Subst. in C8.10c (5.Ü) Vorl. am 16.04.2014

$$\int_{\square}^{\square} \frac{x^2}{\sqrt{(4+x^2)^5}} dx$$

$$\int \frac{x^2}{(x^2+4)^{\frac{5}{2}}} dx$$

Termumformung:

$$\int_{\square}^{\square} \frac{x^2}{(\sqrt{4+x^2})^5} dx$$

$$\int \frac{x^2}{(x^2+4)^{\frac{5}{2}}} dx$$

Das CAS löst die Aufgabe nicht ohne vorherige Vereinfachung.

Eulersubst. Fall 1 (a=2>0) B. S.477

$\frac{x^2}{(\sqrt{4+x^2})^5}=R(x,u)$ mit $u=\sqrt{4+x^2}$ und $R(x,u)=\frac{x^2}{u^5}$

Subst: $\sqrt{4+x^2}=x+t$, d.h. $4+x^2=(x+t)^2$

solve$(4+x^2=(x+t)^2,x)$

$$\left\{x=\frac{-t}{2}+\frac{2}{t}\right\}$$

$$\frac{d}{dt}\left(\frac{-t}{2}+\frac{2}{t}\right)$$

$$\frac{-(t^2+4)}{2\cdot t^2}$$

somit $dx=\frac{-(t^2+4)}{2\cdot t^2}dt$ und

$$\int_{□}^{□}\frac{x^2}{\left(\sqrt{4+x^2}\right)^5}dx=\int_{□}^{□}\frac{\left(\frac{-t}{2}+\frac{2}{t}\right)^2\times\frac{-(t^2+4)}{2\cdot t^2}}{\left(\frac{-t}{2}+\frac{2}{t}+t\right)^5}dt$$

$$\text{expand}\left(\frac{\left(\frac{-t}{2}+\frac{2}{t}\right)^2\times\frac{-(t^2+4)}{2\cdot t^2}}{\left(\frac{-t}{2}+\frac{2}{t}+t\right)^5},t\right)$$

$$\frac{-4\cdot t}{(t^2+4)^2}+\frac{64\cdot t}{(t^2+4)^3}-\frac{256\cdot t}{(t^2+4)^4}$$

PBZ lautet:

$$\int_{□}^{□}\frac{-4\cdot t}{(t^2+4)^2}+\frac{64\cdot t}{(t^2+4)^3}-\frac{256\cdot t}{(t^2+4)^4}dt$$

$$\frac{6\cdot t^4+32}{3\cdot(t^2+4)^3}$$

$\text{ans}\,|\,t=-x+\sqrt{4+x^2}$

$$\frac{6\cdot\left(x-\sqrt{x^2+4}\right)^4+32}{3\cdot\left(\left(x-\sqrt{x^2+4}\right)^2+4\right)^3}$$

Es entsteht eine sehr komplizierte Integraldarstellung!

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Binomisches Integral: B. S. 477 Fall 3

Integraltyp $\int x^m (a+bx^p)^q dx$

mit m=2, a=4, b=1, p=2, $q=-\frac{5}{2}$

und $\frac{m+1}{p}+q=\frac{3}{2}-\frac{5}{2}=-1$ ganzzahlig, Nenner von q ist v=2

Subst: $t=\sqrt{\frac{4+x^2}{x^2}}=\sqrt{4/x^2+1}$

$\frac{d}{dx}\left(\sqrt{4/x^2+1}\right) \mid x>0$

$$\frac{-4}{x^2 \cdot \sqrt{x^2+4}}$$

$$\int \frac{x^2}{\left(\sqrt{4+x^2}\right)^5} dx=\int \frac{x^2}{\left(\sqrt{4+x^2}\right)^5} \frac{x^2 \cdot \sqrt{x^2+4}}{-4} dt$$

$$=\int \frac{x^4}{\left(\sqrt{4+x^2}\right)^4} \frac{1}{-4} dt=\frac{1}{-4}\int \frac{x^4}{\left(4+x^2\right)^2} dt$$

$$=\frac{1}{-4}\int \frac{1}{\left(\frac{4+x^2}{x^2}\right)^2} dt=\frac{1}{-4}\int \frac{1}{\left(t^2\right)^2} dt=\frac{1}{-4}\int \frac{1}{t^4} dt$$

$$\frac{1}{-4}\int \frac{1}{t^4} dt$$

$$\frac{1}{12 \cdot t^3}$$

ans | $t=\sqrt{\frac{4+x^2}{x^2}}$ and x>0

$$\frac{x^3}{12\cdot(x^2+4)^{\frac{3}{2}}}$$

Probe:

$$\frac{d}{dx}\left(\frac{x^3}{12\cdot(x^2+4)^{\frac{3}{2}}}\right)$$

$$\frac{-\left(x^4\cdot(x^2+4)^{\frac{3}{2}}-x^2\cdot(x^2+4)^{\frac{5}{2}}\right)}{4\cdot(x^2+4)^4}$$

simplify(ans)

$$\frac{x^2}{(x^2+4)^{\frac{5}{2}}}$$

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anderer Zugang: rationaler Integrand $R(x,u)=\frac{x^2}{u^5}$

nach B. S. 476: mit $u=\sqrt{a^2+x^2}$ und $a=2$

Subst. $x=2\tan(t)$, $dx=2\left(1+(\tan(t))^2\right)dt$ mit $t>-\frac{\pi}{2}$ und $t<\frac{\pi}{2}$

$$\frac{x^2dx}{\sqrt{(4+x^2)^5}}=\frac{(2\tan(t))^2}{\left(4+(2\tan(t))^2\right)^{5/2}}2\left(1+(\tan(t))^2\right)dt$$

$$\frac{(2\tan(t))^2}{\left(4+(2\tan(t))^2\right)^{5/2}}2\left(1+(\tan(t))^2\right) \mid \tan(t)=\frac{\sin(t)}{\cos(t)}$$

$$\frac{8\cdot\left(\frac{(\sin(t))^2}{(\cos(t))^2}+1\right)\cdot(\sin(t))^2}{(\cos(t))^2\cdot\left(\frac{4\cdot(\sin(t))^2}{(\cos(t))^2}+4\right)^{\frac{5}{2}}}$$

$$\frac{8\cdot\left(\frac{(\sin(t))^2}{(\cos(t))^2}+1\right)\cdot(\sin(t))^2}{(\cos(t))^2\cdot\left(\frac{4\cdot(\sin(t))^2}{(\cos(t))^2}+4\right)^{\frac{5}{2}}}$$

$$= \frac{8}{32}\times\frac{(\sin(t))^2/(\cos(t))^2}{1/(\cos(t))^3}$$

$$= \frac{1}{4}\times(\sin(t))^2\times\cos(t)$$

$t>-\frac{\pi}{2}$ und $t<\frac{\pi}{2}$ ergibt $\sqrt{(\cos(t))^2}=|\cos(t)|=\cos(t)$

$$\int_{\square}^{\square}\frac{1}{4}\times(\sin(t))^2\times\cos(t)\,dt$$

$$\frac{(\sin(t))^3}{12}$$

ans | t=tan⁻¹((x/2)

$$\frac{x^3}{12\cdot\left(x^2+4\right)^{\frac{3}{2}}}$$

Probe:

$$\frac{d}{dx}\left(\frac{x^3}{12\cdot(x^2+4)^{\frac{3}{2}}}\right)$$

$$\frac{-\left(x^4\cdot(x^2+4)^{\frac{3}{2}}-x^2\cdot(x^2+4)^{\frac{5}{2}}\right)}{4\cdot(x^2+4)^4}$$

simplify(ans)

$$\frac{x^2}{(x^2+4)^{\frac{5}{2}}}$$

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anderer Zugang: rationaler Integrand $R(x,u)=\frac{x^2}{u^5}$

nach B. S. 476: mit $u=\sqrt{a^2+x^2}$ und $a=2$

Subst. $x=2\sinh(t)$, $dx=2\cosh(t)dt$

$$\frac{x^2dx}{\sqrt{(4+x^2)^5}}=\frac{(2\sinh(t))^2}{(4+(2\sinh(t))^2)^{5/2}}2\cosh(t)dt$$

Es gilt: $1+(\sinh(t))^2=(\cosh(t))^2$, B. S. 397

$$\frac{(2\sinh(t))^2}{(4+(2\sinh(t))^2)^{5/2}}2\cosh(t)dt=\frac{(2\sinh(t))^2}{((2\cosh(t))^2)^{5/2}}2\cosh(t)dt$$

$$= \frac{2^3}{2^5} \times \frac{(\sinh(t))^2}{(\cosh(t))^4} dt = \frac{1}{4} \times (\tanh(t))^2 \times \frac{1}{(\cosh(t))^2} dt,$$

wobei $(\tanh(t))' = \frac{1}{(\cosh(t))^2}$ gilt.

Damit ist

$$\int \frac{1}{4} \times (\tanh(t))^2 \times \frac{1}{(\cosh(t))^2} dt = \frac{1}{12} (\tanh(t))^3$$

$$= \frac{1}{12} \left(\tanh\left(\text{arsinh}\left(\frac{x}{2}\right)\right)\right)^3 + C$$

Probe:

$$\frac{d}{dx}\left(\frac{1}{12}\left(\tanh\left(\sinh^{-1}\left(\frac{x}{2}\right)\right)\right)^3\right)$$

$$\frac{-\left(x^4 \cdot (x^2+4)^{\frac{3}{2}} - x^2 \cdot (x^2+4)^{\frac{5}{2}}\right)}{4 \cdot (x^2+4)^4}$$

simplify(ans)

$$\frac{x^2}{(x^2+4)^{\frac{5}{2}}}$$

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