. Shape transform of a circle into an isosceles triangle resemblance through DMO

From Figure 3, it follows that

$$\begin{aligned}\text{Area of the circle} &= \text{Area of the isosceles like shape}\\ &\approx \tfrac{1}{2}\,(h)(b)\\ &\approx \tfrac{1}{2}(3r)[(3/9)(\mathbf{2\pi r}) = \boldsymbol{\pi} \mathbf{r^2}.\end{aligned}$$

And similarly, this result holds true when the number of equivalent sectors increases approaching infinity (∞). Thus the shape at the right side of Figure 3 will take the shape of an isosceles triangle.

Epilogue

Further relationships among 2D shapes can be developed through classroom sessions for building conceptual mathematical understanding of the area and measurement concepts of a variety of shapes in geometry. This type of shapes-to-shape interrelationships is directly related to van Hieles (1985) levels of thought development in geometry. As such, these interrelationships are vital and necessary particularly for middle school levels for a deep understanding of measurement in middle school levels such as grades 8 and 9.

The literature suggests that Canada (and the rest of the world for that matter) is currently struggling with numeracy in general, including in particular, the conceptual understanding of the area of a circle. Battista (2003) has expressed concerns on the lack of spatial comprehension that is resulting in an inability for students to apply what they have learned (p. 131). Ma (1999) and Linquist (1987) have stressed the importance for students to explore multiple perspectives. Stigler and Heibert (1999) have reported how the Chinese teachers in their study were considering frustration and struggle as part of the learning process. On the other hand, Baturo and Nason (1996) reported a failure on the part of teachers in North America to provide concrete experiences that involve the development of the area concept in the classroom. The authors stated further that teachers are teaching “to the exclusion of the dynamic of area” (p. 239). Rahim (1986) have introduced examples of how area can be discussed in a dynamic sense in the classroom.

We have shown in this paper that there are a number of ways for students to discover the area of a circle, offering many opportunities for deep conceptual understanding with which students can move forward along the van Hiele’s level of geometric thought development.

The dynamic approach is in no way limited to hands-on manipulations, rather, it can be introduced through the use of Dynamic software too such as the Geometer’s Sketchpad (GSP Version 5 – North America-Based software) and Cabri II and Cabri 3D (Grenoble, France-Based software). Furthermore, the dynamic feature in teaching geometric shapes is not restricted to two dimensional objects; it is suitable to 3D objects too. We believe that with a dynamic approach, students are given the opportunity to see themselves as emerging critical thinkers in the process!

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Creating Desirable Difficulties to Enhance Mathematics Learning

William R. Speer, PhD
Dean of the College of Education
Director, Center for Mathematics, Science and Engineering Education
University of Nevada Las Vegas
Las Vegas, Nevada, USA
william.speer@unlv.edu

Abstract

We can't make our students into seekers if we aren't seekers ourselves. This research-based, practice-oriented paper explores the nature of "desirable difficulties" and the benefits of creating such desirable difficulties to help students shake naïve or loose thinking and to construct "new" knowledge by encouraging transfer of related prior knowledge to new situations. The paper also discusses certain tasks designed to promote the interconnectedness of mathematical knowledge with respect to mathematical concepts from different branches of mathematics and various representations of mathematical concepts.

The intended purpose of this paper is: (1) to help develop instructional strategies that enable all students to engage in reasoning and mathematical discourse about mathematical ideas; (2) to help teachers understand how mathematical ideas interconnect and build on one another to produce a coherent knowledge base of number, geometry, and data analysis, and (3) to recognize and apply mathematics to solve problems, including those in contexts outside of the "apparent" mathematics currently under study.

Introduction

Not everyone can be a mathematician, but everyone can want to be a mathematician. It is important to understand that being good at mathematics is not evidenced by how many answers you know. Instead, being good at mathematics may be best evidenced by what you do when you don't know the answers. We must help students construct and own "new" knowledge in such a manner that they are then able to apply new knowledge in ways that are different from the situation in which it was originally learned.

Albert Einstein may have captured it best in stating that, "Students that can't explain 'it' simply, don't understand 'it' well enough."

Why do some students successfully learn mathematics and others do not? Successful students internalize processes, connect concepts and ideas, generalize across domains, and develop personal understanding. According to the Piagetian concept of reflective abstraction, students engage in interiorization, coordination, encapsulation, generalization, and reversal. A curriculum that emphasizes these actions will lead to increased likelihood of connections and transfer.

Most agree that teaching is a complex practice and hence not reducible to recipes or prescriptions. If we can also agree that true learning is reflected by the ability to readily transfer knowledge, skills, attitudes and values from one situation to another, then certain assumptions about mathematics teaching automatically deserve attention.

These assumptions include:

Mathematics is more than a collection of concepts and skills.
A goal of teaching mathematics is to help students develop mathematical power.
All students can learn to think mathematically.
WHAT content is learned is fundamentally connected with HOW it is learned.
Content should be explored at different levels of abstraction.

There are no simple means to assure that these assumptions are adequately addressed in all situations. However, the fundamental task of a mathematician is to convey to students not only what mathematicians know, but also what they do, and how and why they do it.

Body

Sometimes learners express a reluctance to look at mathematics in an alternative way to their initial exposure to the topic. Pleas of "You're going to confuse me!" may actually signal an unrecognized, and certainly unacknowledged level of confusion that is ALREADY present – not potential confusion on the horizon.

There are many benefits to be gained by actually creating something that we'll refer to as DESIRABLE DIFFICULTIES designed to encourage thinking about mathematics as well as enhancing both long-term retention and transfer. Out of apparent chaos and confusion emerges a deeper understanding and appreciation. In fact, students who are successful in making sense of mathematics are those that believe that mathematics makes sense.

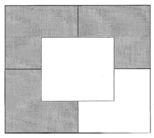
Using the sage principle that "No matter what IT is, the chances of finding IT are dramatically increased if you're looking for IT" we must explore techniques to encourage and reinforce mathematics as a way of thinking. The ideas we gather are like so many pieces of colored glass at the end of a kaleidoscope. They may form a pattern, but if you want something new, different, and beautiful, you'll have to give them a twist or two. You experiment with a variety of approaches. You follow your intuition. You rearrange things, look at them backwards, and turn them upside down. You ask "what if" questions and look for hidden analogies. You may even break the rules or create new ones.

Consider the following incomplete list of students in a class roster:

ANN, Brad, CAROL, Dennis, ???

What name would you propose as the next on this list? Seldom does the question cause "random" answers of "Joe" or Mary" since everyone begins to search for some pattern to apply. Often, the response is along the lines of "Edward" or, perhaps "Edith" – especially those that have self-imposed a rule of alphabetical order. Those suggesting "Edith" may also wish to insist that it should be female to preserve alternating sexes. Then someone chimes in with "Emillie" – with two l's – since the name, whatever it is, should contain seven letters. If the names were also printed in colors, say, black, red, green and blue, then someone else might suggest that the next name should be printed in black to maintain a perceived rotating pattern. Still another would suggest that it should be printed in capital letters. Are we done? How do we know? Have we satisfied EVERY rule? What about the "rule" that calls for three vowels – did you see that? Should we propose “EVELYNN” instead? Does that violate a “rule” that double n’s should appear in every fourth entry? And so on and so on… How does one ever know if their rule (or combination of rules) is the right rule (or combination)? What makes an answer correct? Is there more than one? Are there infinitely many? You’ll know that you failed in this exemplar if, when finished with the discussion, someone still asks, “But what is the CORRECT answer?” The point here is that it isn’t a mistake to have strong views; it is a mistake to have nothing else.

Some “desirable difficulties” challenge us to consider something from a different perspective than initially considered. Such activities shake the foundations of students that believe conclusions “can’t be altered.” Consider the following graphic as a representation of a fraction. What fraction do you see pictured?



If each student is asked to individually identify a fraction based on this picture you’ll find that many different fractions are offered up for consideration. As a challenge for open-mindedness, if student A “sees’ ¾ and student B “sees” 3/5” then challenge each to find the other’s fraction. Many possible fraction representations can be found in this image. Challenge the students to let go of their first impressions and search for something that they didn’t see at first glance. For example, how can this same image also be perceived as 2 ¼?

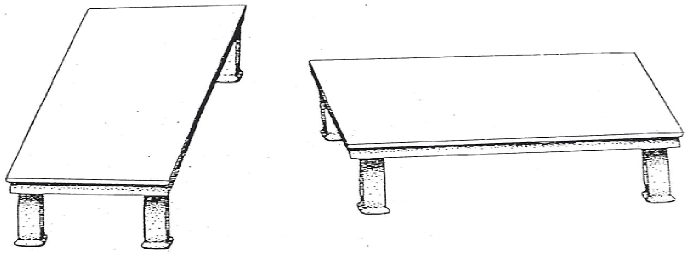
Another example of a “desirable difficulty” is one that involves a situation in which we know WHAT the answer is, we just don’t know WHY that is the answer. For instance, suppose a standard deck of cards is arranged into alternating “red-black” order, and a simple cut of the deck is performed once so that the two “half decks” show alternate colors on the bottom card. Follow this by a simple single shuffle, and ask WHY it is that when the cards are drawn off the top in pairs, there will always be one red card and one black card in each pair? If this is NOT the case, then start over – you’ve done something wrong in the above process. The question is, how can I be so sure? WHY must this always be the case? Why aren’t there any pairs of red or pairs of black? Isn’t a shuffle SUPPOSED to mix the cards?

A common category of “desirable difficulties” is represented by paradoxes. Unlike most DD’s, the main effect of a paradox is that it leaves you scratching your head looking for an explanation. Still, these illustrate the dilemma of unexpected results. A more rewatrding cousin of the paradox is the “desirable difficulty” that has a twist or surprise ending.

Consider the number of segments of length “1” that can be found on a standard 5x5 peg geoboard. A moment’s reflection yields the answer of 40 (20 horizontally and 20 vertically). How many segments of length 2? 30 (15 horizontally and 15 vertically). How many of length 3? 20 (10 horizontally and 10 vertically). How many of length

4? 10 (5 horizontally and 5 vertically) So, how many of length 5 on a standard 5x5 geoboard? The pattern shouts for 0, but, in reality, the answer is 8.

One of the most common exemplars of a “desirable difficulty” may be found in an optical illusion. While such illusions neither hold the attention nor have the rich extensions that subsequent exemplars might have, they, nevertheless, provide the reader with a quick emotion of incredulous surprise. Foe example, consider the following illustration.



A2 TURNING THE TABLES

Despite the appearance, the two tabletops are congruent! I’ll leave it to the reader to cut a shape to match the table on the left and then rotate it to the table on the right.

The feeling of “impossible” or “that can’t be” stays with the reader even after having “proven” the congruence for him/herself. What’s missing, though, is the “so what?” of a richer learning experience.

Concepts, terminology, and symbols are the foundations on which mathematics learning and linkages are built. We learn new concepts, terminology, and symbols in many ways by exploring information that is presented to us in many forms. The notion of "having learned something" implies, among other things, that the learner can readily demonstrate the ability to identify, label, use, and transfer the knowledge, skills, and processes learned. While linking mathematics to “real-world” experiences is an effective method of introducing new concepts and emphasizes the "use and transfer" components of learning, student verbalization of connections through desirable difficulties is another significant means of assisting with the "identification and label" components of transfer.

Student exploration emphasizes opportunities for internalizing and anchoring information by having students verbalize – not blindly, but meaningfully - important facts, identifications, definitions, and procedures. Learners internalize information through continued exposure and concerted effort. Students do not learn by doing – they learn by thinking about what they are doing. Mathematics, therefore, is best characterized as a series of action verbs rather than as the rules that those action verbs produce. Students must engage in Modeling, Analyzing, Thinking, Hypothesizing, Experimenting, Musing, Applying, Transferring, Investigating, Communicating and Solving. These practices anchor information and help students absorb and retain the information upon which critical thought in MATHEMATICS is based.

A healthy combination of student action and linking new material to previously learned mathematics concepts, procedures, and practical experiences will set the stage to help students feel more comfortable in their knowledge and understanding of new concepts or procedures. Understanding gained through concept development and linkages, in combination with meaningful memorization of important terms, concepts, and algorithms, gives students power over mathematics. This power leads to confidence and an increased comfort level in their ability to function and reason mathematically.

Linking new material to previously learned mathematics concepts, procedures, and practical experiences sets the stage to help students feel more comfortable in their knowledge and understanding of new concepts or procedures. Mathematics teachers are cognizant that the concepts and skills they teach today are often used later as building blocks for more abstract ideas. It is just as important to be aware of the benefits of using these links as a means of building a spirit of partnership in learning.

It is as important for us to acknowledge as it is for students to realize that they can do the mathematics that we are about to teach -- before we teach it. Students that believe they possess prerequisite knowledge that leads to a "new" concept play a more active role in learning than those who feel the teacher is the sole provider of the knowledge needed to learn a new concept.

Introducing concepts through linkages enables students to relate new ideas to a context of past learning. Students are then more likely to understand and, therefore, absorb new material. For example, students that are being taught to multiply polynomials should be led to the connection between this algorithm and the standard algorithm for multiplying whole numbers taught in the third or fourth grade. Similarly, the student who recognizes "lining up decimal points when you add decimals" as just an extension of “adding like place values from whole numbers” is far ahead of the student who sees this generalization as a NEW rule to learn, unrelated to prior knowledge.

Connecting mathematics to real-world experiences is another effective method of introducing new concepts through linkages. While students too infrequently link their transactions at the store to mathematics class, they often quickly understand that if one candy bar costs fifty cents, then two will cost a dollar. Thus, buying candy at a store can be linked to such mathematics concepts as ratios, proportions, ordered pairs, linear graphs, patterns and functions. But such “simple” links must also be followed by less direct paths to help students find their way to the richness of connections. For example, how might a telephone book help develop an approximation for pi?

Consider this chain of related concepts:

- Subtraction as a difference [$x_1 - x_2$]
- Distance between two points, x_1 and x_2, on a number line
- Distance between two ordered pairs on a grid
  square root of [$(x_1 - x_2)^2 + (y_1 - y_2)^2$]
- Pythagorean theorem [$a^2 + b^2 = c^2$]
- Equation of a circle [$x^2 + y^2 = r^2$]
- Area of a circle [$A = \pi r^2$]
- Monte Carlo probability model [$P(e) = P(A_c)/P(A_s)$]

Since $$\frac{\text{P(inside circle)}}{\text{P(inside square)}} = \frac{\pi r^2/4}{r^2} = \frac{\pi}{4}$$

We can then use the last four digits of 200 random numbers in the telephone book. Split each of these into two two-digit numbers. Treat the first pair as "x" and the second pair as "y" and then, Determine how many phone numbers out of the 200 selected yield $x^2+y^2 \leq 10000$ and, using the probability formula, we have an approximation for pi from a telephone book!

Conclusion

Understanding gained through concept development and linkages, in combination with memorization of basic facts and algorithms, gives students power over mathematics. This power leads to confidence and an increased comfort level in their ability to function and reason mathematically. Students exposed to mathematics in this manner are more likely to have the ability to set up problems, not just respond to those that have been previously identified for them. They are more likely to value a variety of approaches and techniques; to have an understanding of the underlying mathematics in a problem; to have the inclination to work with others to solve problems; to recognize how mathematics applies to both common and complex problems; to be prepared for open problem situations; and to believe in the value and utility of mathematics.

Ideally, spending valuable instructional time on desirable difficulties will allow us to better address the expectations of a school mathematics program. As teachers, it is important to recognize that our beliefs about the nature of mathematics influence what students learn and what they perceive mathematics to be. Providing students with a rich array of opportunities to construct meaning from experiences that challenge the ways in which student think and act when confronted with unfamiliar mathematics enhances the ability to make the conceptual leap from concrete to abstract reasoning.

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Why don't we make it our business to teach Business Statistics well?
Some parlous practices and some recommended remedies.

Bruce Stephens, Department of Econometrics & Business Statistics,
Monash University, Melbourne, Australia
bruce.stephens@monash.edu

Abstract
"The Grand Vizier of Randomania wants to know the average number of sheep per family ..." "If the ages of the 4 employees of a pizza shop are 19, 21, 19 and 17, calculate the mean, median and mode." "A bank manager wants to estimate the mean balance for a class of accounts. So, she takes a random sample ..." In university teaching, questions like these do not convey to students the importance and relevance of Statistics in business. They signal that Statistics progresses little beyond content learnt late in primary school and that the discipline has yet to discover the computer database. With slapdash or outdated teaching material, students see abundant errors, inattention to detail, perfunctory analysis and solutions, focus on calculation rather than understanding and application, humdrum repetition of school topics, failure to respond to different learning styles and knowledge levels, and failure to keep pace with current computing practices. These problems are exacerbated as research pressures force academics to further sideline teaching – while encountering a larger and more diverse student body. Although we can't create professional statisticians in a semester or two, we should at least aim to produce critical, well-informed consumers of Statistics. This paper discusses some parlous practices and recommends some remedies.

Three recent events prompted me to write this article. When I told a graduate I had just met that I am a statistician, he politely sympathised with my choice of profession because his experiences indicated that it must be so boring. (Another resounding condemnation of Statistics teaching!) My keenly mathematical son, in planning his Science degree, respectfully told me that he wouldn't do any Statistics because he'd already done it every year in high school. Sharing the lecturing of a colleague's subject caused me to reflect (again) on the teaching of Business Statistics. (The newest edition of the textbook very poor and the colleague was uncomprehending of my criticisms as he was unaware that the edition had changed a year ago and was continuing to teach from the superseded edition.)

Although job opportunities have driven many brighter students into Business, rather than Science, degrees Business students remain fairly innumerate and numerophobic – despite the importance of numbers in business, especially those with $ signs attached. (A coursework Masters student returning to study commented to me recently that the hardest topic of the first two weeks of her Statistics subject was Percentages!)

For those students who don't have the mathematics skills (never obtained them or have forgotten them), lecturers should promulgate an inventory of the prerequisite maths knowledge for their subjects, administer a diagnostic test and provide support material for those who need it. Students may be surprised to be reminded that they don't need mastery of the entire school maths curriculum and that a manageable amount of preparation would greatly enhance their confidence and performance. Such recognition of our students' starting point becomes more important as lecturers increasingly must adapt to the diversity embodied in students from overseas, of low Socio-Economic Status, with mature entry, and of lower tertiary entrance scores.

The same approach should apply to the Excel skills that we take for granted. Despite the apparent ubiquity of computers, undergraduates arrive with little spreadsheeting experience. (Over 6 years of science-based secondary education at a well-resourced, private school, my sons barely saw a spreadsheet.) This is a lack which is very limiting in their university studies in Statistics.

Because many Business students are not from a Maths/Science background, they lack a logical, problem-solving approach to the material. They also lack appropriate study techniques for quantitative content - not appreciating the importance of summarising as a simple, but effective technique, and being apparently ignorant of the vital importance of doing (rather than just reading the solutions of) problems. For a subject using a textbook which contains many problems, I review each problem and produce a list of Basics and a list of Extras – to make a student's task more manageable. I continually encourage them to do Problems – with weekly reminders and with a problem log sheet. I think it incredible that such reminders should be necessary, but was gratified by the following unsolicited email from a student. *"I got 83% ... a big thank you to you since ... your advice "do the problems" is in fact very very very useful. I finished most of the problems … I've a much better understanding after doing the problems ... Thank you so much for reminding us to do the problems every week."*

We need to provide students with an expectation of something different from the calculation-fest, received annually at high school, and to convince them that there are useful business skills to be acquired. Because of the abiding skepticism about Statistics ("Lies, damned lies ..."), we start with a difficult task – but an important one - and we will fail to convince if we teach poorly. "Figures don't lie, but liars figure." can be an effective rejoinder. In Week 1, I confront my reluctant MBA (Business Insights from Data Analysis) students with the reflection that, if they can't understand and strategically use the data that's relevant to their business sector, then perhaps their competitors can!

Many current(!) textbooks and on-line courses let us down badly. Firstly, they are a cacophony of calculation – often just a set of context-less numbers awaiting the ministrations of a formula! With most calculation done expeditiously nowadays by electronic means, the task of humans is to do the thinking – are students being trained for this? In the year 2011, why do simple linear regression problems still require calculations (even using the "alternate formulae"!) when interpretation should be the focus? Why do textbooks proclaim, "The (arithmetic) mean is the average ..." when three uses of "average" abound in common parlance? Why do textbooks ask students to repeatedly calculate summary measures when we can instead be discussing how these concepts are commonly (mis)used.

- *"I look forward to the day when all Australian workers earn more than the average wage."*
- *"The average salary was $28,467, that is, most people earned about that amount."*
- *"The average Australian pagan is a female Melburnian under the age of 35 who was born in Australia, has a university degree and lives in a de facto relationship."*
- *"Australia is not a country that follows averages; averages are the exception to the rule."* (From a discussion about rainfall.)
- *"A high variance situation lands you back in the head-in-the-oven-feet-in-the-refrigerator syndrome."* (From a discussion about risk in investment portfolios.)
- *"**Ms Average Australian!** According to the latest census, the average Australian is female."* (From a newspaper report.) Does this make me above or below average?

- *"The condition of this book is classified as 3 stars (average)."* (From a website for collectibles – using a 5 star scale.)

Secondly, errors abound in the body of textbooks and in their problem solutions. (What happened to quality control?) I was recently asked to lecture a colleague's subject, for which a 2010 edition of the textbook (textbook B) had recently become available. The publisher had no errata file when I requested it and this turned out to be the most error-prone book I have ever seen! (As the book had already been adopted, a staff member had to be appointed to re-do all solutions.) Perfunctory, procedure-based, error-ridden, incomplete solutions provide no basis for learning. In some books, the examples in the body of the text are not accompanied by electronic data files. So, to work through the examples, students have to manually enter the numbers into a spreadsheet from a table of data on the page.

One popular Business Statistics offering (textbook B) has, through all six of its US editions (dating back before 1997) and its two Australian adaptation editions, an illustration of basic time series properties via a graph which bends back on itself – giving an "interpretation" of as many as three observations at each time period. How does such laxity persist over so many editions, years and co-authors, and so many countries, lecturers and students? (A colleague dismissed my criticism with, "But <u>you</u> read things really carefully!" I replied, "No, I <u>read</u> things!")

Many textbooks (and an on-line course from a very prominent Business school) persistently have trouble, from edition to edition, with defining a symmetric distribution and distinguishing mode and modal class. *"A data set is symmetric if the data set's histogram has a <u>single peak</u> at the center and 'looks the same' to the left and right of the <u>most likely value</u> of the data."* is one offering. (The underlining is mine.) Students are often asked to calculate the mode of a data set which has several equi-frequent modes, the textbook failing to recognise that Excel reports only the smallest of these. The above-mentioned on-line course states, "Let's return to histograms ..." and displays two time series! Textbook A (in its 5th US edition) doesn't have any of the following words in its index: 'average', 'frame', 'statistic', 'parameter', '(regression) coefficient', 'pie chart', 'confidence level', 'level of significance', 'cyclic behaviour', 'seasonality' or 'expected value'.

With such persistently poor teaching materials from highly experienced authors and institutions, it's little surprise if students become dispirited and fail to take our discipline seriously.

Too often, textbooks are not "business friendly" - a histogram with obfuscating bin midpoints (-0.24945, -0.18845, ..., 0.17758, 0.29959), axes with inadequate or no labels, unnecessarily mathematical phraseology ("Hence, …"), a dearth of explanation of the business context and relevance, and so on. This not how we train our students to analyse data and report results professionally and comprehensibly to a lay audience.

A late 1990s review of a newly released Business Statistics textbook referred to *"a 1990s reprint of 1980s publication, written in the 1970s by a 1960s mind"*. It was referring to just one of many texts (then and now) with anachronistic examples that carry over from the author's first edition. Questions like *"A bank manager wants to estimate the mean balance for a class of accounts. So, she takes a random sample..."* look absurd to today's "computer generation". Why are many tables of the standard cumulative Normal distribution ($P[0 < Z < z]$) not consistent with Excel & current calculators? ($P[Z < z]$)? Why bypass the practical benefits of incorporating the *finite population correction factor* for the archaic justification of simplifying the calculation? Why do the non-heuristic “shortcut” (“alternate”) formulae and, often, their derivation still command attention? These anachronisms (and more!) diminish our credibility.

The credibility of our discipline is further undermined by the use of obviously unreal data sets (*viz*., the Grand Vizier of Randomania). *“If the technique is as important as the teacher claims, how is it that he/she has been unable to find a real example?” … Inventing data serves to reinforce the misconception that Statistics is a science of calculation, instead of a science of problem solving.*[1] There are challenges in creating data that are realistic. For example, textbook A provides customer service times at a fast-food restaurant, in minutes, to 15 decimal places!

Many illustrations of the points that I make here will be provided in my presentation. However, they have been omitted here for copyright reasons.

It seems, from much of the currently available teaching material, that we still have a long way to go in taking Business Statistics away from mere calculation to problem solving and plain-language reporting of the results. To this end, we need to pressure publishers for better textbooks (and ancillaries - preferably correct, complete and mutually consistent). Because of wide variations in the content and presentation of teaching material, busy lecturers will still usually need a textbook (or they will be made busier by having to write their own). Through our communities of practice, we can advise each other on suitable choices. (How much credence can we put in reviews that have been written in return for merchandise from publishers?) Sadly, we can’t rely on the usual set of “cookie cutter” problems (or their perfunctory solutions) in textbooks for effective teaching. However, fortunately, the daily media provide a fecund source of highly applicable and undoubtedly relevant material.

Some see salvation in the use of on-line materials. However, many of these just repeat the well-worn mistakes of the printed page. Perhaps e-books will introduce new frontiers – perhaps for easier updating, maintaining topicality and correcting errors. We should strive to teach with real business data and problems or we will lack credibility. Admittedly, this may be difficult for reasons of privacy or copyright but we should at least strive for realistic business scenarios. Ideally, teaching departments appoint a moderator to support each subject’s coordinator. Part of the moderator’s brief should be to bring Business Statistics teaching at least into the 20th century (preferably the 21st).

With recent graduates informing me that they can’t implement statistical techniques in the workplace because they fear that their bosses will be intimidated, it seems that we still have a long way to go in teaching Statistics to the business world.

<u>References</u>

[1] “A Handbook of Small Data Sets” D.J. Hand, et al, 1994.

Using technology to assist Mathematical Literacy learners understand the implications of various scenarios of loan circumstances when buying a house.

(Workshop Summary)

Joyce Stewart
Kingswood College, Grahamstown, South Africa
j.stewart@kingswoodcollege.com

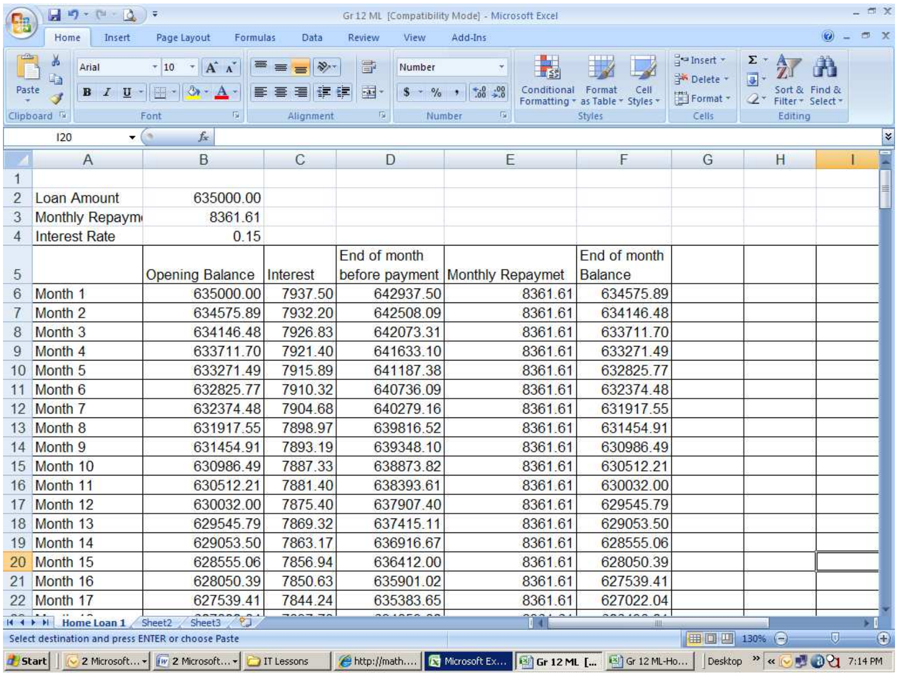
Abstract

Whilst focusing on the contextual topic of buying a house, which is relevant to the South African Mathematical Literacy curriculum. I will be exploring the implications of various scenarios the effects of varying bond terms, changes in interest rates, the impact of additional payments and the consequences of lump sum deposits have on the life of a loan (DoE, 2003). Technology is used to enable the exploration of these various scenarios without the tedious pen and paper method that would be required without an Excel program to extrapolate the data. Excel also allows us to easily change the parameters and see how these changes affect the lifespan of the loan.

Introduction

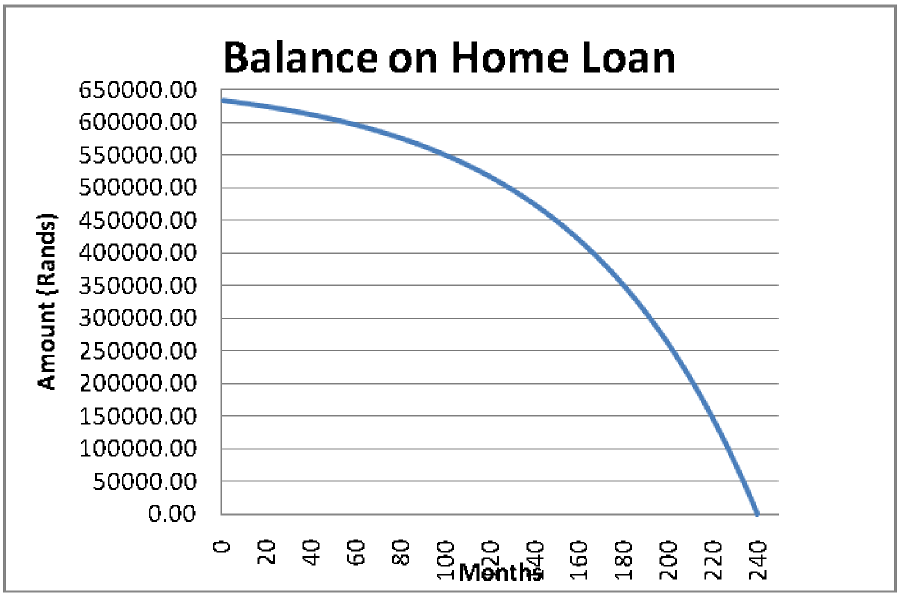
In trying to get learners to understand the intricacies of various scenarios when it comes to buying a house and taking out a bond, I find Excel removes the computational restraints of pen and paper calculations. Alagic (2003) suggests that technology allows learners to engage interactively with problems enabling them to see the immediate effect of changes and therefore comprehending better what happens with parameters are changed.

So the lesson starts with us looking at setting up the Excel spreadsheet, in order to do this the leaner needs to have a basic understanding of the working of Excel and the theory behind how interest and payment calculations work with respect to a loan.



Loan Amount	635000.00				
Monthly Repaym	8361.61				
Interest Rate	0.15				
	Opening Balance	Interest	End of month before payment	Monthly Repaymet	End of month Balance
Month 1	635000.00	7937.50	642937.50	8361.61	634575.89
Month 2	634575.89	7932.20	642508.09	8361.61	634146.48
Month 3	634146.48	7926.83	642073.31	8361.61	633711.70
Month 4	633711.70	7921.40	641633.10	8361.61	633271.49
Month 5	633271.49	7915.89	641187.38	8361.61	632825.77
Month 6	632825.77	7910.32	640736.09	8361.61	632374.48
Month 7	632374.48	7904.68	640279.16	8361.61	631917.55
Month 8	631917.55	7898.97	639816.52	8361.61	631454.91
Month 9	631454.91	7893.19	639348.10	8361.61	630986.49
Month 10	630986.49	7887.33	638873.82	8361.61	630512.21
Month 11	630512.21	7881.40	638393.61	8361.61	630032.00
Month 12	630032.00	7875.40	637907.40	8361.61	629545.79
Month 13	629545.79	7869.32	637415.11	8361.61	629053.50
Month 14	629053.50	7863.17	636916.67	8361.61	628555.06
Month 15	628555.06	7856.94	636412.00	8361.61	628050.39
Month 16	628050.39	7850.63	635901.02	8361.61	627539.41
Month 17	627539.41	7844.24	635383.65	8361.61	627022.04

Secondly we chart the lifespan of the loan (again learners need to be familiar with how to chart graphs using Excel).



Then it is time to investigate what effect changing parameters, such as interest rate, increased payments, lump-sums, additional payments have on the lifespan of the loan. Learners are now able to change these parameters to see what happens to their charts.

Once learners have an understanding of loans I usually then take them onto a banking website to introduce them to the interactive software that can be found there to see what criteria must be met in order to qualify for a loan. It highlights the ratio of earnings to repayments and gives a clear indication of how much one can afford.

This is a good example of real world applications of mathematics through graphical representation of information through the use of technology. It addresses the curriculum criteria of being able to draw graphs by hand/ technological means as required by situations and critically interpreting tables and graphs in real life situations. This enables learners of Mathematics Literacy to critically use their skills to prepare themselves for real life situations. It also moves away from the traditional talk and chalk and becomes more explorative for learners in which to grapple with the real-life problems that they will be confronted with.

Conclusion:

When teaching my grade 12 learners I do feel that the use of the above method helps them gain deeper understanding as to the effects of various scenarios on loans. They themselves can manipulate the loan setup to see what happens when they do what. It also allows them to take the initiative as to what to change to see what happens and if it is as they predicated. This is something that I feel has worked quite effectively in my classroom.

References:

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Department of Education, South Africa (2003). *National Curriculum Statement Grades 10 – 12(General). Mathematical Literacy.* Pretoria: Government Printer.

Developing Skills for Successful Learning

Liz Swersky, I.S.T.D, H.E.D – Edupeg Project Manager, SACTWU Edupeg Project, Roeland Square, Roeland Street, Cape Town, 8001**,** http://www.edupeg.co.za
edupeg@roelandsquare.co.za

Abstract
Many of the teachers in the less advantaged schools in South Africa lack skills, varied teaching techniques and capacity. Multiple changes to the curriculum have left teachers confused about content and application of teaching methods and strategies. Many of these educators are demoralised and confused, which has negatively impacted on their confidence and self-esteem. Many teachers see each support intervention as a new and stand alone entity, and many educators often do not have the skill or experience to either use or integrate the resources that are present in the schools. The focus at Edupeg is to add value to the provincial and national educational endeavour, through facilitating improved curriculum delivery, by increasing material resources, enhancing intellectual capacity and promoting the confidence and self-esteem of both teachers and pupils. Through sensitively exposing teachers to more effective teaching methods and strategies we are able to create opportunities for children to be actively and creatively engaged in their own learning. Our goal is to achieve improved quality of teaching in the classroom, which will lead to improved pupil comprehension, and ultimately, performance.

INTRODUCTION – A practical and interactive session
Many teachers think that "good teaching" consists of a teacher dominating the class proceedings, predominantly through talking, to impart knowledge to the quiet and attentive pupils. A pre-requisite for this teaching style, is that pupils pay attention to what the teacher is saying. As often is observed, the outcome of such an approach is that the children become passive and eventually inert to the extent that they are virtually mute – with virtually no participation in their own learning process. Teachers rely on the more skilled and alert pupils, to answer the occasional posed questions. This process actually masks the inadequacies of the weaker learners, who become further withdrawn due to their lack of participation and their growing perception of their lack of skills and competence. A sense of low levels of confidence and self-esteem tends to become the result of this process. "The biggest obstacle standing in the way of the achievement of young people is what they believe about themselves." Professor Jonathan Jansen, "How to build excellence in a culture of mediocrity", U.C.T. February 2011.

Children should be seen as thinkers, not empty vessels which need to be filled with knowledge. They are indeed active participants in the construction of their own knowledge. It is during these early years, when the first networks of knowledge based on experiences are formed, that the foundations for all later learning are laid. The environments of home and school enhance and support each other in learning. They provide the context in which learning takes place and the opportunity for learning to be effective. These environments obviously influence the child's growth and development, and are the foundation for all learning situations. Extensive studies by Piaget concluded that knowledge is created actively and is not received passively. It is thus vital to create and provide an exciting and stimulating learning environment.

The Edupeg programme provides stimulating, interactive opportunities for young learners themselves to interact with learning resources. While practically engaging with Edupeg activities, children learn to recognise, evaluate and cope with educational and developmental challenges. Edupeg activities also help to develop concentration, perseverance, memory skills, social skills, self-confidence self-esteem, and cognitive skills, including thinking and reasoning.

This comprehensive, learner centred educational programme, is fun and challenging, but not threatening. The activities encourage a positive attitude towards learning and the self-corrective aspect promotes self evaluation, immediate access to right or wrong answers and has obvious multiple benefits in a classroom environment.

There are 22 Edupeg workbooks, which are available in English, Afrikaans, isiXhosa, isiZulu, Sesotho & Pedi. They have no racial gender or cultural bias. The workbooks contain activities which promote and enhance the development of visual perception. They then incorporate Numeracy, Literacy, Language and Life Skills, which extend the development of the child, and increase mathematical awareness and proficiency, as well as promote and improve literacy and communication skills.

The realities in less advantaged classrooms

- Classes are often very large, and many teachers are without assistants, or have assistants who are not effectively used.
- Many classrooms are not big enough to accommodate a mat area, or lack a mat, and so small groups of pupils are not brought forward to work and interact directly with the teacher.
- Pupils have not been trained in routines which keep them actively involved at their tables.
- Classrooms lack sufficient resources, or resources are not used. Also methods of storing, distributing and caring for resources are not taught or practiced.
- There is no differentiation of pupils into ability groups.
- To maintain discipline in the large class, teachers resort to teaching from the front and require all the pupils to be attentive, and not to talk, except for the repetition in unison, required for rote learning.
- Pupils frequently spend long periods of time doing nothing.

The above severely disadvantages the learners because:

- Many learners enter school when they are not school-ready, having inadequately developed perceptual skills. They do not advance easily, until these skills have been developed.
- Educators often do not realise that the children learn skills through “doing”, and the learners are not given sufficient opportunity to be actively engaged.
- Learners often have limited language. Without sufficient encouragements to talk and communicate, language does not develop, self-esteem is low, thinking skills are impeded, and social skills do not develop.
- Learners become very dependent, often copy each other, and do not develop creativity. Thinking skills and problem solving skills are also stunted.
- In many cases, learners adopt survival strategies, which are counterproductive, and difficult to “undo” later.
- In the educator’s haste to “cover” the curriculum, many basic underlying skills are not given the time and attention needed, which then has a follow-on effect of the learners’ battling as they progress, through their schooling.

<u>Edupeg learning values</u>

Edupeg understands that learners learn best when they:

- do things
- explore and discover things for themselves
- have fun
- communicate with each other
- are not afraid of failing
- feel good about themselves

Edupeg also helps learners to develop their skills and abilities. Edupeg develops learners who:

- can communicate
- can solve problems
- are confident
- can work with others
- have the skills they need to cope in life

A strong focus of Edupeg activities is on Numeracy.

Numeracy
Edupeg:
- draws on the child's intuitive and acquired knowledge in number as a stimulus for continued learning
- provides stimulation and enjoyment through the varied activities provided
- encourages confidence, understanding and creative individual thought
- consolidates basic number operations
- increases the ability to communicate mathematically
- builds on knowledge and experience and assists in the understanding of the above in the child's world
- encourages mathematical communication and the correct use of mathematical symbols and terminology
- promotes systematic and accurate written work, including all calculations
-

Language and Literacy play a crucial role in the acquisition of skills.

Language & Literacy
Edupeg:
- encourages familiarity and fluency of language
- supports vocabulary enrichment and extension
- provides exposure to accurate written language, spelling and reading
- creates opportunities for story-telling, including sequencing and ordering of events
- provides practice in and pride of mother tongue
- encourages appreciation of visual stimulation and discussion thereof
- supports interpretation, selection, understanding and processing of given information
- nurtures the development of thinking skills and problem solving techniques
- allows for opportunities to incorporate 2nd/3rd language usage, from the visual stimulation provided

Additionally,

Life Skills
Edupeg encourages and promotes:
- communication
- problem solving
- organisation and planning
- decision making
- comparison of methods and options
- discovery and exploring of relationships
- expanding the awareness of mathematics in it's relationship to human beings
- enjoyment and pleasure

Effective learning resources need to be well conceived and compiled.
Suitability of Edupeg materials
- Content written by experienced South African school educators
- All materials have proved to be durable and the majority, re-usable
- Activities are appealing to the target age group
- Approach and content of materials are curriculum appropriate
- The books contain no cultural, racial or gender bias

- Instructions are simple and clearly understood by learners
- The activities are able to satisfy the needs of learners with differing abilities
- Sufficient variety of activities for stimulation
- Adequate repetition of activities for revision and consolidation
- Simple provision for on-going and continual assessment
- Content, including pictures, diagrams and rubrics stimulate learners' interests and thinking skills, and expand their often limited world
- Activities are enjoyable, varied, challenging and promote independence and self-reliance
- The complexity and depth of activities promote concentration and the active involvement of the learners
- Material available in mother tongue
- Provided with a comprehensive Teacher Resource Book for additional guidance and further ideas

Education needs to be meaningful, and have positive outcomes, where accurate and appropriate teaching resources are utilised.

Benefits of Edupeg

- Learners are engaged in challenging tasks, appropriate for their capacity and level of skills
- There is an improvement in responsiveness of pupils – they are active
- The creation of an active, stimulating, learner centred environment which promotes pupil participation
- Methodical recording of set activities, including the method used to attain and determine answers – promotes concentration and thinking skills as well as neat, accurate recording, including spatial concepts
- Learners improve vocabulary, language, sentence construction and communication skills
- Experiences are provided for children to be active through the asking of open questions, where learners can be right. This builds confidence, self-esteem, fosters creativity and promotes individuality.
- Group work encourages and develops critical thinking skills, verbal communication, active involvement, listening skills, tolerance and problem solving strategies
- Gives learners themselves the opportunity to "find answers" utilising their own talents, and to figure it out rather than to "know" i.e. needing to rely on previous knowledge/rote learning
- Working independently promotes self-reliance and the ability of the learners to work without constant teacher intervention
- Many children fall down in assessment due to their lack of experience and exposure to the format of questioning. Edupeg provides such opportunities.
- Pictures offer experiences that children can own and learners can respond to these. They are not the "teacher's" pictures
- Diminishes the teacher's need to be prescriptive
- Instructions are clear, simple and easy to follow
- Fine motor control is developed
- Diminishes the need for loud/punitive classroom discipline
- Kinetic, Auditory, Tactile and Visual Learning takes place, NOT just Auditory.
- Utilisation of concrete and semi-concrete equipment to assist learners to develop a sound understanding of concepts
- Assists teachers to use and manage resources

A varied range of appropriate classroom practices need to be utilised, and educators need support to become more effective and proficient in the classroom.

Combatting Talk & Chalk!

- Educators are assisted to promote differentiated, learner-centred activities
- A balance can be created between exposition and learner activity
- Learners are actively engaged in tasks, appropriate for their developmental level
- Dialogue, thinking skills, concentration, vocabulary, communication, and tolerance, are all assisted and developed
- Learning takes on aspects of fun and enjoyment
- Learners become enthusiastic and less inert/passive
- Learners are absorbed in tasks and discipline problems diminish
- The pace and content of the lesson is not dictated by the slowest learners
- A handful of learners will not be responsible to "carry" the class, while a large percentage of the class remains mute/inert
- Educators are able to promote more individual questioning
- Teachers can diminish chanting in unison and rote learning
- Educators are able to fulfil continuous assessment, as they can **see** and **evaluate** what learners can **do**

Competent and prepared teachers promote productive learning environments. All teachers are supported by our sensitive, well qualified trainers.

Learn to plan

- Objectives are clearly defined and then the method to achieve these is decided upon. (The "how", "with what", "for how long", "what next"?)
- Suitable resources are selected to fulfil objectives
- Appropriate activities are selected, suitable for level of learner's ability (How many teachers **tell** us that they have a huge differentiation of learners in their class, but teach **all** the **same thing** at the **same time** for the **same length of time**)
- Concrete equipment can be included. (The lack of concrete teaching is a large contributor to the very low level of maths functioning seen in so many schools)
- Active learning enables students to gain skills and hone skills
- Provision is made for learners to have practice in reading instructions, not having instructions read
- Provision is made for learners to methodically and accurately fulfil written activities (NOT just filling in words/figures/completing sentences on photo-copied sheets)
- Written tasks promote concentration, consolidation of knowledge, logic, critical thinking skills, spatial concepts, pride and self-esteem
- Opportunities are created for the educators to assess the set tasks are created

Educators are assisted and encouraged as they seek to improve their skills and capacity.

Constructive classroom support

- Teachers are given practical support, not theoretical workshops
- Teachers who have a limited knowledge and experience of a subject/concept to be taught, as well as a lack of awareness of how best to achieve the desired outcome/ objective of a lesson, are assisted
- Even when teachers appear to understand the content of workshops/training courses, to implement this, **alone**, on their return to school, is **not easy**. Many lose heart and revert to old methods. (Poor system for integrated grade/phase support)
- Teachers become overwhelmed by the task of managing themselves and the learning environment. They need classroom support
- Training is internalised only when it is **used** in the classroom, in **context** and teachers need assistance with this. It is a lonely and difficult process and teachers can easily become discouraged

- Only in the classroom, can a real understanding of concepts and principles be reached
- Relationships of trust need to be established, where support and guidance are received without fear of criticism and censure
- Edupeg helps teachers to promote reasoning, observation and deduction
- Patience, persistence and perseverance are needed as trainers penetrate schools through sound educational practice

The HSRC Study shows that the development of children is hampered by poor teacher input. Assisting educators to become more proficient, more actively engaged in their teaching and more aware of teaching outcomes is crucial to improved learning.

Building confidence and self-esteem

- Assisting educators and learners to realise that if they **understand** concepts and **internalise** them, they can **build** on this knowledge, i.e. they do **not** need to memorise formulae/theories/recipes, etc
- Providing opportunities to "find"/"figure out"/"resolve" a problem using skills, knowledge, reasoning, talents, intelligence and **common sense!**
- Providing practical experiences to perform tasks that involve using concrete equipment, pictures, data, graphs, etc and allowing **time** for answers to be found
- Kinetic, auditory, tactile and visual learning must be encouraged, **not** just auditory
- Relating Maths (and other Learning Areas) to everyday life and real situations
- Reducing the learners' (and educators') anxiety and feelings of inadequacy through providing positive learning experiences. Success to be obtained through self-reliance
- Fostering creativity, language development, sentence construction, communication, thinking skills, opinions, self-expression, etc through adapting an "open" question policy, rather than a "closed", "Yes/No" approach. Children can be right! Educators to enter into dialogue with learners, and to probe the answers given
- Provide opportunities for children to read simple, clear instructions and then to complete the task. Many learners fall down on external assessments due to their lack of experience and exposure to this type of questioning

We all respond to recognition and acknowledgement and teachers are no different.

Addressing low morale

- Encourage teachers who have experienced intervention overload, not to respond with passive resistance
- To develop teachers, who presently feel inadequate
- We need to get teachers "doing", which will help to promote understanding
- Knowledge and awareness will diminish resistance (to change)
- We need to create opportunities for teamwork, experimentation, investigation and practice
- We need to assist teachers to try new methods, which will eliminate their needing to try to control and teach, large, often multi-grade classes from the front, which is **exhausting**
- We need to banish fear
- We need to create and promote hope

The above will be covered in an interactive workshop session, where participants will use the Edupeg resources in a simulated classroom environment.

What I hear, I forget. What I see, I remember. What I do, I understand. ***Confucius***

Teaching Mathematical Modelling to Tomorrow's Mathematicians or,

You too can make a million dollars predicting football results.

Kerry J Thomas, The Southport School, Gold Coast, Australia

kerry.thomas@tss.qld.edu.au

Abstract:

One of the reasons for studying mathematics is to empower us with the tools to enable us to predict with some certainty what will happen in given scenarios. Meteorologists study weather patterns and gather data to produce mathematical models that allows them to forecast the weather, with various degrees of success. Car designers use complicated mathematical models to continually refine automobiles that are stronger, more efficient and more powerful. From sport to demographics to engineering to medicine to business, we are surrounded by mathematicians who are continually modelling the world around us.

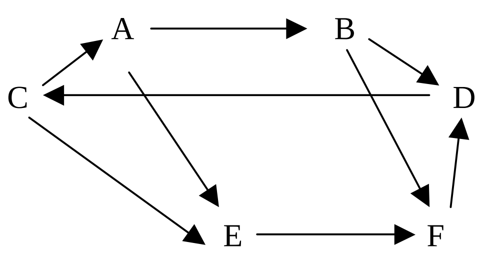
As educators of mathematics we spend the majority of our time on knowledge and content. Teaching the art of Problem Solving is a challenging endeavour indeed. Teaching Mathematical Modelling requires insight and planning to be effective.

This paper will explore an activity that I have devised to allow novice mathematicians to take on a modelling role. They create a model from data, refine the model based on new data, and finally evaluate the strength and weaknesses of their model.

Introduction:

In Queensland, Australia, teachers of years 11 and 12 design and write their own assessment. The assessment is required to be of two types, standard exam and extended response which most people would know as an assignment. There are three criteria all assessment items must address, I Knowledge and Procedures, II Modelling and Problem Solving and III Communication and Justification. I will not elaborate on the first and third of these Criteria, but address certain aspects of the second.

Firstly though, it would be timely to review some basic Dominance Theory using Matrices. This is a technique for ranking teams or players who are playing in a round robin competition and the competition is at the stage where every team has not played each other. Say there are 6 teams in a competition, and so far 3 of the 5 rounds have been completed. The results could be shown like this



Team A has had two wins, one over team B and one over team E but lost to team C. Suppose we were now required to rank the teams from 1 to 6. Clearly 3 teams, A, B and C have had 2 wins and a loss. Can we split these 3 teams to decide the top ranked team?

We could place these results in a matrix, called the Dominance matrix where a “1” represents a win, and a zero represents a loss, or did not play.

$$\begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{Team C vs} \\ \text{D} \\ \text{E} \\ \text{F} \end{array} \begin{array}{c} \begin{array}{cccccc} \text{Team A} & \text{B} & \text{C} & \text{D} & \text{E} & \text{F} \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array} = \boldsymbol{\beta}$$

First order dominance is simply a win. A defeated B so A has first order dominance over B. Second order dominance occurs for A over F as A defeated B and then B defeated F. We could then use this second order dominance to split the three teams on 2 wins. To calculate second order dominance we simply have to square the matrix above.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} = \boldsymbol{\beta}^2$$

This indicates that team A had two second order dominances over team F as 1) A defeated B and B defeated F and 2) A defeated E and E defeated F.

Now to combine these two matrices, it is possible to weight the significance of second order dominance slightly less than first order by scalar multiplication, eg

$$\boldsymbol{\beta} + 0.5\,\boldsymbol{\beta}^2 = \begin{bmatrix} 0 & 1 & 0 & .5 & 1 & 1 \\ 0 & 0 & .5 & 1.5 & 0 & 1 \\ 1 & .5 & 0 & 0 & 1.5 & .5 \\ .5 & 0 & 1 & 0 & .50 & 0 \\ 0 & 0 & 0 & .5 & 0 & 1 \\ 0 & 0 & .5 & 1 & 0 & 0 \end{bmatrix} \text{(Sum rows)} \rightarrow \begin{array}{ll} \text{A} & 3.5 \\ \text{B} & 3 \\ \text{C} & 3.5 \\ \text{D} & 2 \\ \text{E} & 1.5 \\ \text{F} & 1.5 \end{array}$$

As A and C have 3.5 points, and C defeated A we could rank C above A. Likewise E and F have similar points, and E defeated F hence we could rank E above F.
This then allows us to theoretically rank our 6 teams after 3 of the 5 rounds and possibly use this ranking to predict subsequent rounds. The ranking would be C, A, B, D, E and F.
If A was drawn to play D in the next round we would expect A to win and so on. Obviously this does not always happen for a variety of reasons.

The Modelling Process:

There are many ways to describe the mathematical modelling process, but a simplified approach is shown in figure 1. Each of the 7 stages will be explained in greater detail as I present a real life problem that students are required to model.

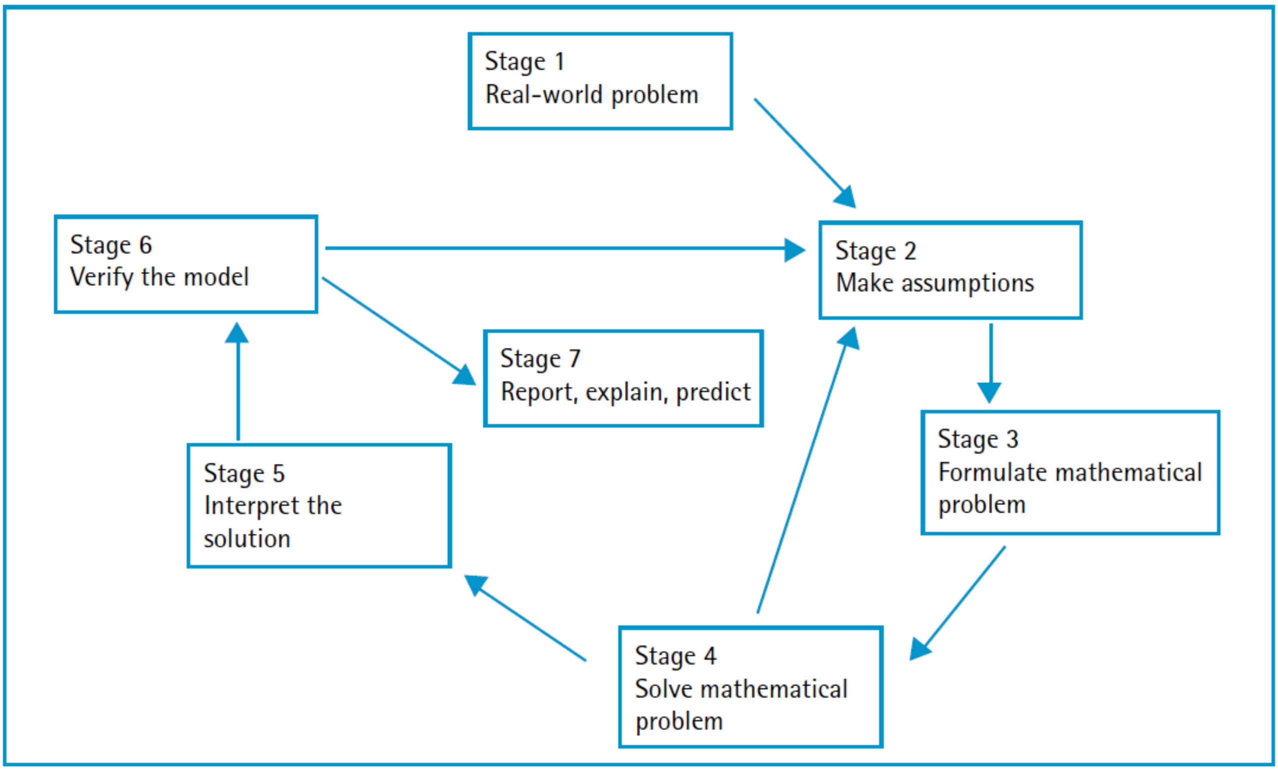


Figure 1. The modelling process

The Modelling Exercise:

Following an introduction to matrices, which also explores some applications of matrices, students are given 8 weeks to develop a dominance theory model for the national Rugby League competition. The task is given to the students a week prior to the commencement of the competition and provides an opportunity for individual creativity in developing a final model. It is made very clear to the students that the end result of picking 8 winners from 8 games is NOT the main objective, but rather students are to concentrate on documenting the stages of the development of their model, and to justify any changes that take place to produce the final working model.

Stage 1: A Real World Problem

The problem is presented simply, free from data. It is a topic (Rugby) that appeals to the cohort (boys). There is no one correct approach, however the first step for all is to collect the data. Dominance Matrix theory has been taught prior to the problem being presented and the students will have practiced simple examples.

Dominance/Supremacy Matrices:

The first 7 rounds of the 2011 NRL competition are listed on the following pages.

- You are to record the results of these matches, along with conditions under which each game was played.

Based on these results,

- Model these results using matrices and dominance procedures to rank the teams.
- Outline any enhancements you have made as you have refined your model as the season has progressed. List strengths and limitations there may be with your technique.
- State any assumptions you are making and any affect these have on the outcomes

(To better fine-tune your system you may like to use the results of the first 3 rounds to predict the winner of the 4th round, and then make any adjustments if necessary to predict the winner of the 5th round and so on. Document each model you develop together with your reasons for any changes you may make in developing subsequent models.)

- Make sure you address the criteria on the attached pages

- Finally,

Use the rankings you have established using your model at the end of round 7 to predict the winners of round 8 below.

Round 8:

Brisbane Broncos	vs.	Canterbury-Bankstown Bulldogs
North Queensland Cowboys	vs.	Manly-Warringah Sea Eagles
St George-Illawarra Dragons	vs.	Parramatta Eels
South Sydney Rabbitohs	vs.	Cronulla-Sutherland Sharks
Canberra Raiders	vs.	Wests Tigers
Melbourne Storm	vs.	Newcastle Knights
Gold Coast Titans	vs.	Sydney Roosters
Auckland Warriors	vs.	Penrith Panthers

Stage 2: Make Assumptions

Done properly, this is a time consuming part of the exercise. A list of all the variables should be made and these modified or simplified before proceeding further. In this example, variables such as whether the game is played in the day or the night, home or away, fine or wet weather, teams are full strength or with injury substitutes etc, could be taken into account. As with any modelling exercise, whenever the real world model is over simplified it is a very strong possibility that the model will lose accuracy when it is used for predicting future outcomes.

Stage 3: Formulate mathematical Problem

A standard Dominance Matrix has a 1 for a win, and a zero for a loss. Students are presented with the opportunity to vary these values. Would it be more appropriate to incorporate home and away wins? For example, 2 for an away win and 1 for a home win…..or 1.5 for an away win? Should a greater value be given for a win margin of more than 20 points? Students are given the freedom to use whatever values they can justify.

Stage 4: Solve the mathematical problem

Once it has been decided what values to use in the matrix, it is necessary to determine what weight needs to be applied to second order, possibly third order dominance to produce the working model. After 3 rounds the teams are then ranked and used to predict the 4th round of matches. From experience, very few students will pick 8 winners from 8 games, so its time to relook at Stages 2 and 3 and adjust the model. Students are encouraged to re-examine their assumptions and develop a new model to predict rounds 5, 6 and 7.

Stage 5: Interpret the solution

The model is further refined until there is some consistency in results. The students are encouraged to revisit the initial problem, and ensure that their model is working within the constraints set. One important aspect of this stage is that they soon realise that their solution is quite clearly governed by the constraints, and is not easily transferred to other situations.

Stage 6: Verify the model

This stage requires each student to look for strengths and limitations of their mathematical model. Reflecting on their model, and the success they have experienced in predicting winners for rounds 4,5,6 and 7, it is clear that their models have limitations caused mainly by a very simplistic approach to the many variables that effect the results of a football match. Just as it is important to identify the variables used it is also of value, and indeed probably more important, to identify the variables that are ignored. This is highlighted in the following Criteria for Modelling and Problem Solving

Criteria for Modelling and Problem Solving

Rating	Description
A	• shows initiative and insight in developing the matrix model • details the assumptions made in the selection and use of their matrix model • details all refinements that took place during the development of the model • explores the strengths and limitations of the matrix model • uses effective strategies to synthesise a matrix model
B	• shows some initiative in developing the model • details the assumptions made in the selection and use of their individualised model • details some refinements that took place during the development of the model • uses effective strategies to obtain a matrix model
C	• develops a basic model and uses some data • uses some effective strategies

Stage 7: Report, explain, predict

In the week leading up to Round 8, students submit their assignments, along with the teams their model predicts will win that round. Students are required to outline the development of their model, and explain all aspects of the process. At the conclusion of round 8, students are handed back their assignments (unmarked), with the results of the round. Under exam conditions students are given 45 minutes to appraise their model based on how well they were able to predict the winner of round 8. They are also encouraged to discuss what future enhancements could be made to their model for the remaining games.

Conclusion:

I have used this learning experience/assessment item on a number of occasions over the last 4 or 5 years. Without a doubt it has proved challenging, rewarding and enjoyable for my students. Other observations are as follows:

- Mathematical modelling is best taught by students “doing”
- It is open ended, which gives students the freedom to explore
- Emphasis is placed on the modelling process and its development rather than predicting 8 winners from 8 games
- It challenges students to critically evaluate, reflect and formulate – a difficult set of important skills to teach in mathematics.
- Above all else the students find it FUN to do.
- I have yet to see a student’s model that consistently predicts winners, so I suppose winning those millions of dollars is still just around the corner.

Teaching and learning high school mathematics through an interdisciplinary approach

Ariana-Stanca Văcărețu, MA
Mathematics teacher
Romanian Reading and Writing for Critical Thinking Association
Cluj-Napoca, Romania
ariana.vacaretu@vimore.com

Abstract

The aim of this paper is to share the results of an implemented action-research project in teaching high schools mathematics through an interdisciplinary approach.
The starting point of the action research project was the high school students' lack of motivation in studying mathematics which had an impact on the students' learning outcomes. Few students learn math simply because they like it. Most students learn math if they understand why they have to learn it, if they have the possibility to apply mathematical concepts and algorithms in real life or within other subjects or various contexts.
The Romanian mathematics textbooks do not support teachers in motivating students' learning. In these textbooks they can find exercises like: "solve the equation …." or "determine the derivative of the function …".
The action research project answers the following questions:

- what are the effects of the interdisciplinary approach on the students' learning in mathematics?
- what are the effects of the interdisciplinary approach on reaching both the specific mathematical aims and mathematical literacy & competence in science and technology - as a key competence domain?

The action research project is documented with results of assessment and evaluation of the mathematical learning. Students' reflections on their interdisciplinary experience, on their learning mathematics through different subjects (physics, chemistry, music, etc.) are included in the paper.
The paper concludes that the interdisciplinary approach enhances both, the students' mathematical learning and their mathematics, science & technology competence.

Introduction

Students' motivation in learning mathematics is decreasing year by year. If students enjoy learning mathematics in the primary school – as during this stage it is more closely related to operations with natural numbers, their enjoyment in learning math is getting lower and lower as the level of abstraction is increasing. Most high-school students learn math only because they have to take the final examination at the end of the 12th grade. As a mathematics teacher, I have often heard the same question asked by different high-school students: "Why do we have to learn …. ?" Lack of intrinsic motivation in learning mathematics has an impact on the students' learning outcomes. Even if the Romanian mathematics curriculum is quite generous, as it states that *the main aims of studying mathematics in high-school are to develop students' skills to reflect upon the world and provide them with knowledge to act upon the world; to formulate and solve problems by using knowledge across different domains*, mathematics textbooks do not support teachers and learners in reaching these aims. Exercises and problems in the math textbooks have a high level of abstraction and they allow for the use of mathematical concepts and algorithms only in mathematical contexts.

The list of key competences for lifelong learning (European Commission, 2007) which *are particularly necessary for personal fulfillment and development, for social inclusion, active citizenship and employment*, includes mathematical competence and basic competences in sciences and technology. Mathematical competence is defined as *the ability to develop and apply mathematical thinking in order to solve a range of problems in everyday situations, with the emphasis being placed on process, activity and knowledge*, while basic competences in sciences and technology refer to the use and application of knowledge and methodologies that explain the natural world. This key competence includes understanding of the changes caused by humans and each individual's responsibility as a citizen. Moreover, this key competence is one of the eight key competences which are all interdependent.

Mathematical thinking is not easy to define. Based on the Principles and Standards for School Mathematics published by the National Council of Teachers of Mathematics (NCTM), mathematical thinking is related to reasoning, problem solving, communicating and connections; it is a thinking process for building mathematical understanding. Needless to say, developing students' ability to apply mathematical thinking to solve everyday life problems is a long process which is based on modeling the real world and developing problem solving skills. During the learning process, students have to learn how to apply mathematical thinking in different contexts.

In autumn 2010, I started teaching mathematics to a class of 31 ninth graders, who focus on studying sciences and English language (bilingual track). At the beginning of the school year, asked why they decided to take the sciences path, the students answered that:

- they wanted to study medicine (45%);
- they wanted to become chemistry researchers (10%);
- they wanted to study some mathematics during high-school, but not as much as those students whose curriculum focuses on mathematics and computer sciences and not as little as those students who focus on social sciences (45%).

I asked the students *what* they would like to learn in their high-school mathematics class; they answered that they would like to learn:

- things that would be useful in their personal and professional life (80%);
- concepts and algorithms that they needed to know to get a good score in the final exam (71%);
- how to reason (26%).

Being asked how they would like to learn mathematics, the students answered that they would like to learn mathematics:

- in as enjoyable a manner as possible;
- without being stressed;
- by interaction;
- with examples of practical/ everyday life problems which stimulate thinking/ reasoning;
- in English.

I carried out the initial/ diagnostic assessment in order to identify the level of the students' mathematical skills and their ability to use mathematical concepts and

algorithms they would have to use during the high-school math. The average score of the initial assessment test was 56.60 out of 100.
Considering all the above mentioned issues, I designed my action-research project in teaching high-school mathematics to this group of students through an interdisciplinary approach. I was, inspired by the results of the ScienceMath Project and the conceptual framework for cross-curricular teaching (Beckmann, 2009) about which I had learnt in the European Science Math teacher training, as well as by the philosophy of inquiry-based learning (Dewey, 1916).

Teaching high-school mathematics through an interdisciplinary approach

As I had to follow the national mathematics curriculum which limits my academic freedom, I integrated the interdisciplinary approach in teaching high-school mathematics in both my lessons and on the e-learning platform I developed for the ninth graders. The e-learning platform is a “moodle” which contains specific tasks for students. The forum on the platform as well as individual e-messaging and the feed-backing option when assessing students’ work allow individual support of learners. Currently, the e-platform contains each student’s e-portfolio.
During the first year of my action research project, the interdisciplinary approach took shape in the following learning activities:

Topic	Involved subject(s)	Students’ tasks	Specific methodology/ strategies used	Mathematic learning objectives
Real numbers	Music	- students compose and play a piano/ guitar piece to represent different rational and irrational numbers.	Music specific methodology	Students distinguish and explain differences between rational and irrational numbers
Number systems	Literature	- students read a passage from Ian Stewart’s book *Nature’s Numbers* using text coding. - students write an essay on Leopold Kronecker’s quote “The dear God has made the whole numbers, all the rest is man’s work.” - students watch Marcus du Sautoy’s videos *From Zero to Infinity, Safety in numbers* and the video *The Teaching Challenge – Simon Singh* (the videos were available on the teachers.tv website) and answer questions (conceptual clarification, probing reasons, perspectives, probing	Reading and writing strategies for understanding texts (reading with text coding, writing essays, Socratic questioning)	Students measure cardinality of number sets, explain the development of the number systems, explain the use of different numbers in everyday life situations and technology, appreciate contribution of different cultures to developments in mathematics.

Topic	Involved subject(s)	Students' tasks	Specific methodology/ strategies used	Mathematic learning objectives
		implications and consequences) which addresses different thinking levels.		
Numerical functions	Sciences (chemistry, biology)	- students collect water samples, measure dissolved oxygen, turbidity, temperature, pH; - students specify relationships among water quality indicators collected data; - students define and graph the numerical functions which described the established relationships.	Inquiry and project based learning Chemistry experiments	Students use different ways of describing a numerical function, analyze practical situations and describe them by using numerical functions.
Quadratic functions	physics	- students watch the video of the freely falling ball experiment and use the freely falling ball graph (position depending on time); - by reading the graph, students identify the position of the ball at different moments, the interval of time between two different positions of the ball, the domain and codomain of the function, axes intercepts and their meaning related to the experiment, coordinates of the vertex point, monotonicity of the function, the image of the function; - students calculate the 2^{nd} degree polynomial which defines the graphed function.	Inquiry based learning	Students interpret functions from graphs, analyze quadratic functions to describe the motion of an object, identify applicability of quadratic functions.
Vectors	Physics, English language	Students explain in writing, in Romanian language, the solution of the two vectors problems (use of addition vectors tools to solve relative velocity); both problems and solutions are provided in	Reading and writing proofs for understanding math concepts and processes	Students use addition of vectors to solve problems in non-mathematical context, analyze and evaluate

Topic	Involved subject(s)	Students' tasks	Specific methodology/ strategies used	Mathematic learning objectives
		English language (the problems were available on Homework helper - Physics and Mathematics website www.jfinternational.com).		written proofs, read scientific text in English language, identify applicability of vectors.

Results of the action research project in teaching high-school mathematics through an interdisciplinary approach

I analyzed the results of the so far implemented action research project by answering my lead questions:

- What are the effects of the interdisciplinary approach on the students' learning in mathematics?
- What are the effects of the interdisciplinary approach on reaching both the specific mathematical aims and mathematical literacy & competence in science and technology – as key competence domain?

Interdisciplinary approach better addresses students' learning styles. For example, two students who are excellent guitar players were very proud about their performance when they represented real numbers through music. When they started the learning activity, they did not remember anything about rational and irrational numbers, even if they had learnt about them in the seventh grade. After six months, they still remembered the learning activity and they easily explained differences between rational and irrational numbers and gave examples of rational and irrational numbers. Learning was a process of "personal discovery of meaning" (Fatzer 1998, p.66) as these students understood real numbers very well; playing the guitar to demonstrate their learning was an important emotionally charged moment for them.

Twelve students (39%) who have predominant verbal/ linguistic intelligence were enthusiastically engaged in learning about the number systems by reading literature and writing essays, and they also obtained excellent results.

The interdisciplinary approach has improved students' motivation to learn mathematics. Their reflections on their interdisciplinary experience are relevant:

"My best work in the grading period is the essay on Leopold Kronecker's quote *The dear God has made the whole numbers, all the rest is man's work.* This is my best work because I took pleasure at writing it. Mathematics does not provide too many opportunities for free-writes. This assignment gave me the freedom to write my own thinking without being afraid that I'm wrong. I enjoyed working with mathematical concepts." (from the semestrial self-assessment report of Adela F., student)

"In the beginning, I didn't enjoy learning the Vectors unit. I thought this unit was useless; it was so different from what we'd learnt in geometry until then. Later on, I found that vectors are useful for solving some geometrical problems – and the proofs were short, and elegant, and that vectors are tools for physics in working with motion and forces. By studying the Vectors unit, I started using vectors to solve geometry problems (which I find difficult) and it has been really easy to solve mechanics problems in physics now I understand the operations with vectors. These motivated me for learning." (from the Learning Journal of Ana A., student)

"I don't like mathematics very much – I'm good at physics, but I have to say that after understanding the vectors applicability in physics I started to think that learning math supports me in better understanding physics. Understanding the concept and the use of operation with vectors helped me to better understand velocities and forces." (from the Learning Journal of Alex K., student)

"From my point of view the most important thing was that we learnt to work in a team and how to cooperate with our classmates. We had a lot of fun while collecting water samples and we wanted to act together to improve water quality. Defining the functions was a bit difficult but we managed to do it together." (from the Feedback sheet – Water Monitoring Activity of Alina S., student)

During the implementation of the action research project 10 students (32%) improved their reasoning and logical thinking, 16 students (51%) used tools, physical models or technology appropriately, and all students were able to make at least one connection to topics outside mathematics.

The average score of the students' assignments is 75.71 out of 100, which shows a relevant improvement in students' learning outcomes (+19.11 as compared with the initial assessment).

Closing remarks

The theme of the South African Conference is *Turning Dreams into Reality: Transformations and Paradigm Shifts in Mathematics Education* has encouraged me "to share innovative and creative ideas for effecting reform and transformation in the area(s) of […] classroom practices". My ideas for transformation may be implemented with no or few adaptations to the learning environment and the specific group of students in mathematics. I do hope that my paper convinced you about the efficiency of the interdisciplinary approach in achieving improved mathematics learning. For me, this approach really is '*Turning Dreams into Reality*' as my dream is to teach mathematics to students that are eager to learn it because they understand it.

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A New Elementary Mathematics Curriculum: Practice, Learning and Assessment Some Classroom Episodes

Isabel Vale
School of Education of Polytechnic Institute of Viana do Castelo, Portugal
isabel.vale@ese.ipvc.pt
Domingos Fernandes
University of Lisbon, Portugal
dfernandes@ie.ul.pt,
António Borralho
University of Évora, Portugal
amab@uevora.pt

Abstract

The aim of this paper is to present the new and innovative Mathematics Curriculum for elementary levels that is being implemented in the Portuguese basic education system (students from 1^{st} to 9^{th} grade) through an overview of an ongoing study of implementation/experimentation of this curriculum. A specific mechanism was implemented in the field to provide scientific and pedagogical support to the development of the new elementary mathematics curriculum (NPMEB) implementation at all grade levels and all over the country. In particular, the NPMEB is being experimented by a set of teachers that teach in their own classes and that have been trained and accompanied along the experience by the different authors of the program. We will focus on some classroom practices, sharing innovative and creative ideas of teachers and students, grounded on some of the tasks used by the teachers. The preliminary results suggest that some improvements are already visible, namely regarding students' attitudes and mathematical competences and teachers' practice.

Introduction

In the current Portuguese education system there is a new and innovative elementary mathematics curriculum (students from 1^{st} to 9^{th} grade), the NPMEB (ME, 2007) that includes a series of changes of the government's responsibility to improve the conditions of the teaching and learning of that discipline. To this have contributed the discontent with the results obtained by students in national external (e.g. standardized tests, examinations) and international assessments (e.g. Program for International Student Assessment - PISA). This curriculum was designed to gather some disperse curricular documents and substitute the current syllabus/curriculum/program, published in the early 90s, but mainly to provide the sustained development of students' mathematical learning focused on the more recent recommendations of mathematics teaching and learning.

A specific mechanism was implemented in the field to provide scientific and pedagogical support to the development of the NPMEB implementation through all the grade levels and all over the country. This approach was not generalized but applied to a sample of classrooms/teachers. In particular, the NPMEB is being experimented since 2008 by a set of teachers that teach in their own classes and whom have been trained and accompanied along the experience by the different authors of the program. At the same time the Ministry of Education named a team of mathematics educators for an evaluation study of the process of implementation/experimentation of the NPMEM for three years. The study was designed in three phases with the following purpose: to describe, analyze and interpret teaching practices and assessment developed by teachers of the experimentation and/or teachers to teach in the process of generalization; to describe, analyze and interpret the involvement and participation of students in developing their learning in the context of the classroom; and to evaluate such practices and other curricular materials applied. It was expected to have three

multiple case studies (one for each cycle of basic education), an evaluative global summary and some recommendations.
We propose to present an overview of some of the tasks used by the teachers and the work of the students in accordance with NPMEB. as well as the strategies used by the teachers. Data has suggested that some improvements are already visible, namely regarding students' attitudes and competences and teachers' practice. Before presenting the method of evaluation of the study and some classroom episodes, we began to identify the main ideas of the NPMEB.

Main ideas of the NPMEB
The NPMEB is no more than a readjustment of the existing program, with nearly twenty years, for grades 1-4 (1st cycle of basic education), 5-6 (2nd cycle of basic education), and 7-9 (3rd cycle of basic education), (ME, 1990, 1991), which points to significant changes on mathematics teaching and learning, and to professional practices of teachers. In our opinion, the most innovative aspect of the NPMEB it was to replace the three existing programs by a single one that involves all grades from 1 to 9 with the same structure and the same mathematical themes. On the other hand transversal skills are addressed in the same way that mathematical topics, i.e. to which were suggested methodological guidelines, resources and examples of tasks.
The aim of the new curriculum (ME, 2007) is to promote, in the students, the acquisition of information, knowledge and experience in mathematics. On the other hand intends to develop the capacity of integration and mobilization in different contexts and also indicate the development of positive attitudes towards mathematics and the ability to appreciate this science of all students. Thus three indissociable basic aspects are pointed out for mathematics education - the acquisition of knowledge, ability to use it appropriately and develop general relationship with the discipline. At a subsequent stage of the program is organized in each cycle, around four major mathematical themes (Numbers and Operations, Geometry, Algebra and Organization and Data Analysis) and three transversal fundamental capacities (Problem Solving, Reasoning and Communication).
The NPMEB also presents several general methodological guidelines, with emphasis on the need for diversification of tasks and giving particular attention to its nature, mainly to the challenge they promote, the role of situations in context, the importance of mathematical representations and the connections in mathematics and with extra-mathematical aspects, the educational value of group work and moments of collective discussion in the classroom, the importance of appropriate use of technology and other materials. It is an opportunity to: value certain features of mathematics that were forgotten or worked in a decontextualized way (e.g. mental computation, number sense, demonstration, visualization, geometric transformations, patterns, algebra, statistics); introduce some topics earlier (e.g. rational numbers, algebra); value mathematical processes (problem solving, reasoning and communication); and value the mathematical tasks and the roles of the teachers and students.
The program outlines a set of general principles for the evaluation and especially emphasizes the importance of curriculum management held at the school level. Such program involves a process of curriculum change and demand for an exploratory teaching-learning with a new kind of classroom culture, where students must be much more active and be part of the construction of new knowledge, where teacher must offer appropriate tasks within a challenging element. The tasks have a crucial importance in this change. It is the teacher who can start by giving a task that uses students' knowledge, while allowing the development of new concepts or processes so that they effectively engage in work and interpret correctly the task proposed. This work can be done in different ways, discussing and arguing ideas, orchestrated by the teacher, in order to avoid repetition, and highlight what is mathematically essential. In this perspective the traditional classroom is replaced by

discovery and the development of higher-order capabilities such as testing, conjecturing, reasoning and proof, can be shared by students and teachers. To face these new challenges it is asked for a change of attitude towards mathematics and its teaching, for proper teacher training programs, appropriate educational materials, as well new organizations in schools.

NPMEB: The Evaluation Study

The Ministry of Education through the Department of Curriculum Development and Innovation (DGIDC) requested under the Process of Experimentation of the NMPEB a evaluation study, that it's ultimate purpose is to produce a set of evaluative synthesis and recommendations that could contribute to regulate and/or enhance the development of NPMEB. The DGIDC devised a plan to implement the new program in basic schools. This plan provided five actions such as: a) experimentation, during 2008/2009, the NPMEB in 40 pilot classes of three cycles of basic education; b) the beginning of widespread NPMEB in the academic year 2009/2010; c) the production and distribution of curriculum materials of different nature (e.g. thematic booklets, assignments for use in classrooms; lesson plans); d) a support structure for the beginning of widespread NPMEB in 2009/2010 (e.g. new program coordinators in each group; set of accompanying teachers); and e) teachers' training. In the training, all teachers experimenters participated in training throughout the school year (50 hours classroom and 50 hours of autonomous work) which, in essence, was of the responsibility of NPMEB' authors. Yet developed a monitoring process (e.g. visits to classrooms and meetings with teachers experimenters) through a coordinator group for each cycle that met on average once a month and all groups three times per year. All teachers had reductions in their school hours and shared their classes with a pedagogical pair. The process of experimentation began in forty pilot classes equally distributed by the 3, 5 and 7th grades.

The First Phase included the design of the conceptual framework of the process of the NPMEB' experimentation. So it was taken into account: 1) the NPMEB framework; 2) the design of the implementation plan; 3) the structure of support for plan implementation; 4) the plan to support the process of experimentation; 5) the management system of the process; and 6) the teachers experimenters. The data was obtained through: *interviews* – to all the 40 teachers experimenters (EP) and 42 others intervenient (from coordination group, consultative council, authors program, and professional and scientific associations, mathematicians and mathematics educators, teachers of the three cycles of basic education, administration) translated into about 110 hours of material recorded and transcribed in full; *document analysis*; *field notes* and a *questionnaire* – to the teachers experimenters.

We can summarize the main results of this first phase of the evaluative study, in the following ideas: 1) An innovative process of implementation 2) A hard beginning of the process of implementation 3) a demanding program (the NPMEB) 4) A well achieved teachers education training. However there were some problems with the educational materials available, the follow-up of the teachers experimenters had some virtues and difficulties, it is expected a difficult generalization so some care will be need.

The Second Phase. In this phase we were mainly concerned with classroom. Taking into account the objectives of the study we intended to describe, analyze and interpret the learning environments, students participation, teaching and assessment of the classrooms in the process of experimentation and generalization of NPMEB. The dynamics of the classroom and its complexity is always difficult to be categorized, because most times there are overlaps and interactions that can not be translated directly into an "instrument". However we designed a assessment matrix where considered three main objects of evaluation and fourteen dimensions, indicated in parentheses, to support the team in the data collection and systematization and also to organize and structure the first report: 1) *Teaching practices* (e.g. teaching planning, organization, resources, materials, tasks, classroom dynamics; role of teachers and students, time management); 2) *Practice assessment* (e.g. integrating, articulation with teaching, predominant assessment tasks; nature, frequency and distribution of feedback; dynamics of assessment; predominant role of teachers and students); and 3) *Participation of students* (dynamics, frequency, nature).

This evaluation study is descriptive, analytical and interpretive in nature and therefore it was decided that the treatment data should follow closely the recommendations of Wolcott (1994). We used for data collecting lessons' observations, semi-structured interviews with teachers and students, and field notes for information from informal conversations with teachers and students. Were also consulted and analyzed various kinds of documents (eg, legislation, guidelines produced by ME; specific bibliography). This study took place over about ten months where participated six teachers, two for each cycle of basic education, that were interviewed and observed their classes and interviewed thirty-eight students. We decide to focus on description, analysis and reflection of what it was listened from teachers and students and of what was observed in the classrooms of each cycle, producing just a narrative by cycle.

Two Classroom Episodes

Despite being quite difficult summarize in few words the work done, we selected two tasks during the observations in the 4th and 6th grades respectively.

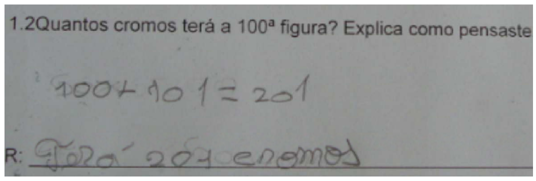
Example 1. This example (part of a math class) intend to illustrate the role played by students and teachers in solving one task, designed under the NPMEB, where you want to develop students own learning. This task is a growing figurative pattern that allowed the exploration of various mathematical topics, with particular emphasis on algebraic thinking that is a new theme for this cycle and used manipulative materials.

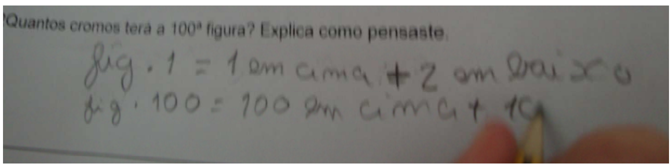
In the third part of the lesson (20 minutes) the students organized themselves into groups of four to solve a pattern task in afigurative sequence context. As a support material the teacher gave each group square to realize the first terms of the sequence.

Emerged as usual in this class, several resolutions and different interpretations and strategies. We conducted a synthesis of all resolutions and each student had the opportunity to share their reasoning with the class.

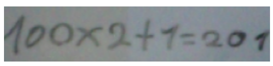
This group was limited to making a statement very concise.

But there were groups that explained their thinking in more detail and clearer.

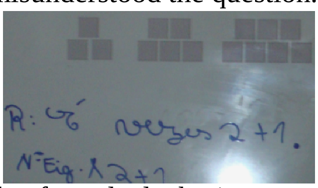




Or through another way to represent the general law of the pattern structure



In the last question, which asked "to determine the number of squares needed to build a picture of any order, "there were many difficulties. A number of students appeared to have misunderstood the question.

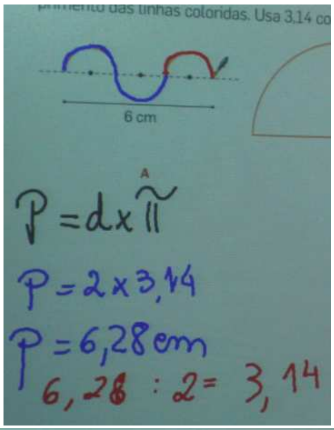


One student, considered the best in the class, explained to colleagues that it was "another two times. " The teacher asked him to complete then he write: "Number figure x 2 + 1".

This, for these young students, it is not easy, it was made *far generalization*, using algebraic thinking and a representation near the formal algebraic expression. Here could have been using the term "double" but in the meantime, the class was nearing the end. (Notes from observation)

If it is true that one of the roles of students is undoubtedly be involved in discussions in small or large group, this is due in very large to the actions of the teachers in order to encourage mathematical communication in classrooms. In fact, the teachers try to provide time and space so that students could present their own resolutions, ask questions and discuss with colleagues. Thus, it is never left to explore issues even in the preparation of the final synthesis.

Example 2. The communication that it was more often observed during classes , it was in the form of dialogue between teachers and students that were always invited to explain and verbalize their thoughts and reasoning. It was noted also that teachers were concerned with questioning on the work and on the arguments presented, providing clues and alternatives in order to supplement and enrich their work. Teacher seemed to have driven discussion of the strategies used in problem solving and presentation of the main findings. The following episode shows that students had opportunities to express themselves, share their thoughts, and present their questions or to submit alternative solutions which could be explored by the whole class.



One student, that usually has problems of being concentrated, read the task. Then the teacher asked to a student:

- How do you think we can find half of the perimeter of a circle?

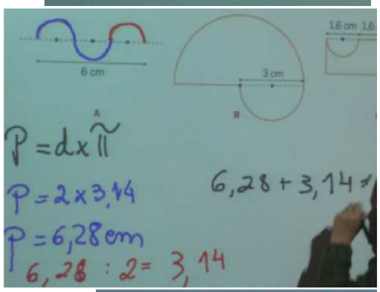
The student answered at once

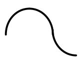
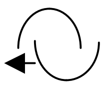
- Radius times pi.

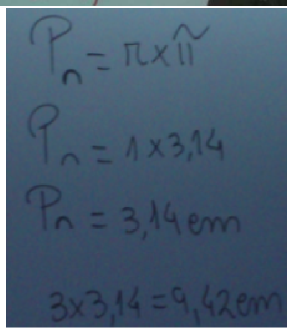
The teacher asked to explain his answer

- What is the diameter of each?
- 2.
- Why?
- Because 2 x 3 = 6.
- Or because 6 : 3 = 2.
- So to know the length of the blue line what I have to do is :

P = d x p ; P= 2 x 3,14 ; P = 6,28



- Anne, do you agree with what William said?
- No, I think it's worth here.  (and makes a draw)
- Why?
- Humm ...
- I wanted to understand ...
- I also wanted to explain ...
- Look, if I took this line and put it here  ...
- Ah! Yeah.
- And now what is missing?
- I think we have to divide 6,28 by 2.
- Let's do the calculation? We can do it but this relationship here (pointing to the calculation 2 x 3,14) ...

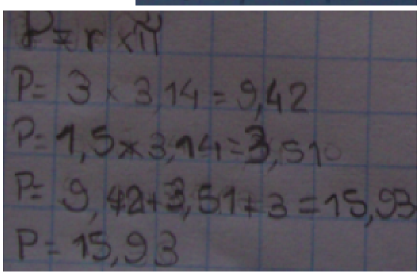


Students painted each line of the drawing with the color of the calculation.

- And now?
- Now we add the two parts.
- Come on. Use another color.

At the end, another student intervened:

- If we make the radius times pi gave half and then we multiple by 3.
- Excellent! Let's register.



Then, the students respond to the other questions by themselves, in pairs. A pair of students presented its solution as shown in the next picture.

This description illustrates the dynamics of work and communication, used often by participating teachers, to help students learn to work in solving problems raised by the tasks presented. However, as mentioned by one of the teachers is not always possible to explore every task or topic with the desired attention and depth, because of the time to carry out all the themes of the program.

Conclusions

The main main conclusions and reflections in this second phase of the evaluative study, can be summarized in the following ideas. 1) *A successful stake* - despite some difficulties it was possible to establish a support system and monitoring which contributed decisively to the creation of new and innovative dynamics in areas such as teacher training and the participation of students; 2) *A well interpretation of the program* - contrary to what appears to be usual, the participating teachers seem to have a well understood on basic content areas of the program, the fact of textbooks are no available may have contributed to the teachers need to study in detail the program; 3) *The planning and collaborative work of teachers* - the planning of lessons, their analysis and discussion led to dynamic collaborative work that contributed to teachers felt more confident of their performance; 4) *The presence of transversal capabilities* - the transversal capacities constant in the program were deliberate and systematic part of the daily concerns of the participating teachers, which eventually settle as a routine in the classroom observed; 5) *Structure of classes and pedagogical routines* - The practices of the teachers were generally well articulated with the methodological guidelines set in the program. Typically, the classes, focusing on tasks followed from in accordance with the next four phases: a) Presentation and appropriation of the task, b) Resolution of the task, c) Discussion of solutions and results, and d) Reflection systematization and synthesis; 6) *Teachers well aware of his role* - all teachers, even with styles and very different experiences and attitudes, seemed to have well-established routines. Perhaps one could say that in general, the teachers participating in this study appear have learned to listen more carefully to a greater number of students; 7) *The problem of time management* - teachers have great difficulty in managing time and when they could, they used the time away from other curricular areas; 8) *The problem of students assesment* - the concepts and practices of teacher assesment participants seemed to be articulated with their teaching practices. For most participating teachers there are issues that are not resolved, such as the very concept of assessment, its purposes, functions, modalities and their nature; and 9) *Student oriented, cooperative and aware of your paper* - students participated with relative ease in the dynamics established in the classroom, were well-oriented by teachers to the tasks to which they were proposed and participated, not all equally, in developing their own learning.

In short, the preliminary results suggest that some improvements for school mathematics are already visible, namely regarding students' attitudes and mathematical competences and teachers' practice.

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Mathematical modelling in classroom: The importance of validation of the constructed model

Michael Gr. Voskoglou, B.Sc, M.Sc., M.Phil., Ph.D
Professor of Mathematical Sciences, Graduate Technological Educational Institute (T.E.I.), School of Technological Applications, 263 34 Patras, Greece, mvosk@hol.gr voskoglou@teipat.gr, URL: http://eclass.teipat.gr/RESE-STE101/document

Abstract

The present paper, where the basic models used to describe the mathematical modelling process in classroom are mainly discussed, is actually the theoretical supplement of a workshop, in which we shall examine the importance of the last two stages of mathematical modelling process concerning the transition from the solution of the model to the real world (validation of the model and implementation of the final mathematical results to the real system) by presenting a number of suitably chosen examples.

Introduction

It is well known that the reformation attempted from the beginning of the 1960's in mathematics education with the introduction of the *modern mathematics* in school curricula, was proved to be a complete failure. The attempt to teach the fundamental generalizations before presenting the objects to be generalized, had as a result the despoliation of curricula from examples and applications connecting mathematics with the real situations of everyday life and the other sciences using it as a tool and giving birth to very many new mathematical problems and theories.

Thus, and after the rather vague movement of the *back to the basics*, considerable emphasis has been placed from the late 1970's on the use of problem as a tool and motive to teach and understand better mathematics, with two coordinates: *Problem-solving*, where attention was given to the use of the proper heuristic strategies for solving pure (mainly) mathematical problems (Polya 1963, Schoenfeld 1980, etc) and *Mathematical modelling and Applications,* a process of solving a particular type of problems generated by corresponding situations of the real world (Pollak 1979, Niss 1987, etc).

Although views appeared later disputing the effectiveness of using problem solving as a "learning device" of mathematics and giving emphasis to other cognitive aspects, like acquisition of the appropriate schemas and automation of rules (e.g. Owen & Sweller 1989), it is generally acceptable nowadays that through problem solving processes one can give students a balanced view of mathematics and can face effectively the false opposition between *learning mathematics* and *learning to apply mathematics* (e.g. Voss 1987, Lawson 1990, Matos 1998 etc). Even Marshall (1995), the introducer of the current schema theory, presented schemas as vehicles for problem solving that can simplify and reconstruct a problem in order to make it more accessible to the solver.

Models for the mathematical modelling process

Concerning mathematical modelling, the transformation from a real world situation to a mathematical problem is achieved through the use of a *mathematical model*, which, roughly speaking, is an idealized (simplified) representation of the basic characteristics of the real situation through the use of a suitable set of mathematical symbols, relations and functions.

Pollak (1979) was the first who described the process of modelling in such a way that could be used in teaching mathematics. He represented the interaction between mathematics and real world with the scheme shown in Figure 1, which is known as the *circle of modelling*. In the *universe of mathematics* classical applied mathematics and applicable mathematics are two intersected, but not equal sets. In fact, they are topics from classical mathematics with great theoretical interest not having (at least for the moment) any visible applications, while they are also branches of mathematics, which are not usually characterized as classical, with many practical applications (e.g. probability and statistics, linear programming, fuzzy sets and logic, uncertainty theory, chaotic dynamics and fractals, etc).

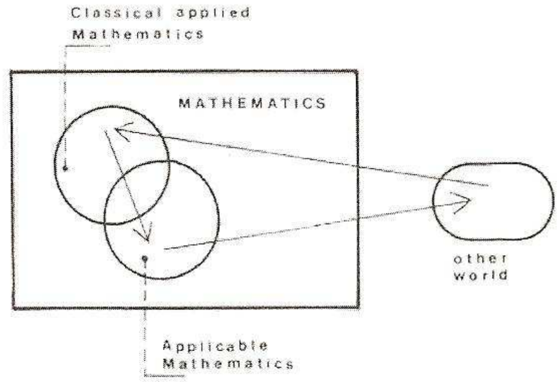

Figure 1: The circle of modelling

The most important feature in Pollak's scheme is the direction of the arrows, representing a looping between the *other world* (including real world, the human activities of everyday life and all other sciences) and the universe of mathematics: Having to solve a problem connected with a real situation we transfer from the other world to the universe of mathematics, where we use or develop appropriate mathematical methods to describe and solve the problem. Then we return to the other world interpreting the mathematical results obtained and implementing them to the real system. If these results are proved to be inadequate in describing properly the real system, we return back to the universe of mathematics making the proper modifications to the model. The modelling circle could be repeated several times until we find the proper solution.
From the time that Pollak presented his scheme in ICME-3 (Karlsruhe, 1976) much effort has been placed by researchers and educators to analyze in detail the process of mathematical modelling. The cyclic representations developed in the late 1970's in undergraduate engineering mathematics courses focussed on student activity at six discrete stages with the addition of a seventh reporting stage (Berry & Davies 1996). Transition between the stages did not at that time receive much attention. Developments in mathematical modelling in schools, since the mid 1980's, prompted a didactical and/or conceptual focus on representations of modelling within a cognitive prospectus. Blomhψj & Jensen (2003) provided a helpful and comprehensive visualization of the mathematical modelling process linking modelling to a perceived reality through six stages, whilst recognizing that theory and data, and associated linking of knowledge, impact on the process. A model similar to that of Blomhψj & Jensen is the modelling cycle of Blum & Leiβ (2007), a key feature of which is its emphasis on understanding the real situation and problem, moving through a situation model and a real model and problem whilst still in the other (extra-mathematical) world. Greefrath's (2007) model is also orientated towards the same direction. The starting point is a real situation that is not the whole reality, but an already structured situation from real life, which obviously must be chosen by someone (e.g. the teacher or the students) to deal with mathematically. This should be transformed into a real model, i.e. a simplified version of the real situation, which

can now be transformed more easily into a mathematical model that should lead to a mathematical result. The transformations between the four stages are not named in this model and are unidirectional. For more details about the above models and graphical representations of them the reader may look at Haines & Crouch (2010) and at Meier (2009).

Summarizing all the existing ideas and in order to make the ideal model a bit clearer for teachers, we could say that the main stages of the mathematical modelling process involve:

- s_1 = *analysis of the problem* (understanding the statement and recognizing the restrictions and requirements of the real system).
- s_2 = *mathematization*, which could be divided to *formulation* of the real situation in such a way that it will be ready for mathematical treatment and *construction* of the model. The former involves a deep abstracting process, in order to transfer from the real system to the, so called, *assumed real system*, where emphasis is given to certain, dominating for the system's performance, variables.
- s_3 = *solution of the model*, which is achieved by proper mathematical manipulation.
- s_4 = *validation (control) of the model*, which is usually achieved by reproducing, through the model, the behaviour of the real system under the conditions existing before the solution of the model (empirical results, special cases etc). A model is valid, if despite its inexactness in representing the real system, gives a reliable prediction of the system's performance.
- s_5 = *Implementation* of the final mathematical results to the real system.

The flow-diagram of the mathematical modelling process in classroom, when teacher gives such problems for solution to students is represented in Figure 2. The modeller starting from s_1, which is always the initial state, moves through s_2 to s_3 .From there, if the mathematical relations obtained are not suitable to allow an analytic solution of the model, he (she) returns to s_2, in order to make the proper modifications to the model. If he (she) finally fails to construct a solvable mathematical model giving a reliable prediction of the real system's performance and being unable to make any other "movements" for the solution of the problem during the time given by teacher, returns from state s_2 to s_1 waiting for a new problem, to be given for solution. Otherwise he (she) returns to s_3, to continue the process. After the solution of the problem within the model the modeller returns to the real system, in order to check the validity of the model (state s_4).

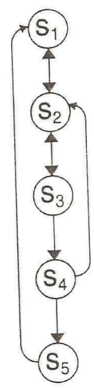

Figure 2: The flow-diagram of the mathematical modelling process in classroom

If the model does not give a reliable prediction of the system's performance, (e.g. if the solution obtained is not satisfying the natural restrictions resulting from the real system, or if it is not verified by known special cases etc), the modeller returns from s_4 to s_2, in order to correct the model. From there he (she) will return through s_3 to s_4 to continue the process. After ensuring that the model "works" well, the modeller from s_4 reaches the state s_5, where he (she) interprets the final mathematical results, "translates" them to the natural language and applies the conclusions obtained to the real system. When the process of modelling is completed at state s_5, it is assumed that a new problem is given by teacher for solution and therefore the process starts again from s_1.

A central object of the educational research taking place in the area of mathematical modelling is to recognize the attainment level of students at defined stages of the modelling process. For this, we have constructed a *stochastic model* describing (Voskoglou 2007), where we introduce a finite Markov chain having as states the main stages of the mathematical modelling process and we calculated its transition matrix in terms of the flow-diagram presented above (Figure 2). Applying standard results from theory of Markov chains we succeeded in expressing mathematically the gravity of each stage of the modelling process (where greater gravity means more difficulties for the students at the corresponding stage) and we obtained a measure for students' modelling capacities. A classroom experiment was also performed to illustrate the application of our model in practice. Two are the main outcomes of this experiment:

- There is a comparison between two groups of students with indications that the teaching of one group might be more effective than that of the other one.
- The analysis shows that mathematization possessed the greater gravity among the stages of the modelling process for both groups of students.

The second outcome was logically expected, since the formulation of the problem involves, as we have already seen, a deep abstracting process, which is not always an easy thing to be achieved by a non expert.

Mathematics does not explain the natural behaviour of an object, it simply describes it. This description however is so much effective, so that an elementary mathematical equation can describe simply and clearly a relation that, in order to be expressed with words, could need entire pages. We believe that this is exactly the main advantage of our stochastic model compared with qualitative methodologies used by other researchers for similar purposes, such as the analyses of questionnaire's collected answers by using respond maps (Stillman and Galbraith 1998), or multiple choice tests (Haines & Crouch 2001) and the related discussion activities, etc.

Models for the mathematical modelling process like the above, and the analogous ones that we have briefly described before, are helpful in understanding what is termed as the *ideal behaviour* (Haines and Crouch 2010), in which modellers proceed from real world problems through a mathematical model to acceptable solutions and report on them. However life in classroom is not like that. Recent research, (Galbraith & Stillman 2001, Doer 2007, Borroneo Ferri 2007), reports that students in school take *individual modelling routes* when tackling mathematical modelling problems, associated with their individual learning styles. Students' cognition utilizes in general concepts that are inherently graded and therefore fuzzy. On the other hand, from the teacher's point of view there usually exists vagueness about the degree of success of students in each of the stages of the modelling process. All these gave us the impulsion to introduce principles of fuzzy logic in order to describe in a more effective way the process of mathematical modelling in classroom (Voskoglou 2010).

In our *fuzzy model* the main stages of the modelling process are represented as fuzzy sets in the set of the linguistic labels of negligible, low, intermediate, high and complete success of students in each stage. The concept of *uncertainty*, which emerges naturally within the broad framework of fuzzy sets theory, is involved in any problem-solving situation, especially when dealing with real-world problems. Uncertainty is a result of some information deficiency. In fact, information pertaining to the model within which a real situation is conceptualized may be incomplete, fragmentary, not full reliable, vague, contradictory, or deficient in some other way.

Thus the amount of information obtained by an action can be measured in general by the reduction of uncertainty resulting from the action. Accordingly students' uncertainty during the modelling process is connected to students' capacity in obtaining relevant information. Therefore a measure of uncertainty is adopted as a measure of students modelling capacities (Voskoglou 2010). Our fuzzy model, apart from the quantitative information, gives also the possibility of a qualitative analysis by providing all the possible students' profiles during the modelling process.

Mathematical modelling appears today as a dynamic tool for the teaching of mathematics, connecting mathematics with our everyday life and giving to students the possibility to understand the usefulness of it in practice. It has also the potential to enhance the performance of students in mathematics generally (Matos, 1998). A special didactic methodology was developed across these lines by De Lang in Netherlands, named by G. Kaiser (Hamburg) as the *application – orientated teaching of mathematics*. But we must be careful! The process of modelling could not be considered as a general, and therefore applicable in all cases, method for teaching mathematics. In fact, such a consideration could lead to far-fetched situations, where more emphasis is given to the search of the proper application rather, than to the consolidation of the new mathematical knowledge!

About the workshop

As we have seen above mathematization seems to be the most difficult among the stages of the modelling process for students. Crouch and Haines (2004, section 1) report that it is the interface between the real world problem and the mathematical model that presents difficulties to the students, i.e. the transition from the real word to the mathematical model and vice versa the transition from the solution of the model to the real world. The latter looks rather surprising at first glance, since, at least for the type of modelling problems solved usually at school, a student who has obtained a correct mathematical solution of the model is normally expected to be able to "translate" it easily in terms of the corresponding real situation. However things are not always like that. In fact, there are sometimes modelling situations, where the validation of the model and/or the final stage of the implementation of mathematical results to the real system, hide surprises that force students to "look back" to the construction of the model and possibly to make (again) the necessary changes to it. The purpose of this workshop is to present some characteristic examples illustrating these situations and to discuss teacher's proper reactions in order to orientate students to act effectively in these cases. We shall see, for example, how it is possible to obtain two mathematically (not practically!) correct solutions when it is asked to construct a channel to run the maximum possible quantity of water by using a given metallic leaf, or why it is possible to find no solution (although there exists!) when it is asked to find the cylindrical tower with the maximum volume among those having the same total surface, etc.

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An Investigation into the design of Advanced Certificates in Education on Mathematical Literacy teachers in KwaZuluNatal

Lyn Webb[1], Sarah Bansilal[2], Angela James[3], Herbert Khuzwayo[4], Busisiwe Goba[5]
Nelson Mandela Metropolitan University[1], University of KwaZulu-Natal[2,3,5]
University of Zululand[4]
lyn.webb@nmmu.ac.za; Bansilals@ukzn.ac.za; Jamesa1@ukzn.ac.za;
hbkhuzw@pan.uzulu.ac.za; gobab@ukzn.ac.za

Abstract
The aim of this paper is to describe an ongoing study into the design of two Advanced Certificates in Education in Mathematical Literacy (ML) offered by two Higher Education Institutions in the KwaZuluNatal (KZN) province in South Africa. Mathematical Literacy is a relatively new subject that has been introduced into the grade 10, 11 and 12 school curricula as an alternate to Mathematics. Mathematics and ML are two distinct subjects with different objectives. There is thus an urgent need to train pre-service teachers and re-skill in-service teachers to teach ML competently in schools, and not to treat ML as a less sophisticated version of Mathematics. This paper looks at the design of the two qualifications and the results of the students who studied for the qualifications.

Introduction
All learners in South Africa have a choice of either Mathematical Literacy or Mathematics for their last three years of schooling. As ML is a relatively new addition to the curriculum, studies on the effectiveness of programmes targeting teacher training are imperative. By comparing and contrasting in-service programmes offered by two Higher Education Institutions (HEIs), methods, strategies and results are brought to light and evaluated. Because of the pressing need to have qualified and effective ML teachers in the classroom, the KZN Department of Education (DoE) tasked two HEIs to deliver Advanced Certificates in Education (ACE:ML) programmes throughout the province. The in-service teachers, who were prospective students, were identified by the DoE and the HEIs, therefore, had no part in specifying mathematics ability, other than the entrance requirements for the qualification. Each programme spanned two years' of study, and extended to a third year, when students were given an opportunity to repeat modules if they had failed to reach the Universities' required standards. The centres dictated by the DoE were in rural, peri-urban and urban areas.

The study described in this paper focuses on approximately 2 200 in-service teachers who have studied ACE:ML on a part-time basis over a period of two years with either of the two HEIs operating in KZN. The method of the broader study (which is preliminary at present) will follow an explanatory mixed method approach where quantitative data on the students' demographics and performance will be collected and analysed to identify any particular trends relating to uptake and success in these programmes. Module templates (providing information about the learning outcomes, modes of delivery, assessment strategies and module evaluation requirements), learning materials and tutor training guides will be analysed to provide information about the nature and design of the two ACE programmes. A questionnaire consisting of closed questions as well as a limited number of open ended items will be sent out to all current students, drop outs and graduates of the programmes. The purpose of the questionnaire will be to gather data about the teachers' perceptions of the effectiveness, relevance and utility of the two programmes as well as to solicit suggestions on how the models of delivery could be improved. Semi-structured interviews will also be conducted with samples of students from each programme to triangulate the data.

ACE programmes from all South African HEIs are currently being placed under the spotlight. According to the November 2010 Draft Policy on the Minimum Requirements for Teacher Education Qualifications selected from the Higher Education Qualifications Framework (HEQF), ACEs will have to be recurriculated into either Advanced Diplomas in Education (ADE) at National Qualifications Framework (NQF) level 7 or Advanced Certificates in Teaching (ACT) at NQF level 6 (Department of Higher Education and Training, 2010). The quandary HEIs are presented with is that the HEQF is under review and there is speculation as to the bredth and depth of content and pedagogical content knowledge that should be included in the proposed qualifications.

Literature overview
Steen (2003) has written extensively about Quantitative Literacy in the United States. He is of the opinion that learners need to be flexibly prepared for life and to this end he suggests teaching a blend of numeracy, mathematics and statistics. De Lange (2003) posits that being mathematically literate has differing definitions depending on the needs of the community; however, in his description of a balanced mathematical literacy curriculum he identifies topics similar to those in the South African NCS, soon to be superseded by an adjusted curriculum:

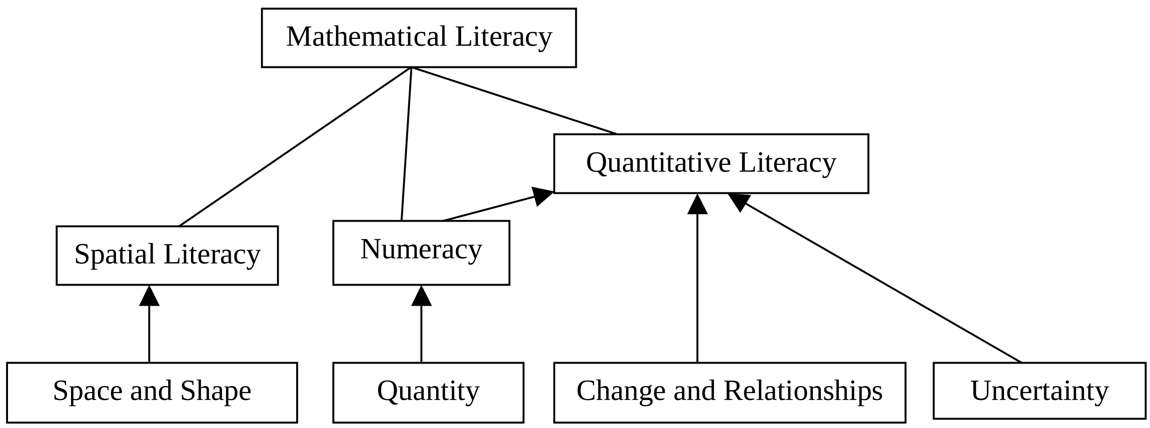

Figure1: Jan de Lange's (2003) conception of a balanced Mathematical Literacy Curriculum

Adler, Pournara, Taylor, Thorne and Moletsane (2009) surveyed the field of mathematics education in South Africa and one of their conclusions was that learning science and mathematics (and, in this instance, mathematical literacy) for teaching is not a simple matter. Many initiatives have not made the required impact on teaching or learner performance. They suggest examining the practices of teacher education with respect to "breadth and depth of domain knowledge; subject content and pedagogy" (p.39)

Schmidt, Cogan and Houang (2011) report on the Teacher Education and Development Study in Mathematics (TEDS-M), an international comparative study of teacher education. The researchers in the TEDS-M study assessed mathematical content knowledge, general pedagogy and mathematical pedagogy as they identified these as three key areas associated with teacher preparation. Although the TEDS-M study focused on the preparation of future teachers, the opportunities to learn are similar to those of in-service teachers who are facing the challenges of a new content area. The TEDS-M study revealed that there is little agreement among HEIs in the United States as to what constitutes teacher preparation, possibly because there is no shared vision of what a highly trained teacher should know (Schmidt et al., 2011). The situation in South Africa with teacher training for ML is similar as there was an urgency to train a massive cohort of teachers before the learning area was implemented in schools, even though the implementation was phased in over three years. HEIs developed programmes in isolation, and perhaps without due regard for Adler et al.'s caveat concerning both breadth and depth of mathematics content. The TEDS-M study further revealed that achievement is related to curricular differences in terms of content cover. Data gathered in the TEDS-M study indicated that the mathematical content offered in teacher training courses was positively related to professional competencies. In the comparison of the two ACEs in this study the majority of modules were ML content-based as professional competencies were deemed to have been acquired through previous teaching experience.

Curriculum contrasts

The requirements for entry to ACE: ML for University A recommended that students have at least a pass for mathematics standard grade at Senior Certificate level, a category C (M+3) teaching qualification at NQF level 5 and at least three years' teaching experience. Similarly, University B required an approved initial teacher education qualification or diploma of at least three years' duration and specified that the prospective students should have at least attempted mathematics at Senior Certificate level. In reality 42% of the 1 048 students studying ACE: ML in KZN from University A and 31% of the 691 students from University B had either no mathematics or failed mathematics in their senior certificate examination.

The curriculum for University A consisted of four 30-credit modules. Three modules focused on ML content and were divided into grade levels – grade 10, grade 11 and grade 12 ML content knowledge. The coherence of the curriculum was addressed by using a commercial ML textbook to underpin the curriculum content. The fourth module was designed as an ML pedagogical module that linked content and context. General pedagogy was deemed to be an aspect of recognition of prior learning (RPL) as the teachers all had a minimum of three years' teaching experience. The emphasis was on reskilling the teachers in a new subject area.

In contrast the curriculum at University B consisted of eight 16-credit modules. The ML content knowledge was covered in four modules that were horizontally aligned to ML curriculum learning outcomes – number and number relationships, functional relationships, space and shape, and data handling. The faculty developed their own study material and utilised school textbooks as supplementary material. There was one module devoted to the ML pedagogical aspects of teaching mathematics and mathematical literacy, while another was a research module designed to improve the teachers' reflective practices. In addition, there were two general pedagogy modules, studied by all ACE students in the faculty regardless of the ACE discipline, and two generic research modules –theory and practice of research.

The ML content modules from both HEIs were devised as a combination of content and context. For example, the one grade 11 examination that was perused was set in the context of a game reserve and the mathematical content. In one question a map was given of the park and a sub-question was,

> "Having watched the elephants at the Hapoor water hole for a while, you realise that it is 17:50 and the game gate closes at 18:30. Sketch on the map the shortest route out of the park estimate the distance in km and calculate how long it will take you if you stick closely to the given speed limit of 40km/h."

Other contextual questions included calculations concerning cell phone billing, loans, interest rates and house renovations. The balance between content and context was similar in all the ML content modules at both institutions.

In both universities delivery was through a cascade method of training tutors to teach students throughout the province. The tutors at University A attended a central block of training then went into the field to tutor on Saturdays whereas University B utilised a mixed delivery approach of block sessions and Saturdays. For both ACE programmes modules were delivered over a semester and there was continuous assessment through assignments and tests throughout the semester. In both programmes all modules, except the practical research module at University B, were summatively assessed with an examination. Class marks contributed towards the final examination mark.

Coherence

The statistical data from the study are in the process of being analysed but a few interesting preliminary results are emerging.

The first investigation was to plot the coherence of the curricula according to Steen and de Lange's recommendations. As the South African NCS seems closely aligned to de Lange's diagram both curricula cover all aspects recommended, even though the University B curriculum is horizontally aligned along learning outcomes whereas the University A is vertically aligned according to grades.

The correlation among the pass rate performance of the students in the three ML content modules at University A ranged between 0.751 and 0.826. Correlation was deemed to be significant at the 0.01 level in a two-tailed test. The correlation of the performance in the mathematical pedagogy module as compared with that of the content modules ranged from 0.712 and 0.769. This indicates that if a student achieved well in one of the four modules, there was a strong possibility that the student would also achieve in the other three modules – and vice versa.

At University B the correlation among the performance in the ML content modules ranged from 0.676 to 0.807; however, the correlation was not as strong between the content modules and the ML pedagogical module (minimum correlation was 0.634), the reflective practice research module (minimum correlation was 0.415), and even less between the content modules and the general pedagogy modules (correlation ranged between 0.385 and 0.567). These correlation statistics imply that students who did well in any one of the ML content modules, performed well in the others. However students who did well in the content modules may not have done well in the general pedagogic modules or the research module, and vice versa. Thus, these statistics indicate that different skills are required for mathematical content, general pedagogy and reflective practice. Thus Schmidt et al.'s (2011) recommendation is strengthened – that mathematical content knowledge, general pedagogy and mathematical pedagogy should be included in teacher training as they focus on different skills.

Further research is required to ascertain whether the two programmes have made an impact on ML teaching in KZN. The objective of the larger study is to investigate teachers' efficacy in the classroom.

Correlation between students' ACE and school leaving results
An interesting aspect of the data collection was the link between the students' achievement in mathematics in the Senior Certificate examination and the probability of passing either qualification in the minimum time period of two years. As all the students, registered at both institutions, were in-service teachers with more than three years' experience, they had all written either Mathematics Higher Grade or Mathematics Standard Grade, or had no Mathematics in their final school examination. Their senior certificate symbol for mathematics was converted to a scale variable ranging from zero (No mathematics or failed mathematics at Senior Certificate level) to 7 (a B-symbol for Higher Grade mathematics at Senior Certificate level.

At both institutions there was a high correlation between the students' mathematics point in the senior certificate examination and their propensity for completing the qualification in the minimum time period. In University A the correlation was 0.805 and in University B the correlation was 0.883. The fact that some students had left school many years before did not appear to affect the data. Not surprisingly, this indicates that a measure of mathematical ability is a prerequisite for success in ML modules.

Conclusion
In this paper we have looked at the composition and student results of two ACE: ML programmes that have been delivered in KZN. It appears that if students are able to achieve well (or vice versa) in a module focusing on ML content from one learning outcome or grade, this is a reliable indication that they will be able to show good progress (or poor) in modules focusing on other learning outcomes or grades. In this study it appears that the students' ML content knowledge was fairly constant over all learning outcomes and grades in both programmes. Those students who achieved in one content module generally achieved in the other content modules as well.

Another outcome of the study is the positive correlation between mathematics marks in the school leaving examination and the students' achievement in the ACE programmes. Results suggest that despite intervening years, school mathematics marks are a strong indicator of success in completing an ML qualification successfully in the minimum time period. Further interviews are planned with the students which may triangulate this supposition.

The expedience and urgency of the delivery of ACE:ML qualifications throughout the province precluded including more than curriculum content knowledge in the ML content modules of both curricula. There is ongoing uncertainty and debate about the extent of mathematical depth required to become an effective ML teacher. Should students who studied ML in their senior certificate examination be excluded from training to become ML teachers? If not, what depth of mathematical knowledge should they explore, considering that they do not have a pure mathematics background? However, if ML learners are excluded from ML teacher training, how will South Africa populate schools with ML teachers if the pool of prospective teachers is limited to those who studied pure mathematics successfully at school?

Both international and national research has indicated that teacher efficacy is a result of balanced teacher training. The implication is that the areas of ML content knowledge, general pedagogy and ML pedagogy should be included in ML teacher training curricula. The issue of the breadth and depth of domain knowledge has not been addressed; however, it is a pertinent issue to discuss during the forthcoming qualifications recurriculation process.

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USING A MODELLING TASK TO ELICIT REASONING ABOUT DATA

Helena Wessels
Research Unit for Mathematics Education
Stellenbosch University, South Africa
hwessels@sun.ac.za

Abstract
Completing modelling tasks not only develops prospective teachers' mathematical knowledge and problem solving competencies, but also prepares them to implement mathematical modelling later in their own practice. I present the preliminary findings of one investigation in a longitudinal project in South Africa where 188 prospective teachers completed a modelling task requiring reasoning about data. Responses show a variety of models on different levels of sophistication. Analysis has not been completed yet.

Introduction and theoretical background

Mathematical modelling is increasingly recognised as feasible teaching and learning perspective in schools (Mousoulides, 2009; Wessels, 2009). Mathematical modelling means "applying mathematics to realistic, open problems" (Maaß & Gurlitt, 2009) and engages students in open, non-routine problems that elicit powerful mathematical models which are extended and refined into systems that can be generalised for use in other contexts (Lesh & Doerr, 2003). The mathematical modelling process entails a number of steps that students iteratively go through, usually jumping between the different stages in a non-cyclic way (Ärlebäck & Bergsten, 2007). Each of these phases are also characterised by multiple cycles of "interpretations, descriptions, conjectures, explanations and justifications that are iteratively refined and re-constructed by the learner, ordinarily interacting with other learners" (Doerr & English, 2001). Mousoulides (2009) describes six processes in the solving of modelling problems: the understanding of the specified task with its constraints and alternatives; the identification of the relevant constraints; exploring and representing possible alternatives; choosing between alternatives; evaluating the choice; and communicating and defending the decision. The reality of the real-world problem is therefore progressively cut away to convert the real-world problem into a mathematical problem. The mathematical solution is then in the end evaluated against its usefulness in the reality of the situation. Modelling problems require students to make sense of the situation and use quantities and operations that they understand and find useful (Doerr & English, 2001). Students' exposure to mathematical modelling tasks cannot be a once off experience – they need multiple experiences to explore mathematical constructs and to use their models in new contexts to be able to generalize it.

Open, non-routine or model-eliciting problems (MEA's) provide the opportunity for students to access and process in the classroom complex mathematics problems at different levels of intellectual sophistication and solve these problems through the interaction between their informal and more formal mathematical knowledge. The paradigm shift from traditional teaching and learning of mathematics to a problem centred approach and a mathematical modelling perspective represents a shift to a more equitable situation in mathematics education. Usable models can originate from solution strategies on different levels, affording achieving students, non-achieving students and students from disadvantaged school environments the same opportunity to build successful models. Students who are exposed to MEA's often change their beliefs about mathematics positively and enjoy these activities, resulting in a shift to positive dispositions.

International reform movements in mathematics have shown that mathematical modelling may be an effective instrument in bringing about change. We know from research (Clarke, Breed & Fraser 2004; Kim, 2005; Riordan & Noyce, 2001; Schoen, 1993) that mathematical modelling students do at least as well, and often better, on standardised tests; are more able to transfer mathematical ideas into real world; are more confident in mathematics; value communication in mathematical learning more highly than students in conventional classes; and, developed a more positive view about the nature of mathematics than their counterparts. A mathematical modelling perspective therefore brings us a step nearer to turning the dreams of children, parents and teachers into reality where all can achieve in and enjoy mathematics. The use of mathematical modelling tasks in teacher education affords prospective teachers the opportunity to learn worthwhile mathematics while they develop their ability to apply the mathematics they already know in the development of powerful mathematical constructs (Niss, Blum & Galbraith, 2007). Prospective teachers are at the same time prepared to implement mathematical modelling in their future practice. The extension of a mathematics modelling perspective to the education of prospective Foundation Phase (K-3) teachers is not common. In traditional classrooms the focus is on what the teacher teaches and not necessarily on what children understand. Problem solving abilities are usually also not focused on in the early years. A modelling perspective in the Foundation Phase (FP) represents another paradigm shift as it emphasises the importance of mathematics education that fosters understanding as well as the development of problem solving abilities in the primary years.

To be able to implement a modelling perspective in the classroom, it is crucial that prospective teachers are exposed to mathematical modelling in their own education at undergraduate level (Garcia, Maaß & Wake, 2009). Successful implementation of a mathematical modelling perspective further depends on teachers

- being familiar with the key concepts of mathematical modelling;
- having appropriate beliefs about the nature of mathematics education; and
- being aware of their own competency to implement this perspective in practice (Maaß & Gurlitt, 2009).

Prospective teachers need to be made aware of the nature and spectrum of modelling competencies and how they are used in problem solving. Modelling competencies involves cognitive, metacognitive and affective competencies which are applied in an integrated way in the solving of mathematical modelling tasks (Biccard, 2010).

Method

The investigation described here is part of a longitudinal research project to prepare prospective FP teachers to implement a mathematical modelling perspective in their classrooms when they start teaching; to determine whether and how they implement this approach; and what the reasons and challenges are for not implementing MEA's. This paper reports on the modelling cycles prospective teachers went through and the models they created while solving the second of the MEA's they were exposed to during the project. Hundred and eighty eight prospective Foundation Phase teachers completed the MEA as part of their mathematics education module. Second, third and fourth year undergraduate prospective teachers (n= 88, 75, 25 respectively) solved the modelling task in groups of two to six in class during 3 to 4 lectures (three to four hours).

The task, "Making Money" (Lesh, Amit & Schorr, 1997), is about an entrepreneur employing nine vendors to sell popcorn and drinks in an amusement park during the summer months. She has to cut the number of vendors employed and asks for recommendations of which six vendors she should rehire full-time and part-time for the next summer. She supplies tables showing the number of hours worked and the money earned by each vendor when business was busy (high attendance); steady and slow (low attendance). The task is to make recommendations of who she should rehire in a letter, describing in detail how the vendors were evaluated and giving clear explanations so that she can decide whether the method is a good one for her to use.

The task met all six requirements for a MEA, which differs radically from traditional textbook word problems (Lesh, Amit & Schorr, 1997):

- The *reality principle*: the task focused on a 'real' problem that needs to be addressed – not a contrived cleaned-up school textbook problem
- The *model construction principle*: the task required the construction of a model and the description of assumptions and conditions in the justification of decisions made while constructing the model
- The *self-evaluation principle*: the task explicitly stated for what purpose and by whom the results were needed to enable the student teachers to evaluate their own solutions and decide whether it needed improvement
- The *model-documentation principle*: the task required explicit explanations of their thinking about the situation and the solution paths they followed
- The *model generalisation principle*: the model(s) created could be adjusted and applied to other situations and contexts
- The *simple prototype principle*: the problem was designed to elicit the creation of a model while still being as simple as possible.

Mathematical modelling essentially involves group work as ". . . competencies of the group are likely to be greater than those of individuals" (Biccard, 2010; Mousoulides, 2009; Zawojewski, Lesh & English, 2003). Individuals verbalise and criticise ideas more spontaneously in a small group setting, resulting in a higher level of creativity and the development of more sophisticated systems.

Data sources included the researcher's field notes, student teachers' rough notes, mind maps and workings as well as their final reports describing their recommendations, their solutions paths and reasons for specific decisions made along the way. The analysis is still in progress.

The project is being conducted with problem centred teaching and learning as context – the researcher did not suggest any strategies or solution paths, neither did she prompt student teachers to use the different conceptual models. She provided resources and gave ample time for students to complete the activity. They were allowed to use technology such as pocket calculators and computers.

When student teachers in the project had to solve a MEA for the first time, they could not believe that they were not going to be told how to solve the problem as was the case in school mathematics. Some groups only started working productively on the task after almost an hour and most students took 3 to 4 hours to complete the task, some even took longer. With the second MEA students did not start right away working on the task, but this time they knew that they had to do it on their own and that the researcher would not give hints or prompt strategies and solutions. By the time they had to solve the third MEA, they immediately started working on the problem and most groups finished after about two and a half hours.

Preliminary findings

Responses fall into two big categories (See Table 1). In the first category models were developed without combining the two data sets, i.e. by totalling earnings and totalling number of hours worked (n=6) and focusing on averages (N=1). Two groups calculated only totals for

income and for hours worked, while 4 groups totalled income an hours worked for the different shifts (busy, steady and slow times) and ranked the vendors accordingly. One group used average income and average hours worked to develop their model.
In the second category the data set with number of hours worked was combined with the data set with earnings to calculate rand-per hour (n=41). This category showed 12 different interpretations of the problem (see Table 1), with groups inventing decision-making rules on different levels of sophistication.

Table 1: Analysis of responses

	Category	**Description**	**Number of group responses**
No combining of the 2two data sets	**Total income and total number of hours worked** N=6	Totals only	2
		Totals for different shifts, & rankings	4
	Average income and average number of hours worked N=1	Average hours & average income	1
Combining the two data sets	**Income per hour (R/h)** N=41	R/h for months separately	3
		R/h over 3 months	15
		R/h for different shifts & R/h overall	6
		R/h for different shifts & R/h overall & ranking	4
		R/h for different shifts & R/h overall & ranking using ratio	1
		R/h & consistent high scores for different shifts	1
		R/h & R/h for slow times	1
		R/h & R/h in busy times (for deciding part time vendors)	2
		R/h & weighting (points system)	5
		R/h & % of average R/h of all vendors together	1
		R/h & average of all vendors together – deviation from average	1
		R/h & average of all vendors together – above this average	1
		Total	48

Prospective teachers mostly used tables and bar graphs to represent their interpretations of the task, although a few (inappropriate) broken line graphs and pie graphs were used.

Discussion

Realistic problems require working with and understanding different kinds of quantities and operations than in traditional tasks (Doerr & English, 2001). The “Money Making” task entailed statistical analysis and included multiple views and representations of the data (mostly tabular and graphical), measures of central tendency (averages and deviation from averages), as well as the combining of data and analyses of trends. Operations needed in the task therefore differ from operations in traditional problems: additional to adding and dividing quantities and calculating averages, student teachers also needed to sort, organise, select, combine and transform the entire data set rather than just working with single data points.

Groups had to describe in detail the solution paths they followed in the process of developing the model. They were required to identify different alternatives and evaluate and justify their choices of alternatives. One procedure was to brainstorm together and then trim away less useful suggestions to pursue one or two ideas. Another procedure was that each group member had to come up with an own idea and do the preliminary working out. Ideas were then compared and useful alternatives further developed. A third modus operandi was to first explore just one idea initially and then work towards other alternatives. Group 38 for example used income per hour to identify the best six vendors and then compared income per hour for the slow shifts because ". . . if one shows that they can bring in the most money in the slow times then they are good vendors. Not everyone enjoys working slow times and this suggests that Thandi, Jose and Maria are willing to work and are more dedicated to their jobs than the rest. They should be hired full-time". Group 41 ranked vendors by a weighting scheme and remarked: "Using this method, each vendor's performance can be clearly compared against that of the others, and a fair decision (one that takes into account all factors – the number of hours they worked, how much money they collected, as well as when they worked) can be made".

Different solution paths for similar interpretations were common. Some groups for example first worked out the average income per hour of each vendor for each shift per month and then added it all together to calculate an overall average for the individual while other groups worked out total earnings and total hours for each individual and then calculated the total average.

Although the prospective teachers did not receive specific instruction in these ideas, some groups developed quite sophisticated models. Group 47 for example combined the two data sets by calculating income per hour per shift of each vendor, but also focused on the average productivity (income per hour) of the whole group. They ranked vendors according to deviations from this average to decide which vendors to recommend for full-time and which to recommend for part-time employment.

Groups who used only totals or averages of income and/or hours did not critically evaluate their solution paths and stuck with one idea. Most other groups explored the usefulness of different models before deciding on a specific one. A few groups compared conclusions reached through the use of earlier models to check the conclusions reached with a later more sophisticated model.

Summary

The models described above were developed without any interference or guidance from the researcher. Most groups went through a number of modelling cycles through which they progressed from focusing on subsets or isolated pieces of the data to considering combined data sets and underlying trends and regularities.

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Comparing the Use of Virtual Manipulatives and Physical Manipulatives in Equivalent Fraction Intervention Instruction

Arla Westenskow
Doctoral Candidate, Mathematics Education
Utah State University
arlawestenskow@gmail.com

Abstract

This paper describes a study designed to identify differences in learning trajectories related to the use of virtual and physical manipulatives during equivalent fraction intervention instruction. Changes in recent years in the approach to mathematical intervention have increased the need for research which identifies effective methods for the teaching of students with mathematical learning difficulties. Recommendations made from current intervention research emphasis the importance of students developing a proficiency in the use of representations through the use of manipulatives. Research indicates that virtual and physical manipulatives are effective tools of instruction. However, each manipulative type has its own unique affordances which affect the learning of specific concepts. The purpose of this study is to identify differences in students' learning trajectories related to the type of manipulative (virtual and physical) used during intervention instruction of equivalent fraction concepts.

Background

Due to the globalization of markets, advances in technology, and the overwhelming spread of information in today's society, mathematical skills have becoming increasingly important for success. Yet, there are a number of students who fail to acquire the needed mathematical skills during regular classroom instruction and the number of students needing special education services is increasing (Singapogu & Burg, 2009). As a solution, many programs, to more effectively support students with mathematical learning difficulties, have begun to shift from the traditional remediation approach to a response to intervention (RtI) approach. As an intervention approach, RtI focuses on the early identification and support of students who are struggling academically. The most common form of RtI intervention is a three tiered design. The first tier of intervention takes place in the regular classroom setting where all students receive research proven effective instruction. It is expected that at least 80% of the students will master the concepts taught during Tier I instruction (Fuchs, Compton, Fuchs, Bryant, & Davis, 2008). In Tier II intervention, students who did not achieve mastery in Tier I are given additional assistance. Tier II intervention is content specific and typically conducted by the classroom teacher in small group settings. Students who do not respond to Tier II intervention are considered to be "non responders" to intervention and will receive additional Tier III special education services (Fuchs et.al. 200).

Preliminary research results of RtI for mathematics intervention, although limited has been positive (Glover & DiPerna, 2007). Yet, developers of RtI programs have reported that the lack of available Tier II instructional materials and tools is limiting program implementation and research (e.g.: Fuchs, Seethaler, Powell, Fuchs, Hamlett, & Fletcher, 2008; Gersten et.al., 2009; Glover & DiPerna, 2007). This study focuses on the use of virtual and physical manipulatives in equivalent fraction intervention instruction.

Representations

In a review of RtI literature, Gersten, et al.(2009) made eight research based recommendations for setting up effective RtI programs. The fifth recommendation reads:

> Intervention materials should include opportunities for students to work with visual representations of mathematical ideas and interventionists should be proficient in the use of visual representations of mathematical ideas (p. 30).

Gersten, et al. (2009) explained that one of the most common difficulties experienced by students with mathematical learning difficulties is the ability to connect abstract symbols to visual representations. They suggested that in regular classroom instruction, representations are not emphasized strong enough nor presented systematically enough to facilitate the scaffolding of learning for students with mathematical learning difficulties.

The purpose of using external representations (e. g. manipulatives, drawings, mathematical tables, etc.) in instruction is to aid students in their development of internal representations (Behr, Lesh, Post & Silber, 1983). Students' internal representations are: a) verbal /syntactic images, b) mental images, c) formal notation, and d) affective images including emotions, attitudes, beliefs and values (Goldin & Shteingold, 2001). As student's conceptual understanding of mathematical concepts develops, the power and flexibility of their internal representations grows. Students who have developed only partial internal systems of representations often experience difficulties in learning new concepts (Goldin & Shteingold, 2001). Physical and virtual manipulatives are tools of representation used to scaffold students' learning of mathematical concepts.

Physical and Virtual Manipulative Effectiveness

Physical manipulatives are concrete objects which students use to explore mathematical concepts through the students visual and tactile senses (McNeil & Jarvin, 2007). Virtual manipulatives are tools which are "interactive, web based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (Moyer, Bolyard & Spikell, 2002, p. 373). Advocacy for the use of manipulatives stem from the learning theories of Piaget, Bruner and Montessori that students develop and build knowledge as they move from concrete experiences to abstract thinking (McNeil & Jarvin, 2007). Piagetian theory suggests that children need to physically manipulate objects and then should be encouraged to reflect upon the meaning of the results of their physical actions (Baroody, 1989). These theories are supported by the research of studies such as the Rational Number Project, a project focusing on the development of representations through the use of manipulatives. Study results indicate that students using manipulatives significantly outperformed students taught using the more symbolic approach (Cramer & Post, 2002). The researchers described four ways manipulatives facilitated students development of fraction understanding: 1) students used the manipulatives to develop mental images of fraction meaning, 2) comparing manipulative objects helped students form correct methods for compare fractional sizes, 3) students use the manipulatives as references when justifying their answers, and 4) students using manipulatives developed less misconceptions.

The effectiveness of using physical manipulatives in mathematical instruction has been the focus of a large number of research studies. A review of the literature identified three meta-analysis reports. Suydam and Higgins (1977) reported that 11 of 23 studies reported finding significant differences in student achievement favoring the use of manipulatives, two studies favored not using manipulatives, and ten studies reported no significant differences between use and non use of manipulatives. Parham (1983) analyzed 64 studies and obtained 171 effect size scores comparing the use of manipulatives with non use on student achievement. Their analysis yielded an average mean effect size of 1.03, indicating a large effect size favoring manipulative use. Sowell (1989) reported, in their analysis of 60 studies that although manipulatives generally are found to be more effective than other types of instruction, but that there is a lot of variability in results, with some studies yielding large effect size scores while a few yielded negative effect size scores.

Moyer-Packenham, Westenskow and Salkind (2011) conducted a meta-analysis evaluating the effect of virtual manipulatives on student learning. The analysis of 70 effect scores obtained from 29 studies yielded a moderate average effect size of 0.37 when compared with the use of other methods of instruction. When virtual manipulatives were used alone as the primary tool of instruction and compared with instruction using physical manipulatives and with traditional classroom instruction, the averaged effect scores were a small effect of 0.18 (33 effect scores) and moderate of 0.73(18 effect scores) respectively. Their analysis of the qualitative data suggested that virtual manipulatives have affordances of focused constraint, creative variation, simultaneous linking, efficient precision and motivation.

Students with Mathematical Learning Difficulties.

A search of the literature identified five studies investigating the effectiveness of instruction using physical manipulatives with students having mathematical learning difficulties. In two studies, Butler, Miller, Crehan, Babbit, and Pierce's (2003) and Witzel, Mercer, and Miller (2003), students with mild to moderate mathematical disabilities who participated in instruction using physical manipulatives scored significantly higher than students who did not use manipulatives. Similarly, results from three studies involving students with learning disorders' reported that achievement scores improved after students participated in instruction using manipulatives (Cass, Cates, Smith, & Jackson, 2003; Maccini & Hughes, 2000; Moch, 2001).

Four studies investigating the effectiveness of instruction using virtual manipulatives with students having mathematical learning difficulties were identified. Suh, Moyer, and Heo (2005) reported that when using virtual manipulatives, higher achieving students were more efficient and used more mental processes for finding answers while lower achieving students were more methodical in their use and more dependent on using the visual models when scaffolding between the pictorial and symbolic. Three studies reported positive effects when students receiving special education services used virtual manipulatives and two of the studies reported that the students using virtual manipulatives outperformed the students who did not use the manipulatives (Guevara, 2009; Hitchcock & Noonan, 2000; Suh & Moyer-Packenham, 2008). In summary, all of the identified studies in which manipulatives were used with students of differing abilities reported that students with mathematical learning difficulties benefitted from the use of manipulatives.

Combining Use of Virtual and Physical Manipulatives

Although limited, research indicates that there may be an advantage to combining the use of physical and virtual manipulates in instruction. In a meta analysis, Moyer, Westenskow and Salkind (2011) identified 26 effect size cases of instruction combining the use of virtual and physical manipulatives. When instruction with combined use was compared with instruction using only virtual manipulatives, only physical manipulatives, and traditional classroom instruction, results produced a moderate averaged effect of 0.26 (9 effect size cases), a small effect of 0.20 (11 effect size cases), and a moderate averaged effect of 0.69 (6 effect size cases) respectively. These results indicate combining the use of virtual and physical manipulatives may be advantageous to student achievement.

Physical and virtual manipulatives have distinct affordances and disadvantages (i.e. many virtual manipulatives have explicit symbolic pictorial links, physical manipulatives involve tactile senses). Several researchers have reported that the affordances of each type of manipulative have resulted in variations of learning unique to the type of manipulative (Izydorczak 2003; Moyer, Niezgoda, & Stanley, 2005; Takahashi, 2002). As suggested by Behr et al. (1983) while a manipulative may be the most effective tool to use in teaching one concept, it may when used to teach a different concept impede student learning. They suggest more research is needed to identify which manipulative will best facilitate the learning of each concept. The purpose of this research study is to identify differences, related to manipulative type, in students' learning of the concepts of equivalent fractions. The research questions guiding the research are:

1. Are there variations in student achievement unique to the use of different instruction tools (virtual manipulatives, physical manipulatives or a combination of virtual and

physical manipulatives) in the learning of equivalent fraction concepts by students with mathematical learning difficulties?

2. Are there variations in the learning trajectories unique to the use of different instructional tools for intervention (virtual manipulatives, physical manipulatives or a combination of virtual and physical manipulatives) in the learning of equivalent fraction concepts by students with mathematical learning difficulties?

Methods

To answer these research questions, a study will be conducted providing preliminary Tier II intervention to fifth grade students who have not mastered equivalent fraction concepts in the regular classroom. The study uses a mixed methods approach of triangulating evidence from quantitative and qualitative data collected from pre/post/delayed post tests, session assessments, instructors' logs and session artifacts (task sheets, explore papers and video tapes). Data analysis will focus on the development of learning trajectories to be used as models of the progress made by students as they construct equivalent fraction understanding through the use of virtual and physical manipulatives. Not only will the overall effect of equivalent fraction learning be determined from the analysis of pre and post treatment data, but the researcher will also analyze data at the concept level, the individual lesson level, thus making it possible to identify effects of manipulative tool on the spectrum of students' development of equivalent fraction understanding.

Participants will be selected through a screening process. All students in the fifth grade classes of the participating schools will complete an equivalent fraction pre test. Students, except those participating in special education services for mathematics, scoring 70% or lower on the pretest, will be invited to participate in the intervention. Using an ability stratified method, students will be assigned to three groups, virtual manipulatives alone (VM), physical manipulatives alone (PM) or virtual and physical manipulatives combined (PM/VM). Intervention instruction groups will consist of two to four students. Participants will receive 5 days of equivalent fraction instruction, during which they will use their assigned treatment tool. Daily sessions will be approximately 45 minutes in length. The same lesson structure and content will be used for all three treatment groups, with minor adaptations made for tool differences. Each lesson will consist of five phases; pre-assessment, explore, apply, practice and lesson assessment. The lesson assessment consists of two parts; three lesson concept questions and eight cumulative fraction knowledge assessment questions. Lessons will be videotaped and all task sheets and student work will be collected for further analysis. At the conclusion of the intervention treatments students will complete a post test.

Analysis of the data will follow two paths: 1) analysis to identify differences in student achievement at the lesson, concept and summative levels, and 2) the development of learning trajectories showing knowledge and skill growth including the resolution of misconceptions and errors. The learning trajectories will be compared to identify differences related to manipulative type.

The following hypotheses are anticipated outcomes of this intervention study:

- Differences in the type of manipulative (physical or virtual) used creates differences in student's learning trajectories of equivalent fractions.
- Affordances of manipulative (physical and virtual) use are specific to each concept within the content domain of equivalent fractions.
- The type of manipulative (physical or virtual) used in instruction affects the occurrence and resolution of misconceptions and errors which are frequently experienced by students in their development of fraction understanding.

When developing intervention curriculum, teachers and curriculum developers must make important decisions about the type of instructional tools to be used. Although the research literature indicates that both physical and virtual manipulatives are effective tools of instruction, research showing the affordances of each manipulative type for the learning of specific concepts is limited and does not answer the question of how to effectively combine the use of physical and virtual manipulatives in intervention instruction. The purpose of this study is to provide teachers and curriculum developers of intervention instruction, research that can be used in their selection of manipulatives in the different phases of students' equivalent fraction learning trajectories.

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Workshop title: A new rational approach to the teaching of trigonometry in schools and colleges

Assoc Prof N J Wildberger UNSW (PhD Yale 1985) n.wildberger@unsw.edu.au

Workshop summary: I will outline a new approach to the teaching of trigonometry and geometry in schools and colleges. This approach has much in common with the work of the ancient Greeks, in paritcular Euclid's Elements. The workshop will introduce you to the main concepts of rational trigonometry, which are purely algebraic and much simpler than conventional trigonometry, and show you how to apply it in a variety of practical situations. The series of YouTube videos WildTrig found at user njwildberger goes into a lot more detail, this workshop can be viewed as an introduction to that series.

Comprehensive indicators of mathematics understanding among secondary school students.

Noor Azlan Ahmad Zanzali, M Sc, Ph D (University of Wisconsin)
Professor of Curriculum and Instruction, Faculty of Education, Universiti Teknologi Malaysia
azanzali@utm.my
Abdul Halim Abdullah, Norulhuda Ismail, Associate Professor Aziz Nordin, Dr. Johari Surif
Faculty of Education, Universiti Teknologi Malaysia

Abstract

The Malaysian science and mathematics curriculum has undergone several significant changes within the last two decades. Correspondingly, the approach to learning and teaching science and mathematics has also changed drastically. From the perspectives of science and mathematics education, the teaching of science and mathematics has shifted from the normative to descriptive (naturalistic) view of mathematics. That is, from the absolutist (behaviorist) tradition to the constructive tradition. However, what students have actually acquired in terms of problem solving, science process skills, communication, reasoning, thinking skills and abilities in seeing the interconnectedness of ideas, as stated in the mathematics and science curriculum are still not clearly articulated or defined. Therefore, there is a need to study what our students have actually acquired based on the above aims (or standards). Nevertheless, if these standards are not achieved, there is also a need to study the gap that exists between those ideals and those attained by our students. This paper reports the research study that aims to identify the levels of mathematics understanding amongst secondary school students, related to their abilities in terms of problem solving, communication, understanding the interrelatedness of mathematical ideas, and mathematical reasoning. It takes a comprehensive look and simultaneously explore into students' attainment both in terms of skills and levels of understanding using the School Science and Mathematics Indicators Program (SSMIP) designed by the research team. The methodology and procedures of the research project consist of document analyses of curriculum materials and guidelines to produce some indicators on the levels of understanding students are expected to attain as they proceed through the schooling system, conducting the tests to be used in describing the levels of achievement in specified subject areas, conducting task analyses based on the questions designed by the research team and finally conducting in-depth clinical interviews on selected students. Started in June 2010, the research is still ongoing and expected to be completed in another year. Initial findings indicated that there seems to be significant differences between curriculum expectations and students' levels attainment and understanding as defined by the curriculum standards.

Introduction

Within the perspectives of science and mathematics education, the teaching of science and mathematics has shifted:

- From the perspective normative to descriptive (naturalistic) view of mathematics.
- From the absolutist tradition to the constructive tradition (from the behaviorist to the constructivist approach).

One can safely conclude that the approach to learning and teaching of mathematics has changed drastically (Noor Azlan Ahmad Zanzali, 1995) over the last five decades. Thus the question of how much the students have benefited from these improvements in the curriculum is relevant. Several studies that aimed to look at what the students have acquired from the curriculum have been conducted. These studies, however, have been generally looked at skills acquired at specific area or level and thus are limited in scope.

This proposed study aims a more comprehensive look and simultaneously probe into students' attainment both in terms of skills and levels of understanding. The inquiry into and creating the School Science and Mathematics Indicators Program (SSMIP) will produce comprehensive and computerized guidelines on school-leavers achievement indicators in both science and mathematics. Potential users will include all higher institutions of learning both public and private institutions, and individual science and mathematics educators. While this type of school achievement indicators are quite common in developed countries, they are, however, new in the Malaysian scenario.

The basic principles of assessment

The word assessment refers to the process of collecting and using evidence about students' learning. Assessment and evaluation both describe the processes of collecting and interpreting evidence for some purpose. They both involve decisions about what evidence to use, the collection of that evidence in a systematic and planned way and the interpretation of the evidence is to produce to produce some of judgment (Harlen, 2007, Harlen2008, Khodori, 200; Salvia, J. &Ysseldke. J. E. 2001).This description is illustrated by the following diagram:

In a nutshell "Educational assessment is formal attempt to determine students' status with respect to educational variables of interest (Popham, 2006; pg 6)

In recent years educators have been urged to broaden their conception of testing so students' status determined via a wider variety of measuring devices – a variety extending well beyond the traditional paper-and-pencil tests. Thus they are many worthwhile learning outcomes not best measured by paper-and-pencil tests. Assessment is a broader descriptor of the kinds of educational measuring teachers do – a descriptor that, while certainly including traditional pap er and pencil tests, covers many more kinds of measuring procedures.

Noor Azlan Ahmad Zanzali (2005) emphasized assessment must be based and address several critical issues of teaching and learning. They are

Issue 1:	Underlying assumptions about the philosophy and goals of the curriculum
Issue 2:	Assessment must be in consonance with current learning and instructional considerations
Issue 3:	Specifications of performance standards
Issue 4:	Developing authentic tasks
Issue 5:	Assessment should measure status and growth
Issue 6:	Scoring and what form?
Issue 7:	Reporting –making public

Objectives of the Research

This research program aims to identify

- the levels of mathematics understanding amongst secondary school students, corresponding to their levels of schooling students' abilities to acquire a variety of mathematical concepts,

- The abilities to carry out a variety of mathematical procedures and to use them to solve problems in both familiar and unfamiliar situations.
- The skills and understanding that students are expected to acquire as stipulated by the curriculum expectations.
- The levels of problem solving abilities, communication, understanding the interrelatedness of mathematical and science, and mathematical and scientific reasoning.

A complete guideline on the levels of students' understanding in science and mathematics according the levels of schooling will be produced.

For each component (mathematics and science) students will be assessed at three levels. Each level is related to the understanding of concepts, application, and problem solving in the respective content areas.

- Level 1 – describing the early stages of mathematical and scientific knowledge typical in a secondary school.
- Level 2 - describing the mathematical and scientific knowledge acquired in the intermediate years of secondary schooling.
- Level 3 - describe knowledge and skills acquired by students who have completed the full range of mathematics and scientific courses typical of a secondary school education

Methodology

The research team will consist of 4 groups, each looking at the areas of mathematics, physics, chemistry and general science areas.

The methodology and procedures of the research project are as follows:

1) Document analyses of curriculum material and guidelines to produce guidelines on the levels of understanding students are expected to attain as they proceed through the schooling system
2) Designing the tests to be used in describing the levels of achievement in specified subject areas.
3) Conduct task analyses based on the questions designed in (2).
4) Conducting in-depth clinical interviews on selected students.

Both the quantitative and qualitative methods will be used.

- **Quantitative Method**: Sets of mathematical questions to be answered by students for each level. Coding procedures will be used to assess levels of understanding. Tests of questions will be designed by each group.
- **Qualitative Method**: Qualitative procedures such document analyses, interviews, observations, task analyses and small group interactions of selected of students

Initial Results:

At this juncture, the process of collecting is still at the initial stages.

Document analyses are seen through the content, the psychological and the pedagogical perspectives. Our initial analyses do indicate that the intended curriculum places heavy emphasis on naturalistic view of mathematics based on constructivist nature of teaching and learning. The use of problem solving, communication, integration and reasoning of mathematics ideas are heavily emphasized. Two questions need to be addressed. First are the elements emphasized in teaching and learning? Second, do the students attain those elements as they go through the teaching and learning processes? The answers to the first questions are discussed in my earlier papers. This research attempts to address the second question

We conducted a qualitative survey on the views and attitude of the students, who are our subjects. Initial findings indicate that they do attain above average attitude as expected by the curriculum.

The results of other procedures is still being conducted and we hope to be able to collect and analyze at the end of the year,

Conclusion

The need to assess students' achievements in terms of the elements as emphasized curriculum, but not evaluated by the paper and pencil tests in the high-stake public examinations, is still very important. This will indicate the attainment of students as expected and regarded as the key elements of learning mathematics by the curriculum.

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The Use of Graphic Organizers to Improve Student and Teachers Problem-Solving Skills and Abilities

Alan Zollman
Associate Professor of Mathematics Education, Mathematical Sciences Department,
Northern Illinois University, DeKalb, Illinois, USA, zollman@math.niu.edu

Abstract
This paper reports on the use of graphic organizers to improve student mathmatical problem solving. Graphic organizers are visual and graphic displays that spatially depict the relationships between facts, terms, concepts and ideas within a learning task. Graphic organizers have widespread, and successful use in the areas language arts and special education in communication skills and comprehensions abilities (Hall & Strangman, 2008; DiCecco & Gleason, 2002; Goeden, 2002; Griffon, Malone, & Kameenui, 2001; National Reading Panel, 2000; Ritchie & Gimenez, 1996; Robinson, 1998). In this manuscript, I coalesce the findings of three separate, but related studies using a graphic organizer, *four-corners-and-a-diamond,* specifically designed to aid students in mathematical problem solving. Results found elementary teachers' comfort and self-confidence levels increased dramatically. Results also found third, fourth and fifth-grade students results on open-ended measurement problems increased. Lastly, results found significant improvement on open-ended geometric and algebraic problems with sixth, seventh and eighth-grade students.

Introduction
Think of how you would begin to put together a commercial bathroom vanity cabinet kit. Would you first read the instruction manual? Or would you check the inventory first to see if all parts were included? Would you begin by studying the picture of the completed cabinet? Would you just ask for help from someone more knowledgeable? Would you not even try? Your response to the problem situation above is analogous to student reactions to problem solving. Your previous experiences greatly influence your persistence and approach to a problem. Similarly, we have to accept and utilize student tendencies to increase their problem-solving skills and abilities.

As problem solving is a major, *if not the major*, goal of learning mathematics, how do we assist students develop their skills and abilities (National Council of Teachers of Mathematics, 1989; 1995; 2000)? One approach is to teach Polya's four-step problem-solving heuristics, namely first understand the problem; devise a plan; carry out the plan; and look back (Polya, 1944). However students, and teachers, many times misinterpret the four-steps to be a linear step-by-step procedure, not a process. So teachers and students try to make problems into sequential procedures, which does not work with most problems. And students quit working on problems if they get stuck on a step. This manuscript describes a specific approach to mathematical problem solving derived from research on reading and writing pedagogy, specifically, research indicating that students use graphic organizers to organize their ideas and improve their comprehension and communication skills (DiCecco & Gleason, 2002; Goeden, 2002; Griffon, Malone, & Kameenui, 2001; National Reading Panel, 2000; Robinson, 1998).

Graphic Organizers

Graphic organizers are visual and graphic displays that spatially depict the relationships between facts, terms, concepts and ideas within a learning task (Ellis, 2004). Graphic organizers have widespread, and successful use in the areas of language arts and special education in communication skills and comprehensions abilities (Hall & Strangman, 2008;

DiCecco & Gleason, 2002; Goeden, 2002; Griffon, Malone, & Kameenui, 2001; National Reading Panel, 2000; Ritchie & Gimenez, 1996; Robinson, 1998). Graphic organizers allow (and even expect) the student to:

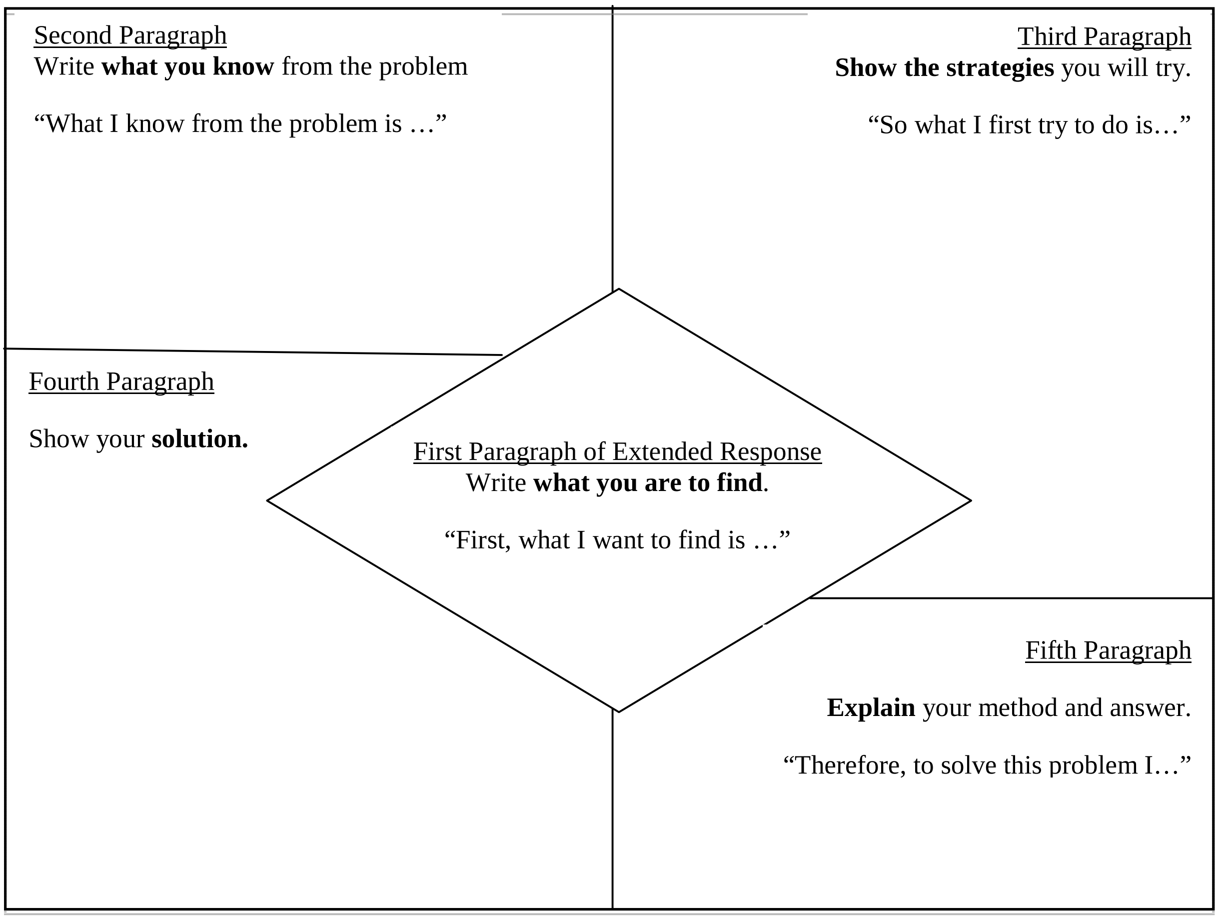
- sort information as essential or non-essential;
- structure information and concepts;
- identify relationships between concepts;
- organize communication about an issue or problem; and
- allow students to utilize their previous tendencies and experiences as a starting point of the problem-solving process (Zollman, 2011; 2009a; 2009b).

Figure 1 depicts the *four-corners-and-a-diamond* mathematics graphic organizer. This graphic organizer is modeled after a *four squares* writing graphic organizer described by Gould and Gould (1999). This *four-corners-and-a-diamond* mathematics graphic organizer (Zollman, 2011; 2009a; 2009b) has five areas: What do you need to find? What do you already know? Brainstorm possible strategies to solve this problem. Try your ways here.

What things do you need to include in your response and what mathematics did you learn by working this problem?

Figure 1

Four-Corners-and-a-Diamond Mathematics Graphics Organizer



The *four-corners-and-a-diamond* graphic organizer encourages students to begin working on a problem before they have identified a solution method. The extended-response written answer is not begun until there exists information is in all five areas. As in the four square writing method, students then organize and edit their thoughts by writing their solution in the traditional linear response, using connecting phrases and adding details and relationships. For the extended-response write up (as used in large-scale assessments) students usually first state the problem, identify the given information, next they propose methods for solving the problem, then show their mathematical work procedures, and finally present their final answer, conclusions and learning. (Zollman, 2011; 2009a; 2009b).

Methodology

These one-year research studies were initiated to improve student mathematical problem solving. There were 240 elementary and 186 middle school students in the studies. The elementary students utilized graphic organizers on the topic of measurement (area and perimeter); the middle school students utilized graphic organizers on the topics of algebra and geometry. None of the students, or their teachers, had previous experience using graphic organizers in mathematical problem solving.

Results

Using the graphic organizer had a substantial impact on the elementary teachers practices. Initially 50% of the elementary teachers reported being uncomfortable teaching problem solving. Afterwards, 100% of the teachers reported being comfortable (80% being very comfortable) teaching problem solving with their students. All the teachers subsequently reported changing their instruction to include much more writing in mathematical problem solving.

On achievement in area and perimeter problem-solving items, the 3rd grade students rose from 55% to 78%. The 4th grade students increased from 40% to 52%. And the 6th grade students improved from 40% to 62% from pretest to posttest scores.

Likewise, the middle school students also significantly improved their achievement scores from pretest to posttest on algebra and geometry problem-solving items. On a four-point scoring rubric, the students went from a pretest score of 0.83 to a posttest score of 2.93 on mathematical knowledge (Z-score -19.8849, $p<0.001$; effect size 1.94). On strategic knowledge, the student scores rose from 1.52 to 2.79 (Z-score -11.66049, $p<0.001$; effect size 1.16). Similarly, their explanation scores grew from 0.83 to 2.67 (Z-score -15.3907, $p<0.001$; effect size 1.51).

Teachers studying their own students reported the use of graphic organizers in mathematical problem solving to be efficient and effective for students at all achievement levels. Teachers saw students who normally would not attempt open-response problems now had partial written solutions. Students who normally did well on problems now had an efficient method of writing and communicating their thinking in logical, complete arguments.

Conclusions

These research studies found that the proper use of the mathematics graphic organizer *four-corners-and-a-diamond* to be an extremely useful instructional method in the middle school mathematics classroom. Students should improve their problem solving abilities with any instructional intervention. The teachers specifically attributed the increase in student performance in mathematical problem solving to the student use graphic organizers. Lastly, the teachers also changed their instructional pedagogy during the study. This may be due to teachers viewing the *four-corners-and-a-diamond* graphic organizer a comfortable method to include writing-across-the-curriculum into their mathematics lessons, and subsequently including more extended-response problem solving into their lessons.

Teaching about problem solving in a hierarchy of procedural steps is neither efficient nor effective. This study concurs with other problem-solving research – teaching *via* a problem solving process is the crucial instructional development (Lester, Masingila, Mau, Lambdin, dos Santon, & Raymond, 1994).

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