School-Mathematics all over the World – some Differences

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Abstract:

The lecture is devoted on some differences of definitions in school-mathematics. In other countries we have sometimes other definitions compared with Germany. Sometimes we can discover even in Germany some differences, because we have in Germany 16 smaller federal countries and every country has its own ministry of education.
On the other hand every professor in any university is free and independent concerning the research and education. He could use his own definition in special fields of mathematics. We would have no problems, if we respect different definitions of different teachers and in different software:

1. What is the set of natural numbers?
   {0, 1, 2, …} or {1, 2, 3, …} and what means the symbol **N**?
2. What means $3^1/_2$? Is it $3 + {}^1/_2 = 3.5$ or $3 * {}^1/_2 = 1.5$?
3. Where the real function $y = \mathrm{f}(x) = x$^(1/3) is defined? x must be non-negative or $x \in \mathbf{R}$?
4. What means the notation $y = \tan^{-1}(x)$? Is this the arctan-function or the cot-function?
5. What is $y = \log(x)$? We know $y = \log_a(x)$. What is the base a in $y = \log(x)$?
   Without any a we know $y = \lg(x)$ or $y = \ln(x)$ or $y = \mathrm{lb}(x) = \mathrm{ld}(x)$ (older notation).
   Sometimes we find the notation $y = {}_a\log(x)$. Why different notations?
6. What is the main argument φ of a complex number in the Gaussian plane?
   $\varphi = \arg(-2\text{-}2\mathbf{j}) = -3\pi/4$ or $\varphi = \arg(-2\text{-}2\mathbf{j}) = 5\pi/4$?
7. What is the 3rd main root of the complex number -2-2**j** ? What is (-2-2**j**)^(1/3) ?
   What is (-8)^(1/3) ? Draw the real function $y = \mathrm{f}(x) = x$^(1/3) ≈ x^0.33333333 !
8. What is the α-quantile x_α of a probability distribution of a random variable X?
   x_α is a number with $P(X < x_\alpha) \le \alpha \le P(X \le x_\alpha)$ or $P(X < x_\alpha) \le 1\text{-}\alpha \le P(X \le x_\alpha)$
9. Is the distribution function $y = F(x)$ of a random variable X right continuous or left continuous? Is $F(x) = P(X < x)$ or $F(x) = P(X \le x)$? Consider the binomial distribution!

We have to solve in this context two basic problems:

A. If students change the school or university (this is the mobility of our young people) and go to another country, than they remark or not remark that some differences exist in mathematical definitions. In the examination sometimes we observe some mistakes of our students and the reasons are some differences in education.

B. We observe sometimes differences between teaching and used software in calculators or PC or used books of several authors.

In the lecture we will discuss about the stated problems and show some examples by the help of CASIO **ClassPad330** (operating system 3.06, published 2011).
The well known software package **Mathematica** (Version 8) by Wolfram could be the basic for all teachers in the world to work with standard definitions which are used in Mathematica.
In Germany the "**German Institute for Standardization**" (Deutsches Institut für Normung, **DIN**) offers stakeholders a platform for the development of standards as a service to industry, the state and society as a whole. DIN has been based in Berlin since 1917. DIN's primary task is to work closely with its stakeholders to develop consensus-based standards that meet market requirements. By agreement with the German Federal Government, DIN is the acknowledged national standards body that represents German interests in European and international standards organizations. Ninety percent of the standards work now carried out

by DIN are international. Standards play a major deregulatory role. DIN's goal is to develop standards that have validity worldwide.
ISO (the **International Organization for Standardization**) is the world's **largest developer** and publisher of **International Standards**. It has its headquarters in Geneva, Switzerland. ISO considers this trend of utmost importance and believes in the fundamental contribution that educational institutions can give on teaching what international standardization is and what can be achieved through it. Cp. chapter 10, **SANS** (South African national standard).

References:
http://en.wikipedia.org/wiki/International_Organization_for_Standardization
http://www.din.de/cmd?level=tpl-home&languageid=en
http://www.iso.org/iso/home.htm
http://edu.casio.com/products/classpad/
http://www.wolfram.com/

1. What is the set of natural numbers?

There are two conventions for the set of natural numbers: it is either the set of positive integers {1, 2, 3, ...} according to the traditional definition; or the set of non-negative integers {0, 1, 2, 3, ...} according to a definition first appearing in the 19^{th} century.

Have a look in **DIN5374**(logic and set theory, symbols and concepts) or **DIN1302**(general mathematical symbols and concepts) or **ISO31-11**(quantities and units – part 11: Mathematical signs and symbols for the use in physical sciences and technology, revised by **ISO 80000-2**:2009): **N**={0, 1, 2, …} and **N***={1, 2, 3, …},
see: http://en.wikipedia.org/wiki/ISO_31-11

e.g. in the online-book
MATHEMATICAL PREPARATION COURSE BEFORE STUDYING PHYSICS
http://www.thphys.uni-heidelberg.de/~hefft/vk_download/vk1e.pdf (March 10, 2011)
you can read:
"We begin with the set of natural numbers {1, 2, 3, …}, given the name **N** by mathematicians and called "natural" because they have been used by mankind to count within living memory." Here $\mathbf{N}_0$ ={0, 1, 2, 3, ...} (in Germany: ISBN 978-3-8274-1638-4)

2. What means $3^{1}/_{2}$?

http://lamar.colostate.edu/~hillger/faq-images/faq-frac.jpg

Using metric measurements, you don't need to do arithmetic with mixed numbers.

What means $3^1/_2$? Often our students are not sure is it $3 + {}^1/_2 = 3.5$ or $3 * {}^1/_2 = 1.5$?
For our students a problem of the mixed notation is that it can be misinterpreted as a product.
A **mixed number** is the sum of a whole number and a proper fraction.

In the expression $\mathbf{a}\ {}^{\mathbf{b}}/_{\mathbf{c}}$ is omitted the operator. Here it is a multiplication, because in terms with symbolic variables other arithmetic operators can not be omitted, cp. DIN1302.

The multiplication of quantity symbols (or numbers in parentheses or values of quantities in parentheses) may be indicated in one of the following ways: ab, a b, a·b, a×b.
When the dot is used as the decimal marker as in the United States, the preferred sign for the multiplication of numbers or values of quantities is a cross (that is, multiplication sign) (×), not a half-high (that is, centered) dot (·).
Example: Write 15×72 but not 15·72 , cp. http://physics.nist.gov/cuu/pdf/sp811.pdf

In some countries such as France, the mixed number is unusual.
See: http://en.wikipedia.org/wiki/Fractions and http://de.wikipedia.org/wiki/Bruchrechnung

You may, of course, ***say*** "three and a half" — 0.5 is often read aloud as "one half" — but you should always ***write*** it as a decimal fraction.
http://lamar.colostate.edu/~hillger/faq.html
http://lamar.colostate.edu/~hillger/decimal.htm

Why Decimal?

A room measures **15**ft. **3-3/4**in. by **21**ft. **7-1/2**in. (**4.667**m by **6.591**m).
Questions:

What is its floor area in *square yards*?
What is its floor area in *square meters*?

Answers:

36.79sq.yd., or **30.76**m^2

3. Where the real function *y* = f(*x*) = *x*^(1/3) is defined?

The 3rd root is a special case of exponentiation (real power with a fraction), cp.
http://en.wikipedia.org/wiki/Exponentiation

We know:
The exponentiation operation with integer exponents requires only elementary algebra.
By definition, raising a nonzero number to the −1 power produces its reciprocal.
Raising a positive real number *x* to a power that is not an integer, say 1/3, can be accomplished in two ways.

a) Rational number exponents can be defined in terms of n^{th} roots, and arbitrary nonzero exponents can then be defined by continuity: *x*^(1/3) ≈ *x*^0.333333333 = (*x*^333333333)^(1/1000000000) = (*x* ^(1/1000000000))^333333333 (the exponent is only a limit of a sequence with finite decimal numbers). Thus it seems, *x* can't be negative.
b) The natural logarithm can be used to define real exponents using the exponential function: *x*^(1/3) = exp(ln(*x*)/3). Thus it is clear, *x* can't be negative (and not zero)

In DIN1302: *y* = *x*^(1/n) is the n^{th} root **with a positive *y*** such that *y*^n=*x*. Here n∈**N*** and *x* is a nonnegative real number. In many **German school books** we find this definition of the n^{th} root in the domain of the real numbers.
e.g. Definition 1.6 in ISBN 978-3-427-21503-5(2007: Mathematik für Berufliche Gymnasien)

Another question is the real solution of the equation x^3 = -8.
The real solution is -8^(1/3)= -(8^(1/3)) = -2 but not (-8)^(1/3) or (-8)^0.333333333.
The last numbers are not real but complex (cp. principal value, complex main root in **C**).

In US-school books a negative base is allowed:
For any real numbers *a* and *b*, and any positive integer *n*, if $a^n = b$, then *a* is the n^{th} root of *b*.
Here (-8)^(1/3) = -2, where from the point of view of **C** the value -2 is not the principal root.

http://www.farmersville.k12.ca.us/aztecs/Department/Math/spradling/Algebra%20II/Alg%202%20unit%207/Review%20&%20tests/Ch%20review%20answers.pdf

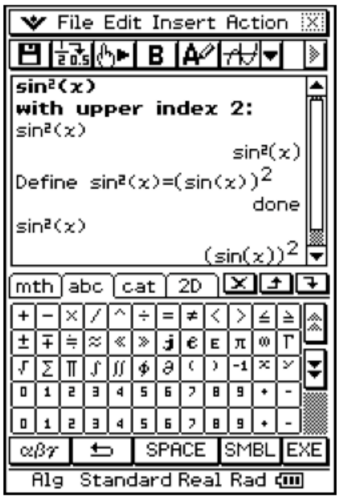
ISBN 0-618-39478-8 (2005: Precalculus with Limits – A graphical Approach)
ISBN 0-471-48273-0 (2005: Calculus)

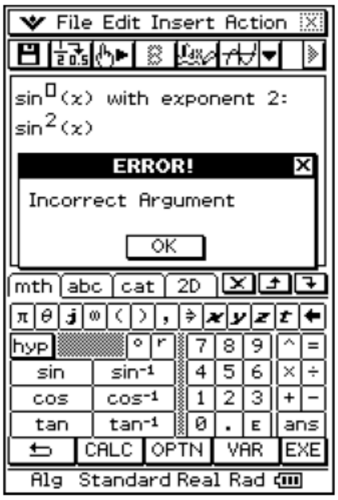
4. What means the notation $y = \tan^{-1}(x)$?

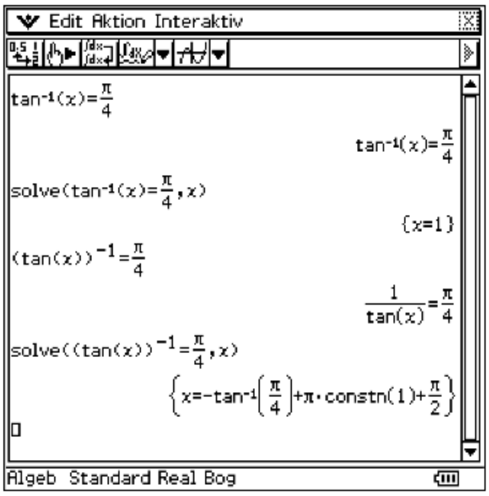
Is this the arctan-function or the cot-function?
For a given function y = f(*x*) we have the notation $y = f^{-1}(x)$ for the **inverse function**.
However the notation $(f(x))^{-1}$ means 1/f(*x*). Here we have the **exponentiation with -1**.

In textbooks we often read e.g. $\sin^2(x)+\cos^2(x) = 1$ (trigonometric Pythagoras)
http://www.qc.edu.hk/math/Certificate%20Level/Trigo%20Py%20Th.htm

If you work with a calculator: the input $\sin^2(x)+\cos^2(x) = 1$ is not defined. You have to write $(\sin(x))^2+(\cos(x))^2 = 1$ or you have to define the symbol $\sin^2$ and $\cos^2$ (a name with four characters!). On the other hand the symbol $\tan^{-1}$ is clear for the calculator.







Only in the case if the symbolic exponent equals -1 our students have problems e.g. with $y = \tan^{-1}(x)$. The calculators use the notation $\tan^{-1}(x)$, but in written form the students should use the notation arctan(*x*) (DIN1302) and not $\tan^{-1}(x)$ to avoid problems and misunderstandings.

5. What is *y* = log(*x*) ?

We know $y = \log_a(x)$. What is the base *a* in *y* = log(*x*)?
Without any *a* we know *y* = lg(*x*) or *y* = ln(*x*) or *y* = lb(*x*) = ld(*x*) (older notation). Sometimes we find the notation $y = {}_a\log(x)$. Why different notations? We should use modern notations given by DIN and ISO. We should avoid the old notations ld(*x*) and ${}_a\log(x)$. In calculators log(*x*) means lg(*x*). Why not a lg-key instead of log-key? In ISBN 978-3-8274-1638-4 (and English and Spanish translation 2011) again the old notation, e.g. $ld(x) = {}_2\log(x)$ – why ?

6. What is the main argument φ of a complex number in the Gaussian plane?

$\varphi = \arg(-2-2\mathbf{j}) = -3\pi/4$ or $\varphi = \arg(-2-2\mathbf{j}) = 5\pi/4$?

In every calculators and PC-software and DIN1302 the main argument is in the interval]-π, π] = (-π, π], but in many school books we can read: the main argument is in the interval [0, 2π[= [0, 2π).

Mathematica: In[1]:= Sqrt[-1] Out[1] = **i** In[2]:= Arg[-2-2**i**] Out[2] = -3π/4

ClassPad: √ (-1) = **j** arg(-2-2**j**) = -3π/4

We have two notations for the complex unit: **i** in mathematics and **j** in engineering.

In the engl. translation (2011) of the book ISBN 978-3-8274-1638-4 you can read:

"for the (only modulo 2π determined) argument of a complex number:
$0 < \varphi = \arg z := \arctan(y/x) \leq 2\pi$"

Here $z = x + y\mathbf{j}$, i.e. (x, y) are the Cartesian coordinates of z. Why here not include $\varphi = 0$?

We know the real function f(t) = arctan(t) has its values in the interval]- π/2, π/2[= (- π/2, π/2), thus the definition $\mathbf{0 < \varphi = \arg z := \arctan(y/x) \leq 2\pi}$ is not correct!

In the German online script http://www.thphys.uni-heidelberg.de/~hefft/vk_download/vk1.pdf (March 11th, 2011) the author has added a remark:

„**Einschub: Alternative Phasenkonvention:** Natürlich kann man auch ein um den Ursprung symmetrisches Intervall der Länge 2π für die Argumente der komplexen Zahlen wählen:
$\mathbf{-\pi < \varphi = \arg z := \arctan(y/x) \leq \pi}$, was allerdings später die Herleitung der Wurzelfunktionen etwas komplizierter macht. Nach Prof. L. Paditz von der HTW Dresden empfiehlt die DIN diese Konvention."

Here the author is again not correct: He has to write $\mathbf{-\pi < \varphi = \arg z \leq \pi}$ whithout the arctan-function! Furthermore he remarks that with the DIN convention for the argument φ the definition of the root-function will be more difficult – this is not true! The definition of the root based on the main argument $\boldsymbol{\varphi} = \mathbf{arg}\ z$ is very easy:

The main root is $z_0 = z^{\wedge}(1/n) = |z|^{\wedge}(1/n) * \exp(\arg(z)\mathbf{j}/n)$
and the other roots are $z_k = z_0 * \exp(k*2\pi\mathbf{j}/n), k = 1, 2, \ldots, n-1.$

This definition of the main root and the other roots is implemented in all calculators and PC-software.

ClassPad: arg(x+yi)|x<0 yields the correct main argument **$\tan^{-1}$(y/x)+signum(y)*π**

Spanish version (March 11th, 2011) of the considered online book „MATHEMATICAL PREPARATION COURSE BEFORE STUDYING PHYSICS":

CURSO DE MATEMÁTICA PREPARATORIO - para el estudio de la Física
traducido por Prof. Dr. LAUTARO VERGARA, Universidad de Santiago de Chile

http://www.thphys.uni-heidelberg.de/~hefft/vk1/k0/000s.htm

"Argumento el número complejo: $\mathbf{0 < \varphi = \arg z := \arctan(y/x) \leq 2\pi}$ (sólo determinado módulo 2π)"

http://www.thphys.uni-heidelberg.de/~hefft/vk1/k8/814s.htm

Thus an obvious error goes around the world!

7. What is the 3rd main root of the complex number -2-2j ?

What is (-2-2j)^(1/3) ?

What is (-8)^(1/3) ? Draw the real function $y = f(x) = x$^(1/3) ≈ x^0.33333333 !

The main root is z_0 **= z^(1/n) = |z|^(1/n) * exp(arg(z)j/n) = 1 – j** if z =-2-2**j**

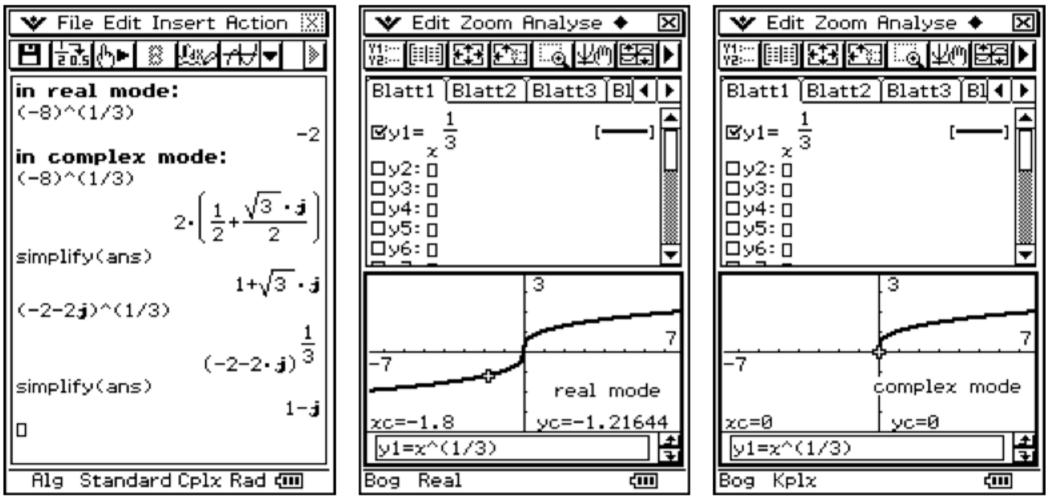
Here we use the main argument **arg(z)** = -3π/4, i.e. **arg(z)j/n** = -π/4.

If we follow the other point of view with **arg(z)** = 5π/4, i.e. **arg(z)j/n** = 5π/12, we would get the main root (-2-2**j**)^(1/3) = **√(3)/2 – 1/2 + (√(3)/2 + 1/2) j** and not **1-j.**

The same with (-8)^(1/3). The main root is 2*exp(πj/3) = **1 + √(3)j** and not **-2.**

ClassPad: in complex mode: (-8)^(1/3) = **1 + √(3)j, in real mode:** (-8)^(1/3) = -2.

In the real mode the ClassPad follows the US-definition of a root, thus we get here different graphics of $y = f(x) = x$^(1/3) in the x-y-plane, in dependence on real mode or complex mode:



However in real mode or complex mode we get the same graphics for $y = f(x) = x$^0.33333333!

8. What is the α-quantile x_α of a probability distribution of a random variable X?

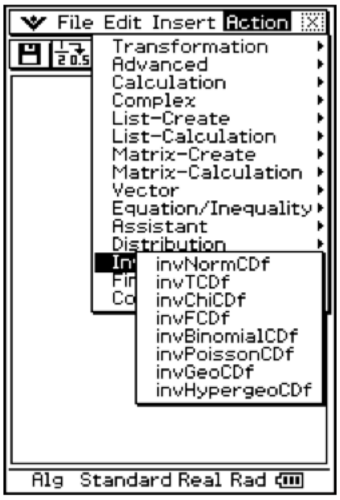
x_α is a number with $P(X< x_\alpha) \le \alpha \le P(X \le x_\alpha)$ or $P(X< x_\alpha) \le 1-\alpha \le P(X \le x_\alpha)$?

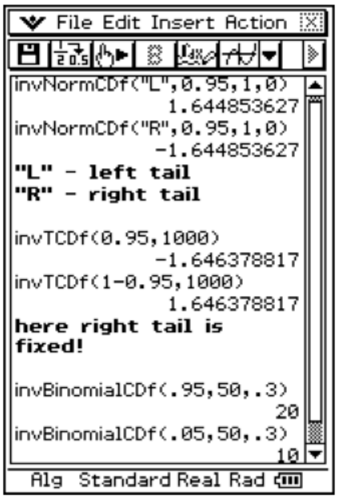
Follow **Mathematica "Quantile[dist, α]** is equivalent to **InverseCDF[dist, α]**" we get the definition: $P(X< x_\alpha) \le \alpha \le P(X \le x_\alpha)$, i.e. according to DIN ISO 3534-1 (2009) (Statistics – Vocabulary and symbols – Part 1: General statistical terms and terms used in probability) based on ISO 3534-1 (2006) (http://it.wikipedia.org/wiki/ISO_3534).

For a continuous distribution of X the inverse CDF at α is the value x_α such that **CDF[dist, x_α]= α.** For a discrete distribution of X the inverse CDF at α is the smallest integer x_α such that **CDF[dist, x_α] ≥ α.**

cp.
http://reference.wolfram.com/mathematica/ref/Quantile.html
http://reference.wolfram.com/mathematica/ref/InverseCDF.html
http://reference.wolfram.com/mathematica/ref/CDF.html
http://reference.wolfram.com/mathematica/ref/Probability.html

However ClassPad only follows this definition for discrete distributions but for continuous distributions ClassPad works with the other definition $\mathbf{P(X< x_\alpha) \leq 1\text{-}\alpha \leq P(X \leq x_\alpha)}$.





The last picture shows a good service during the computing the quantile of a discrete distribution. If the assumed probability (say $\alpha = 0.05$) is no value of the distribution function, than the computed quantile is the smallest integer x_α such that $\alpha \leq P(X \leq x_\alpha) = F(x_\alpha)$, i.e. for x_α-1 we have already $F(x_\alpha-1) = P(X \leq x_\alpha-1) = P(X< x_\alpha) < \alpha$. The “WARNING!” is connected with the last digit after the decimal point of the assumed probability prob = α and will not appear, if a change in the last digit by one step will not change the computed quantile.

9. Is the distribution function $y = F(x)$ of a random variable X right continuous or left continuous?

Is $F(x) = P(X< x)$ or $F(x) = P(X\leq x)$? Consider the binomial distribution!

For a continuous distribution both definitions of $F(x)$ are equal, but for discrete distributions we get here a left continuous function with $F(x) = P(X< x)$ and a right continuous function with $F(x) = P(X\leq x)$. The last definition is in context with **DIN ISO 3534-1 (2009)** based on **ISO 3534-1 (2006)** and all software, e.g. Mathematica or ClassPad.

http://en.wikipedia.org/wiki/Cumulative_distribution_function
http://en.wikipedia.org/wiki/Quantile_function

However in Russian school books we have the other definition $F(x) = P(X< x)$. This was the Russian standard during the time of the former Soviet Union, which is sometimes not compatible with DIN or ISO.
http://www.toehelp.ru/theory/ter_ver/3_2/

However in the Russian Wikipedia we find the new definition too
http://ru.wikipedia.org/wiki/Функция_распределения
http://www.exponenta.ru/educat/class/courses/tv/theme0/2.asp

In the Polish Wikipedia http://pl.wikipedia.org/wiki/Dystrybuanta e.g. you can read:

“Niech P będzie rozkładem prawdopodobieństwa na prostej. Funkcję F: R→R daną wzorem **F(t) = P((-∞,t])** nazywamy dystrybuantą rozkładu P. … Niekiedy w definicji dystrybuanty stosuje się przedział otwarty: **F(t) = P((-∞,t))** Dystrybuanta jest wówczas funkcją **lewostronnie ciągłą** (w przeciwieństwie do przypadku gdy w definicji stosuje się przedział prawostronnie domknięty i dystrybuanta jest funkcją **prawostronnie ciągłą**).”

Here the hint that both definitions exist – but what is the recommendation?

10. More references on ISO 3534-1 and ISO 31-11 around the world:

International standard, ISO 3534-1: Geneve, Switzerland : ISO, 2006.
Statistics - vocabulary and symbols. Part 1, Probability and general statistical terms = Statistique - vocabulaire et symboles. Partie 1, Probabilite et termes statistique generaux.

DIN ISO 3534-1, Berlin, Beuth-Verlag, 2009,
Titel (deutsch): Statistik - Begriffe und Formelzeichen - Teil 1: Wahrscheinlichkeit und allgemeine statistische Begriffe (ISO 3534-1:2006); Text Deutsch und Englisch

SANS 3534-1: Pretoria, South African national standard, 2007:
"This national standard is the identical implementation of **ISO 3534-1:2006** and is adopted with the permission of the International Organization for Standardization."
Cancels and replaces ed. 1 (SANS 3534-1/ISO 3534-1:1993), ISBN: 9780626189983

UNE-ISO 3534-1: Madrid, Asociación Española de Normalización y Certificación (AENOR) 2008, estadística : vocabulario y símbolos. Parte 1, Términos estadísticos generales y términos empleados en el cálculo de probabilidades

PN-ISO 3534-1: Warszawa : Polski Komitet Normalizacyjny 2002, Statystyka – Terminologia i symbole - Część 1: Ogólne terminy z zakresu rachunku prawdopodobieństwa i statystyki, ISBN: 9788323676690

SIST ISO 3534-1: Slovenski standard. Ljubljana: Slovenski inštitut za standardizacijo, 2008, Statistika - slovar in simboli. 1. del, Splošni statistični izrazi in izrazi v zvezi z verjetnostjo,

UNI ISO 3534-1 è la versione in lingua italiana, c.p.
http://it.wikipedia.org/wiki/UNI_ISO_3534-1

International standard, ISO 31-11, Geneve, Switzerland: ISO, 1992. Quantities and units = Grandeurs et unités. Part 11. = Partie 11, Mathematical signs and symbols for use in the physical sciences and technology = Signes et symboles mathématiques à employer dans les sciences physiques et dans la technique.

SIST ISO 31-11, Slovenski standard. Ljubljana: Urad Republike Slovenije za standardizacijo in meroslovje, cop. 2008.Veličine in enote. Del 11, Matematični znaki in simboli za uporabo v fizikalnih in tehniških vedah : (istoveten ISO 31-11:1992)

PN-ISO 31-11, Warszawa: Polski Komitet Normalizacyjny 2001, Wielkości fizyczne i jednostki miar - Znaki i symbole matematyczne do stosowania w naukach fizycznych i technice

New International standard, ISO 80000-2 (1st ed. 2009-12-01), Geneva, Switzerland: Quantities and units – Part 2: Mathematical signs and symbols to be used in the natural sciences and technology.
This edition cancels and replaces ISO 31-11 (1992), which has been technical revised.
Four clauses have been added, i.e. “Standard number sets and intervals”, “Elementary geometry”, “Combinatorics” and “Transforms”.

"Mathematics online and mathematics mobile – where is all this going?"

Douglas Butler debutler@argonet.co.uk
iCT Training Centre, Oundle (UK)
www.tsm-resources.com

Douglas Butler will explore the Aladdin's cave of online resources that busy teachers can now call on to add a sparkle to their lessons. But now students can explore these too on their mobile devices, so the learning process is no longer restricted to the classroom. Exciting and challenging times! But how to ensure secure understanding?

Since around 1995, both the quantity and quality of online resources for mathematics teachers have risen dramatically. Faced with this staggering choice, teachers naturally need a lot of guidance. The TSM Resources website is one solution, breaking the massive spectrum into digestible categories, and this presentation will give a whistle-stop tour of some of the possibilities, including:

- web broadcasting: straight into the classroom
- resources for the busy teacher
- calendar of training events
- useful excel files for mathematics
- integer lists
- resources for dynamic software
- mathematics links and data from around the world

But wait – there is another player in this equation: increasingly, the students themselves have their own access to all this on a device that sits in their pocket. And they are creating their own online resources and sharing their learning experiences online.

This has the potential to change the dynamic of teaching and learning forever! And does the subject of mathematics stand to be affected by all this more than other academic subjects? What other subject comes close to expecting a detailed and proactive knowledge of spread-sheets, dynamic software, computer algebra systems and internet resources? What are the implications on In-service training?

Finally there is the burning question: does all this innovation allow teaching and learning to be more efficient and more effective? How can we be sure that the learners are not cutting corners? Is their understanding secure? After all, a spell checker is generally not much use if you can't spell ...

SETTING MATHEMATICS LABORATORY IN SCHOOLS

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ABSTRACT

Mathematics as a subject is indispensable in the development of any nation with respect to science and technology since mathematics itself is the language of science. In this 21st century where virtually all attentions are shifted towards technological advancements and the mathematics education into the 21st century project is waxing stronger in objectively achieving all its goals in which mathematics is a veritable tool. Hence, this paper explicitly discuss the concept of mathematics and education, mathematics laboratory and its numerous advantages, mathematical equipment/materials in an ideal mathematics laboratory and how to set a mathematics laboratory. Pictorial representations of mathematics laboratory with some mathematical instructional materials were used to substantiate the paper.

INTRODUCTION

Mathematics involves thinking logically and reasonably so as to understand how formulae are derived and their applications. In order to enhance learners' mastery and meaningful learning of mathematics, it is necessary to reduce to the bearable minimum its level of abstraction with the use of instructional materials. Adenegan(2010) testified to this that instructional materials, when properly used in the teaching and learning situation, can supply concrete bases for conceptual thinking, high degree of interest for students in making learning more permanent.

According to Oyekan (2000), "instructional materials are those things that can facilitate effective teaching and pleasant learning that is teaching aids through which learning process may be encouraged and motivated under the classroom situation". These enhance the teaching learning process when adequately and appropriately used.

To this end, this paper focuses on setting mathematics laboratory which directly houses instructional materials. Specifically, this paper aims at defining a mathematics laboratory, listing and identifying mathematical equipment/materials that can be put in a mathematics laboratory and enumerating ways of setting mathematics laboratory in the school.

Concept of Mathematics and Education

Mathematics is the study of numbers, set of points and various abstract elements together with relation between them and operation performed on them. In the beginning, mathematics curriculum in school was arithmetic, since people were just able to calculate, but by the early 1950's the concept of mathematics in the school as subject had developed and was being taught in three different sessions as arithmetic, geometry and algebra.

One of the objectives of teaching mathematics in all strata of education, from primary school level upward is the attainment of an understanding of the nature of the subject within the umbrella of a science education in relation to everyday activities of one's life as asserted by Adenegan (2003). Mathematics leads people into discovering things. However, new discoveries cannot be made unless it is effectively taught through application of adequate and efficient human and physical facilities.

Mathematics cannot be pushed aside in our day to day activities, yet mere mentioning the name of the subject sends cold chill round majority of the students' spine. Nervousness and fear that followed are better seen than imagined! The mathematics students shiver and fear wrinkles up their youthful faces. Then one wonders why this repulsive and uncheerful attitude towards mathematics in our schools, (almost at all levels; primary schools not excluded), in a period when the government desires a technological breakthrough. Could this be attributed to ineffective and inefficient handling of mathematics or inadequacy and non availability of instructional facilities? It is, however, important to note that for pupils to develop interests and do well in mathematics, being the language of science, and a tool for their future career, care must be given to what is taught and how they are being taught in the various schools.

Education can be defined as the process of imparting and acquisition of knowledge through teaching and learning especially at a formal setting such as schools or similar institutions (Alao, 1997). Thus, education can be perceived as a process whereby a person learns how to learn. It actually begins at birth and ends at death. In fact, education is an age-long concept. Mathematics as a subject is part of the curriculum content taught and learnt at different educational strata. Education enriches man with information and when one is not informed, one is at the risk of being deformed.

Importance of Mathematics

Mathematics is a model for thinking, for developing scientific structure, for drawing conclusions and for solving problems. It is a subject that deals with facts. As a result, Olademo (1990) opined, "this subject-mathematics should be given much consideration and let no man think of it as abstract or as untrue". As posited by Balogun et al (2002), "Mathematics instruction is a training of logical thinking. It is a means of solving many problems. It is confronted with finding solutions to problems that have not been provided by a similar type. Its greatest virtue is its flexibility and the high esteem at which it is held as a tending discipline is partly due to its illustrious pedigree".

People who have become more and more skeptical towards mathematics saw it as discipline that pursues needless complications, inventing unrealistic problems and prescribing solving methods within the frame of elementary mathematics. To this end, Adenegan (2003) highlighted Mathematics importance under four broad functions-utilitarian, cultural, social and personal functions.

- Utilitarian functions: It is useful in everyday life that is it serves as a functional tool in studying individuals everyday problems; it is useful as a tool to other discipline, that is, serves as a hand maiden for explanation of quantitative situations in other subjects such as economics, physics, navigation, finance, biology and even the arts. This service of Mathematics is exceedingly important to future scientists, engineers, technologists, technicians and skilled mechanics; it is useful for national income and budgeting and useful for laying foundation for further education.

- Cultural functions: It is useful for calculation in local languages and useful for naming objects.

- Social functions: It is useful in voting, games and lotteries, birth and death rates and population census.

- Personal functions: It encourages correct or accurate thinking, allows for cooperation with others to achieve common goals, allows for character building (patience, persistent and perseverance) and remarkably, it makes one to be happy.

  In a nutshell, “Mathematics is now an enormously useful science which, in order to attain this status, has had to cross a desert of usefulness where Mathematics was nursed tenderly as a science of mind” (Balogun et al 2002)”. Astronomy is a practical science of Mathematics. It is used to foretell the calendar, feast, eclipses, wars, pestilence, whirlwinds, storms and the future of nation and even of individuals. It is a useful application of Mathematics and would link on for at least the next two millennia.

The diverse applications of mathematics abundantly establish that mathematics, as a discipline, is fit for purpose, as mathematics continually drives the expansion of the frontiers of other disciplines through their progressive formalization and symbolization and the building of mathematical paradigms of real world systems.

In Nigeria, a credit in mathematics is required for admission to countless programmes of study at the tertiary level of education. Ekhaguere (2010) asserted that in view of this fate-determining place of mathematics in the nation’s educational system, a policy must be formulated and implemented toward ensuring that no child is left behind in mathematics at the pre-tertiary level of education.

The Mathematics Laboratory

As defined by Adenegan (2003), the mathematics laboratory is a unique room or place, with relevant and up-to-date equipment known as instructional materials, designated for the teaching and learning of mathematics and other scientific or research work, whereby a trained and professionally qualified person (mathematics teacher) readily interact with learners (students) on specified set of instructions. The picture below is an example of a mathematics laboratory where the children are seen playing with educational toys under the supervision of their teacher.

figure 1: *A typical mathematics laboratory with educational toys for children*

In a related term, a current version (miniature) of mathematics laboratory is the "mathematics corner". This indeed is still a new concept. In a school where there is no mathematics laboratory, the teacher together with the students can readily improvise and create what we call the mathematics corner in the classroom as can be found in the picture below. The teacher can start by creating a corner in the class as mathematics corner where he can be depositing periodically mathematics equipment or ask the pupils to bring, with pride and boldness, local mathematics materials like different geometrical shapes so as to facilitate a successful take off and unhindered success of the establishment. The mathematics corner can contain some of the equipment found in the mathematics laboratory but will not be as full and well organized and assembled as what we found in the later.

Figure 2: *A typical mathematics corner at Mathematics Department, Adeyemi Coll. Of Edu. Ondo.*

The materials or equipment that can be found in the mathematics laboratory include, among others constructed (wooden/metal/plastic made) mathematical sets, charts and pictures, computer(s), computer software, audio-visual instructional materials such as projector, electronic starboard, radio, television set, tape recorder, video tape, etc, solid shapes (real or model), bulletin board, three-dimensional aids, filmstrips, tape photographs, portable board or whiteboard, abacus, cardboards, tape measure, graphics, workbooks, graphs, flannel boards, flash cards, etc.

Figure 3: *Teacher/Students-Made Instructional Materials:Solid Shapes and Geometrical Objects during a workshop for Junior Secondary School principals at Ebonyi State, Nigeria by the Author.*

Mathematics laboratory is relatively new in the teaching and learning of mathematics. It is a practical oriented classroom or place where materials useful for the effective teaching and learning of mathematics are kept. It is the latest design to make mathematics real. The term "laboratory method" is commonly used today to refer to an approach to teaching and learning of mathematics which provides opportunity to the learners to abstract mathematical ideas through their own experiences, that is to relate symbol to realities. It is uncommon in our schools today possibly as a result of lack of fund or the absence of any government policy on the provision of such laboratory facilities. In short, its non-existence in our schools is one of the major contributory factors to mass failure in mathematics. Thus, as highlighted by Adenegan (2003), the functions of mathematics laboratory include the followings:

- Permitting students to learn abstract concepts through concrete experiences and thus increase their understanding of those ideas.
- Enabling students to personally experience the joy of discovering principles and relationships.
- Arousing interest and motivating learning.
- Cultivating favourable attitudes towards mathematics.
- Enriching and varying instructions.
- Encouraging and developing creative problems solving ability.
- Allowing for individual differences in manner and speed at which students learn.
- Making students to see the origin of mathematical ideas and participating in "mathematics in the making"
- Allowing students to actually engage in the doing rather than being a passive observer or recipient of knowledge in the learning process.

SETTING MATHEMATICS LABORATORY

Having already discussed extensively the mathematics laboratory, we will proceed to itemize how to set a befitting and remarkable mathematics laboratory in the school.

1 Identify the necessary materials required in the laboratory by labeling them with name tags.
2 Put or assemble all related equipment or materials on the same side/place. e.g. geometric objects should not be placed where audio-visual materials are positioned.
3 Put the bulletin board close to the entrance door in case of any information display.
4 Arrange the benches and tables to allow for free movement in the laboratory.
5 Hang relevant pictures and charts on picture rails and boards.
6 The starboard or white board must be positioned where every student can readily see it.
7 Shelves can be constructed for keeping and demarcating materials.
8 Electronic materials such as projector, television, etc, should be properly displayed.
9 Electrification of the laboratory should be professionally done to allow for safety use.
10 Display materials on tables in an organized manner.
11 The laboratory should be set in such a way that it must be well ventilated.
12 Handy materials that can be easily destroyed or lost can be kept in a cabinet or separate shelve.
13 Arrange the materials in places (on tables, shelves, board, etc) in a way that they can be easily accessed when needed and returned appropriately after use.

CONCLUSION

It is expected that the 21st century mathematics educators/teachers should be readily acquainted with the modern day technique of teaching mathematics in our schools and possibly facilitate their teaching pedagogies with the aid of modern mathematics laboratories to be able to achieve the objectives of the mathematics education into the 21st century project. This paper hereby strongly *recommend* to all school teachers to liaise with their school principals/heads to facilitate the establishment of a mathematics laboratory or for a start, mathematics corner in their schools.

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Technology: The Bridge to Facilitate Learning of Adult Learners of Mathematics

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Abstract

With the advent of many adults returning back to college, math professors all over the country have been trying to find ways to facilitate adult student learning. Many students have been out of school for over ten or more years and are now required to take college algebra (in an accelerated format). However, many adult learners have forgotten facts of basic math. Although some adult students will need heavy remediation of the subject matter regardless of the venue, there must be a way to help adult learners recall basic math and algebra facts.

This paper seeks to explore whether or not blended courses using MyMathLab/MathXL are effective in expediting learning in developmental math courses such as basic math, elementary and intermediate algebra. Research concerning adult learners and technology is significant because learning new technology along with mathematics would seemingly pose an even bigger challenge than just recalling and learning basic math and algebra facts alone.

Why Technology for Adult Learners of Math?

Many adult math learners have not seen math in ten (10) or twenty 20 years. For their given degree program, college algebra is required, but they cannot remember basic math facts. With the notion of andragogy, some adult students are misplaced in accelerated (5 and 8 week) courses because previous life experiences, however this does not correlate in most math courses. Most adult learners need excessive remediation in order to fulfil the requirements of College Algebra due to no prior knowledge of the subject matter, lack of prior knowledge of the subject matter, inaccurate knowledge relating to the subject matter. There are misconceptions, preconceptions and illogical reasoning that adult learners must overcome. Many adult students have math phobias and math anxiety. Some adult students have little or no exposure to the computer and technology related to the math courses that they are taking (e.g, graphing calculators, pda's) In addition, many adult math learners lead busy lives and do not have extra time to study in order to fill in gaps in their learning.

Adult learners of mathematics face the same challenges that a typical math student may face. For example, they may know the material, but cannot remember it. Since there have been removed from an academic setting for so many years, many adult learners of mathematics may have poor study habits and note taking skills.

Other challenges that adult learners of mathematics face include time management and finding what Elayn Martin-Gay calls "teachable, math learning moments."[1] Moreover, working adult learners do not retain much of the material being presented in night classes, especially those with busy schedules in the day. Adult Students can follow the professor during class, but cannot retain the same level of understanding once they attempt the homework assignment outside of class.

As previously stated, adult student learners of mathematics face the same challenges as a typical math student. Victory University provides a robust college learning environment which admits over 600 students annually. The average age of a Victory University student is 35. Thus, specifically addressing the needs of the adult learner is essential. This paper will access the current status of adult learners of mathematics taking developmental courses at Victory University as well as consider the use of mathematical educational software as a means of enhancing the performance rates of adult learners of mathematics in developmental courses.

Methods

The analysis in this paper considers data taken from 433 students of Victory University (formerly known as Crichton College) who have taken math courses between the fall of 1991 until the fall of 2011. Student age and final grade has been taken into consideration. There were a few sections of the LE Basic Math Courses were given Satisfactory/Unsatisfactory assessments at the end of the course. These grades were tabulated as either 2.0= C for Satisfactory or 0=F for Unsatisfactory. Students who withdrew from the course or received an FA grades were translated as 0=F.

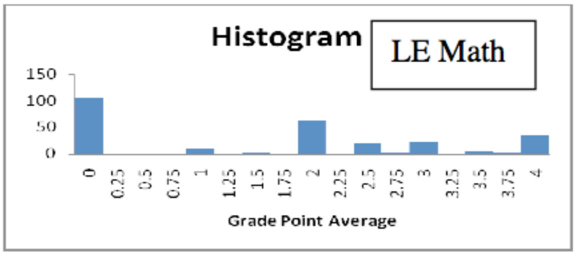
Hypothesis: Adult learners between the ages of 45-80 will have lower grade point averages than the traditional 18-26 year old student. In addition, the grades will be reviewed from two LE0114 Basic Math Courses taught by the author of this paper. The first course was given in the Fall of 2008 and the second course was given in the Spring of 2011. The course in the Spring of 2011 was a hybrid course that included MyMathLab mathematical learning software. The course given in the Fall of 2008 was given in a traditional format. The final component of this research paper examines individual adult student learners of mathematics. In particular, four students over 59 years of age were observed over the course of a semester in basic math, elementary and intermediate algebra. Two students were observed between the ages of 40-45, who openly shared their challenges in intermediate algebra.

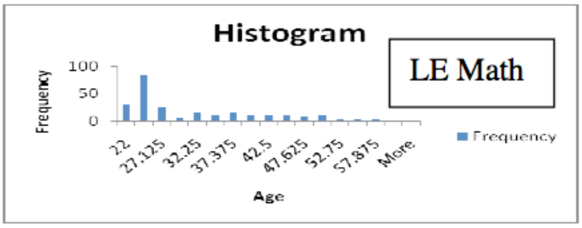
Results

From the fall of 1991 until the fall of 2011, there were 260 students who had recorded grades for LE Basic Math at Victory University. The average age of the students taking LE Basic Math courses is 32 and unfortunately the average grade is 1.61 (D). The following histograms provide the age and grade demographics for each student.

[1] Martin-Gay, E. (2009). *Beginning & Intermediate Algebra* (4th ed.). University of New Orleans: Prentice Hall.

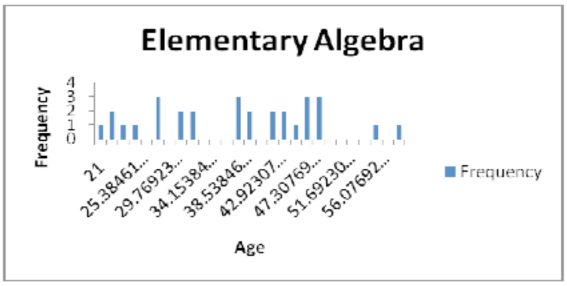
Because the regression coefficient r = .0412, the data also reveals that there is no

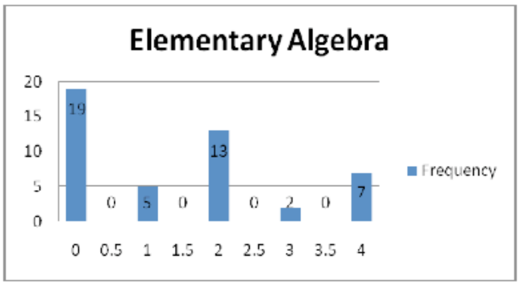


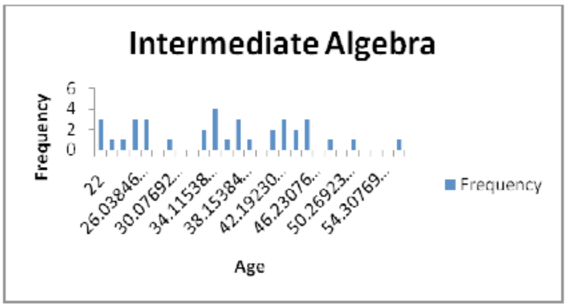


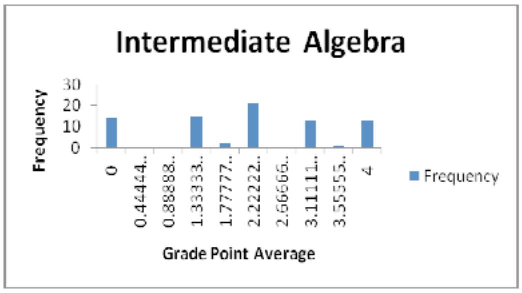
correlation between age and the grade in the course. Since there is no correlation, it is erroneous to assume that older students would perform better or worse in LE mathematics. Yet the grade point average for students 45 and older in LE mathematics, which is 1.72, is slightly higher than the group average. These results reveal that most students taking LE Basic Math have challenges passing the course.

The results for the Elementary and Intermediate Algebra courses are similar. Of the 30 recorded students who have taken elementary algebra from the Fall of 2008 until Fall 2009, the average age is 37 and the average grade is 1.43 (D). The correlation coefficient is r =.25—this implies that there is no relationship between the student's age and grade in Elementary Algebra. Yet again, the grade point average for students 43 and older is 2.0 (C), which is slightly higher than the average student. Of the 36 recorded students who have taken Intermediate Algebra from the Spring of 2006 until the Fall of 2011, the average age is 36 and the average grade is 1.95 (C). Moreover, the correlation coefficient for the student ages and grades of Intermediate Algebra students is r=.35—this also reveals that there is no relationship between student age and the grade that he/she will make in intermediate algebra. Note that .35 is closer to 1 than .042, so age is has a closer relationship to grades in intermediate algebra than in LE Basic Math. The graphs are given below:









Although this paper does not focus on non-developmental math courses, it is interesting to note that 70% of all students (433 students) taking math courses (developmental and regular courses) at Victory University earned an average grade of 2.0 (C).

The LE Basic Math Course in the Fall of 2008 had 11 students with an average age of 34 and an average grade of 1.09 (D). Similarly, the LE Basic Math Course in the spring of 2011 had 8 students with an average age of 48 and an average grade of 2.25 (C). For each course, the correlation coefficient r is under .8 (.34 and -.27 respectively) so one cannot make a strong case for a relationship between the student's age and academic performance. However, the spring of 2011 class has the higher grade point averages and the average age is also higher.

Three students over the age of 59 and two students over 40 were given extra reinforcement within MyMathLab/MathXL. Though these students are not strong in mathematics, they were able to pass their respective developmental math courses with the grade of a C. These students noted that the view an example feature and the help me solve this feature helped them to study concepts that they did not understand from the lecture. Unfortunately, students in the fall of 2008 LE Basic Math class were not afforded this option in the traditional format course.

Discussion

The sole purpose of observing the data above is to help older learners of mathematics excel at a faster rate. However, the data does not substantiate the hypothesis that older students will perform worse than the average/traditional age student. History reveals that older students have performed slightly better. Yet the average age of the students taking developmental math courses at Victory University is 36. What the data does reveal is that the average student taking a developmental math courses is an adult learner (not between the traditional ages of 17-26) who will struggle to pass the course with a grade of C or better. Adult learners in developmental math courses need more support in order to be successful in the course. Looking closely at the data suggests that there are other factors beside age that are instrumental in student performance in developmental math courses:

1. Time devoted to study
2. Aptitude
3. Study Skills
4. Comprehension
5. Utilization of teacher/tutor for outside support
6. Completion of homework and other classroom assignments

The hybrid course reinforces the aforementioned factors within the design of the course. Traditional courses place the burden of excelling these areas on the student with little or no help. With MyMathLab/MathXL students have access to an online tutor more hours of the day than their instructors. Students in the hybrid format can get help with several problems in a more time efficient manner.

Students who are returning back to school after being out of school ten (10) years or more usually are more challenged in recalling basic math facts. Though the data suggests that being out of school for many years does not automatically assume failure, it does not suggest that adult learners of developmental math could not use extra support.

Conclusions

All adult learners of mathematics could use help with mastering the material presented in developmental math courses at Victory University. As in the LE Basic Math courses observed in this study, students who are exposed to learning technologies outperform students within traditional class settings. Helping adult learners to use technology necessary for completing math assignments can be well worth the investment. However, there are some disadvantages to using the math learning technology.

The six students observed in this study needed a lot of help initially with understanding how to operate the computer application. Major disadvantages include:

- Helping students to understand MyMathLab/MathXL takes away from curriculum time
- Challenging internet connections and servers not being available is sometimes discouraging to the students
- Requiring an additional cost to the student sometimes is not favoured by colleagues
- Logging in with the access code and course id usually delay the course's start date by 1 or 2 weeks for most adult learners who have been out of school for several years
- Learning the new technology takes about 2 to 3 weeks in order to become proficient
- Some students will find learning both technology and mathematics overwhelming and drop the course

Although there are several disadvantages, they are not insurmountable. In fact, the benefits compensate for the disadvantages list above—the result is accelerated levels higher learning. The fact that the average student in the MyMathLab/MathXL based LE course has an average grade of 2.25 (C) versus the average grade of 1.09 (D) in a traditional LE math course reveals the potential usefulness math learning technology within the classroom.

Of the students observed, they all claimed that they were able to identify the areas that they are deficient in and spend the time on the computer "filling in the gaps" of their learning. For a couple of the students, MyMathLab/MathXL assignments were more effective than one on one tutoring, because they could play and re-play the explanation to the problems until the concept was understood. Of course, MyMathLab/MathXL is not a replacement for necessary tutoring, but should be utilized before the student access a tutor. MyMathLab/MathXL allows students to use homework time more effectively, which increases their likelihood of mastering the material. Moreover, 4 of the 5 students observed believed that using MyMathLab/MathXL to complete assignments helped them to develop confidence in solving math problems.

Implications

Assuredly, professors still need to enforce the learning and understanding of mathematical concepts of mathematics while using technology. However, more research is necessary to determine if technology can support true understanding of mathematical concepts vs. building basic skills. Another desire of many math professors is to give students feedback in real-time, while they are working on the math problem. Recommendation: Installation of real time software that assesses the student at each level of the problem and remediates the student accordingly. The "Help me solve this feature moves in this direction—precisely because it breaks the problems down into smaller chunks. However, more scaffolding is needed. At the level at which the student is unable to complete the problem, the student will be able to click a link to learn the missing concept via view an example, PowerPoint lesson, animation and/or video lecture. Currently, this technology is not available.

More research is necessary to ensure the viability of developmental math learning with the use of technology over learning within the traditional classroom lecture format. Yet there is a lot of research that supports this claim.[2] For example, if one were to track the amount of time spent in MyMathLab/MathXL by the average adult developmental math learner, would there be a correlation to the grade point average attained in the course? For what percentage of students does technology provide a hindrance for developmental math learning? These and other related questions provide implications for further research.

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USING A VALUES-BASED APPROACH TO PROMOTE SELF-EFFICACY IN MATHEMATICS EDUCATION

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Abstract

This study examines the effects of a values-based approach to teaching mathematics on promoting self-efficacy amongst university mathematics education students (n=130) registered for a BEd degree course at the Nelson Mandela Metropolitan University. Data were generated using students' written reflections (disposition statements), interviews, student journals, and observation of classroom practice video-clips. These data were analysed and triangulated to provide insights into aspects of the 'social reality' in terms of the moral values that participants brought to their mathematics education classes, how they perceived their experience of making explicit moral values in mathematics education classes, and whether these experiences influenced their mathematics self-efficacy. The data suggest that the strategy raised self-efficacy levels and that the participating students recognized the importance of their lecturer practicing core values in her classes, and that they also recognized the importance for themselves as future teachers.

Introduction

This study on a values-based approach to mathematics teaching and learning emanated from an interest in values and morals in education and broader concerns around low self-efficacy amongst many pre- and in-service teachers (Nieuwenhuis, 2007). The 'new scholarship' approach of Whitehead (1989) and McNiff and Whitehead (2006), which is premised on the view that teaching is about participatory learning and the living nature of educational enquiry, provided a framework for the intervention used and the analysis of the data generated.

The intervention used firstly introduced the 130 participating pre-service mathematics education students to a 'values wheel' (honesty, fairness, respect, accountability and compassion bounded within a ring (rim) of integrity, and turning on an axle of trust), after which whole-class discussions were held to provide opportunities to clarify their thinking about these values, as well as the possible influence that a values-based approach might have on their learning. The lecturer (first author) then informed them that she would attempt to live out the values of respect, fairness, accountability, honesty and compassion, and encouraged them to hold her accountable. She explained that she believed that by doing so a respectful and trustful relationship could be developed, which would provide opportunities for them to experience feelings of success and wellbeing, which are essential for the development of self-efficacy (Woolfolk, 2004). As such, the ultimate aim of the intervention was to assist students to adopt a self-efficacious and life-affirming view of themselves as individuals and as mathematics teachers (McNiff, 2002). Data on the effect of this approach was generated via interviews, written student reflections (affective-disposition statements), journal entries and video-recording of an aspect of the intervention.

Values and self-efficacy

Kidder (2006) refers to core values as the moral values which constitute humanity's common moral framework. Based on research conducted by the Institute for Global Ethics these core values are acknowledged as "the core moral and ethical values held in highest regard in diverse communities around the world, given the global diversity of culture, ethnicity, race, religion, gender, political persuasion, economic disparity and educational attainment" (p. 42-43). The core values are the five moral ideas of honesty, fairness, respect, responsibility and compassion (Kidder, 2006, p 64). Nieuwenhuis (2007) asserts that values are consciously and unconsciously at work in all our interpersonal interactions and are an integral part of all human actions, thus making education by its very nature a value-based and value-laden phenomenon. He further notes that there is general agreement that values are important concerns that education institutions will have to deal with if they are serious about issues of quality and effectiveness. He argues that we choose our behaviour based on our personal and socially constructed values, assumptions and beliefs, which in turn inform our understanding of what is morally right and morally wrong and of the type of conduct that would be just and ethical (Nieuwenhuis, 2007). Furthermore, values need to be clarified, discussed, refined and reinvented through a process of active deliberation, debate and the provision of concept clarifying within the socio-cultural milieu of the classroom and community.

Woolfolk (2004) points out that without students' trust, respect and cooperation even the best teaching materials and methods can fail, and she believes that enabling students to feel valued and respected plays a strong role in developing their sense of self-efficacy. The intervention used in this study attempted, via examining moral values in the classroom, to make these relationships explicit and tangible for students. It also attempted to provide opportunities for them to experience success and feelings of well-being and, as such, to create learning environments that foster positive self-efficacy.

The critical outcomes of the South African curriculum suggest that the government has accepted that inculcating values associated with being a 'good citizen' is a key objective in order to develop responsible citizens for the 21st century (Department of Education, 2002; 2007). It is this apparent acceptance that motivates and provides context for this study.

Research design and methodology

Seventy nine first-year and 51 second-year (n=130) mathematics education students, who came from diverse social, cultural, economic and political backgrounds, and who were registered for a B Ed degree at the NMMU in Port Elizabeth, South Africa, participated in the study. The participants were informed as to the aims of the project, that the data generated would be used for research purposes, and that they could withdraw from the project if they wished to do so.

Prior to the intervention, focus-group interviews were held with the students and they were required to provide written reflections (disposition statements) as to how they 'felt' about mathematics. The data generated by the post-intervention affective-disposition statements were used to probe the degree to which students had enjoyed participating in the values-based mathematics education strategy and whether their engagement had influenced their levels of mathematics self-efficacy. Pre-, mid- and post-intervention semi-structured interviews with focus groups of students generated similar data. The data generated by these interviews were classified into broad themes and analysed within the framework of the literature reviewed.

The students kept a personal journal in which they were asked (i) to respond to five formal prompts which were introduced sequentially throughout the semester, and (ii) to make *ad hoc* informal entries throughout the intervention. The prompts included questions as to whether they thought a values-based approach would benefit their teaching, what they enjoyed or did not enjoy about the approach, the influence of the strategy on their thinking and beliefs, their perceptions of what problem solving entailed, and their thoughts on the strategy for their own teaching using a values-based approach. The process of journaling allowed the students to record their perceptions, challenges and experiences, and their own 'transforming self-beliefs' (McNiff, 2002) in their ability to do mathematics. Video recordings of lessons also provided data as to the educational influence the strategy had with regard to students' self-efficacy and belief in their own ability to do and teach mathematics. Overall, the data generated, when triangulated, provided insights into aspects of the 'social reality' that participants brought to their mathematics education classes in terms of moral values, how they perceived these experiences, and whether they influenced their mathematics self-efficacy.

Results

Affective disposition statements

The results of the pre- and post- affective disposition statements suggest an improvement in self-efficacy over the course of the intervention. Twelve percent of the first year students and 20% of the second-year group indicated that they were more positive towards mathematics. Although no students from the second-year group initially expressed a desire to improve their mathematics skills, 15% responded in the post-intervention statement that they believed that they had in fact improved their mathematics skills during the course of the mathematics education lectures.

'Fear of failure' was expressed by 31% of second-year students and 7% of first-year students in the pre-intervention responses. However, in the post-intervention responses, only 17% of second-year students expressed reservations regarding their mathematics capabilities, i.e. that they were still fairly dependent on assistance in solving mathematics problems. No first-year student indicated that 'fear of failure' was a concern for them. These results suggest that both groups of students improved their self-efficacy levels during the intervention.

Interviews

Student responses from both groups indicated that they believed that their initial mathematics self-efficacy levels were directly linked to their success and achievement at mathematics, or lack thereof, during their school years. The results also indicate that a major motivating factor for enjoyment in mathematics was 'experiencing success' This finding concurs with Pajares' (2002) belief that students who perform well in mathematics are likely to develop a strong sense of confidence in their mathematical capabilities, whereas poor performance generally weakens students' confidence in their capability. Student responses clearly indicate that 'fear of failure' was a dominant cause of lack of enjoyment in mathematics. However, by the end of the semester, the data indicated that the intensity levels of students' 'fear of failure' had dissipated significantly, suggesting increased self-efficacy.

The majority of participating students (99%) regarded values as making a positive contribution to their engagement in mathematics education. Only two students felt that a focus of values in mathematics teaching was not important, and stated that they believed that teaching values at university level was too late to make a difference.

Reflective journals

Student responses indicated that the use of journals in their mathematics education classes was a major motivating factor in terms of reflecting and internalising values, and highlighted the role they had played in contributing to social responsibility and social cohesion, especially when shared in class. They appreciated the time the lecturer had spent responding to each of their journal entries and believed that the written communication with her had enabled them to feel that the she had a personal interest in them, despite the anonymity of the journal entries (students dropped off and collected their anonymous journals at agreed and predetermined times and places).

Observation

Observation of the video-clips suggest that the lecturer was able to make the core values explicit at appropriate times, that the values were evident in her attempts to guide the students towards experiencing success in the problem-solving activities, and that these activities elicited a positive response from most students who engaged in the process.

Discussion

Trustworthiness is defined by Kidder (2006) as the manner in which an individual acts in order to engender trust and merit the confidence of others. He describes the warm, solid 'gut feeling' you get from trust – from counting on yourself and in trusting and being trusted by others – as one of the great enablers of life. All student responses for both the mid- and post-intervention interviews suggested that they felt that they had developed a sense of trust in their lecturer. Journal entries and interview responses indicated that the development of a trust relationship was not only due to their perception of her personal commitment to both the students and her teaching, but predominantly to the fact that she had responded to every journal entry that they had submitted within a two-day period of time, as promised. Students' verbal and written responses and discussions suggest that her behaviour in promoting and 'living out' the core values also encouraged the development of a lecturer-student trust relationship.

As noted above, the data suggest that the students believed that their lecturer lived out the core values adopted, and that not only did they recognise the importance of her practising core values in their classes, but that they also recognised the importance of making values explicit for themselves as future teachers. They appreciated the opportunity to practise values in their mathematics education classes and believed that values enrich and "make better people". They also stated that they also believed that the implementation of a values-based approach to teaching and learning promotes more positive attitudes, improves performance and creates opportunities for the development of positive relationships. The importance of values for the wider community and society at large was also acknowledged.

It was the responses in which students suggested they trusted that their lecturer would not ridicule or ignore their contributions to the learning process that prompted her to believe that 'valuing students' opinions' had played an important role in the development of their student-lecturer trust relationship. This supports Llewellyn's theory (2005) of encouraging student inquiry as active participants in the learning process, as well as Brooks and Brooks'

(1993) model of building a classroom community by getting students to believe that their ideas count, promoting a trust relationship within a learning community, and thereby influencing their self-efficacy. We believe that without the continued purposeful recognition, acceptance and practice of the five core values which underpin this research study, the desired classroom climate of integrity (which depends on all of the core values being upheld) would have remained an unattainable goal. For this reason, we believe it was important to address issues which appeared to be in contradiction of the core values that the lecturer was trying so hard to uphold.

Although this research study is limited, and that more in-depth interrogation of students' conceptions of the role of values would be profitable in the search for greater understanding of the role of values in mathematics education, we believe it has a contribution to make in terms of informing teacher educators, teachers and policy-makers about perspectives and interventions which may contribute to pre-service teachers' mathematics self-efficacy levels, and provide pointers to strategies for developing teachers who will be able to contribute to the integrity and vitality of the teaching profession. These strategies include the role that a values-based approach to the teaching and learning of mathematics may have on influencing students' self-efficacy levels, both mathematically and within other subjects. We also believe that the findings of this study can contribute to the on-going debate about the process of quality teaching and learning in terms of the possible gains which could emerge from a values-based approach to teacher education.

The data suggest that, in spite of large classes of multi-cultural and diverse students with differing world views, the first author was able to encourage her students to commit themselves to devoting time and energy to the task of using values to develop an innovative educational approach, and to accept that such an approach has the potential to promote the development of their knowledge, competencies, and personal and professional values as they strive to reach their full potential and help shape a fairer, equitable and more just society in the 21st century (Department of Education, 2002; 2007). An underlying, but implicit intention of this study was to influence stakeholders at policy and other levels to consider and embrace the possibilities that a values-based approach to mathematics education may have on students' levels of mathematics self-efficacy, and to include these notions in the design of mathematics curricula for the 21st century. As such, we believe that the findings of this research study point to the potential to challenge teacher educators to re-assess, adapt and improve their teaching strategies, where necessary, and revise the assumptions about teaching and learning on which they are based.

Considering the current crisis in mathematics and science education in South Africa, as well as the crisis in responsible citizenship, it is imperative that we seriously consider the devastating consequences of teacher-education programmes which are not underpinned by sound value systems and effective approaches to teaching and learning. The challenge is to implement a curriculum which is relevant in content and context to South African educational demands for a strategic, but often controversial, reform processes. In attempting to promote such a paradigm shift, it is important that policy-makers, education departmental officials and teacher educators take cognisance of the reality that students consistently refer to the quality of their own teachers as the primary reason for their achievements and choice of careers.

Concluding remarks

Working in collaboration with students interrogating of her own practice was a new experience for the first author. She had always valued the opinions of others, but felt hesitant about giving her students the responsibility for holding her accountable for living out the values identified in this research study. However, she realised the importance of giving them a voice in finding meaningful ways of improving her teaching practice, which in turn would motivate their own mathematics self-efficacy levels. She was aware that students (as they did in this study) often refer to the quality of their own teachers as the primary reason for their achievements, or the lack thereof, and for the choice of their career, and was gratified that the findings indicated that they felt that her "good teaching" contributed to their positive feelings about mathematics; and hence, their self-efficacy levels.

For her the significance of this research study was largely that she had learned to live her values more fully in her own practice. She developed greater insights into the issues she was investigating, and came to understand how her work has the potential to influence her students in new ways. In the case of this study, the new way was through using a values-based approach to teaching and learning, an approach which positively influenced her students' levels of mathematics self-efficacy.

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Problem-centred teaching and modelling as bridges to the 21st century in primary school mathematics classrooms

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Abstract

Moving mathematics classrooms away from traditional teaching is essential for preparing students for the 21st century. Rote learning of decontextualised rules and procedures as emphasized in traditional curricula and teaching approaches have proven to be unsuitable for the development of higher order thinking. The 'dream' is to have skills (that employer seek for the 21st century) such as being able to make sense of complex systems or working within diverse teams on projects [2: p. 316] fostered in mathematics classrooms, even at a primary school level. In this paper it will be shown that the problem solving perspective that modelling emphasizes includes competencies and skills that are essential in developing authentic mathematical thinking and understanding. Results of a study on modelling competencies [1] will be presented to highlight the growth of a problem solving mode of thinking. We will therefore explain that modelling achieves important aims for mathematics education in 21st century. Modelling fosters students' abilities to actualise existing (but not yet explicit) knowledge and intuitions; to make inventions; to make sense and assign meanings; and to interact mathematically [10: p. 176], thereby developing authentic mathematical thinking. The aim is to provide a perspective that shows how modelling meets the challenge of changing mathematics classrooms.

Introduction

Students need to learn mathematics with understanding since 'things learned with understanding can be used flexibly, adapted to new situations, and used to learn new things. Things learned with understanding are the most useful things to know in a changing and unpredictable world' [11: p.1]. Adaptability and flexibility in using mathematical knowledge is particularly important when students solve contextual problems. Mathematical problem solving has many faces and requires definition. Schroeder and Lester's [3: p. 32, 33] three main descriptions of problem solving are used for this paper. In a traditional sense, problem solving means solving 'word' problems as an extension of routine computational exercises. This can be seen as teaching *for* problem solving – teaching of procedures takes place first and then problems specifically related to the taught concepts are solved. In some progressive programs, students are taught *about* problem solving and are taught to employ various methods or heuristics as options when faced with a problem (e.g. drawing a table or graph etc). When students learn *via or through* problem solving, problems are used to teach important mathematical concepts. When students interact with modelling problems, they solve the problems in their own way with mental tools that they already have available to them. The teacher facilitates by connecting different ideas that allow students more sophisticated understandings through these connections. It is by solving problems first and then building by connections between student ideas and representations that students become adaptable and flexible and move toward a problem solving mode of thinking. Modelling allows students to learn via problem solving and can be appreciated as a significant mathematics teaching and learning opportunity.

The Problem-centred approach and modelling

Hiebert, Carpenter, Fennema, Fuson, Human, Murray, Olivier & Wearne [4] build a problem-centred approach on Dewey's principles of reflective inquiry. They work from the assumption that understanding is the goal of mathematics education. Students solve problems at the outset of a mathematics lesson and the process of solving, collaborating, negotiating and sharing leaves 'behind important residue' [4: p. 18]. As expressed by Human [5: p. 303] problems are used as vehicles for developing mathematical knowledge and proficiency together with teacher-led social interaction and classroom discourse. The problems are opportunities for students explore mathematics and come up with reasonable methods for solutions [11: p. 8]. The role of the teacher and the student changes which means that the classroom takes on a different culture. It is in the problem-centred classroom culture where the real benefits to student learning lie. Students have regular opportunity to discuss, evaluate, explain, and justify their interpretations and solutions [6: p. 6]. This can only be achieved if the teacher allows this discussion to take place without presenting or demonstrating set procedures to solve problems. It is this change in focus in the classroom away from 'teacher thoughts' to 'student thoughts' [13: p. 5] that epitomizes student-centred methods such as the problem-centred approach and modelling. The classroom culture now takes on what Brousseau [7: p. 30] termed an 'adidactical situation'. The teacher, in an adidactical situation, does not attempt to tell the students all. Brousseau also explains that the 'devolution' [7: p. 230] of a problem is fundamental in adidactical situations. This happens when the teacher provokes adaptation in the students by the choice of problems put to them. The problems must also be such that the student accepts them and wants to solve them. The teacher refrains from suggesting the knowledge, methods or procedures he/she is expecting or wanting to see. The teacher seeks to transfer part or all of the responsibility of solving the problem on the student. It is this adaptation to the adidactical situation that allows students to learn meaningfully. It is this adaptation to a problem-centred approach that allows a growth of mathematical understanding, adaptability and flexibility. This in turn promotes a problem solving mode of thinking that is so necessary in the 21st century workplace.

The problem-centred approach in mathematics education allows us to understand modelling and its place in effective mathematics teaching and learning. Modelling goes beyond problem solving in that the important questions of *when* and *why* problems are solved as well as *whose* thoughts ideas and constructs are used when solving problems. In defining what a problem solving capacity or mode of thinking entails, the constructs of [10] are used. When students solve problems, they should be provided with opportunities to: actualise existing (but not yet explicit) knowledge and intuitions; make inventions; make sense and assign meanings, and interact mathematically [10: p. 176]. These four constructs encompass what it means to solve problem with understanding and flexibility. It is often difficult for teachers to elicit existing knowledge from students since there is an array of different understandings and levels of thinking in a single classroom. Modelling allows students to verbalise their current ways of thinking and improve on these ways of thinking. In making inventions, students are able to use their own ways of thinking in constructing a response to the problem. Modelling tasks allow students to produce meaningful solutions that keep the context of the problem in sight. While students work collaboratively on modelling tasks they do make sense and assign meaning since they have to communicate their thinking and ideas while interacting mathematically with each other in order to make progress. These four constructs underline what it means to develop a problem solving mode of thinking since they encompass student understanding, adaptability and flexibility in solving problems. It also underlines what students need to learn meaningful mathematics in the 21st century.

Mathematical modelling goes beyond problem solving since students 'create a system of relationships' [12: p. 110] from the given situation that can be generalised and reused. Although students are solving problems when modelling, a modelling approach means that students must display a wider and deeper understanding of the problem. Modelling goes beyond problem solving because students structure and control the problem – not only solve it. The aim of this paper is to show that modelling tasks allow students and teachers access to significant problem solving that bridges student understanding and student problem solving abilities. The development of a problem solving mode of thinking that results from student involvement with modelling tasks is presented in this paper. Problem solving and modelling problems specifically hold a reciprocal interdisciplinary relationship with other knowledge fields. Modelling problems for mathematics classrooms are applicable to and can be sourced from fields outside mathematics such as engineering, architecture, commerce and medicine to name a few.

The study

The main study [1] investigated the development of modelling competencies in grade 7 students working in groups. Partial results will be presented in this paper. Twelve grade 7 students were selected to work in three groups of four students in each group. The results of only one of the groups working on the first (of three) task are presented in this paper. This group comprised students whose mathematics results in a traditional setting the previous year were considered "weak". The groups solved three model-eliciting tasks over a period of 12 weeks in weekly sessions of about one hour. The results from this group's discussions around the task – Big Foot is presented.

Task 1: Big Foot taken from [9: p. 123]. Example of footprint (size 24) given to students. Groups had to find the height/size of this person and also provide a 'toolkit' on how to find anyone's height/size from their footprint. Supporting material: rulers, tape measures, calculators

Table 1: Task Instruction for Big Foot

Students solved and presented their solution as a group with minimal teacher/researcher intervention. These students had not been exposed to a problem-centred approach nor had they solved modelling tasks before. Each group presented their solution to the other groups and students were encouraged to question each other's models. The contact sessions were audio recorded and transcribed. Transcriptions were coded for each competency for the main study and coded again for the results presented in this paper. The competencies identified for the main study were: understanding, simplifying, mathematising, working mathematically, interpreting, validating, presenting, using informal knowledge, planning and monitoring, a sense of direction, student beliefs and arguing. How students develop and refine a problem solving mode of thinking is highlighted in this paper. The four constructs [10: p. 176] were used to code the data from the transcriptions and to structure the discussion in the next section. This assisted in establishing to what extent students working in groups were engaging a problem-centred paradigm when solving model-eliciting problems.

Results

The results presented are from the group's solutions processes for Task 1: Big Foot. This was their very first modelling task so it exhibits the impact modelling tasks have on students thinking. Furthermore it highlights the mathematical learning opportunities that are implicit in a modelling task. In the transcripts R stands for researcher.

Actualising existing knowledge and intuitions
This group had an intuitive idea that there was a universal foot to height ratio although this took place in the second session. The first session was taken up by a seeming avoidance of mathematising the task. Once they had decided to take action on their own intuitions they were successful in producing a model for this task.
M: Ok wait, why don't I take my foot and divide it by my height, times a 100
R: why do you times it by 100?
M: Because that is how you find your percentage so we can find out …I am saying that when we do his (Big Foot) *then we must get the same…*
The students introduced the idea of a percentage on their own accord, but it later transpired that they used the 'times by 100' to remove the decimal number that resulted from their division.
This group also had an intuitive idea that Big Foot had to be very tall and they were able to use this to interpret and validate their progress which allowed them to make progress in their solution process. They had taken a number of measurements including across their hips which they called their 'width'.
M: 58 (a group member's height) *divided by 2 is 28*. Similar to his width - they measured 27 as this person's 'width'.)
N: So he is 30 inches!
M: No he can't be, that's too short.
If student do bring their own ideas and constructs forth and they act on these ideas it is clear to see how 'making inventions' is possible. This would not be possible if students are offered methods or procedures by the teacher.
Making inventions
This group 'invented' their model to assist them in resolving Big Foot.
M: I divided my feet to my height and I timesed it by 100 and I got 15.
N: Yes…
M: So now I have to try get 16 times by what to get 15 again because it is a human. Then that will be right.
On their presentation sheet (see Fig 1) they had written:
The solution is to take his foot size and divide it by an estimated number, multiply that by 100. The result should be 15-20.
Although they did not see a connection here between multiplying and dividing (surprising for this year of their schooling), they invented a way around this of 'estimating' the multiplicand so that the result would be 16. Once this group had found the foot length to height ratio of all their group members, they had four different (although very close) ratios. They then realized they needed more data and tried more people. After trying three more people they found that 16 seemed to be a common ratio. Although they never used the term 'mode', this is a construct that they 'invented' by understanding that they needed this from their set of data. When questioned:
M: R is 16 and N is 16. Then we must use 16.
R: why did you decide that N and I have the right measurements?
M: Because you are the most.
Making sense and assigning meaning
After calculating that 15 was one of the group member's foot/height ratio, they continued to work through the rest of the group, other people in the room as well as continuing this at home and with other students at school. They were clearly in control of this 'method' or model

although it was not an elegant approach it was meaningful and they were able to assign meaning to other areas of the model.
M: Divided by (known height) *and times 100 and let hope it equals something nearby 15 and 20.*
N: 62 divided by 11, ag no sorry the other way; 11 divided by 62 times 100 is
M: Yes I told you. 17. So I equal 15 and you 17.
M: OK it (the quotient) *might be 15, 16 or 17. So he* (Big Foot) *might be: 98, 97 or 96.*
When looking at their presentation sheet- they understood that if the ratio was 15, then their estimated height was too short, or it the ratio was 19, then their estimated height was too tall. They were able to assign meaning to a fairly complicated model which is surprising since they achieved lower mathematics results than average in a traditional setting.

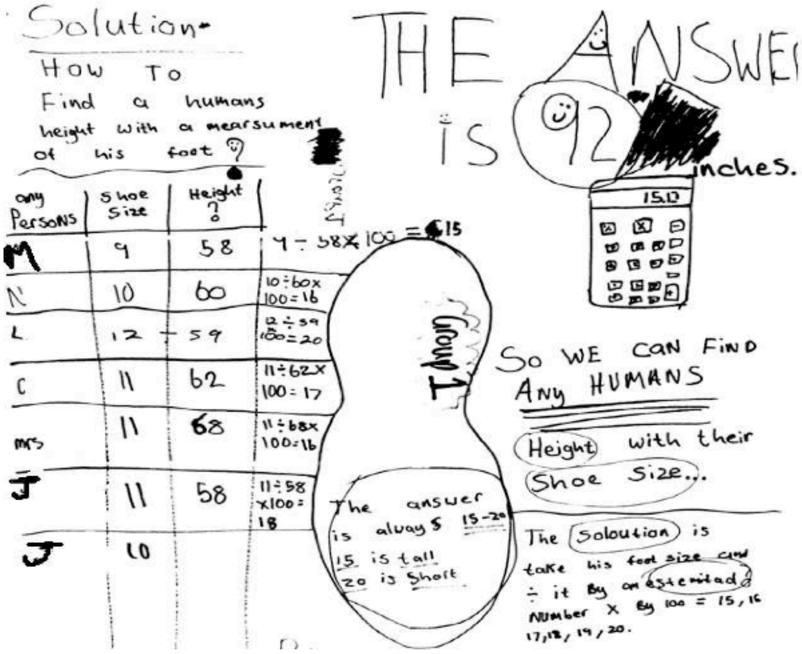

Fig 1: Group presentation sheet

Interacting mathematically

The following excerpt from the transcripts for this group shows how one group member explains a fairly inelegant yet complicated model for Big Foot to the other members.
M: Look, I take your foot (length) *right; the foot is 12* (inches)*, then I divide it by any* *__estimated number__, like I will take, a number will come in my head and I will divide it* (by the foot length) *and then multiply by 100. Probably* (the result will be) *over 20 or below 15. If it's below 15, it means the person is taller, if it's over 20 it means the person is a bit shorter. Then you estimate a bit lower until you get 15, 16 or 17.*

Conclusion

The confluence of the problem-centred environment and modelling tasks present mathematics education with a 'developmental space' for the learning of essential, meaningful mathematics [1: p. 37].
The data presented in [6] suggested that a problem-centered instructional approach in which the teacher and students engage in discourse that has mathematical meaning as its theme is feasible in the public school classroom [6: p. 25]. The results of [8] suggest that a problem-centered approach together with a change in teacher beliefs is a viable for reforming mathematics classrooms. Furthermore, a modelling approach assists in developing student competencies in

problem solving, modelling and mathematics. Modelling tasks present an arena for teaching and learning that assists teachers in understanding a problem-centred approach and to simultaneously apply these principles in teaching. Modelling tasks can be used successfully by teachers and students unfamiliar to problem solving or a problem-centred approach.

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iMath- Reaching the iGeneration in the Mathematics Classroom

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Abstract

The major aims of this paper are to: define a new generation of tech-savvy children that are in our math classrooms known as the iGeneration, discuss how instruction should be adapted to meet the needs of this new learner, and illustrate the potential of this generation's i-devices in the math classroom. Though educational technology is not a new concept, these i-devices, such as the iPod touch and iPhone, bring a new dimension to instruction that can offer a classroom greater "motivation, engagement, conceptual understanding, and problem solving skills" (Manzano, 2009, para. 11). This paper targets teachers and schools in an effort to shed light on these fairly inexpensive, readily available technologies that have tremendous instructional potential in mathematics.

Introduction

Generations in the American culture have been named since the early 1900's with the development of the Western World. Significant events like the boom in births in the 1940s, the Baby Boomers, or the creation and widespread use of the Internet, Millennial Generation, have defined groups of Americans based on cultural shifts. A new generation has cropped up that is having a major impact on how we think, teach, live, and learn. These individuals are known as the iGeneration. The formation of this generation came with the creation and widespread use of "i" devices such as iPods, iPhones, and iPads. This generation is now in elementary and secondary school. They are described "as being solidly committed to the ubiquitous use of mobile technologies, most commonly MP3 players, Smartphones, and similar devices." (Rosen, 2010, 229) This generation thrives with individualized technologies and has them as a daily part of their lives. They have had the Internet since they were old enough to read and never knew a time without cell phones.

Technology has distinctively influenced life for all ages, not just those in the iGeneration. We are more apt to Google a topic than looking in an encyclopedia. We book flights online rather than go through travel agencies. Information is emailed more often than sent hard copy through the postal service. With this undeniable shift to a more technology-saturated world, it is time to reflect on today's classroom and how we go about instruction. In the area of mathematics, the National Council of Teachers of Mathematics (NCTM), the largest math education association in the United States, highlights technology as a vehicle for the teaching and understanding of mathematics in their Principles and Standards of Mathematics (2000). Their vision of mathematics includes classrooms where 21st century technologies are readily available and used as a tool to engage and challenge children. The International Society for Technology in Education echoes these sentiments emphasizing the need for children to "think critically, solve problems, and make decisions" through the use of various technologies (ISTE, 2000, para. 2).

Technology is nothing new to the American classroom. Within the last ten years we have seen widespread use of such tools as interactive whiteboards, mathematics education software, laptops, and graphing calculators. The time has come, however, to make room for a new technology that is not necessarily seen in the classroom. The iGeneration uses them all the time. You might see a young boy with an iPod while mom shops or a teenager texting a friend on their iPhone. These i-devices provide many of the capabilities of a computer in the palm of the hand. This paper will focus on embracing this new generation and their technology in the mathematics classroom.

The Growing iGeneration

A striking reality is the research done on the boom of i-devices. Rosen (2010) in his book about the "iGen" speaks of research he did with 1,500 parents. Half of 9- to 12-years olds of these parents had cell phones and two thirds had some form of MP3 player or iPod. Even in the 5- to 12-year old category, more than half of them had video or handheld games in their bedrooms. These children are in our classrooms. With every passing day, the way these children are motivated and engaged changes.

The iGen are those born beginning in the 1990's. This generation has never known a time without technology and sees new technologies cropping up every year. Parents of these children are embracing this technology allowing more and more of it to be a part of their child's life. Rosen argues that this has bearing on why many children report school as boring or dull (2010). Though advances have been made with technology in classrooms, Rosen reports that it is not catching up with this generation. A teacher may use a Power Point presentation as part of a lecture but they don't necessarily utilize the kinds of technology tools they are motivated and engaged by. This paper focuses specifically on the i-devices that have helped characterize this generation including the iPhone, iPod, and iPad. These tools have many capabilities that can be educational and a meaningful addition to the mathematics classroom.

The Tools of the iGeneration

The focus of this paper is on just one of the many categories of technology that has quickly grown in popularity with the iGeneration or "iGen". The iPod Touch or iTouch, iPhone and iPad all fall within the category of wireless mobile devices (WMD). These i-devices are hand-held computers providing users with instant access to the Internet, communication tools, and a great variety of applications. They dominate the technology landscape of youth. For instance, Apple reports 160 million i-devices sold as of April 2011 (Ogasawara, 2011). Of those, 55 million are iPod Touch users. 46% of these users are under 18 years of age (Loechner, 2009). I-device features that make them so enticing to youth include touch screens, colorful graphics, real-time video, and quick access to Internet and applications. According to Rosen, i-devices have "great promise in education" (2010). However, these educational tools have not been used beyond small, isolated studies (Rosen, 2010).

One of the most popular and well known features of these i-devices is the applications, known by iGen simply as "apps". Apps are a piece of software like software that can be bought for a desktop computer or a videogame for a gaming system. These apps number in the hundred thousand with ten thousand plus specifically in the education category (Go Rumors, 2010). Applications are often free or inexpensive costing on average about $3-$5. They span a wide range of topics including mathematics. A quick search of the word "math" in the Apps Store on these devices will find anything from a simple flash card to graphing calculator app. Teachers are beginning to see the potential of these tools for the math classroom. A We Are Teachers blog article (2010), for example, offers educators a list of math education apps ranging from early learners to secondary school children. The popular McGraw-Hill education publishing company has created a collection of apps focusing on math teachers and their math curriculum (PadGadget, 2011). Even books published on apps such as Best iPad Apps feature apps suitable for the math classroom (Meyers, 2010).

Adapting Math Instruction for the iGeneration

With a new breed of learners comes a fresh perspective on how we teach them. Rosen's book Rewired (2010) has inspired many educators to reflect and consider how the classroom can be adapted to better meet the needs of the iGen learner. In the paragraphs to follow, major traits are considered that motivate the proposal to utilize i-devices in the math classroom.

iGens are said to be a "creative, multimedia" generation" (Rosen, 2010, p.218). They thrive in environments that blend sights, sounds, video, colors, etc. This doesn't mean that a clever Power Point presentation is enough to appease the iGen. Teachers need to consider varied digital modalities to reach them (Wood, 2010). Teachers shouldn't be afraid to use audio, video, or web-based information to teach with or to blend these approaches. iGens are multi-taskers so let them possibly use headphones while working on an assignment or have an iPod while they do a group activity. The more opportunities these learners have to be independent and active with digitally rich sources the more engaged they are likely to be.

Another area where iGens thrive is in creativity. These children "live to create" (Rosen, 2010). "Teachers can engage students by giving them choices in demonstrating content-area knowledge through different tools, whether it be a movie, podcast, digital poster, or webpage" (Wood, 2010, para. 1). iGens want access to all the tools they have, digital and traditional. Teachers should provide them with choice and flexibility when given a major assignment to work on.

Freedom to work at their own pace is preferred by iGens in school. It's said to be a product of their fast-paced lives. iGens can easily shift from task to task and work well under pressure. With a few carefully set guidelines, these learners want to be set free to approach the task (Rosen, 2010). Wood (2010, para. 1) warns in his blog that this would likely work best with check points to ensure learners are on target with their progress.

iGens have added a fourth R to the well known three R's- reading, writing, arithmetic, and realistic technology (Rosen, 2010; Wood, 2010). Teachers are encouraged to embrace technology as the fourth R and "use it to teach and augment the original three" (Rosen, 2010, p.225). There is also the "realistic" aspect of technology. Technology offers the chance for teachers to bring the real world into a lesson and the lesson into the real world (Shuler, 2009).

Teachers shouldn't fear this new generation of learners. However, there is an undeniable call to consider adapting the more traditional models of instruction. I-devices and their apps offer much of what the iGeneration seeks as part of their learning and is seen as a "key to the future of education" (Rosen, 2010, p.203). A professor who was sharing thoughts on a recently national report called *Pockets of Potential* (Manzo, 2009) compares the potential impact mobile devices can have on learning to that of Sesame Street on television. Just as this TV program convinced viewers that TV could be more than a tool for amusement, mobile devices are likely to some day be seen as a tool for engaging and activating learners in the classroom (Manzo, 2009).

Apps for the Mathematics Classroom

The iGeneration sits in today's classrooms. With a sense of who they are and what they need, the teacher has the task of determining how to adapt what they do to bring the "i" into their math instruction. I-devices feature inexpensive apps that can be loaded and played with only a few screen touches. These apps are a way to bring the characteristics of learning that an iGen hopes for in their classroom.

When looking at apps for i-devices in the area of mathematics education, the list grows daily. As of May 2011, a search done in the Apple App Store of an iPad using the key word "math" within the education category results in 2,533 apps for the iPhone and iPod as well as 1,006 for the iPad. These apps cover a wide variety of categories and each have their own unique spin on learning and practicing mathematics.

For this paper, apps were reviewed and categorized based on a number of features to reduce the thousands of apps listed down to a manageable set of quality math related apps for possible classroom use. One feature considered was the style of the app. iGens love multimedia so apps that offer a blend of strong audio and visuals are a good choice. These apps look and feel like a video game. HyperBlast, for example, offers an arcade like feel with 3-D graphics and three game play levels. While battling aliens, children can practice their basic facts in addition, subtraction, multiplication, and division. Other apps utilize mathematical thinking to play the game. Blokus, for example, has players using transformations to place geometric pieces on a game board. In Cut the Rope a player must decide how to cut swinging ropes so a piece of candy can be sent to the hungry animal waiting for it. These types of apps are "game-based" offering mathematical practice while playing a game.

Real world connection is another important iGen characteristic that apps can offer a classroom. Lemonade Tycoon is a perfect example. The player in this app manages his/her own lemonade business with the goal of making a profit. The player has lots of options including the choice of location, recipe, and weather. A more simplistic app for younger players is Pizza Fractions. This app uses pizza to illustrate fractional parts. In any of these real world apps, the player experiences real life application of mathematics in some form. i-Gens appreciate this connection because it establishes a link between what they are learning in school and its relationship to their everyday life.

Many of the apps that can be found offer children a chance to practice a math skill. There are also apps that teach skills and offer tutorials to help understand mathematical concepts. Fractions Helper is one such app offering the user a step-by-step walk through on how to solve addition problems with fractions. Another app called Cheater Pants can offer students a step-by-step visual of the solving process for doing basic arithmetic problems.

Other features that are cropping up among the many apps are the ability to reward the user and provide progress reports. In the app Cash Cow, for example, you can earn virtual money to purchase items and complete tasks on your own virtual farm. The app Academy rewards correct answers with stickers that can be used to decorate selectable backgrounds. Froggy Math app provides a "report card" to let the player know how they are progressing at each level of play. Both of theses types of features add a more personalized feel that appeals to the iGen learner.

Apps offer a wide variety of options and mathematical topics to the classroom as a whole. The chart, found at www.tinyurl.com/iMathBoakes, lists a total of 55 apps selected for the many features discussed. Beyond this analysis, each of these apps received a user approval rating of at least 4 out of 5 stars through the Apple App Store rating system. Apps are organized in alphabetical order and are categorized based on the NCTM (2000) mathematical standard area, grade level of the user (lower elementary, upper elementary, middle, and secondary), as well as features including price, game based/real world, practice/tutorial, reward, and progress report. With some creative new thinking, teachers, parents, and iGens themselves can use these apps to enhance and develop understanding of mathematical concepts and skills.

Conclusion

The focus of this paper was to highlight the ever-changing composition of the math classroom. When it comes to technology, the pace of advances is even more pronounced. This new generation of learners known as the iGeneration is cause for pause and reflection on how we approach instruction. iGens think and learn differently. They thrive in media-rich environments especially those that include the i-devices they are aptly named for. Studies over the past five years by the Research Center for Educational Technology at Kent State University in Ohio have found that using these hand-held devices in the classroom can improve students' motivation, engagement, conceptual understanding, and problem-solving skills. (Manzo, 2009, para. 11).

The applications of i-devices discussed are a simplistic way to begin capitalizing on the iGen's interests in the math classroom. Apps span all grade levels and mathematical standard areas. They can provide many of the characteristics that an iGen seeks in their learning from realistic applications to an engaging, multimedia style. With the prevalence of hand-held technologies and inexpensive versatile apps, it is time for all those who teach to consider this possibility as a way to reach today's learner.

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Physicists use mathematics to describe physical principles and mathematicians use physical phenomena to illustrate mathematical formula – Do they really mean the same?

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Abstract:
Physics and mathematics educators often find themselves in an absurd situation. Students have problems in thinking in an abstract way. In order to help them to overcome these problems mathematical formulae in math classes are often illustrated with physical phenomena which are introduced as authentic applications of mathematics in every day life. However, students in physics classes have also difficulties in gaining a deep understanding of physics. They try to overcome their lack of understanding by memorising facts or using formula (without understanding their meaning).

Given these problems in abstract thinking and understanding of physical phenomena the issue arises what might be the potential confusions regarding a physical phenomenon after having experienced mathematics and physics lessons on this topic? To address this issue we chose an example of geometrical optics – the mirror image.

We generate and evaluate a heuristic framework for describing and exploring the process of understanding a physical phenomenon. This heuristic framework differentiates several scientific models (e.g., physical, mathematical) which are necessary for understanding and explaining the phenomenon. Furthermore, it integrates these models in a multiperspective instructional model (i.e. didacticised model). Using this heuristic framework we analysed the problems in understanding which occur when students have to understand the mirror image.

Introduction

The mirror image is one of the phenomena most studied and most misunderstood in early physics education. Results of numerous studies (e.g. Blumör and Wiesner (1992a), Galili, Bendall, and Goldberg (1993), Jung (1981), La Rosa, Mayer, Patrizi, and Vicentini-Missoni (1984), Gropengießer(1997)) show that the understanding of the mirror image phenomenon is quite unsatisfactory. What is the problem?

At first we have to consider that the misunderstandings are not to be found in physics – but the learners think, that they are not able to understand a physical phenomenon! To explain the mirror image an optical and a non-optical (human) argumentation is needed – and have to be linked to each other. But both sides of this medal cause problems. Already early researchers like AlHazen and Euler said that for the explanation of the mirror image both different kinds of modelling are needed. The first one – the physical argumentation - means the explanation of the mirror image with the help of geometrical optics, and the second one - the non-physical argumentation - is our human interaction with light- the interpretation of the picture on the retina through our brain.

In this area of conflict one answer to the question for problems with the mirror image can be identified. The non-optical argumentation plays a marginal role in school lessons. But there is still another point of view – the mathematical modelling of the mirror image has an important influence on the understanding of the mirror image. This important role is being discussed in this paper.

The starting point is the question: "How do the pupils understand a physical phenomenon?". We have to look at the process of constructing mental models of a phenomenon. In order to understand how students handle mathematical and physical knowledge it is necessary to examine their modelling process in physics in a more detailed way.

Understanding physical models

According to Stachowiak's theory of modelling (1973) nobody can describe the real world – only a model of the real world. In particular novices have problems to understand science because they do not understand that teachers are talking about models of the reality and not about reality itself. This is one of the most important problems in modern science teaching – physical theories should be taught in a way acknowledging that these theories are models of the real world. In shaping this processes science teachers have to pay attention on insights in how students acquire scientific concepts (i.e. epistemic processes). The process of constructing mental models during the acquisition of physical knowledge plays an important role in understanding physical phenomena.

To describe and explore the process of understanding physical phenomena through science instruction we developed a heuristic framework which differentiates several scientific models (e.g., physical, mathematical). We assume that in order to help students understand a physical phenomon these different models have to be explained and integrated in science teaching. That is teachers have to develop a didacticised model of the phenomenon which addresses the various scientific models. Hence, the heuristic framework contains five main parts: (1) the phenomenon, (2) the scientific models necessary for explaining and understanding the phenomenon, (3) the teacher, (4) the didacticised model of the phenomenon and (5) the learner which interacts with both, the phenomenon and the didacticised model.

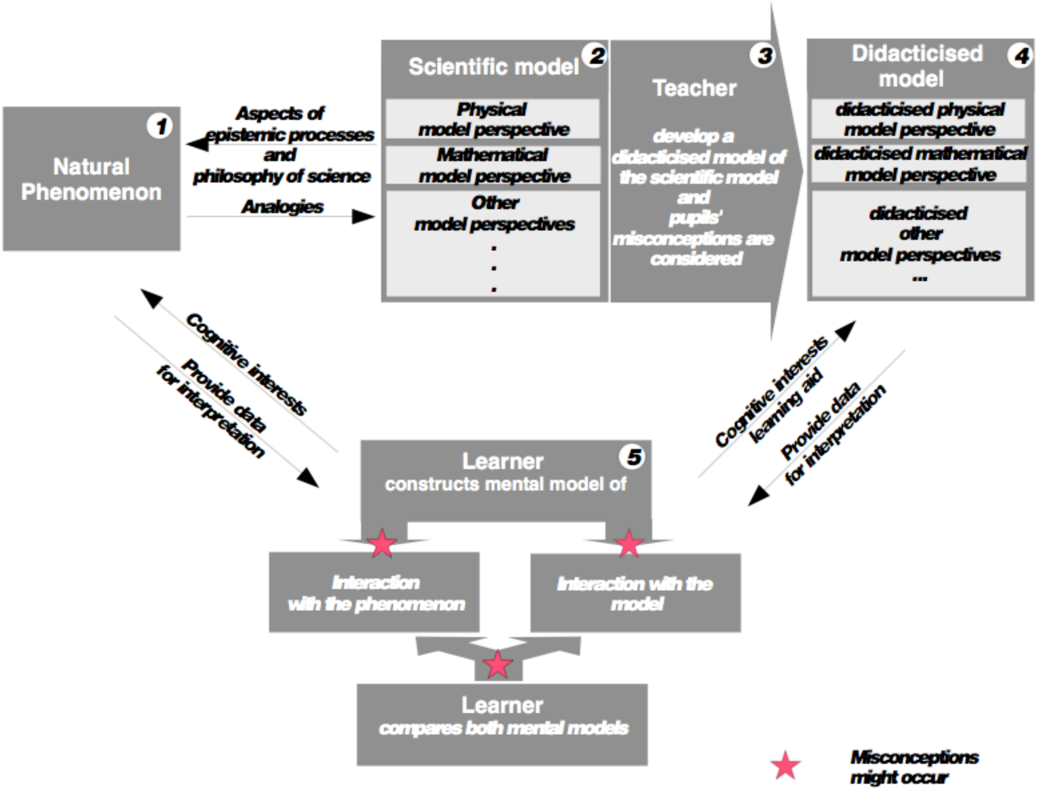


FIGURE 1: Heuristic framework for describing and exploring the process of understanding a physical phenomenon through science instruction (Böhm, Pospiech, Körndle, & Narciss, 2010b, p. 148)

According to this framework, in understanding a physical phenomenon the learner has two models to handle with: (1) the own mental model and (2) the didacticised model of this phenomenon. Hence, in science education a learner is not approaching a physical phenomenon in the way researchers do it. Researchers develop a scientific model, which describes in the phenomenon as detailed as possible. This scientific model is then examined by experiments and by applying it to the real world.

Learners are mostly taught a didacticised model which a teacher developed especially for the teaching and learning process. The challenging task of the learners is to combine these models with their own models of the phenomenon. This process includes a lot of interactions and is the main cause for misunderstandings in the learning process.

The new idea is to divide the scientific model into different parts of the model according to different science areas. Only all model parts together can explain the phenomenon correctly. If only some (not all) model perspectives are used, the learner is not able to understand the phenomenon in a correct way. With this model it is also possible to discuss the role of the mathematical model perspective in the process of understanding natural phenomena. According to Greca and Moreira (2002) the model of a natural phenomenon is divided into two model perspectives: (1) the physical and (2) the mathematical model perspective. For Greca and Moreira the physical model of a theory is described with linguistic symbols and the mathematical model is described with mathematical symbols; understanding physics in school is achieved if it is possible to predict a physical phenomenon from its physical models. To understand complex physical phenomena (like the mirror image) other perspectives besides the physical and mathematical perspective of modelling are necessary for understanding. The three model perspectives are shown in Figure 2.

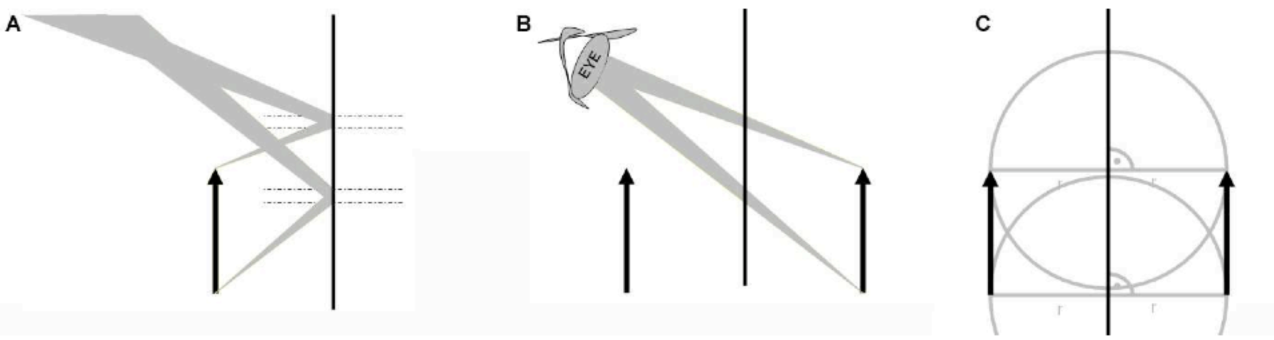


FIGURE 2: (A) physical model (B) "human" model, (C) mathematical model

Understanding mathematical models

The way of thinking in mathematics is totally different to the one in physics. Devlin (1994) describes the core of mathematics in recognizing a pattern. We define abstract objects and look for patterns. At this stage there is no connection to the real world, math is an abstract world. In mathematics abstract definitions and logical consequences of the definition are learned. Mathematicians formulate theorems and find arguments to prove them. Everybody can follow the strict logical rules (if he really wants to) in mathematics. Mathematics is abstract thinking without being linked to the real world.

This, however, does not hold true if mathematics is applied to real situations. This case is described by the modelling circle: (1) mathematical modelling of a physical system, (2) mathematical processing, (3) interpreting the mathematical representation and (4) evaluating the solution by comparing the physical system and the original system.

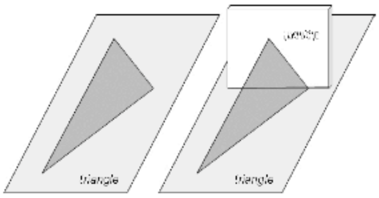
But we can also adapt the heuristic framework for describing and exploring the process of understanding (see Figure 1). Normally abstract problems have to be illustrated by the teacher, so that the learner wants to solve a given problem - the learner should develop a cognitive interest. Depicting a line reflection e.g. is motivated by folding tasks or using a mirror (see Figure 3). An often used mathematical explanation of the mirror symmetry in beginners' lesson is: "A way to think about it is: if the shape of a figure were to be folded in half over the axis, the two halves would be identical: the two halves are each other's mirror image." However, if we think physically – the mirror image comes into being only in the mind of the observer. What does it then mean when we talk about 'each other's mirror image'? Mathematical knowledge itself, however, is not gained from the illustration of mathematical contents, but from a mathematical discourse.

Problems occurring when linking mathematical with physical models and the impact of prior knowledge

Normally learners have mathematics from the beginning of their school career and only some years later science teaching starts. Thus the learners have a lot of mathematical previous knowledge, which they can use in the physics lessons. How can we succeed in cross linking previous knowledge in Mathematics with new ways of physical thinking? The model of a light beam is used e.g. in beginners' classes to represent optical paths. The model is a single light ray – a half-line in geometry. In contrast to mathematics, models in physics are essentially needed to gain physical knowledge about reality.

The first step to solve this was to understand how students handle their previous knowledge in Mathematics when they start with lessons in Physics. We carried out an investigation to elaborate this process (Böhm, Pospiech, Körndle, & Narciss, 2010). During the process of evaluation we found two very interesting examples of fundamental problems by using the axial symmetry for modelling the mirror image: (1) understanding of the virtual image and (2) left and right conversion of the mirror image.

At first mathematics schoolbooks were analysed with respect to their definition of symmetry.



A figure is symmetrical if one half is the mirror image of the other half. That is why the axis of symmetry of a figure is called mirror axis.

FIGURE 3: The definition of axial symmetry by using the mirror image

This example does not only contain abstract mathematical content to define axial symmetry. It is not an exact mathematical definition like: "A plane figure is symmetrical if it has at least one identical image which is mapped on itself by e.g. line reflection and rotation.

In using the mirror image to define axial symmetry we forget that the mirror image is not existing – it is 'only' a stimulus on our retina and its interpretation through the brain. The mirror image is not really existing like the other image in line-reflection. But this is not mentioned in math lessons. Thus physics teachers should not be surprised that students have problems to understand the virtual image when axial symmetry for modelling the mirror image is used – without mentioning the eye as a mapping system.

The second problem is, that not in every case we are able to see the mirror image. In Figure 4 two cases are demonstrated: Only cases comparable to Figure 4, picture A, looking diagonally onto the mirror we can see an image. In case Figure 4, picture B, if we looking perpendicular at the axis of the mirror we can not see an image – but mostly exactly this case is used in physics lesson to model the mirror image by using the former knowledge of the learners from the math classes – the axial symmetry.

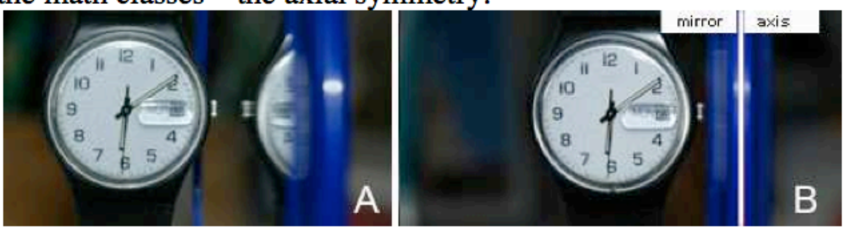


FIGURE 4: The definition of axial symmetry by using the mirror image

It is absolutely strange that for modelling the mirror image the only case of looking on the mirror is taken in which we can not see an image – and this is not told to the learners! When case B in Figure 4 is used pictures like the one in Figure 5 (in physics lessons) can be drawn.

In both drawings the eye does not play any role at all. But in this case, the mirror image does not exist – it only exists in the mathematical case – in line-reflection.

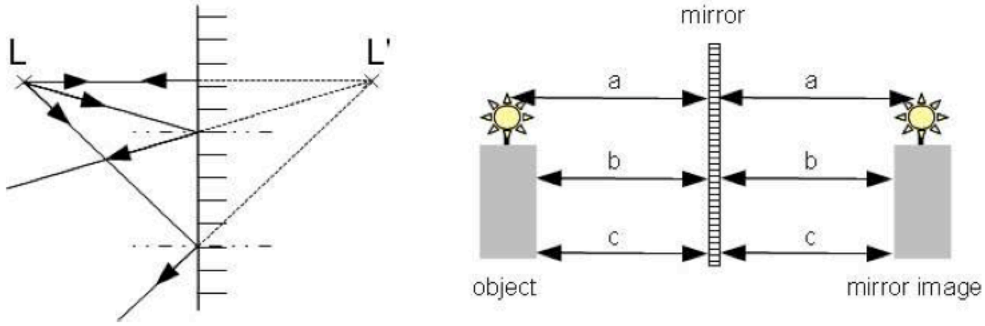


FIGURE 5: Drawings taken from physics text books to explain the mirror image.

If we look perpendicular at the drawing, we see that the left and right sides of the object and the mirror image are changed. But the mirror does not change left and right, the mirror changes front and back! This is easy to accept, if we know that we model the only case when no mirror image can be seen. We have to turn our head and look into the mirror – so the change of left and right becomes a change of front and back.

Conclusions

Every time models are used in physics education. The fact has to be taken in account that not reality itself, only a model of reality is being described. The model should fit the reality very well. We must pay attention to the role the used model has in its original meaning. On the other hand, if we are using examples to illustrate abstract structures in mathematics education we think about the physical understanding of the real phenomenon. In the case of the mirror image mathematical and physical explanations do not go hand in hand. The learner has no chance to construct adequate mental models in each subject.

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Moving from Diagnosis to Intervention in Numeracy – turning mathematical dreams into reality

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Abstract

When students experiences difficulties in mathematics, they require assistance to overcome the misconceptions or inappropriate ways of thinking they have developed. Such difficulties should be identified early and assistance provided to address the needs revealed by a *Diagnosis* of their ways of acting and thinking. *Intervention* can then focus on providing support within a teaching sequence as soon as a difficulty is noted or anticipated as an essential element of ongoing learning. In this way, students be can helped to develop the conceptual understanding and fluent processes needed to acquire and use mathematics. This paper will provide an overview to the intervention process, from determining underlying causes of any difficulties, leading a student to see inadequacies in their ways of proceeding and thus appreciate a need to change, to implementing means of building appropriate ways of thinking, generalising and applying mathematical ideas. A series of numeracy screening tests (Booker 2011) will be presented.

Introduction

> Children should have a robust sense of number … this includes an understanding of place value, meaning for the basic operations, computational facility and a knowledge of how to apply this to problem solving. A thorough understanding of fractions includes being able to locate them on a number line, represent and compare fractions, decimals and per cents, estimate their size and carry out operations confidently and efficiently.
>
> Final Report of the National Mathematics Advisory Board, 2008 pp 17 & 18

In a society "awash in numbers" and "drenched in data" (Orrill 2001), numeracy must be considered an essential goal of education for all (National Numeracy Review p.xi, 2008). It is no longer enough to simply study mathematics; mathematical knowledge needs to be able to be used in an ever-widening range of activities. Indeed, those who lack an ability to think mathematically will be disadvantaged, unable to participate in high-level work and at the mercy of other peoples' interpretation and manipulation of numbers and data. As Steen (1997) predicted "an innumerate citizen today is as vulnerable as the illiterate peasant of Gutenberg's time"; both numeracy and literacy are critical components for living full lives in the 3rd millennium. An ability to solve problems, communicate the results and methods used to obtain solutions, to interpret and use the results of mathematical processes and making sense are all essential to being numerate. However, many students fail to achieve even minimal standards of numeracy (DETYA, 2000; National Numeracy Review, 2008; OECD, 2004) and even those who do frequently say that they are 'no good at maths', feel inadequate and are unable to use the elementary mathematics that they have 'acquired'.

Diagnostic assessment

Assessment is integral to teaching and learning. While it can be used to grade students or measure them against National or International benchmarks, more importantly, well-focused assessment can reveal how students think and provide guidance to plan ongoing teaching. It also supplies information about how students are dealing with the mathematical tasks they are exposed to and feedback on particular programs or learning activities, whether they are suited to the students and content in question or whether they need to be modified to produce the expected learning. At the same time, assessment can provide information to the student, parents or caregiver and other teachers about the student's mathematical capabilities and potential. It may also highlight different outcomes across different groups of students, across different classes and schools when the results are discussed with other teachers.

While it is important to know *what* students know, of even more importance is *how* they know – is their knowledge simply memorised routines, or is there a deep understanding based on well-understood concepts and fluent, meaningful processes applied in appropriate ways? Hence the need for *Diagnostic assessment* that is designed to reveal not only what a student knows but also how they know, not to see what they do not know but to reveal what they need to know. Most importantly, it may reveal gaps in a student's mathematical knowledge – critical ideas may not have become part of a student's way of thinking or may not have been included in the steps used to build up a topic. For example, there are many programs where the aspects of renaming needed for computation and other number processes are not developed as an extension of place value or to provide a complete understanding of larger numbers.

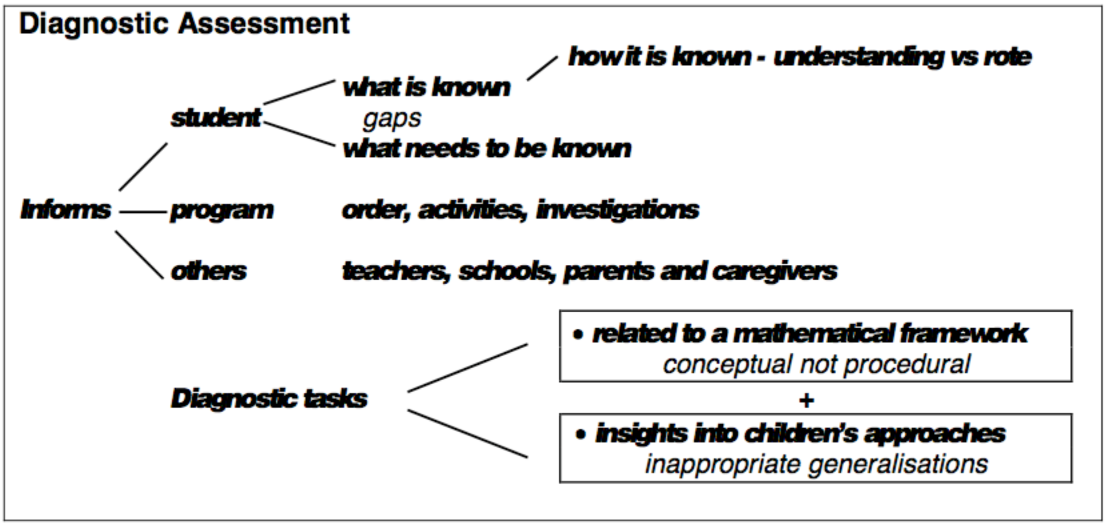


Figure 1: Components of Diagnostic Assessment

In this way, diagnostic assessment highlights the strengths that a leaner brings to a topic and also the weaknesses in their prior knowledge that may cause errors or difficulties.

A diagnostic test is different from other forms of testing. It needs to provide insight into all the steps required to develop facility with a topic and not just measure how well a final outcome or particular points in the development have been attained. However, the sequence of questions posed cannot simply mimic those used in teaching a topic as this may allow a student to refresh their knowledge and mask aspects that have not become central to their thinking. It is often difficulties with steps at the outset of developing facility with a topic that cause deficiencies rather than just an inability to or apply the final process. Misconceptions or errors might also arise from insufficient understanding of mathematical aspects that were developed outside of a particular topic but are essential to the processes. For example, many difficulties with computation are in fact due to underlying aspects of numeration such as zero, place value and renaming.

	87 -40 40	86 +42 148	702 -348 264
Difficulties with	*zero*	*place value*	*renaming & zero*

Similarly, difficulties with measurement may be due to insufficient understanding of spatial properties or of the significance of zero or decimal fractions in the way measuring instruments are applied.

Many errors are based in confusion with and among the rules that have been acquired or else in insufficient understanding of the numbers being worked with. A further source of difficulties are *inappropriate generalisations*, where something that worked in one situation is taken to another setting where the conditions that permitted it no longer apply. For example, *additive thinking* is often used within multiplicative situations for computation, fractions or measurement.

	46 x58 248	46 x58 408	35 x47 425 200 625
Multiplying only	ones by ones, tens by tens		*renaming* difficulties as well

Strengths and weaknesses revealed in diagnostic assessment then need to be stated in terms of the mathematical ideas that underlie them, reasons need to be proposed for why they came about and the underlying causes of the difficulties identified so that appropriate teaching strategies to overcome difficulties can be planned, Close observation is critical to provide insight into the ways of thinking being applied. Mostly it will require a task chosen to elicit the ways in which a student is acting then systematically exploring the possible forms this takes. Consequently, any initial attribution of reasons can only be an assumption, usually based on experience with this type of behaviour. Nonetheless it may not be the actual thinking being evinced and must be treated as only a possibility that needs to be investigated further. This will require further probing to determine what is in fact occurring and then analysis of the likely underlying causes.

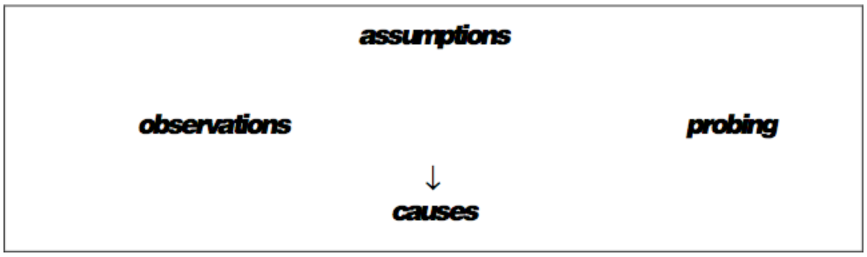


Figure 2: Cycle of Diagnostic assessment

Often several possibilities for an error may need to be considered and the process of probing and observing continued in order to first dismiss one or more before the likely reason or reasons can be determined. In this way, a cycle of observations, assumptions and probing will eventually lead to an understanding of the underlying causes and suggest what is needed to overcome them.

From Diagnosis to Intervention

Diagnostic assessment is critical in planning how to teach or re-teach those essential aspects of mathematics that underpin numeracy. When misconceptions, difficulties and gaps in a student's knowledge have been identified, means to intervene in the learning can be planned and implemented in a manner appropriate to both the learner and the way in which concepts and processes are best established and consolidated. Both what is known and needs to be known must be identified and described in terms of the underlying mathematical ways of thinking rather than rules or procedures that may be followed in a less than meaningful way.

The most effective way of constructing new ways of thinking will mostly begin with the use of materials to draw out the patterns on which the ideas are developed, linking to a language that provides meaning and only moving to the symbolic expressions that express what is happening succinctly when the learner has adopted the way of thinking as his or her own. Simply showing a student what to do using a written procedure is rarely successful in replacing procedures that have led to errors. At best they will try to copy and remember a teacher's approach but it may be that the link to what they do know is not apparent in the purely recorded form. Rather, engaging and different practice activities, often in the form of games in which learners willingly participate, are an essential part of learning to bring a concept to the forefront of a learner's mind and enable a process to become fluent.

In this way, intervention can build from the understandings that are essential for the development of further concepts and processes and provide the links needed to extend the ideas to enable applications to new situations, means of solving a range of problems and make possible extensions to further mathematics. This process can be summarised:

The process of intervention

1. **the identification of** *understandings and errors* **and the description of them in terms of the underlying mathematical concepts and processes**
2. **uncovering** *sources of difficulties* **– not only inappropriate thinking but also the degree of understanding of why processes and responses are correct**
3. **revealing** *inadequacies in thinking* **to a child in order to build an appreciation of a need for change**
4. **the implementation of means of** *constructing or re-constructing* **appropriate ways of thinking**
5. *practice* **that is focused and motivating to allow a way of thinking to become secure and provide a basis for generalisation to more complex problems and applications or to the development of further mathematics**

Figure 3: The process of intervention

Case study

A student was observed to have difficulties with decimal fractions involving tenths:

Write the numbers that are 3 tenths more:

8.4	*11.4*	**9.7**	*12.7*	**6**	*9*	**0.8**	*3.8*	**2.9**	*5.9*

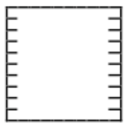
In this case the student has read the question as 'write the numbers that are 3 more' either because they have simply focused on the number 3 and used the cue word *more* when reading the question or because *place value* is not secure for decimal fractions. For instance, many students say '8 point 4' for the number 8.4 without any emphasis given to the place values involved. 8 and 4 are simply seen as digits separated by '.' rather than 8 ones 4 tenths which is then read as '8 and 4 tenths. In either case, no meaning for the '3 tenths' requested in the question would have been present.

To increase a number by 3 tenths means that the digit in the tenths place is increased by 3 which may also involve renaming 10 tenths as 1 one:

8.4	__*8.7*__	**9.7**	_*10*__	**6**	__*6.3*__	**0.8**	_*1.1*_	**2.9**	_*3.2*__

In order to provide intervention on the underlying difficulties the diagnosis has revealed, the fraction concept and place value for decimal fractions need to be built up. This requires teaching to

- revisit the model for fractions using rectangles to link model and names based on ordinal numbers
- use a square with the beginning lines and have the student connect the lines to see there are 10 equal parts - tenths



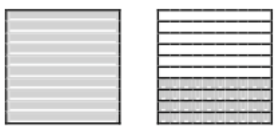
10 equal parts - tenths

- name fractions with ones and tenths

2 ones and 7 tenths – 2 and 7 tenths

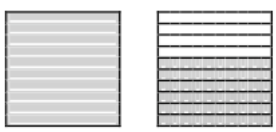
ones	***tenths***	
2	***7***	***2.7***

- shade ones and tenths diagrams to match symbols and read names



1 and 4 tenths 1.4

- ***shade decimal fractions that are 2 tenths more and name them***



1 and 6 tenths 1.6

- write the symbols for the initial fractions and the fractions that are 2 tenths more

2 tenths more than 1.4 is 1.6

Note that the example chosen to (re-)establish the way of thinking is different to those that were completed incorrectly so as to focus on the underlying ideas rather than simply be seen to correct an example that was answered incorrectly. When several examples like these have been examined and the student has developed an understanding of what is needed, then the fractions originally answered can be looked at again to see that the student can not only obtain correct results but is able to see what was done incorrectly when he was first asked these questions.

Conclusion

Building numeracy in all students is a critical aspect of contemporary schooling. Understanding how concepts and processes are constructed and connected provides a basis for overcoming misconceptions and inappropriate ways of thinking that may have developed (Hiebert & Grouws, 2007, p.391). Appropriate intervention programs can then be planned and implemented to build students' competence and confidence with fundamental mathematical ideas. In this way, students will be prepared to engage with further mathematical ideas and be inclined to use their knowledge of mathematics in the many everyday and work contexts where reasoning and sense making will be required.

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PROFESSIONAL LEARNING COMMUNITIES AND TEACHER CHANGE

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Professional learning communities are increasingly seen as a sustainable and generative method of professional development in mathematics education. However links between the actual work of the community and changes in teachers' practices are rarely made. In this paper I examine part of a journey of one teacher in a professional development project that focused on teachers' learning to engage with learner errors in their teaching. I show how her classroom practice and the conversations in the community informed each other, and supported her to change her teaching.

Professional Learning Communities

Professional learning communities are increasingly seen as a sustainable and generative method of professional development in mathematics education (Clark & Borko, 2004; Jaworski, 2008). Professional learning communities support teachers to use their experience, evidence from their classrooms, their own and their colleagues' insights, and knowledge from research to decide what they need to learn and how they can learn it. Teachers monitor their own and their learners' learning in ongoing ways (Boudett, City, & Murnane, 2008), thus engaging in deepening cycles of analysis, reflection and action, interrogating current practice and exploring alternatives. The collective nature of professional learning communities is important - teachers collaborate and learn together about how their learners' needs can influence and improve their practice.

There are strong theoretical arguments for professional learning communities and some evidence that they do produce improved teaching practices and learner achievement (Boaler & Staples, 2008; Katz & Earl, 2010). There has been extensive research on how successful communities work and the difficulties in sustaining them (McLaughlin & Talbert, 2001). Less research has been done on explicitly connecting the actual work of the professional learning community to shifts in teachers practices, or as Kazemi and Hubbard (2008) argue, how teachers' development in both the professional community and the classroom co-evolve. They suggest (2008, p.453):

> One methodological entry point … is for researchers to identify a practice that is the focus of the PD effort, track how teachers reason and work with that practice as it travels to their classrooms, and track how they reason with that practice when they return to PD

Professional learning communities typically investigate artefacts of practice, such as tasks, student work, lesson plans and classroom teaching (Clark & Borko, 2004). However, we cannot assume that what teachers learn in the community, outside of the classroom, travels intact to the classroom and vice versa (Kazemi & Hubbard, 2008). Different aspects of the learning in the community will be salient in different ways for different teachers and links between teachers' experiences in the community and their developing practices need to be established.

The focus of our professional development project was learning to work with learner errors. Our position on errors and the main learning point of the program is that errors are evidence of reasonable and interesting mathematical thinking on the part of learner (Borasi, 1994; Prediger, 2010) and errors do not signal a deficit in teaching or learning, in fact they are a normal part of learning mathematics. At the same time, teachers can learn to work with learner errors in better ways. Many teachers tend to work away from errors: avoiding them through narrowing tasks; pretending to work with them through the use of leading questions; or re-teaching concepts with the assumption that errors mean that learners haven't learned "properly" the first time. We work with the notion that teachers can embrace learner errors in ways that can advance learners' mathematical thinking by understanding the validity in the reasoning behind the errors and the source of the errors as over-generalisations of previously successful ideas (Smith, DiSessa, & Roschelle, 1993).

In this paper I examine part of a journey of one teacher in a professional development project that focused on teachers' learning to engage with learner errors. I show how analyses of the teacher's classroom practice in the professional learning community informed her subsequent classroom practice.

The empirical site and data analysis

The data for this paper comes from a three-year professional development program, located in Johannesburg, South Africa. The project engaged teachers in a number of activities including test and curriculum analyses, interviews with learners, readings and discussions on learner errors in particular topics, lesson planning and lesson reflections. All of the activities focused on building teachers' understandings of and engagement with learner errors. The teachers worked in small grade-level groups of 3-4 teachers, with a group leader who was member of staff or post-graduate student at our university. At particular times in the programme, the small groups presented their ideas to a larger group that was facilitated by one of the project leaders, the author of this paper. In this paper I focus on the learning of one Grade 8 teacher, Andrea[1], through the first two years of the project, by analysing her teaching before the project began and through two cycles of the project.

In each of the two cycles, each small group planned and taught lessons to develop learners' understanding of the relational meaning of the equal sign or learners' use of visuals in solving problems. They had read and discussed papers on these topics and analysed tests results and the curriculum in these areas. One teacher from each group volunteered to teach the lessons and the teaching was videotaped. The small groups discussed the videotapes and then the teacher presented two episodes to the larger group: one where the small group thought the teacher had dealt well with learner errors and one where the small group thought the teacher had not dealt so well with learner errors. The presentation included a brief background to the episodes to contextualise them within the set of lessons and a justification for why the group had chosen each episode. Each group gave a 10-minute presentation and about 50 minutes were given for discussion where other groups could comment, question, challenge and give feedback.

The main data for this paper are the presentations on the two concepts and the teacher's lessons – two lessons before the project started, two on the equal sign and one on problem solving. Secondary data are the lesson plans, interviews with the teachers, and the teacher's written reflections after the presentations. Analyses of the lessons were done with the Mathematical Quality of Instruction (MQI Plus) instrument (Learning Mathematics for Teaching Project, 2010). Coding is done according to a set of categories in eight-minute episodes, on a scale of 1 to 3. A second, more qualitative analysis of the lessons listed all the learner errors made in the lessons and the teacher's responses to these. These analyses showed changes in the teaching and also confirmed that the episodes chosen for presentation to the larger group did in fact illuminate important issues relating to the teacher's engagement with learner errors that came up throughout the lessons. Detailed summaries of the presentations were analysed by looking for how the teachers accounted for their actions to each other and what counted for them as important to speak about at each point. This meant looking for who said what and when, presences and absences in what they said, and tracing comments back to previous comments and forward to subsequent ones.

[1] All names in the paper are pseudonyms.

Changes in teaching practice

The two MQI Plus categories that are most relevant to teacher learning in our project are: working with students and mathematics and student participation in meaning making and reasoning. Each main category has subcategories, the first has remediation of student errors and difficulties; and responding to student mathematical productions, while the second has students provide explanations; student mathematical questioning and reasoning; and enacted task cognitive demand. Table 1 below gives the number of episodes in each of the three sets of lessons that were coded 1, 2 or 3 for each subcategory.

	Prior to project 11 episodes coded			Equal sign 12 episodes coded			Problem solving 7 episodes coded		
	1	2	3	1	2	3	1	2	3
remediation of student errors and difficulties	11			12			5	2	
responding to student mathematical productions	8	3		11	1		3	2	2
students provide explanations	11			12			4	3	
student mathematical questioning	11			12			6		1
enacted task cognitive demand	9	2		12			6		1

Table 1: Shifts in Andrea's practice

Table 1 shows that Andrea's practice remained unchanged for the most part in the first set of lessons that she taught while in the project (equal sign) but shifted significantly in the second set of lessons (problem solving). The qualitative analysis of the shift shows that the teacher's responses to learner errors in the first two sets of lessons were predominantly leading questions, while in the third she moved towards asking probing questions.

Conversations in the professional community

The shifts in Andrea's practice can be linked to conversations about her practice in the two communities, the small group and the larger group. Her group chose the following episode to present on the equal sign. There were two expressions written on the board:

$$(+1) + (+1) + (+1) + (-1) + (-1) + (-1) = 0$$

$$(+1) + (+1) + (+1) + (+1) + (+1) + (+1) + (-1) + (-1) + (-1) + (-1) + (-1) + (-1) = 0$$

Andrea had asked the learners whether the left hand sides of both equations were equal to each other. The following interaction with one learner, Carol, occurred:

Andrea: Put up your hands if you say yes…[a number of learners put up their hands] put up your hands if you say no…[only Carol puts up her hand] Carol, very brave my darling well done, why do you say no

Carol: Mam, I said no because um, the numbers are different like there three negative ones there and there are six negative ones there, and there there are three negative ones there, I mean three positive ones, and there there are six positive ones, mam

Andrea:	Well, except that here I've got six negative ones and six positive ones and they equal to zero that's one number
Carol:	Yes mam
Andrea:	So does it matter how many numbers we've got on each side
Carol:	No
Andrea:	What matters
Carol:	Um, if they equal to zero
Andrea:	If they equal to zero, so is this top line gonna equal to zero
Carol:	Yes mam [nods head]
Andrea:	In this bottom line here, equal to zero
Carol:	Yes mam [nods head]
Andrea:	So we can say they are equal to each other

Initially the small group had thought that Andrea had dealt well with Carol's error but after some discussion with the group leader they came to see that she had led Carol to the answer without asking her to explain her thinking and there was no evidence that Carol had in fact understood the difference between what the expressions looked like and their values. The group named what Andrea had done as using "leading questions".

Two key questions structured the conversation in the larger group about this episode. The first question was asked about all the errors that teachers presented – what might the learner be thinking in order to make the error. The second question related specifically to Andrea's teaching - what is the role of leading questions in engaging with errors. After some discussion the teachers reached consensus about Carol's thinking, expressed here in Renee's words:

> I would imagine what was going through her mind was that she's thinking that equals means the same and that in some contexts that is correct but we we've got to be very careful that they've got to understand the difference between quantity and value, when we say equal we talking about the value of something.

Renee pinpointed both the validity in Carol's thinking and where her error lay. She showed that the error is sensible given the meaning of "the same" in some contexts but that a full understanding of equivalence required a different meaning – in relation to the value of the expressions rather than the quantity of terms. Andrea took the discussion further saying:

> The misconceptions are so deeply ingrained in them that actually they take them for granted and they don't think about, and it doesn't even cross their minds to change their thinking. I mean some of the stuff that I said, its pretty deeply ingrained in me as well. If I listen to some of the language that I use, I use things like the answer and all of that kind of stuff

In relation to the leading questions, Andrea argued that:

> I had not allowed her to let me know that she had fully grasped the concept, I just kind of barked out the orders … so I don't actually know that she can actually do these questions by herself now

So Andrea articulated a direct link between her practice and her engagement with learner errors, indicating her group's understanding of the teacher's role in working with learner errors. The notion of "leading questions" became a depiction of practice (Kazemi & Hubbard, 2008) that was spoken about in a number of ways throughout the program. In this session a number of teachers contested the idea. Tawana argued:

> I don't see why her questions are leading towards a solution. She was using a model previously and they've now left the model behind and are looking at the numbers and equating things, so if someone had a misconception, it was important for them to refer back to something that they'd agreed upon

Tawana's argument was that Andrea had explained the ideas previously, using conceptual resources, the model, and that it was appropriate for her to expect that this had been understood and she could therefore use leading questions to remind learners of what they had learned previously. Even though her leading questions were supported by other teachers, Andrea stood her ground and argued that she could act differently by asking more questions such as "why do you think that", "how do you come to that", take me through your thinking" to support learners to articulate their ideas. Such questions are referred to in the literature as probing questions or press questions (Kazemi & Stipek, 2001). Many of the teachers could not distinguish the difference between leading and probing questions at this point. Subsequent discussions articulated more clearly the differences between leading and probing questions. Andrea's word "ingrained" also became a shared term in the community. As the facilitator, I was able to pick up on the link that Andrea made, talking about teachers' "ingrained" practices such as leading questions, and how those are as difficult to shift as learners' misconceptions.

In Andrea's reflection she wrote that she had learned that leading questions were not useful in eliciting and engaging learner errors and that she was going to try to use them less in her teaching. Table 1 shows that she was more successful in engaging with learners' errors and reasoning in the next set of lessons. Her group chose one of a number of possible episodes, which shows some of her success. An analogue clock drawn with the hands at three o'clock was on the board and Andrea asked the question: how many ninety-degree angles will there be in twelve hours? A number of learners made conjectures. In the chosen episode, Rabia conjectured that "every hour can have a ninety degree angle" and when Andrea asked for an example she said "like um two o'clock and eleven o'clock". Millaine challenged Rabia's claim saying, "at two o'clock and eleven o'clock it won't necessarily be a right angle" which led Rabia to change her statement, saying that she meant five to two (i.e. the hands would be on eleven and two). Leanne listened to this interchange and then offered the possibility that "every right angle has two hours in between" and when Andrea asked her to elaborate she used the example of three o'clock on the board and Rabia's example to explain.

Andrea's responses were completely different from those in her interaction with Carol in the first episode. She asked for examples and elaborations and as she said in her presentation:

> I didn't refuse any of their answers, I put them on the board and I kind of left them there and I didn't say no you're wrong or wow you're amazing you're right, I just kind of put them there and we moved on, and talked about it a little bit and then put more up, and then some kids would say but these are wrong because of the following reasons, so the learners kind of deepened their understanding through their own questioning

This was not only the case in this one episode but in many others, and her teaching supported the learners to make key breakthroughs at later points – first that both hands move so that, for example, at five to two the hour hand will not be exactly on two, and second to get close to an answer, some arguing for 22 and some arguing for 24 and then deciding how to work out the answer. Andrea did not only work as she described above, she also made inputs where necessary, and asked some "leading questions" where appropriate, for example, she used leading questions to make the point that a ninety degree angle can be in a number of orientations. Her more flexible use of different kinds of questions supported me as the facilitator to make links between the different kinds of questions, to further support her learning and the learning of the other teachers.

Conclusions

The observed quantitative and qualitative shifts in Andrea's teaching can be linked to the discussion of her teaching in the small and larger group. The notion of "leading" questions came up in the smaller group and was further discussed in the larger group. As a "depiction of practice" these structured Andrea's work and the work of some of the other teachers who began to distinguish between these kinds of questions, try them out in their classrooms and bring them back to the community for discussion.

The project's methodological entry point (Kazemi & Hubbard, 2008), learner errors, supported this shift in Andrea's teaching. The discussions of her teaching of the equal sign pinpointed the validity of Carol's thinking that Andrea missed because of her use of leading questions. As she shifted her use of leading questions in the second set of teaching she also shifted her view of learner errors to conjectures (Borasi, 1994) and was happy to leave errors on the board as inputs to the discussion, to be taken up by other learners. Andrea's reflections on her own language and practices as "ingrained" also provided a point of focus for discussions about how to link new teaching practices to the teachers' developing understanding of errors.

So the two contexts, the community and the classroom came together and interacted with each other to produce learning in both contexts. Andrea was able to talk about her practice with others, build on or challenge their thinking, accept challenges from them and take forward what she learned into her classroom. At the same time, her classroom practice created possibilities for further discussion and for other teachers' learning.

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Numbers: a dream or reality?
A return to objects in number learning

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Abstract
The complexity of mathematical concepts and practice may easily lead to teaching practice that results in mathematical objects being seen as an abstract dream. This paper explores ways that mathematical objects may be seen as real, objective entities, while still acknowledging this complexity. It develops a systemic view of mathematical objects as mental objects constituted in the intersection of three major systems of the child's experience: the physical; social and technical (mathematical) systems. This development is fleshed out in examples relating to the teaching and learning of both whole numbers and rational numbers.

Introduction
A child's learning of numbers at school involves the mastering of a large number of technical details. Details that include different mathematical operations, different representations and models of numbers and operations, and different relationships between numbers, operations and representations. Together these form a complex technical system of facts and skills. In learning this system, children often get lost among the myriad of details, and come to see numbers as an abstract dream. Particularly when most of these details are expressed in terms of rules and symbols.
For this reason, is deemed important to teach in ways that develop conceptual understanding (Hiebert, 1986; Kilpatrick, Swafford and Findell, 2001), to lend meaning to what is learned and link these technical details into a web of meaningful relations. According to Kilpatrick et. al. (2001) conceptual understanding

> refers to an integrated and functional grasp of mathematical ideas. Students with conceptual understanding know more than isolated facts and methods. They understand why a mathematical idea is important and the kinds of contexts in which is it useful. They have organized their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know. (Page 118)

Yet, even with this, many children do not become proficient with this complex system and numbers remain a dream.
The Oxford English Dictionary (Oxford Dictionaries, 2008), sees a concept is an "idea of a class of objects". The importance of the mathematical idea is clearly evident in the above quote, but no mention is made of the class of objects to which these ideas refer. Without a referent, children may experience these ideas as meaningless, or imaginary. In this case, conceptual understanding may itself become meaningless, abstract and confusing. It is the reference object that provides a point of focus for the concept and grounds it in reality. This paper investigates the possibility of reinvigorating the status of mathematical objects, as objective entities that form the referents of mathematical concepts. Particularly in the case of school mathematics, when the child is still developing the capacity for mathematical abstraction at the level needed to enable effective engagement with the formal and abstract systems of higher mathematics . It proposes a framework for understanding mathematical objects as mental, or psychological objects that, for functional and structural reasons may transcend the boundaries of individual subjectivity and take on the status of objective entities.

This understanding will exemplified through examples from the learning of whole numbers and rational numbers.
This object framework arose as part of a research project into rational number learning and the examples from rational number learning will be drawn from the project. But the focus of this paper is the object framework — detailed research results will be reported elsewhere.

Informing Frameworks

Psychological theory of concepts: Prototype vs Definition

Gabora, Rosch and Aerts (2008) review the development of psychological theories of concepts since 1950. The classical view, is that a concept denotes a category and is mentally represented by specifying common attributes of the members of the category. Research into the psychological structure of concepts demonstrated that whether an object is seen as a member of a category is not generally seen in all or nothing terms, but rather in terms of graded degrees of membership and that these degrees of membership were strongly context dependent. This gave rise to the view that concepts are represented as rich, experiential prototypes, that are formed first by identifying and internalizing 'basic level objects' and then expanded through relations to more specific (subordinate) and more general (superordinate) objects. Conceptual prototypes are highly flexible and vary according to context and the manner in which the person wishes to interact with the context.

Cognitive Science: Distributed vs discrete models of thinking

Two major models of thinking and currently prevalent in cognitive science. The first (Newell, 1990) sees thinking systems as physical symbol systems and thinking as the manipulation of symbols. According to this view, concepts comprise of discrete symbols, that are linked to form complex networks. Thinking involves operating on these discrete symbols according to well defined rules. The power of this model lies in the possibility of identifying a concept by means of a single symbol and the control and insight this provides into manipulations and operations on the symbol/concept. The second approach (Rumelhart McClelland and the PDP Research Group, 1986) models thinking systems as neural networks and thinking as the spreading of patterns of activation in neural networks. A concept would thus relate to a pattern of activation that is distributed throughout the network. The mental effects of this concept would occur as a result of links between nodes activated in this pattern and other nodes in the network, whose activation is either stimulated or inhibited by these links. This model enables flexible and robust recognition of concepts, and an enhanced ability to coordinate thinking with detailed structural features of the environment and the interaction between person and environment.

Conceptual grounding: Everyday vs Scientific Concepts

Vygotsky (1987) identified two different types of concepts: everyday and scientific concepts. Everyday concepts arise spontaneously in everyday experience. They are closely linked to this specifically experience and attain personal meaning through this grounding. On the other hand, scientific concepts are general, abstract and organized to form a conceptual system. These concepts do not arise spontaneously, but are learned in response to some form of implicit, or explicit teaching. The generality and organization of scientific concepts gives them power and flexibility. But without appropriate experience to relate them to meaningful everyday concepts, this generality would be empty of meaning.
Building on Vygotsky's work, three different perspectives on everyday experience were developed by Leontiev (1978). Two of which yield valuable insight for this work. **Activities** are active social experiences that are given significance because they enables the group (or individual) to satisfy some motivating need or desire. **Actions** involve the controlled

performance of an instrumental task in order to attain some goal. Attaining the goal requires one to understand the structure of both the situation and the task.

A number of recent issues in cognitive science relate to the interaction between everyday and scientific knowledge. For instance, Barsalou (2008) argues that semantic thinking is not merely amodal symbol processing, but is instead grounded in experience — occurring through the re-enactment of perceptual experience. Situated cognition (Clarke, 1997) also makes a strong case for the importance of the environment and one's interaction with the environment in structuring our thinking and understanding.

What constitutes a mental, mathematical object?

This paper draws on these different perspectives and formulates a mental object as an system of interrelated elements that incorporate the active experience of the individual, in both physical and social spheres. The system is given an characteristic identity through a specific way of viewing it as a whole. Examples of these elements will be given for two different number objects in a child's developing understanding of numbers: whole, or counting numbers; and rational numbers. To simplify our reference, we will abuse our notation and refer to rational numbers as 'fractions'.

1. Grounding instances

A grounding instance is an element of the person's experience that arises when engaging in personally significant activities, in the sense of Leontiev (1978). Similar instantiating activities give rise to the basic object of a prototype concept, which is grounded in the experiential context provided by the activities. A grounding instance emerges through repeated and varied participation in a particular activity. And an object will often emerge from multiple grounding instances in a number of different activities.

A number of different grounding instances can be identified for counting numbers:

- When a young child engages in the activity of counting in order to master the basic number sequence. Here the significance arises from the act of achieving competent social participation.
- When a child wishes to precisely describe the size of a collection of discrete objects, or the number of repeats of an action or event, and generates a number to do this by using 1-1 correspondence between counting sequence and object, action or event.
- When a child uses counting and the resulting cardinal number, to compare the sizes of two collections, or two sequences of events or actions.
- When a child accesses the passing of time, or forms and describes rhythms using counting and cardinal numbers or number patterns.

In a similar fashion, rational numbers arise from a number of grounding instances:

- Fair sharing, where a child wishes to share something (that can be broken up to share) in a fair and equal manner between a number of people. Fractions arise from a consideration of how much of the full amount each person received, or how much of a whole object each person received.
- Allocation, where a child wishes to subdivide and allocate something to a number of people, based on a reasonable, but unequal allocation principle. For example, when sharing a chocolate bar between three boys who were first, second and third in a race.
- Packing or filling involves the packing of a number of objects into containers which each have the same capacity. Here fractions arise from comparisons between the total number of objects, the number in a container and the number of containers packed.
- Exchanging occurs commonly in shopping, or swopping. We come into contact with fractions when we compare the quantities of the two things being exchanged.
- Converting or constructing involves starting with some raw material and then using it

to make something different. Fractions again are useful for comparing the quantities of input and output materials.

- Measuring may relate to selecting a desired amount of material, or to determining the amount of material present. Fractions may arise through comparing the total, the unit and the measurement, but also through the choice of fitting subdivisions of the unit and the effect of different subdivisions on the final value for the measurement.

2. *Structural experience in the interaction*

This relates to how we interact with the object in each instantiation, thus developing a more detailed knowledge of the object and our interactions with it. As we repeatedly engage in the grounding activities, certain regularities in the way the object emerges and in the way we interact with it become evident. Our knowledge of these regularities may be explicit, or we may deliberately focus on them (often through the mediation of others) and so develop an explicit understanding of this structure. Regularities which can be identified as units of structure or interaction, occurring across multiple grounding instances, become incorporated as basic structural elements of the object. Such unitary elements include:

a. Invariant elements in our interaction with the object, that occur in a number of instantiations. These provide different perspectives of the object in different contexts.

b. Relational elements that interrelate the different invariant elements and perspectives to form a coherent, integrated structure.

Some examples of structural elements of counting numbers are:

- The counting sequence itself.
- Setting up a 1-1 correspondence between numbers and objects, or actions..
- Using the last number in a count as a cardinal number to describe the ‘size’ of what was counted (seen in counting responses such as: “One, two, three. Three balls.”)
- The invariance of addition and subtraction bonds when combining, separating and comparing collections, and when calculating results both mentally and in writing.

Some structural elements that may be identified for fractions are:

- Equal subdivision of a whole into a number of equal parts.
- Forming composite groups containing a number of discrete items.
- Constructing or identifying units. Units can be either wholes (1 pie), parts (½ a pie) or composite (a 2-pie group, one ¾ pie).
- Linking chosen units of two quantities and comparing linked units, or quantities. For example. If 7 apples are needed to bake 2 pies and we have 21 apples, how many pies can we make? Forming linked groups of 7 apples and 2 pies, we get:

$$\begin{array}{rcr} 7 & \longrightarrow & 2 \\ 7 & \longrightarrow & 2 \\ \underline{7} & \longrightarrow & \underline{2} \\ 21 & \longrightarrow & 6 \end{array}$$

So we can make 6 pies. The comparison between the linked units results in the ratios 7:2, or 21:6; and fractions 7/2, 2/7, 21/6 and 6/21.

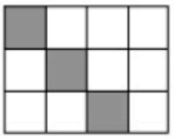
3. *Presentational and representational tools*

These are structured tools, signs and symbols that we use to physically and mentally present the object to ourselves and to represent the object for physical and mental analysis and manipulation. They include representational drawings, schematic diagrams, graphical models, mathematical symbols, and language, words and text. The majority of these representational tools are not developed by the individual, but are pre-existent in the community and are socially presented (in physical or verbal form) and aligned with the given practice. In the process of this social mediation, the person internalizes the tool and so constructs

corresponding mental symbols. Whether these symbols are seen as discrete and fundamental entities that can be simply 'linked in' by the individual, (as in the physical symbol system model) or as more complex constructions (for the distributed model), organized systems of symbols are important to both structure and enable our reflective capacity.
Some common representational tools for counting numbers are:

- Numerical representations using the base 10 place value system.
- The mathematical symbols for the four basic operations.
- The number line.
- Drawn collections of objects (or schematic dots), grouped by line borders.
- The standard symbolic formats for vertical performance of the four basic operations.

Representational tools for fractions include diagrams such as:



Descriptions such as: "Stretch by 3, shrink by 7"; and "Three out of 12"
and symbols such as: 3/7, 4:9 and 3.5

4. Constitutive / characteristic perspective

This element is the stable perception of an object which may be reliably distinguished within each instantiating activity; conforms to the identified structural elements; and is fittingly presented and represented by the learned cognitive tools. This element serves to objectify the concept by constitute it as an object in the person's experience. Note that this is not a full description of the properties of the object, or a definition. Rather, it is a way of looking at things that allows the object to come to the fore as a coherent, discernable, entity. This element provides a strong, unifying focus, that enables the person to transcend the view of the interrelated elements as merely a conceptual system, and instead see them as a conception of a definite, identifiable object. In the case of numbers, a relational object, such as a father.

Because of its unifying and constituting function, there is only a single perspective for each object. For counting numbers, a strong candidate for this perspective would be:

- A number as a completed count.

For rational numbers, this perspective is no longer sufficient and a better candidate would be:

- A rational number as a rational comparison of two quantities.

Both of these perspectives are evident in the examples given above.

Mental objects and objectivity

The elements discussed above, combine to give weight to a person's subjective experience that such a mental object has an objective status. This final section tentatively discusses possible relationships between this subjective status as a psychological object and the objective experience of the person.

Experience of the impersonal other, and objectivity

Here we take the impersonal other to refer to both the physical world, and also to interactions with other people where interpersonal relationships do not co-constitute the interaction. For example, a predominantly instrumental interchange, such as renewing a motor vehicle license at the Traffic Department, will be considered as experience of the impersonal other. An important aspect of our experience of the impersonal other in grounded contexts, is that our subjective experience is irrelevant to the response of the other — we are only able to influence the response through our instrumental actions in the interchange. This separation between our self and the other, warrants the experience of objectivity in these interactions. For it is more fitting with our experience to respond to this entity as if it were separate and objective, than as if it were connected and subjective.

For example, consider the activity of sharing a chocolate bar fairly between two people. Equivalent shares will only occur if the measuring and cutting are precisely done. The precision required to form equal halves thus becomes an important structural consideration in a child's developing understanding of halves and general fractions. And this precision is necessary for a fair sharing, irrespective of the child's subjective experience.

Social Presentations and Objectivity

It is important to note that experiences such as the above will often be guided and mediated by relating to the personal other in community — where interpersonal relationships do co-constitute the interaction. Social objectivity may be seen as arising through relating to the personal other individually, in small groups or in larger communities. These interactions mediate the presentations and representations that we develop for the object, our structuring of the object and our grounding experience of the object. In this way, conceptual metaphors, such as those described by Nunez (2006), arise. These are socially presented and socially shared perspectives on grounding experience, that bring out or preserve a certain structure. Objectivity is warranted through the achievement of a socially consensual perspective.

The two forms of objectivity balance each other. As demonstrated by Nunez, different communities may hold different perspectives on a single concept, resulting from incompatible metaphors that preserve different structural aspects of the object. A perspective that incorporates both aspects of the object will then unify these incompatible metaphors.

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CORRELATED SCIENCE AND MATHEMATICS: A NEW MODEL OF PROFESSIONAL DEVELOPMENT FOR TEACHERS

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Abstract

This article describes a new professional development (PD) model of linking science and mathematics instruction called *Correlated Science and Mathematics* (*CSM*), which enables teachers to integrate science and mathematics curriculum. Integration usually implies that science teachers use mathematics as a tool or mathematics teachers use a science as an application of a mathematics concept. Although both the science and mathematics standards recommend integration, unless there are effective PD models that will facilitate it, broad-scale integration is not likely to occur. The *CSM* model of PD links science and mathematics more thoroughly and uniquely than the traditional integration model. Each discipline is taught with seven fundamental goals: (a) teaching for conceptual understanding, (b) using each discipline's proper language, (c) using standards-based learning objectives, (d) identifying the natural links between the disciplines, (e) identifying language that is confusing to students, (f) identifying the parallel ideas between the disciplines when possible, and (g) using a 5E (Biological Science Curriculum Study [BSCS], 1989) inquiry format for science and mathematics when appropriate. Use of the *CSM* PD model resulted in grades 5-8 science and mathematics teachers planning and teaching integrated lessons and using the proper language of each discipline.

Introduction

Research has successfully demonstrated that teachers make a significant difference in how well students learn (Hiebert & Grouws, 2007; Loucks-Horsley, Stiles, Mundry, Love, & Hewson, 2010). Continued professional development (PD) provides teachers with an avenue to better understand the content of their discipline, as well as improve their pedagogical strategies that will enhance the students' conceptual understanding of the content. The literature on the importance of PD in improving the quality of teaching is substantial (Darling-Hammond, 1998; Office of Educational Research and Improvement, 1998; Porter, Grant, Desimone, Yoon, & Birman, 2000; Wenglinsky, 2000). Reform-based PD providing opportunities for active learning, attending as a group, study groups, mentoring relationships, and teacher networks have a positive impact on teaching practice (Cohen & Hill, 1998; Desimone, Porter, Garet, Suk Yoon, & Birman, 2002; Sowder, 2007).

Connecting Science and Mathematics

Both the National Council of Teachers of Mathematics (NCTM) and the National Academy of Sciences (NAS) advocate linking the disciplines of science and mathematics. NCTM (2000) states, "The process and content of science can inspire an approach to solving problems that applies to the study of mathematics" (p. 66). Likewise, NAS (1996) acknowledges the indispensable role that mathematics plays in scientific inquiry when they state that "mathematics is essential to asking and answering questions about the natural world" in scientific inquiry (p. 148). The integration of science and mathematics is one way to improve teacher achievement in each discipline. Basista, Tomlin, Pennington, and Pugh (2001) report significant gains in the participating teachers' understanding of content and confidence to implement integrated science and mathematics. Similarly, Basista and Matthews (2002) report increased understanding of content and confidence. Additionally, they reported increased pedagogical knowledge of teachers, increased administrator awareness of the science and mathematics standards, and increased administrative support for teachers.

The *Correlated Science and Mathematics* Model

Integrating science and mathematics traditionally means linking the two disciplines in some manner (Davidson, Miller & Metheny, 1995). Science integrates mathematics by using mathematics either as a tool to work science problems (e.g., solving genetics or rate problems) or to actually teach a science concept (e.g., using ratios in the equation to teach photosynthesis). Similarly, mathematics integrates science by using science applications to explain or practice mathematics concepts or to employ the science to reinforce students' interest in mathematics or to enable the students to recognize the broad utility of mathematics. The need to infuse mathematics and science more completely was recognized by West and Tooke (2001) and termed *Correlated Science and Mathematics* (*CSM*). The *CSM* PD model was developed in 2006 and has been continually revised and refined. The *CSM* model is unique in that it integrates science and mathematics in a more comprehensive manner than other integration models. Each discipline is taught with seven fundamental goals: (a) teaching for conceptual understanding, (b) using each discipline's proper language, (c) using standards-based learning objectives, (d) identifying the natural links between the disciplines, (e) identifying language that is confusing to students, (f) identifying the parallel ideas between the disciplines when possible, and (g) using 5E inquiry format in science and mathematics when appropriate (see Figure). The *CSM* approach to PD is "centered in the critical activities of the profession-that is, in and about the practices of teaching and learning" (Ball & Cohen, 1999, p. 13).
A new continuum is proposed that spans from pure mathematics to pure science with the midpoint now representing a correlated lesson including each of the seven goals of *CSM*.

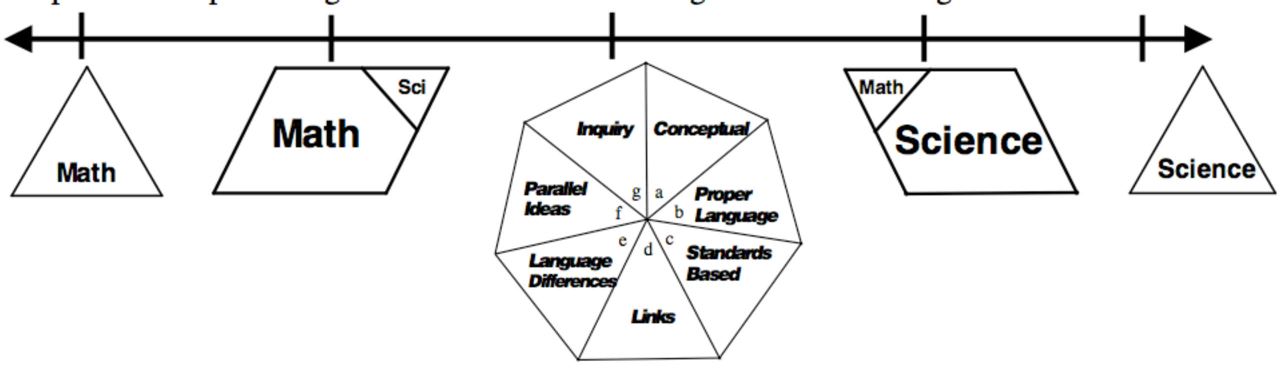

Figure. *CSM* Continuum of mathematics and science correlation.

Components of the *CSM* PD Model

The seven components of the *CSM* PD model are described as follows. (a) Conceptual understanding: Developing a deep understanding of a one's discipline is a critical attribute for teaching for conceptual understanding (Darling-Hammond, 1998; Devlin, 2007; National Research Council, 2001). Conceptual understanding is defined as the ability to apply concepts to new contexts or connect new concepts to existing information or to use general principles to explain and justify (Malloy, Steinthorsdottir, & Ellis, 2004; Wiske, 1997).; (b) Proper language: Each discipline has its own language or vocabulary (Jacobs, 1989). Content expertise, as well as content pedagogy expertise, is required of the instructional team to achieve the *CSM* goal of proper use of each discipline's language.; (c) Standards: National and state standards are purposefully considered as the *CSM* lessons are planned and the science and mathematics learning objectives are identified. Also, many of the national PD standards are met with the *CSM* model. As stated by the National Staff Development Council (NSDC), "It is essential that staff development leaders and providers select learning strategies based on the intended outcomes and their diagnosis of participants' prior knowledge and experience" (n. p.).; (d) Linkages between disciplines: Science uses some mathematical concepts and skills as tools, and mathematics uses science to explain mathematics concepts.; (e) Confusing language: In the *CSM* model, language that may be confusing to students is identified and clarified. An example of confusing language is the difference between the meaning of distance in mathematics and science. In mathematics, distance is the linear length from one point to another, whereas in science the same concept is called displacement; (f) Parallel skills, ideas, processes or concepts: Concepts, etc. that are similar because they behave similarly in both science and mathematics or share many characteristics.; (g) 5E: The *CSM* science lessons developed generally follow the 5E model for inquiry. The *CSM* instruction is team-taught by a science expert and a mathematics expert, both of whom are well versed in both content and content pedagogy. The instructors design lessons purposefully to meet the defined goals for the PD participant. *CSM* is implemented with middle school science and mathematics teacher teams and their principals. Each project consists of 70-hour summer institutes and 30 hours of academic year (AY) trainings for the teachers. The principals received 12 hours of summer training and an optional 30 hours of AY training. According to the NSDC (2001), "An extended summer institute with follow-up sessions throughout the school year will deepen teachers' content knowledge and is likely to have the desired effect" (n. p.). To follow the *CSM* model, the first five goals must be addressed whether the lesson or project is primarily focused on science or primarily focused on mathematics. The last two goals of identifying parallel ideas and using inquiry are met when appropriate since not every lesson contains parallel ideas between science and mathematics or is suitable for inquiry. For example, a skill in science, such as using a piece of equipment, is more effectively taught using a direct instruction model rather than inquiry and does not share parallel ideas with mathematics.

Research Questions and Methods

Several research questions concerning the *CSM* projects arose as the *CSM* projects were developed and revised over time. This paper contains data from the 2009-2010 project. Data concerning improved student achievement is under analysis but not available at this point in time. However data concerning improved teacher content knowledge and integration of lessons is available and will be addressed in this paper. Research questions: (1) would CSM training increase teachers' content knowledge in physics and mathematical reasoning, (2) would *CSM* training enable science and mathematics teachers to plan integrated science and mathematics lessons, (3) would *CSM* training enable science and mathematics teachers to implement integrated science and mathematics lessons?

Teachers' content knowledge was accessed with pre and post-tests in physics and mathematical reasoning at the beginning and end of the two-week summer institute. Teachers' increase use of integrated lessons in their classrooms was assessed using level four of Gusky's (2002) critical levels of professional development evaluation. Three data sources concerned with Guskey's level four, a change in professional practices, were utilized: (1) AY classroom observations in conjunction with teacher and principal interviews, (2) observations of sample integrated lessons taught by the participants during the AY Saturday sessions, and (3) examples of integrated lessons reported by teachers during interviews and during AY sessions. Twenty teachers (ten mathematics, ten science) participated in this study. The teachers worked in teams to create lessons that included science and mathematics concepts. A minimum of two classroom observations of each participant were conducted during the fall and spring of the AY following the summer institute. Interviews of both the teacher and their principal were conducted after each set of classroom observations. During each AY session, teachers taught lessons to their fellow participants and used planning time to create lessons.

Data Analysis

Teacher Pre and Post-tests: Participant teachers were administered pre and post-tests in physics and mathematical reasoning at the beginning and end of the two-week summer institute. A related-samples t-test was conducted for each content test to determine whether teacher content knowledge of physics and mathematics significantly improved as a result of the summer institute. Results indicate that participant teachers did significantly improve their knowledge of both physics and mathematical reasoning. The mean score on the physics post-test ($M = 62.29$, $SD = 12.970$) was significantly higher than the mean score on the physics pre-test ($M = 45.76$, $SD = 11.377$, $t(16) = 4.992$, $p = .000$). The effect size was large ($d = 1.211$), and 61% of the variance was accounted for by the treatment ($r^2 = 0.609$). The 95% confidence interval for the mean difference between pre and post-test physics scores was 9.51 to 23.55. The mean score on the mathematical reasoning post-test ($M = 82.00$, $SD = 12.817$) was also significantly higher than the mean score on the mathematical reasoning pre-test ($M = 73.50$, $SD = 13.135$, $t(15) = 2.984$, $p = .005$). The effect size was medium ($d = 0.746$), and 37% of the variance was accounted for by the treatment ($r^2 = 0.372$). The 95% confidence interval for the mean difference between pre and post-test mathematical reasoning scores was 2.43 to 14.57.

Through observations and interviews, three major findings were revealed. Classroom observations disclosed an increased ability to create and teach integrated science and mathematics lessons and to use the proper language of each discipline. Teachers reported that prior to the training they never even thought about teaching integrated lessons, whereas after training they were teaching from one to eight integrated lessons per month. Teachers reported that the integrated lessons went well and they were planning to use the lesson again the next year. During each afternoon of AY training the teams were given time to design an integrated science lesson and an integrated mathematics lesson. During the planning, each teacher taught their partner the content and proper language of the lesson. At the end of the day, the teams shared their written plans with the entire group. Providing joint planning time is an integral part of *CSM* PD. As stated by NSDC, "teachers are likely to adapt their instruction to new standards-based curriculum frameworks through the joint planning of lessons".

Principals reported seeing science and mathematics teachers collaborating more and creating integrated lessons. Also, the principals stated that prior to training they looked for general instructional strategies, not science or mathematics specific ones. They reported that they now see the connections between mathematics and science so that when observing a science or mathematics lesson they ask themselves whether the lesson could be integrated. Moreover, they are asking themselves if some manipulatives or science equipment could be used in hands-on strategies during the lesson.

Contribution to Teaching and Learning of Science

The integration of science and mathematics has long been advocated by both disciplines and their national standards, *National Science Education Standards* and the NCTM *Principals and Standards for School Mathematics*. Although the discussion for developing an integrated science and mathematics curriculum has been ongoing, teachers have not been trained to develop an integrated curriculum. Moreover, no PD model was structured and described well enough to support trainers in facilitating teachers to integrate science and mathematics. Research indicates that teachers are better able to help their students learn mathematics when they have opportunities to work together to improve their practice, time for personal reflection, and strong support from colleagues and other qualified professionals (Brown & Smith, 1997; Putnam & Borko, 2000). The critical attributes of *CSM* can enable teacher teams to effectively teach integrated science and mathematics. The CSM model of PD seems to support teachers in the integration of mathematics and science. As stated by a recent *CSM participant*,

> Through intense hands-on instruction, numerous investigations, use of manipulatives, and the instructors' questioning strategies and modeling of collaboration, I was able to work side-by-side with a fellow teacher to develop wonderful integrated lessons for our students. This team-teaching approach to lesson planning and construction will help to keep my students' interest in math and science alive. (*CSM* participant, 2008)

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