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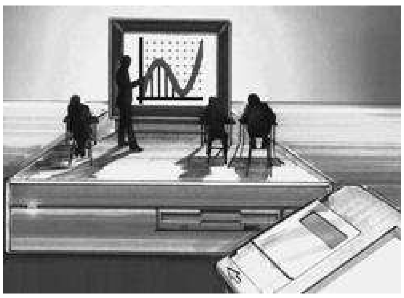

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Foreword

This volume contains the papers presented at the International Conference on "Models in Developing Mathematics Education" held from September 11-17, 2009 at The University of Applied Sciences, Dresden, Germany. The Conference was organized jointly by The University of Applied Sciences and The Mathematics Education into the 21st Century Project - a non-commercial international educational project founded in 1986. The Mathematics Education into the 21st Century Project is dedicated to the improvement of mathematics education world-wide through the publication and dissemination of innovative ideas. Many prominent mathematics educators have supported and contributed to the project, including the late Hans Freudental, Andrejs Dunkels and Hilary Shuard, as well as Bruce Meserve and Marilyn Suydam, Alan Osborne and Margaret Kasten, Mogens Niss, Tibor Nemetz, Ubi D'Ambrosio, Brian Wilson, Tatsuro Miwa, Henry Pollack, Werner Blum, Roberto Baldino, Waclaw Zawadowski, and many others throughout the world. Information on our project and its future work can be found on Our Project Home Page http://math.unipa.it/~grim/21project.htm

It has been our pleasure to edit all of the papers for these Proceedings. Not all papers are about research in mathematics education, a number of them report on innovative experiences in the classroom and on new technology. We believe that "mathematics education" is fundamentally a "practicum" and in order to be "successful" all new materials, new ideas and new research must be tested and implemented in the classroom, the real "chalk face" of our discipline, and of our profession as mathematics educators. These Proceedings begin with a Plenary Paper and then the contributions of the Principal Authors in alphabetical name order. We sincerely thank all of the contributors for their time and creative effort. It is clear from the variety and quality of the papers that the conference has attracted many innovative mathematics educators from around the world. These Proceedings will therefore be useful in reviewing past work and looking ahead to the future. For further information about the work of our Project, please email alan@rogerson.pol.pl

We wish to thank especially Fayez Mina and Rainer Heinrich for all their support and hard work without which this conference, and these Proceedings, would not have been possible. A special thanks are also due to all our very generous major sponsors, the Saxony Ministries of Education and Science & the Arts and the City of Dresden.

Ludwig Paditz Alan Rogerson

Presented Papers

Keynote Plenary Address

A Paper Accepted for the Proceedings but not Presented at the Conference

Plenary Address: Language and Mathematics, A Model for Mathematics in the 21st Century

David K. Pugalee (David.Pugalee@uncc.edu)
University of North Carolina at Charlotte, Charlotte, NC, USA

"Human language and thought are crucially shaped by the properties of our bodies and the structure of our physical and social environment. Language and thought are not best studied as formal mathematics and logic, but as adaptations that enable creatures like us to thrive in a wide range of situations" (Feldman, 2006, p. 7).

Language and Mathematics: A Complex Symbiotic for Learning

In order to know how to use this language correctly requires an integrated knowledge of multiple facets of communicative competence and mathematical knowledge. Vosniadou and Vamvakoussi (2006) suggest that if knowledge is viewed as a process instead of a product that the emphasis of teaching shifts from one focused on subject-matter content to thinking and learning skills. They further assert that mathematics is not only a process in which one participates but the knowledge products of complex social interaction. It can be argued then that there exists a complex interplay between language and mathematics, both as processes and products. For Vygotsky (1978), "thought is not merely expressed in words; it comes into existence through them" (p. 218). Learning is dependent upon this relationship between thinking and language. To improve our conceptions of learning requires exploring the complex questions about the mediation between thought, language, and mathematics.

Walshaw and Anthony (2008) argue that student outcomes are dependent on an array of cultural scripts and imperatives that are part of a pedagogical activity system. They challenge that classroom discourse, the means of constructing mathematical knowledge, will expand only when there is a viable cohesion between all the elements and interrelated contingencies. The model presented here is a response to that challenge: an effort to describe this cohesion between all the elements and interrelated contingencies. I concur that this understanding is essential if we are to develop the capacities to affect this multidimensional system.

Explicating a Model

Language and competence in mathematics are not separable. The model that is presented in this paper [See Figure 1] is intended to be an emerging work to stimulate discussion and to provide opportunities for dialogue about the complex nature of this relationship. The work of Erik DeCort has been substantive in moving this discussion forward. De Cort (2007) eloquently articulates five components that are necessary for developing competence in mathematics:

(1) A well-organized and flexibly accessible domain-specific knowledge base involving the facts, symbols, algorithms, concepts, and rules that constitute the contents of mathematics as a subject-matter field.

(2) Heuristic methods, i.e., search strategies for problem analysis and transformation (e.g., decomposing a problem into subgoals, making a graphic representation of a problem) which do not guarantee, but significantly increase the probability of finding the correct solution.

(3) Meta-knowledge, which involves knowledge about one's cognitive functioning (metacognitive knowledge; e.g., knowing that one's cognitive potential can be developed through learning and effort), on the one hand, and knowledge about one's motivation and emotions (metavolitional knowledge; e.g., becoming aware of one's fear of failure when confronted with a complex mathematical task or problem), on the other hand.

(4) Positive mathematics-related beliefs, which include the implicitly and explicitly held subjective conceptions about mathematics education, about the self as a learner of mathematics, and about the social context of the mathematics classroom.

(5) Self-regulatory skills, which embrace skills relating to the self-regulation of one's cognitive processes (metacognitive skills or cognitive self-regulation; e.g., planning and monitoring one's problem-solving processes), on the one hand, and skills for regulating one's volitional processes/activities (metavolitional skills or volitional self-regulation; e.g., keeping up one's attention and motivation to solve a given problem), on the other hand. (p. 20-21).

These five components describe prominent features that elucidate the link between mathematical competence, language, and thought. In the model offered in this paper, the importance of these components is evident. The proposed model is intended to identify the multifaceted interaction of complex features that impact thinking in the communication circuit or loop with the goal of demonstrating the dependency of multiple components in creating an effective multidimensional communicative process.

External factors. In the communication loop, the receiver is influenced by both external and internal factors. While internal factors represent those facets that are most complex and abstract, the role of external factors in this process should not be downplayed. The features and nature of the communication provides the first interaction between the message and the receiver. In written communication, features of text comprise a linguistic register and include phonological, lexical, grammatical, and sociolinguistic elements (Scarcella, 2003). The dense conceptual level of mathematical texts and the technical register present problems for the reader (Pugalee, 2007). These difficulties are further compounded by a lack of metalinguistic awareness that supports the reader as he reflects on the structural and functional features of the text as decisions are made about how to communicate information and manipulate units of language (MacGregor & Price, 1999; Pugalee, 2007). Metacognition or self-monitoring and application of learning strategies assists the reader in comprehending the text. Oral language also must pass through similar filters. Oral language must be considered within the broader framework of the content features and the quality of the interaction (Bussi, 1998).

Internal Factors. As previously presented, DeCort (2007) provides a rich discussion of salient features of the cognitive elements that are necessary in any model for which mathematical competence is the core. Also of interest is the sociocultural dimension which represents a complex interplay between the physical world and how those in discourse communities construct meaning through language. The sociocultural dimension involves norms, values, beliefs, attitudes and practices of language within cultural settings which includes the learning environment. Mathematics might be thought of as a "cultural activity that involves inventing, using and improving symbols" (Cobb, 2000, p.20). In the sociocultural sense, mathematical discourse comprises perceiving and doing in additional to speaking and writing (Sfard, 2000; Dorfler, 2000). Communication in this symbol rich environment of mathematics is challenging both in terms of executing effective communication but also in terms of interpreting information and constructing knowledge within this environment.

> "There is also the question of whether symbolism can be used as a tool for cutting through the relevant noise during the abstraction process or whether it can only be used to formulate what has already been abstracted. If the latter, then every learning situation will have a ceiling determined by (1) the amount of noise generated, and (2) the amount of noise the learner is able to cut through" (Dienes, 1963, pp. 160-161).

Ernest (1997) argues that the physical world is only describable through linguistic means (categories) and that such description is the result of interpretation. Language and the construction of mathematical meaning are social phenomena mediated by environmental and individual factors.

The cognitive processes that are required to give students access to mathematics cannot be separated from the linguistic aspects of the information received and how that communication interacts with multiple variables as a result of cognition. Higher order thinking, strategic competence, and metalinguistic awareness (Scarcella, 2003) along with metaknowledge, metacognition, heuristic and procedural knowledge, and conceptual understanding (DeCort, 2007; Scarcella, 2003) are included in the cognitive register. The student's strategic competence will impact their ability to formulate, represent, and do mathematics. Their capacity for adaptive reasoning will influence the quality of the product or output as they reflect, explain, and justify their thinking. The student's procedural and conceptual understandings further arbitrate the degree to which they are capable of producing meaningful and acceptable results of their thinking. All of this is mediated by language – and produces a cognitive load that some students are ill equipped to handle.

Some Concluding Thoughts

The progress of students depends on the advancement of our thinking about the relationship between language and the learning and teaching of mathematics. Anna Sfard (2001) posits that communication is the heart of mathematics education and should be viewed “not as a mere aid to thinking, but as almost tantamount to the thinking itself” (p. 13). Consider the model presented in this paper. How can it inform our thinking about the complexities of our practice? What is the nature of student’s failure and success in mathematics? Is it possible to have mathematical learning with understanding void of a thoughtful and deliberate consideration of the role of language and communication?

The model offered in this paper is an attempt to represent complex and multifaceted processes that affect mathematics teaching and learning. Models are but an attempt to simplify complex activities so that they can be better conceptualized. Understanding communication and language is essential in understanding mathematics learning. Consider the model; discuss it; critique it. Accept the challenge to inform our practice as mathematics educators through the consideration of how our work is both informed and constructed by language.

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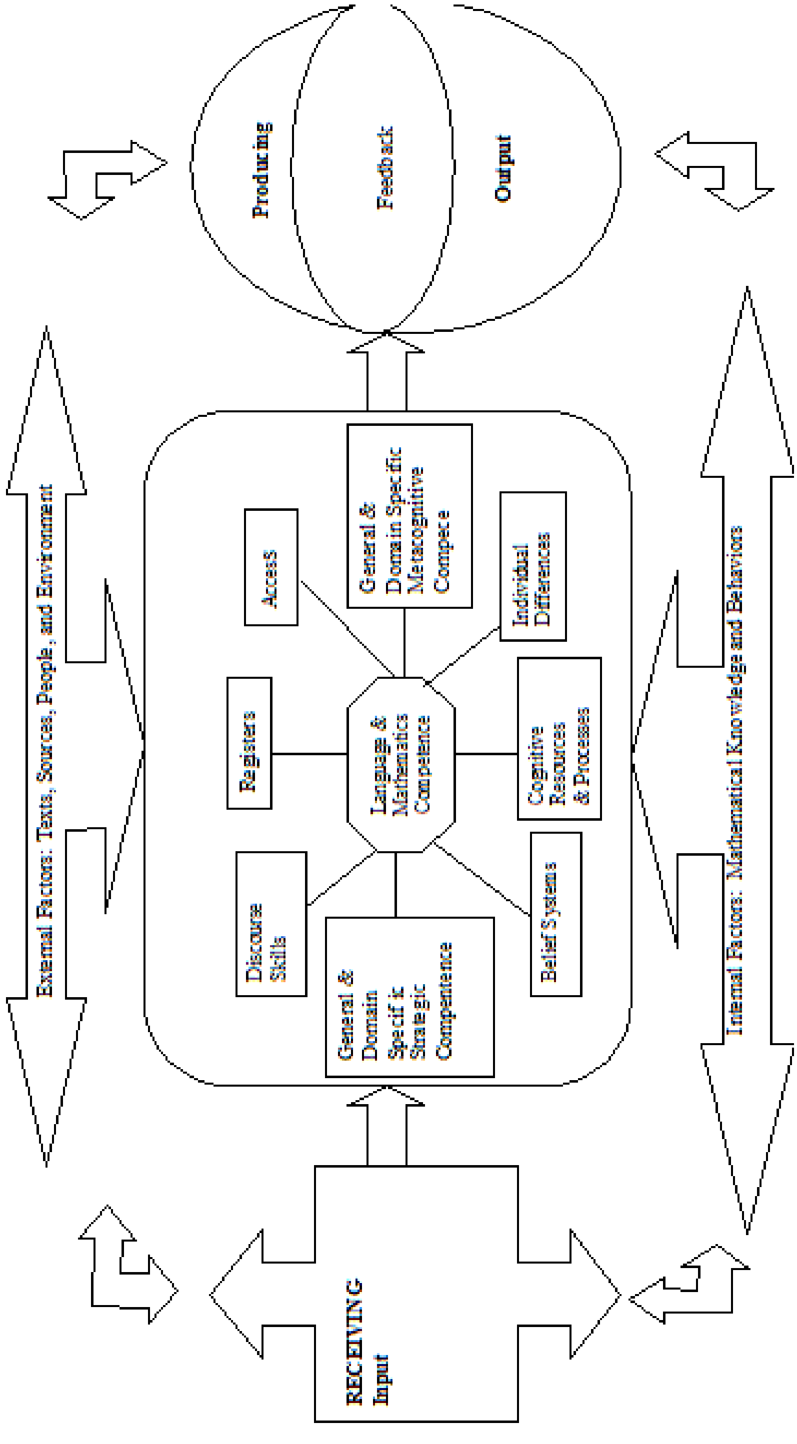

igure 1. Model of Communication and Mathematics

Proofs and "Puzzles"

*Buma Abramovitz, D.Sc. abbuma@braude.ac.il
*Miryam Berezina, D.Sc. berezm@braude.ac.il
**Abraham Berman, PhD berman@tx.technion.ac.il
*Ludmila Shvartsman, PhD sluda@braude.ac.il
*Department of Mathematics, ORT Braude College, Karmiel, Israel
**Departments of Mathematics and of Education in Technology & Science, Technion
Haifa, Israel

Abstract
It is well known that mathematics students have to be able to understand and prove theorems. From our experience we know that engineering students should also be able to do the same, since a good theoretical knowledge of mathematics is essential for solving practical problems and constructing models.
Proving theorems gives students a much better understanding of the subject, and helps them to develop mathematical thinking. The proof of a theorem consists of a logical chain of steps. Students should understand the need and the legitimacy of every step. Moreover, they have to comprehend the reasoning behind the order of the chain's steps. For our research students were provided with proofs whose steps were either written in a random order or had missing parts. Students were asked to solve the "puzzle" – find the correct logical chain or complete the proof.
These "puzzles" were meant to discourage students from simply memorizing the proof of a theorem. By using our examples students were encouraged to think independently and came to improve their understanding of the subject.

Introduction
It is well known that in mathematics one has to understand and learn the theory to be able to solve problems. Our experience of teaching mathematics to engineering students has taught us the importance of providing students with a good theoretical basis. It is not an easy task for a lecturer when some students come from schools where they were taught mathematics as a set of algorithms needed to solve problems.
An important part of Calculus is learning theorems, and it is often difficult for students to understand the meaning of a theorem. We tried to develop a special approach for improving students' understanding of a theorem (see our papers of 2007 and 2009). . However, as we taught students how to understand theorems we came across another difficulty: proofs. Many students simply do not grasp the logic behind building a proof. Many aspects of this problem were researched in the noteworthy papers of Sowder & Harel (2003) and Tall (2005). Often students are not taught proofs at all, so they have to believe that a theorem is correct. We think that it is important even for freshmen to study at least some proofs. By proving a theorem students learn where in the proof a given condition of the theorem is used. The objective, however, is not to have students to memorize the proof of a theorem as they would a poem.
The problem was how to provide students with the means to learn a theorem's proof. Accordingly, we formulated the given proofs as a chain of steps and presented students with exercises to solve. For instance, in the "scattered puzzle" the steps are randomly ordered and students are required to "puzzle out" the correct order. In the "fill in puzzle"

students are asked to fill in the missing parts of the proof. Such "puzzles" were given to students as homework exercises over the Webassign system (see www.webassign.net).
In this paper we continue the subject of teaching proofs to engineering students that we started in our papers (2004, 2006). There we discussed the problem of checking students' ability to prove theorems using multiple choice exams. During our research we realized that the "puzzle" proof, as a kind of theoretical exercise, could be used in teaching students to learn and understand proofs.

"Scattered puzzles"
In the two following examples the steps of the proofs are written in a scattered way.

<u>Example 1</u>

<u>Theorem:</u> *The function* $f(x)=\log_a x$ *is differentiable for* $x>0$, *and* $f'(x)=\dfrac{1}{x\ln a}$.

<u>Proof</u>

Step 1: $f'(x)=\lim\limits_{\Delta x\to 0}\left(\dfrac{1}{\Delta x}\log_a\left(1+\dfrac{\Delta x}{x}\right)\right)$.

Step 2: $f'(x)=\lim\limits_{\Delta x\to 0}\log_a\left(\left(1+\dfrac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}\right)^{\frac{1}{x}}$.

Step 3: $f'(x)=\lim\limits_{\Delta x\to 0}\dfrac{\log_a(x+\Delta x)-\log_a x}{\Delta x}$.

Step 4: $f'(x)=\log_a e^{\frac{1}{x}}=\dfrac{1}{x}\log_a e=\dfrac{1}{x\ln a}$.

Step 5: $f'(x)=\lim\limits_{\Delta x\to 0}\dfrac{\log_a\dfrac{x+\Delta x}{x}}{\Delta x}$.

Step 6: $f'(x)=\lim\limits_{\Delta x\to 0}\log_a\left(1+\dfrac{\Delta x}{x}\right)^{\frac{1}{\Delta x}}$.

The correct order of the steps in the proof is: 3, 5, 1, 6, 2, 4.

<u>Example 2</u>
<u>Theorem:</u> *If* $f(x)$ *is a continuous function on the interval* $[a,b]$ *and* $h(x)$ *is a primitive of* $f(x)$ on $[a,b]$, *then* $\int_a^b f(x)\,dx=h(b)-h(a)$.

<u>Proof</u>

Step 1: $g(a)=h(a)+C=\int_a^a f(t)\,dt=0$.

Step 2: Define $g(x)=\int_a^x f(t)\,dt$. Then $g(x)$ is a primitive of $f(x)$ on $[a,b]$, so

$g'(x)=f(x)$.

Step 3: $C=-h(a)$.

Step 4: $g(b)=\int_a^b f(t)\,dt=h(b)+C=h(b)-h(a)$.

Step 5: Let $h(x)$ be another primitive of $f(x)$ on $[a,b]$. Then there exists a constant C such that $g(x)=h(x)+C$.

The correct order of the steps in the proof is: 2, 5, 1, 3, 4.

In the above two examples the proofs are simple and short. There is only one answer to the puzzle. So, it was easy to check students' homework by using an online checker. In the case of a longer and more complicated proof we either did not use the Webassign system to check homework or we connected several steps into a block and asked students to find the correct order of the blocks.

"Fill in puzzles"

In our opinion, the "fill in puzzle" is comparatively more advanced and demands a deeper understanding of a proof.

In Examples 3 and 4 some parts of the proofs are missing. Students were provided with a list of the missing parts and some additional unnecessary ones.

<u>Example 3</u>

<u>Theorem:</u> *If* $f(x)$ *is a continuous function on* $[a,b]$, *and* $g(x)=\int_a^x f(t)\,dt$, *then* $g(x)$ *is differentiable on* $[a,b]$ *and* $g'(x)=f(x)$ *at every point* x *in* $[a,b]$.

<u>Proof</u>

By the definition of the …… (answer 7) $g'(x)=\lim_{\Delta x\to 0}\dfrac{g(x+\Delta x)-g(x)}{\Delta x}$.

$g(x+\Delta x)-g(x)=\int_a^{\ldots(answer\ 4)} f(t)\,dt-\int_a^x f(t)\,dt=\int_x^{\ldots(answer\ 4)} f(t)\,dt$ using the …(answer 9).

Suppose $\Delta x>0$ (in the case of $\Delta x<0$ the proof is similar). By the … (answer 11) there exists a point $c(\Delta x)$ such that $x<c(\Delta x)<\ldots\ldots(answer 4)$ and

$\int_x^{x+\Delta x} f(t)\,dt=f(c(\Delta x))\Delta x$.

Hence,

$$g'(x)=\lim_{\Delta x\to\ldots(answer 3)}\frac{g(x+\Delta x)-g(x)}{\Delta x}=\lim_{\Delta x\to\ldots(answer 3)}\frac{f(c(\Delta x))\Delta x}{\Delta x}=\lim_{\Delta x\to\ldots(answer 3)} f(c(\Delta x)).$$

When Δx converges to 0, then $c(\Delta x)$ converges to …… (answer 1).

$g'(x)=\lim\limits_{\Delta x\to 0} f\left(c(\Delta x)\right)=f(x)$ by the definition of a …… (answer 8).

The list of the missing parts:
1) x; 2) a; 3) 0; 4) $x+\Delta x$; 5) Δx; 6) $c(\Delta x)$; 7) derivative; 8) continuous function; 9) properties of a definite integral; 10) Rolle's Theorem; 11) Mean Value Theorem.

<u>Example 4</u>

<u>Theorem:</u> *If* $f(x)$ *is a twice differentiable function on* $[a,b]$, *fulfilling* $f(a)=f(b)=0$, *then for every* $x\in[a,b]$ *there exists* $c\in[a,b]$ *such that* $f(x)=(x-a)(x-b)\dfrac{f''(c)}{2}$.

<u>Proof</u>

Let x_0 be an arbitrary point in the interval $[a,b]$. We have to prove that there exists a point c in the same interval, such that $f(x_0)=(x_0-a)(x_0-b)\dfrac{f''(c)}{2}$.

If $x_0=a$ or $x_0=....$ (answer 2) the statement is trivial.

Let x_0 be an arbitrary point in the interval (a,b) and let define the function $\varphi(x)=f(x)-k(x-a)(x-b)$. By computation we obtain that $\varphi(....(answer\,1))=\varphi(b)=0$.

In order to obtain $\varphi(x_0)=0$, k may be chosen $k=....(answer\,13)$.

Since $\varphi(x)$ is …… (answer 9) on $[a,b]$, $\varphi(x)$ is continuous on $[a,b]$.

On the interval $[a,x_0]$, $\varphi(x)$ is differentiable and $\varphi(....(answer\,1))=\varphi(x_0)=0$, so the conditions of …… (answer 11) are fulfilled, and there exists a point d_1 in the interval (a,x_0) such that $\varphi'(d_1)=....(answer\,14)$.

For the same reason on the interval $[x_0,b]$, there exists a point d_2 in the interval (x_0,b) such that $\varphi'(d_2)=....(answer\,14)$.

$\varphi'(x)$ is differentiable on $[d_1,d_2]$ and $\varphi'(d_1)=\varphi'(d_2)=0$. The conditions of Rolle's Theorem are fulfilled for the function ……(answer 17) , so there exists a point c in the interval (d_1,d_2) such that $\varphi''(c)=....(answer\,14)$.

By computation we obtain that $\varphi'(x)=....(answer\,15)$ and $\varphi''(x)=....(answer\,16)$.

Now, $\varphi''(c)=f''(c)-2k=0$, such that $k=\dfrac{f''(....(answer\,3))}{2}$.

On the other hand, $k=\dfrac{f(x_0)}{(x_0-a)(x_0-b)}=\dfrac{f''(....(answer\,3))}{2}$ and we obtain

$f(x_0)=(x_0-a)(x_0-b)\dfrac{f''(c)}{2}$.

The list of the missing parts:
1) *a*; 2) *b*; 3) *c*; 4) d_1; 5) d_2; 6) x_0; 7) *k*; 8) *1*; 9) differentiable; 10) continuous;
11) Rolle's Theorem; 12) Lagrange's Theorem;
13) $k = \dfrac{f(x_0)}{(x_0 - a)(x_0 - b)}$; 14) 0; 15) $f'(x) - k(2x - a - b)$; 16) $f''(x) - 2k$;
17) $\varphi'(x)$.

The "fill in puzzle" can also be presented without the list of missing parts; then students can complete the proof without this aid.

Conclusions
"Puzzle" proofs can be used at different students' levels of mathematical knowledge.
Usually we constructed "puzzle" proofs for theorems proved in the classroom, but the method is also suitable for theorems that are part of homework assignments (see Example 4).
In our opinion, "puzzle" proofs are appropriate not only for Calculus theorems, but also for theorems from other mathematical subjects, as well as for high school mathematics.
Based on our teaching experience we realized that students need theoretical exercises, which provide them with an opportunity to practice the theoretical part of Calculus. Many problems available to students allow them to practice their computing skills but few problems help them improve their ability to prove theorems.
By using "puzzle" proofs students take the first step in the right direction of proving and better understanding theorems. Our hope is that, with practice, they will be able to prove simple statements independently.

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Some Initiatives in Calculus Teaching

*Buma Abramovitz, D.Sc. abbuma@braude.ac.il
*Miryam Berezina, D.Sc. berezm@braude.ac.il
**Abraham Berman, PhD berman@tx.technion.ac.il
*Ludmila Shvartsman, PhD sluda@braude.ac.il
*Department of Mathematics, ORT Braude College, Karmiel, Israel
**Departments of Mathematics and of Education in Technology &Science, Technion
Haifa, Israel

Abstract
In our experience of teaching Calculus to engineering undergraduates we have had to grapple with many different problems. A major hurdle has been students' inability to appreciate the importance of the theory. In their view the theoretical part of mathematics is separate from the computing part. In general, students also believe that they can pass their exams even though they do not have a real understanding of the theory behind the problems they are required to solve. In an effort to surmount these difficulties we tried to find ways to make students better understand the theoretical part of Calculus. This paper describes our experience of teaching Calculus. It reports on the continuation of our previous research.

Introduction
From our experience we know that many students have difficulties in learning theory. To try to solve the problem we developed an approach called "self learning method" (SLM). We originally described the method in an earlier paper (2007). The method is intended to help students better understand the meaning of a theorem by involving them in the learning process, step by step. The way in which the material is presented encourages them to take part in formulating, discussing and proving a theorem. The following is a description of the process of learning a theorem:
1) *Reviewing concepts:* The first step is to have students review the concepts, definitions and theorems that are needed to learn a specific theorem.
2) *Formulation of a conjecture*: Normally, teachers present a theorem in a way that is clear to mathematicians, but is too formal for students. The SLM method aims to turn students from passive receivers of knowledge into active partners in the learning process. Students are given self learning material that includes assignments. The aim is for students to learn the concepts under review by completing the given assignments in an informal way and as a result, to be ready for formulation of a theorem, hopefully on their own.
3) *Formulation of the theorem*: At this stage of the learning process we state the theorem in a schematic way, emphasizing what is assumed and what is concluded.
4) *Exploring assumptions and conclusions*: We provide students with exercises and problems that focus on the following questions: What are the assumptions of a theorem and what are the conclusions? What is the geometrical meaning of a theorem? What happens when one or more of the theorem's assumptions are not fulfilled? What assumptions are necessary and which are sufficient? Generally speaking, what does the theorem mean?
5) *Proving the theorem*: This is the final step of the process. We use different ways to check whether the students really understand the material or whether they have simply memorized it.

In our earlier papers (2007 and 2009) we discussed the fourth SLM step using the three basic theorems of the differential part of Calculus: Lagrange's Mean Value Theorem, Fermat's Theorem and Rolle's Theorem. We had three groups of about one hundred engineering students in total try out the instructional materials. We were satisfied with the results of our experiment, as shown in the first paper. We, accordingly, decided to develop SLM for other Calculus theorems. We know from our teaching experience that students have many problems in understanding the Basic Theorem of Calculus:

If a function $f(x)$ *is continuous on the interval* $[a,b]$, *then the function* $F(x)=\int_a^x f(t)dt$ *is differentiable on the interval* $[a,b]$ *and* $F'(x)=f(x)$.

Our students have the following problems with this theorem:

1) In the definition of the definite integral there are constant limits of integration. When we define the function $F(x)$, one of the limits is a variable; this confuses students.
2) Students see the function $F(x)$ as very abstract, and do not understand its practical meaning.
3) Students get used to elementary functions, and it is difficult for them to accept other types of functions.

The above problems underscore how important the first and the second steps of SLM are. Students need to deal with the relevant concepts before formulating a theorem. This is in line with Tall's idea (2004) about the three worlds of mathematics used to distinguish among different modes of mathematical thinking. The three worlds are the embodied world, the proceptual world and the formal world. They reflect a development from just perceiving a concept through actions to formal understanding of a concept. Individuals go between the worlds as their needs and experiences change and mental representations of concepts are formed and altered. If we, the teachers, want to take students immediately to the formal world, they are very likely to get lost on the way. Therefore, we should invite the students into the embodied world in order to become acquainted with relevant concepts and statements. We concluded that the best way to implement this is to teach or learn "with examples" (Hazzan & Zaskis, 1999).

In this paper we demonstrate how we used SLM Steps 1 and 2 to teach the Basic Theorem of Calculus.

Reviewing concepts

The following concepts and statements were given to students to review:

1) Continuous function $f(x)$ at a point $x=a$ (on interval $[a,b]$);
2) Differentiable function $f(x)$ at a point $x=a$ (on interval $[a,b]$);
3) Antiderivative;
4) Definite integral;
5) Intergrable function $f(x)$ on interval $[a,b]$.

Statement 1. *If function* $f(x)$ *is constant (* $f(x)\equiv C$ *), then* $\int_a^b f(x)dx=C(b-a)$.

Statement 2. *If function* $f(x)$ *is integrable on interval* $[a,b]$, *and* $f(x)\geq 0$, *then the integral* $\int_a^b f(x)dx=S(D)$, *here* $S(D)$ *is the area of the domain:*

$D=\{(x,y)\mid a\leq x\leq b, 0\leq y\leq f(x)\}$.

Statement 3. *If function* $f(x)$ *is integrable on interval* $[a,b]$, *then for every point c ,* $c\in[a,b]$,

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$$

Statement 4. *If function* $f(x)$ *is integrable on interval* $[a,b]$, and

$$g(x)=\begin{cases} f(x)\ , & x\neq c \\ C\ , & x=c \end{cases},\ a\leq c\leq b,$$

then $g(x)$ *is integrable on interval* $[a,b]$ *and* $\int_a^b g(x)dx = \int_a^b f(x)dx$.

It must be noted that the students had not yet learned the Newton-Leibniz Theorem.
In order to introduce the integral function $F(x)$ from a geometrical point of view, we gave the students the following definition:
Let $f(x)$ *be an integrable function on interval* $[a,b]$, $f(x) \ge 0$ *, then the area function* $S(x)$ *is the area of the domain* $D = \{(t,y) \mid a \le t \le x, 0 \le y \le f(t)\}$.

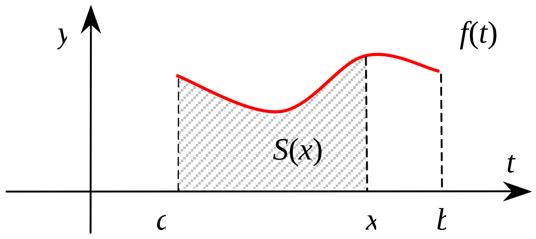

Fig. 1. The Area Function

Remark: As follows from Statement 2: $S(x) = \int_a^x f(t)dt$, $a \le x \le b$.

The following assignments were given to the students.

Assignment 1.
Below are several examples of functions defined on the interval $[0,2]$. For every function fill in Table 1 and try to find out the formula of the area function. We recommend that you apply the statements given above.

$f(x)=1$	$S(0)$	$S(0.5)$	$S(1)$	$S(1.5)$	$S(2)$	$S(x) =$ ______
$f(x)=\begin{cases}1, & x \ne 1\\ 2, & x=1\end{cases}$	$S(0)$	$S(0.5)$	$S(1)$	$S(1.5)$	$S(2)$	$S(x) =$ ______
$f(x)=\begin{cases}1, & x \le 1\\ 2, & x>1\end{cases}$	$S(0)$	$S(0.5)$	$S(1)$	$S(1.5)$	$S(2)$	$S(x)=\begin{cases}\_\_\_\_, & x \le 1\\ \_\_\_\_, & x>1\end{cases}$
$f(x)=x$	$S(0)$	$S(0.5)$	$S(1)$	$S(1.5)$	$S(2)$	$S(x) =$ ______
$f(x)=\begin{cases}x, & x \le 1\\ 1, & x>1\end{cases}$	$S(0)$	$S(0.5)$	$S(1)$	$S(1.5)$	$S(2)$	$S(x)=\begin{cases}\_\_\_\_, & x \le 1\\ \_\_\_\_, & x>1\end{cases}$
$f(x)=\begin{cases}x, & x \le 1\\ 2, & x>1\end{cases}$	$S(0)$	$S(0.5)$	$S(1)$	$S(1.5)$	$S(2)$	$S(x)=\begin{cases}\_\_\_\_, & x \le 1\\ \_\_\_\_, & x>1\end{cases}$

Table 1. Examples.

The students were also provided with graphs of the functions and area formulas for a rectangle, a triangle and a trapeze.

Table 2 shows the answers to the questions, which were also given to the students. This table can also be used for the next assignment.

Example 1	Example 2	Example 3
$S(x)=x$	$S(x)=x$	$S(x)=\begin{cases} x & , x\le 1 \\ 1+2(x-1) & , x>1 \end{cases}$
Example 4	Example 5	Example 6
$S(x)=\dfrac{x^2}{2}$	$S(x)=\begin{cases} \dfrac{x^2}{2} & , x\le 1 \\ \dfrac{1}{2}+(x-1) & , x>1 \end{cases}$	$S(x)=\begin{cases} \dfrac{x^2}{2} & , x\le 1 \\ \dfrac{1}{2}+2(x-1) & , x>1 \end{cases}$

Table 2. Answers.

Assignment 2.

Answer the following questions (see Table 1):

1) Which $f(x)$ functions are integrable on the interval $[0,2]$?
2) Which $S(x)$ functions are continuous on the interval $[0,2]$?
3) Can the following statement be true?

Statement:

If function $f(x)$ is integrable on $[a,b]$, then the function $S(x)$ is continuous on this interval.

Formulation of a conjecture

We used the above examples and the results of the assignments for the next SLM step, in which the students continued to learn about the relation between the functions $f(x)$ and $F(x)=S(x)$. We also provided additional examples for the integral function in the general case.

Assignment 3.

Using the previous assignments, fill in Table 3. Answer: “Yes” or “No”.

Example	$f(x)$ is continuous on $[0,2]$	$S(x)$ is differentiable on $[0,2]$	$S'(x)=f(x)$ on $[0,2]$
1	Yes	Yes	Yes
2	No	Yes	No
3	No	No	No
4	Yes	Yes	Yes
5	Yes	Yes	Yes
6	No	No	No

Table 3. Assignment 3 and Answers.

Assignment 4.

Which of the following statements cannot be true?

1) *If function $f(x)$ is integrable on $[a,b]$, then the function $S(x)$ is differentiable on this interval.*
2) *If function $f(x)$ is continuous on $[a,b]$, then the function $S(x)$ is differentiable on this interval and $S'(x)=f(x)$.*
3) *If the function $S(x)$ is differentiable on $[a,b]$, then the function $f(x)$ is continuous on this interval.*

At this point the students were able to answer the above questions easily since they had worked their way through enough examples, including counterexamples to false statements.

Assignment 5.

Definition: *Let* $f(x)$ *be an integrable function on interval* $[a,b]$ *, then the integral function* $F(x)$ *is defined in the following way:* $F(x)=\int_a^x f(t)dt$.

<u>Remark</u>: If $f(x)\geq 0$, then $F(x)=S(x)$. In the general case $F(x)$ is not an area of a domain.

Find the function $F(x)$ for each of the following functions:

Example 7	Example 8	Example 9
$f(x)=-1$	$f(x)=\begin{cases}-1\,,\ x\leq 1\\ -2\,,\ x>1\end{cases}$	$f(x)=\begin{cases}-1\,,\ x\leq 1\\ 1\,,\ x>1\end{cases}$

Table 4. Additional Examples.

We recommend using the questions of Assignment 2 and the properties of the definite integral. Table 5 provides the answers to Assignment 5.

Example 7	Example 8	Example 9
$F(x)=-x$	$F(x)=\begin{cases}-x\,,\ x\leq 1\\ -2x+1\,,\ x>1\end{cases}$	$F(x)=\begin{cases}-x\,,\ x\leq 1\\ x-2\,,\ x>1\end{cases}$

Table 5. Answers.

Assignment 6.

Formulate two statements, similar to the statement in Assignment 4, about the integral function: a false one and a true one.

Conclusions

The main points of our approach are:

1) Students are given a bank of comparatively simple examples, in which they can also find counterexamples, a task which they usually find to be difficult.
2) When solving seemingly simple problems the students also solve theoretical problems, and arrive at theoretical conclusions. They are thus co-opted into the research process.
3) Students' learning becomes motivated by their success.

By using the above examples we tried to introduce the concept of an integral function before attempting to formulate the Basic Theorem of Calculus. The SLM method allowed the students to absorb the concept of an integral function into their embodied world, so they were able thereafter to draw it into the formal world.

Our hope is that, in the end, students will be able to formulate a theorem independently.

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A Study On Problem Posing-Solving in the Taxicab Geometry and Applying Simcity Computer Game

Tuba Ada
Anadolu University,Eskişehir, TURKEY tyuzugul@anadolu.edu.tr
Aytaç Kurtuluş
Eskişehir Osmangazi University,TURKEY agunaydi@ogu.edu.tr

Abstract
Problem-posing is recognized as an important component in the nature of mathematical thinking (Kilpatrick, 1987). More recently, there is an increased emphasis on giving students opportunities with problem posing in mathematics classroom (English& Grove, 1998). These research has shown that instructional activities as having students generate problems as a means of improving ability of problem solving and their attitude toward mathematics (Winograd, 1991). In this study, teaching Taxicab Geometry which is a non-Euclidean geometry is aimed to mathematics teacher candidates by means of computer game-Simcity- using real life problems posing. This studies' participants are forty mathematics teacher candidates taking geometry course. Because of using Simcity computer game, this game is based on Taxicab Geometry. Firstly, students had been given Taxicab geometry theory for two weeks and then seperated six each of groups. Each of groups is wanted to posing problem and solving from real life problems at Taxicab geometry. In addition to, students applied to problem solving at Simcity computer game. Studens were model into Simcity game. They founded ideal city, healty village, university campus, holiday village, etc. interesting of each others.

Introduction

Problem-posing is recognized as an important component in the nature of mathematical thinking (Kilpatrick, 1987). More recently, there is an increased emphasis on giving students opportunities with problem posing in mathematics classroom (English & Hoalford, 1995; Stoyanova, 1998). These research has shown that instructional activities as having students generate problems as a means of improving ability of problem solving and their attitude toward mathematics (Winograd, 1991). Nevertheless, such reform requires first a commitment to creating an environment in which problem posing is a natural process of mathematics learning. Second, it requires teachers figure out the strategies for helping students posing meaningful and enticing problems. Thus, there is a need to support teachers with a collaborative team whose students engage in problem-posing activities. This can only be achieved by establishing an assessment team who support mutually by providing them with dialogues on critical assessment issues related to instruction. Problem-posing involves generating new problems and reformatting a given problems (Silver, 1994). Generating new problems is not on the solution but on creating a new problem. The quality of problems in which students generated depends on the given tasks (Leung & Silver, 1997). Research on problem posing has increased attention to the effect of problem posing on students' mathematical ability and the effect of task formats on problem posing (Leung & Silver, 1997).

Problem posing is becoming recognized in the United States as a necessary component of mathematics teaching and learning (NCTM, 2000; Silver, 1994). Allowing students to pose their own mathematics problems can influence, among other things, attitudes towards mathematics, ownership of mathematics and mathematics achievement (Brown, S. I. & M. I. Walter. 1993, Silver, 1994). As stated by Silver, "contemporary constructivist theories of teaching and learning require that we acknowledge the importance of student-generated problem posing as a component of instructional activity (Silver, 1994)." Researchers and educators have begun to incorporate problem posing into mathematics teaching and learning Brown, & Walter, 1993; Silver, 1994). Leung and Silver showed that prospective elementary school teachers were able to pose mathematical problems but in many cases their problems lacked mathematical complexity. As problem posing is beginning to be incorporated into mathematics classrooms it is important to continue to document students capabilities as problem posers. Problem solving is widely regarded as playing a fundamental part in the learning and understanding of mathematics (NCTM, 2000; Schoenfeld, 1985; Polya, 1973). NCTM defines problem solving as "engaging in a task for which the solution method is not known in advance," and suggests that "solving problems is not only a goal of learning mathematics but also a major means of doing so" (2000, p. 52). Schoenfeld (1985) points out, what might be a significant task for one student could be routine or second-nature for another. Here lies one challenge for the mathematics teacher:

picking quality problems whose solution strategies are not immediately known to each student, but which are within each student's grasp (cited in, Perrin, 2007). Freire contrasted problem-posing education with teacher-dominated education, which he deemed as "banking" education (Lewis at all, 1998). The core of mathematical investigations and scientific research entails problem posing and solving activities. Beside this problem posing being an important component of problem solving process lies at the hearth of mathematical activities (Kilpatrick, 1987). Problem posing is defined as reformulating given problems and generating new problems (Silver, 1994). Problem posing is not limited to generating new problems from given mathematical situations or by changing the conditions of given problems**.** Problem posing also entails reformulating given problems and generalization for the solution. Problem posing is not independent from problem solving (Silver,1994; Cai & Hwang, 2002; English, 2003; Silver & Cai, 1996; Lewis, 1998). There is a close relationship between problem posing and solving as a cognitive process (Lowrie, 2002). Problem posing is most closely associated with the “looking back” stage of Polya’s four steps to problem solving. This is considered by Polya to be the most important step (Silver, Mamona-Downs, Leung, Shukkwan, & Kenney, 1996). In scientific inquiry formulating a good problem can be more important than discovering a solution for the problem (Einstein & Infeld, 1938; Cai & Hwang, 2003). One of the important consensuses in mathematics education is to provide opportunites to students in mathematics lessons for developing their problem posing skills (Brown & Walter, 2005; NCTM, 2000).

In this study, teaching Taxicab Geometry which is a non-Euclidean geometry is aimed to mathematics teacher candidates by means of computer game-Simcity- using real life problems posing. Simcity is used as a tool is based on Taxicab geometry. Maxis has published a set of resources for teachers on its website, he said that *SimCity* 3000 could be used in the classroom to enhance just about any instructional unit. It could stand alone as an enrichment computer activity, or it could be used as a pivotal activity connected to other activities and projects done before, during, or after using the computer program. Use the lessons in this guide to integrate *SimCity 3000* into your curriculum, with minimal preparation, or to create custom lessons to suit your needs. As Doug Church commented at the 2002 Electronic Entertainment Exposition, most people who have played SimCity recognize that it can be an excellent resource for understanding urban planning, most people would also not want to live in a real city designed by someone who has only played SimCity. As urban planner Kenneth Kolson points out, *SimCity* potentially teaches the player that mayors are omnipotent and that politics, ethnicity, and race play no role in urban planning (Kolson, 1996).

In this study of teacher candidates to see Simcity game to provide the educational side and can be used in teaching Taxicab geometry is a suitable tool is intended to realize.

Objects of the study

1) Exploring differences between Euclidean geometry and non-Euclidean geometry.

2) can be realized while Euclidean Geometry appears to be a good model of the “natural” world, Taxicab Geometry is a better model of the artificial urban world that man has built.

3.) Setting up to perform research and activities of information gathering required problem posing.

4) to transfer problem solving at Simcity computer game.

Process of the Study

Researchers had described Taxicab geometry to mathematics teacher candidates for two weeks. E F. Krause has defined a new geometry, the *Taxicab Geometry* in 1975, by using the metric $d_T(A,B) = |a_1 - b_1| + |a_2 - b_2|$ *f*or , $A = (a_1, a_2), B = (b_1, b_2)$ the Euclidean plane. This equation means count of blocks that would have to travel horizaontally and vertically to get from A to B. In taxicab geometry, all the streets are assumed to run straight north and south or straight east and west; streets are assumed to have no width; buildings are assumed to be of point size. By using the geometry, the teacher candidates had been asked to do posing ,solving and modeling a problem , in real life a living space. Students were divided into six groups. Finding the solution using Taxicab and Euclidean geometry were expected to place on Simcity game. These groups of teacher candidates from each other interesting, creative living areas are modeled. In particular, consider the needs of society , the ideal cities, holiday resorts, university campuses, completely equipped health resorts have been prepared by students were given examples of these models. One of the examples is given below :

For the establishment of an ideal holiday island for 50 blocks and 50 blocks of the size of an island in the shape of the taxicab circle with tender was received by the company Island. The company Island, a 6-person team of mathematicians has wanted to be. The 3-4-5 team was established for. This team reviews the results of the blood reaching the ground was not solid. Therefore, multi-storey buildings will be constructed not. Holiday island by using the taxicab geometry was decided to prepare the layout plan. There will be three pools and holiday island will be constructed for them.

1) Pools which in the holiday island will be installed are purple, red and black, and this name is designated as the area is divided. The center of the purple pool (-12, -20), the red center of the pool (16.8), black pool center (-4, -20) coordinates were established. Accordingly, let it KEMBID holiday island that you go to each client's own the nearest swimming pool and construct point to the administration building of finding the region 's place.

2) We limitted Holiday island with a taxicab circle. What are the coordinates of the holiday island KEMBID accordingly?

3) This holiday island is building 15 hotels, 1 bar, 1 health center, 1 football, 1 basketball, boutique, children's playground, cafe, discotheque, health center, 3 security center, 1 meeting room, 1 fitness and 1 beauty salon will be constructed. You construct this center coordinates the most appropriate place.

4) Buket and her family have come to the island KEMBID holiday. Becauseof Buket in this holiday on the island of meeting her father to perform the job, there have chosen. Moreover, because of her mother's health problems will remain to be close to the hotel's health center. for a vacation and more comfortable, health centers and meeting rooms is equal to the distance and the nearest the pool they want to stay at a hotel. What are the coordinates of this hotel? Accordingly which pool Buket should go to?

5) Melek taking a holiday vacation island KEMBID prefer not like to walk. Therefore, the of the hotel will be 15 blocks from the beauty center and the pool is to be at a distance of no more than 5 blocks. Accordingly, management should direct it to the hotel?

6) Demet, Emel and Kubra is 3 friends. Demet, entertainment tooth, , Emel and Kubra, prefer the pool to swim in the sea instead. Accordingly, the hotel will be 8 blocks away to the disco and at the nearest point to the sea must have. Accordingly, what is the coordinates of this hotel?

7) A sports school building was decided at a distance equal to sports field (tennis courts, basketball and football field). Set your coordinates.

8) To easily take advantage of the holiday island equal to the distance to the center and meeting rooms in library is required to be established. Determine the location of the library

9) Irmak, with the family to spend holiday vacation, have come to the island. Her son wants to play basketball and Irmak wants to use the library for complete her thesis work They want to put in to a hotel, for this, equal to the distance from the library and the sports school. Which hotel should they prefer?

10) The water plant will be constructed in three to meet the needs of the island and the pool water. These water plants asked to be placed distance equal to two pools and away from the center of the island. Where these plants should be established?

11) You solve these questions in the Euclid geometry and compare with the results of Taxicab geometry.

Solving of Problem

The above posing problems and solving the student group at the taxicab geometry is done in Euclidean Geometry. These solutions have drawn graphics. Figure 1 and Figure 2 in 7th solution of the problem as an example of the graph is given.

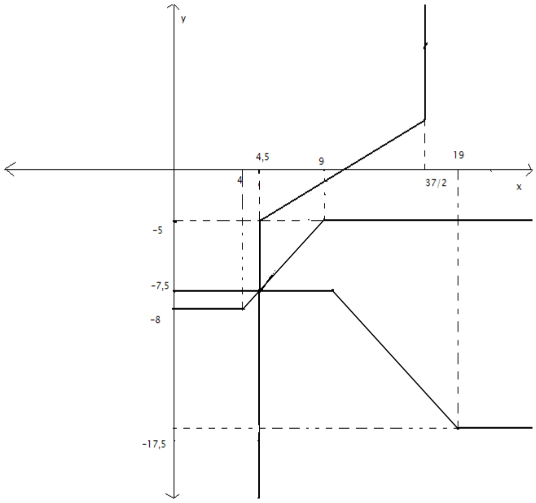

Figure1: graphy of solution the question 7 in Taxicab geometry

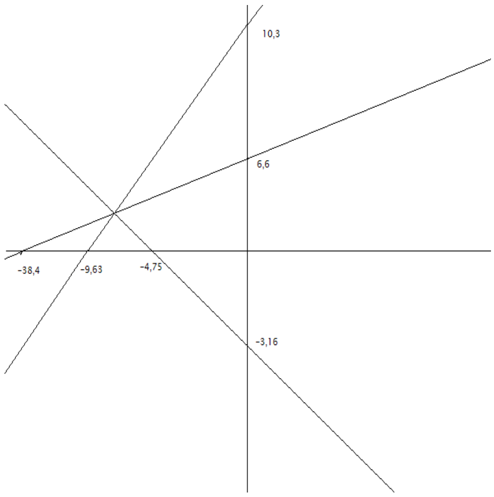

Figure2: graphy of solution the question 7 in Euclidean geometry

Moreover, 7th question on the model of the problem with SimCity game in Figure 3 can be seen. All problems resulting from the graphical model is given in Figure 4 also.

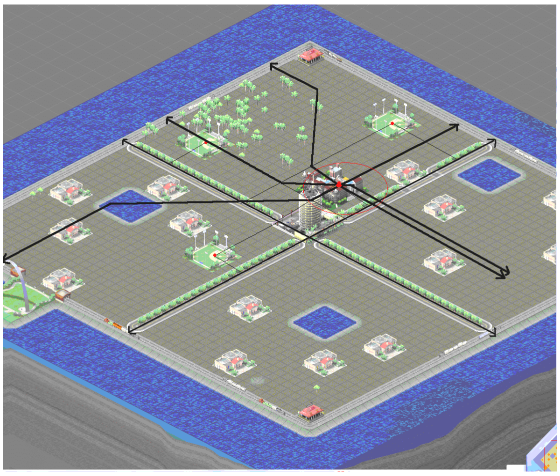

Figure3: graphy of solution the question 7 in Simcity

Figure4 : graphy of solutions all of the questions in Simcity

Conclusions

At the and of the study, mathematics teacher candidates learnt a new geometry “Taxicab geometry” different from Euclid Geometry and saw difference between Taxicab and Euclid geometry using Simcity computer game. By means of this study, mathematics teacher candidates realised Simcity is a educational game based on Taxicab Geometry.

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An innovative model for developing critical thinking skills through mathematical education

Einav Aizikovitsh & Miriam Amit
Department of Science Education, Ben Gurion University of the Negev Israel
einav.aizikovitsh@gmail.com & amit@bgu.ac.il

Abstract

In a challenging and constantly changing world, students are required to develop advanced thinking skills such as critical systematic thinking, decision making and problem solving. This challenge requires developing critical thinking abilities which are essential in unfamiliar situations. A central component in current reforms in mathematics and science studies worldwide is the transition from the traditional dominant instruction which focuses on algorithmic cognitive skills towards higher order cognitive skills. The transition includes, a component of scientific inquiry, learning science from the student's personal, environmental and social contexts and the integration of critical thinking. The planning and implementation of learning strategies that encourage first order thinking among students is not a simple task. In an attempt to put the importance of this transition in mathematical education to a test, we propose a new method for mathematical instruction based on the infusion approach put forward by Swartz in 1992. In fact, the model is derived from two additional theories., that of Ennis (1989) and of Libermann and Tversky (2001). Union of the two latter is suggested by the infusion theory. The model consists of a learning unit (30h hours) that focuses primarily on statistics every day life situations, and implemented in an interactive and supportive environment. It was applied to mathematically gifted youth of the Kidumatica project at Ben Gurion University. Among the instructed subjects were bidimensional charts, Bayes law and conditional probability; Critical thinking skills such as raising questions, seeking for alternatives and doubting were evaluated. We used Cornell tests (Ennis 1985) to confirm that our students developed critical thinking skills.

Introduction

In his monograph "Smart Schools," (Perkins, 1992) the author focuses on how to change our teaching to enable children to learn more meaningful information. One of his suggestions is that we should have a thoughtful school, which means: teachers should teach by using a language of thinking. In this paper, we are focusing on the language of critical thinking. We begin by first explaining the term critical thinking (CT). Actually critical thinking is not a new concept; we can find it in ancient times. Socrates, as reported by Plato, used to the streets of Athens asking people all kinds of philosophical questions about the purpose of life, morality, justice, etc apparently for the purpose of stimulating a form of critical thinking. These questions and answers were collected and published in the "Socratic dialologues" (eg The Meno, and The Protagoras). In the field of education, it is generally agreed that CT capabilities are crucial to one's success in the modern world, where making rational decisions is increasingly becoming an important part of everyday life. Students must learn to test reliability, raise doubts, and investigate situations and alternatives, both in school and in everyday life. There have been many attempts to define it, but we would like to focus on three of them. Mecpeck defines critical thinking as "skills and dispositions to appropriately use reflective skepticism" (Mecpeck, 1981). Lipman claims that critical thinking is "thinking which enables judgment, is based on criteria, corrects itself, and is context sensitive" (Lipman, 1991). The third definition is the one we have based our research on. Ennis (1962) defines it as "a correct evaluation of statements". Twenty-three years later Ennis broadened his definition to include a mental element, so now the improved definition is "reasonable reflective thinking focused on deciding what to believe or do" (Ennis, 1985). Our research is based on three key elements: a CT taxonomy that includes CT skills (Ennis, 1987); the learning unit "probability in daily life" (Liberman & Tversky 2002); and the infusion approach of integrating subject matter with thinking skills (Swartz, 1992). Ennis' Taxonomy (Ennis, 1987), the learning unit "probability in the daily life" (Liberman & Tversky 2002) and the infusion approach (Swartz,

1992) collectively provide a firm basis for our thesis. In light of his definition, Ennis developed a CT taxonomy that relates to skills that include intellectual aspect as well as behavioural aspect. In addition to skills, Ennis's (1987) taxonomy also includes dispositions and abilities. Ennis claims that CT is a reflective and practical activity aiming for a moderate action or belief. There are five key concepts and characteristics defining CT: practical, reflective, moderate, belief and action. In our novel learning unit, which is a part of the formal syllabus of the Ministry of Education, the student is required to analyse problems, raise questions and think critically about the data and the information. The purpose of the learning unit is not to be satisfied with a numerical answer but to examine the data and its validity in order to. In cases where there is no single numerical answer, the students are required to know what questions to ask and how to analyse the problem qualitatively, not only quantitatively. Along with being provided with statistical instruments, students are redirected to their intuitive mechanisms to help them estimate probabilities in daily life. Simultaneously, students examine the logical premises of these intuitions, along with misjudgments of their application. A combination of the two theories was put forward by the infusion approach. There are two main approaches to fostering CT: the general skills approach which is characterized by designing special courses for instructing CT skills, and the infusion approach which is characterized by providing these skills through teaching the set learning material. According to Swartz, the Infusion approach aims for specific instruction of special CT skills during the course of different subjects. According to this approach there is a need to reprocess the set material in order to combine it with thinking skills. In this report, we will show how we combined the mathematical content of "probability in daily life" with CT skills from Ennis' taxonomy, restructured the curriculum, tested different learning units and evaluated the subjects' CT skills.

Methodology

Our methodological challenge is to investigate the development of the language of critical thinking" through critical thinking skills incorporated into a structured mathematics lesson, based on the proposed models.

Setting, Population, and Data**:** Fifty-five children between the ages of fifteen and sixteen participated in an extra curriculum program aimed at enhancing the critical thinking skills of students from different cultural backgrounds and socio-economical levels. An instructional experiment was conducted in which probability lessons were combined with CT skills. The study consisted of fifteen 90 minutes lessons, spread out over the course of an academic year, in which the teacher was also one of the researchers.

Data sources were students' products, pre and post questionnaires, personal interviews and class transcriptions. Students' products (papers, homework, exams etc.) were collected:. Personal interviews were conducted randomly. Five students were interviewed at the end of each lesson and one week after. The personal interviews were conducted in order to identify any change in the students' attitudes throughout the academic year.

All lessons were video-recorded and transcribed. In addition, the teacher kept a journal (log) on every lesson. Data was processed by means of qualitative methods intended to follow the students' patterns of thinking and interpretation with regards to the material taught in different contexts. As already mentioned, the probability unit combines CT skills with the mathematical content of "probability in daily life". This new probability unit included questions taken from daily life situations, newspapers and surveys, and combined CT skills. Each of the fifteen lessons that comprised the probability unit had a fixed structure: a generic (general) question written on the blackboard; the student's reference to the question and a discussion of the question using probability and statistical instruments and; an open discussion of the question that included practicing the CT skills. The mathematical topics taught during the fifteen lessons were: Introduction to set theory, probability rules, building a 3D table, conditional probability and Bayes theorem, statistical connection and causal connection, Simpson's paradox, and judgment by representative. The following CT skills were incorporated in all fifteen lessons: A clear search for an hypothesis or question, the evaluation of reliable sources, identifying variables, “thinking out of the box,” and a search for alternatives (Aizikovitsh & Amit, 2008). Each lesson followed the same four part structure.

1. Given text :In the beginning of the lesson the teacher presented a short article or text.
2. Open class discussion in small groups: Discussion in small groups about the article and the question.

- Initial suggestions for the resolution of the question
- No intervention by the teacher

3. Further discussion directed by the teacher: Open class discussion. During the discussion the teacher asked the students different questions to foster the students' thinking skills and curiosity and to encourage them to ask their own questions.

- Various suggestions from students in class.
- Interaction between groups of students.
- Reaching a consensus across the whole class (or just across the group).

4. Critical thinking skills and mathematical knowledge (teaching)

The teacher referred to the questions raised by the students and encouraged CT, while instilling new mathematical knowledge: the identification of and finding a causal connection by a third factor and finding a statistical connection between C, and A and B, Simpson's paradox and Bayes Theorem.

Case study- The Aspirin Case

Below, I have provided a detailed description of one lesson called the Aspirin Case. Following the description, I outline the analysis of the lesson using the following techniques: referring to information sources, raising questions, identifying variables, and suggesting alternatives and inferences. The lesson topic was conditional probability. The CT skills practiced in the lesson were evaluating source reliability, identifying variables, and suggesting alternatives and inference.

1. A Given Text

Your brother woke up in the middle of the night, crying and complaining he has a stomachache. Your parents are not at home and you don't know what to do. You gave your brother aspirin, but an hour later he woke up again, suffering from bad nausea and vomiting. The doctor that takes care of your brother regularly is out of town and you consider whether to take your brother to the hospital, which is far from your home. You read from a book about children's diseases and find out that there are children that suffer from a deficiency in a certain type of enzyme and as a result, 25% of them develop a bad reaction to aspirin, which could lead to paralysis or even death. Thus, giving aspirin to these children is forbidden. On the other hand, the general percentage of cases in which bad reactions such as these occur after taking aspirin is 75%. 3% of children lack this enzyme.

(Taken from “probability thinking” p. 30+slight changes made by researcher)

2. Open Class Discussion in Small Groups

Discussion in small groups about the generic question:

Should you take your brother to the emergency room? What should you do?

Can aspirin consumption be lethal?

3. Critical Thinking Skills and Mathematical Knowledge (Teaching)

This phase of the lesson focused on encouraging critical thinking and instilling new mathematical knowledge (Bayes formula) statistical connections by referring to students' questions and further discussion.

A teacher-led discussion focused on methods of analysis using such Critical Thinking skills as: Source identification: Medicine book; Source reliability: High; Variable identification: A – enzyme deficiency, D – adverse reaction to aspirin; Mathematical Knowledge: Data: $P(D/A)=0.25$ $P(D)=0.75$ $P(A)=0.03$, To prove: $P(A/D)=?$

Using Bayes formula (or a two dimensional matrix) the result is:

Lesson Conclusion is that only 1% of the children without the enzyme develop an adverse reaction to aspirin, thus there is no need to go to the hospital.

Even so, is it worth taking the risk? What do you think? (question to the class).

Discussion

Research analysis according to critical thinking skills in this case study through the model of the infusion approach, students practice their CT while acquiring technical probability skills. In this lesson, the following five skills are exercised: raising questions – asking question about the article and probing on the main question about the connection between aspirin and death; referring to information sources and evaluating the source's reliability - the text took from Medicine book; the students skepticism, and identification of variables – students identified the enzyme deficiency and adverse reaction to aspirin. Following these skills, another skill, searching for alternatives, was presented. In class the teacher and the students spoke about suggesting alternatives, not taking things for granted, but examining what had been said and suggesting other explanations. Hence, the skills that were practiced in the described lesson were: raising questions, evaluating the source's reliability, identifying variables, and suggesting alternatives and inference. In order to understand and monitor the students' attitudes toward CT as manifested by the skills specified above, interviews were conducted with five students after the aforementioned lesson. In these interviews, the students acknowledged the importance of CT. Moreover, students were aware of the infusion of instructional strategies that advance CT skills. Results of our study should be considered with some caution, as this case study presents one lesson which was designed in a fixed pattern – a generic question, a discussion of the question, the practice of statistical connection, introduction to causal connection and experiencing the use of CT skills such as: raising questions, evaluating the source's reliability, identifying variables, and suggesting alternatives and inferences. On the basis of the interviews conducted and questionnaires that were qualitatively analyzed, it is not known, at this stage, whether these skills had been acquired. Skill acquisition will be evaluated at a later phase in this study, using quantitative measures – the Cornell Critical Thinking Scale and the CCTDI (Facion, 1992) scale. This case study provides encouraging evidence of the effectiveness of this approach and further investigation in this direction is needed. The small scale research described here constitutes a small step in the direction of developing additional learning units within the traditional curriculum. Current research is exploring additional means of CT evaluation, including: the Cornell CT scale (Ennis, 1987), questionnaires employing various approaches, and a comprehensive test composed for future research. The general educational implications of this research suggest that we can and should lever the intellectual development of the student beyond the technical content of the course, by creating learning environments that foster CT, and which will, in turn, encourage the student to investigate the issue at hand, evaluate the information and react to it as a critical thinker. It is important to note that, in addition to the skills mentioned above, in the course of this lesson the students also gained intellectual skills such as conceptual thinking and developed a class culture (climate) that fostered CT. Students practiced critical thinking by studying probability. In this lesson, the following skills were demonstrably practiced: referring to information sources, encouraging open-mindedness and mental flexibility (all questions), a change in attitude and searching for alternatives. A very important intellectual skill is the fostering of cognitive determination – to be able to express one's attitude and present an opinion that is supported by facts . In this lesson, students could be seen to be searching for the truth, they were open-minded and self-confident. In other words, they practiced critical thinking skills. A new language was being created: the language of critical thinking.

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The “Kidumatica” project - for the promotion of talented students from underprivileged backgrounds.

Miriam Amit, Ph.D
Professor of Mathematics Education, Head, Deparment of Science and Technology Education, Ben-Gurion University amit@bgu.ac.il

“There is nothing more unequal than equal treatment to unequal people" -- Attributed to Thomas Jefferson

Abstract: This article describes ‘Kidumatica’ – a highly successful project for the promotion of talented students from underprivileged backgrounds. In its 11 year run, Kidumatica has evolved into a way of life for its many students, allowing them opportunities to realize their potential, enter advanced academic studies, and successfully enter a society rich in knowledge and achievement. Kidumatica is based on academic research in the fields of excellence, cognition and mathematics education, and on the social principle of equal opportunity for all and one’s right to self-realization and aspiration, regardless of ethnic background and socio-economic status. Beyond these social/educational purposes, Kidumatica is also a research model and laboratory for testing new programs and teaching methods for gifted students. The following are the basic premises of the Kidumatica model, its goals and how they are achieved, including the recruitment of club members and the mathematical content.

Background

The economic, scientific, and technological future of a society depends upon identifying and nurturing a cadre of talented youth. In light of this statement, the Kidumatica Youth Mathematics Forum was established in 1998 by the Center for Science and Technology Education at Ben-Gurion University with the aim of advancing the mathematical education of talented youth in the southern region. This region suffers from the highest rate of unemployment in the country, the highest percentage of new immigrants and the lowest average in academic achievement.

Kidumatica, a play on words in Hebrew meaning "advancing math," provides 400 talented pupils from 70 schools and diverse socio-economic backgrounds with academic empowerment and social and cultural support they may not receive otherwise. The participants of Kidumatica comprise an intriguing mosaic of ethnic backgrounds. They are immigrants from North Africa, Asia, Ethiopia, India, Europe, North and South America and the republics of the former Soviet Union, Bedouin Arabs and native born Israelis. Kidumatica helps pupils to become aware of the existence of cultural and social diversity as expressed through clothing, language, mentality, and family background.

All Kidumatica's activities take place in a University campus, where the pupils have access to a wide variety of materials, equipment, computers, and libraries.

Because of its innovation and uniqueness, Kidumatica has rapidly gained a reputation for excellence, and is highly regarded in the professional community.

Kidumatica’s Working Assumptions

1. Students with mathematical potential require special attention and one cannot assume that they will simply ‘manage on their own’ (just as the talent of a promising athlete or musician requires nurturing). This project aims to address this need.
2. Surveys show that the relative percentage of gifted students from underprivileged backgrounds whose talents are fostered is extremely low. This project addresses these students in the region.
3. In the case of the particular area described here, it is important for talented students to remain in their ‘natural’ environment, especially because of their potential contribution to the other students.

4. On the other hand, it is important for gifted students to be among peers, whose company is socially and intellectually stimulating. Kidumatica therefore does *not* cut students off from their school environment, but draws them together from their various places of residence to create and *additional* social framework, specially designed for the talented.
5. Kidumatica does not compete with the school curriculum. The students continue to study as usual with no premature advancement, and receive in Kidumatica only enrichment and expansion of their formal schooling (see course list).
6. Teaching gifted students requires a specialization that combines a high level of content knowledge, an appreciation of the particular characteristics of gifted students, and a great deal of creativity and enthusiasm. The Kidumatica instructors are trained mathematicians, who, in addition to their extensive practical teaching experience, are knowledgeable in research of both mathematical and general giftedness.
7. Learning, in addition to being attractively presented in a problem based learning approach, must also be fun, so Kidumatica also incorporates activity days, scientifically and socially oriented field trips, the writing of a newspaper, etc.
8. Why teach mathematics? Because mathematics is, in today's society, an indicator of ability and success. Moreover, learning mathematics develops an ability for logical and critical thinking that is applicable to other fields, as well as being relatively non-based in culture and therefore equally suitable for students from different cultural backgrounds.

Theoretical Background on Mathematically Gifted and Talented

Scientific literature differentiates between three types of excelling students: "good learners", "talented students", and "extremely talented students" (Greenes, 1981; Ream & Zollman, 1994). Kidumatica's goal is to reach into the varied communities populating southern Israel, to find and unite excellent students of all three types, promoting each according to his or her own abilities. In achieving this goal, the math club not only provides gifted students with advanced mathematics education, but also provides a social framework for students of varied socio-economic and ethnic backgrounds, uniting them around their shared interest in deepening their mathematical knowledge.

Psychologists agree that academic talent is comprised of a combination of high cognitive abilities, creativity and motivation (Renzulli, 1986; Wertheimer, 1999). This combination is the main pedagogical guideline for the Kidumatica programs. Cognitive aspects include encouraging interest in solving complex problems and elegant solutions, and developing abstraction and generalization abilities (Freiman, 2006; Krutetskii, 1976; Wieczerkowski, Cropley & Prado, 2000; Amit & Neria, 2008). The creative aspects include developing flexible thinking and using a wide-range of mathematical strategies in non routine ways (Koichu & Berman, 2005; Sriraman, 2005). The motivational aspects include developing determination in performing hard tasks, developing ambition, strengthening concentration ability and encouraging willingness to face learning challenges (Middletone & Spanias, 1999).

Kidumatica Activities

All participants share a strong commitment and desire to learn more about math. The Kidumatica after-school activities include: a.Workshops held in the form of twenty groups of approximately 20 students each, twice a week in the afternoon in the university campus, for four hours. b. **"**special activity day" held every fifth week, in which the groups are blended and activities include games, competitions, guest lectures, and a Club Newspaper.c.Research projects with individual guidance. d. Field trips, museum visits, national Olympiads and competitions.

Mathematical Content

Pupils are divided into small groups of about fifteen to twenty students each, who participate during two activity days, taking two courses-workshops per day. The courses are comprised of specially designed teaching materials, tailor-made for the students' needs and conveyed through a pedagogy based on openness and support. This teaching method, though designed to be "friendly," is by no means carried out at the expense of a strict insistence on mathematical rigorousness and accuracy. The mathematical couses ares extracurricular in order to avoid "competition" with school, and include:

Probabilistic and critical Thinking – Deals with analyzing examples taken from our daily life and developing probability intuition and a critical perception.

Logic – Deals with developing logical thinking in and out of pure mathematics.

Inventiveness – Deals with developing original and inventive thinking, developing imagination and providing techniques in a variety of fields.

Algebraic Laboratory – Deals with the development of quantitative thinking and skills for algebraic techniques and intuition.

The Theory of Numbers – A course designed to develop mathematical thinking in general and the resolution of unconventional problems in particular.

Optimization – Provides a base for multi-directional unconventional thinking, incorporating geometrical methods for solving algebraic problems, and includes the history of mathematics.

Quantitative Sense – Develops a numerical sense and exposes the pupils to methods that are not taught in school.

Geometry as a Tool for Developing Spatial Vision – Deals with the development of spatial and combinatory modes of thinking and three-dimensional visualization.

Problem-Solving Strategies – Deals with structured means for solving non-conventional problems and analyzing model complex situations.

Algorithmic Thinking – Examines the regularity in a series of simple games, devising algorithms for a winning strategy and discovering the regularity behind events.

Data Analysis and Decision Making – Designed to develop the ability to collect, analyze and derive conclusions from real-life situations

Research Projects-Young Researchers: Veteran pupils have conducted research projects, individually supervised by instructors. They either choose topics from a list proposed by the teachers, or alternatively, develop their own research topic pending approval from the supervisor.

Special 'Peeking' Days: Every five weeks, Kidumatica holds special 'Peeking Days', which include group competitions, games, mathematical parties, and mathematical journal writing. On these days, pupils are encouraged to leave their original groups and 'peek' at any other group according to their areas of interest. This opportunity provides students with the added social bonus of getting to know other pupils of various ages.

Field Trips: Pupils participate in educational outings to science museums and research institutes. All of these field trips include a guided tour, hands-on experiments, and various social and educational activities.

Scientific 'Guest' Lectures: Lectures, delivered by leading academic and industrial experts, include such topics as: ecology, environmental science, geology, basic economics, aeronautics, robotics and the connection between music and mathematics.

Optimal Promotion of Excellence: Research shows that mathematically gifted children are interested in challenging assignments, as long as they maintain a delicate balance between the challenge and the achievement (Diezmann & Watters, 2002; Stein, Grover, & Henningsen, 1996, Wieczerkowski, Cropley, & Prado, 2000). Choosing mathematical assignments based on the degree of difficulty allows pupils to experience a sense of triumph and overcome frustration. In order to optimize excellence Kidumatica is divided into 20 distinctive groups based on age, mathematical aptitude and motivation.

Profile of Kidumatica Members

As of 2009, Kidumatica's 470 students, reflect the ethnic diversity of the Negev's population: our students originate from 23 different countries (among them Uzbekistan, Ukraine, Azerbaijan, Belarus, Algiers, Morocco, Tunisia, Argentina, Chile, Romania, France, and India) and speak at home a total of about 11 different languages and dialects (Hebrew, Arabic, Russian, English, Farsi, French, Spanish, Hindi, and more). This diversity is evidence of Kidumatica's spirit of equal opportunity. The Kidumatica students include 57 Bedouin students from various Negev tribes. These students receive special tutoring to help them to overcome any language barriers that could prevent them from competing equally with the Hebrew speaking members of the club ((Amit, Fried, & Abu-Naja, 2007; Neria, Amit, Abu-Naja, & Abo-Ras, 2008).

The Admission Process - Equal Opportunity for All

The process of selecting the students is unique and is based on the social and educational ideology of its directors. Research has shown that scholastic achievement does not always overlap academic talent ((Borland, & Wright, 2000; Ford, Baytops, & Harmon, 1997). There are talented pupils who are unable to express their capabilities within formal frameworks It is also known from enrichment programs in other countries that the population composition in such programs does not always mirror that of the general population. One of the reasons for this is that admission tests usually do not take into consideration cultural distinctions among the various pupils.

To provide an equal opportunity for all the candidates, a three stage process and a unique instrument were developed to examine the pupil's ability to cope with mathematical issues regardless of their school's quality. The validity of this unusual classification process has been proven over the years. The drop-out percentage is negligible (less than 5%) and is due partially to logistical difficulties or overlapping extracurricular activities.

Achievements in Competitions and Further Studies

Competitions are a part of the mathematical community's culture. The benefits for the participants include the gratification of winning, academic growth and development, increased motivation, as well as acquired insights on how to cope with frustration and failure. Over the last years Kidumatica pupils won about 50% of the national and some international prizes. These achievements are unheard of in any mathematical forum and especially in an underprivileged area such as the one in which Kidumatica operates.

University Studies: About 20 Kidumatica junior and high school pupils registered as full-time students in the University's department of Mathematics. In order for them to succeed, they needed mentall and academic support given by the Kidumatica teachers.

Closing Remarks

The Kidumatica model, the popular success of which is evidenced by its longevity and the continual rise in its demand throughout its 11 year run, has proven itself successful academically as well. Its members consistently win major math prizes in state competitions and are significantly represented in state teams for international competitions. The importance of this national and international success is compounded by the fact that Kidumatica's participants hail from underprivileged sectors of the society. In addition to the Kidumatica's cognitive contribution on both the personal and the social level as a nursery for the cultivation of talented minds, it serves the added and equally significant function of a social melting pot for region's diverse population. Once a student enters Kidumatica, ethnic and socio-economic differenced disappear, replaced by challenges to be faced and goals to be reached by all of the club members together.

Kidumatica has not only changed the lives of many of its members, but has created an expanded community of lecturers, researchers, school- teachers, pupils and parents, who have pooled their energies and their resources into the promotion of mathematics education in this relatively disadvantaged area. And – in the words of one of the Kidumatica's more seasoned veterans (five years in Kidumatica) – "we started at the bottom, and today we are at the top of the mathematical pyramid." There is much left to do and to learn, but every long journey begins with a small step. The Kidumatica Youth Forum is a significant step in the right direction.

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How do rabbits help to integrate teaching of mathematics and informatics?

Agnis Andžāns, Dr.habil.math, professor of mathematics,
Faculty of Physics and Mathematics, University of Latvia, Zeļļu Street 8, Rīga, Latvia
agnis.andzans@lu.lv
Laila Rācene, scientific assistant, Faculty of Physics and Mathematics,
University of Latvia, Zeļļu Street 8, Rīga, Latvia laila.racene@gmail.com

Abstract

Many countries are reporting of difficulties in exact education at schools: mathematics, informatics, physics etc. Various methods are proposed to awaken and preserve students' interest in these disciplines. Among them, the simplification, accent on applications, avoiding of argumentation (especially in mathematics) etc. must be mentioned. As one of reasons for these approaches the growing amount of knowledge/skills to be acquired at school is often mentioned.

In this paper we consider one of the possibilities to integrate partially teaching of important chapters of discrete mathematics and informatics not reducing the high educational standards. The approach is based on the identification and mastering general combinatorial principles underlying many topics in both disciplines. A special attention in the paper is given to the so-called "pigeonhole principle" and its generalizations. In folklore, this principle is usually formulated in the following way: "if there are $n + 1$ rabbits in n cages, you can find a cage with at least two rabbits in it". Examples of appearances of this principle both in mathematics and in computer science are considered.

Introduction: education and competitions

As we can learn from many written and oral sources the number of lessons devoted to exact disciplines in school is decreasing in many countries, especially in post-"socialist" ones. Of course, in some sense this can be compensated by introducing new technologies into education. Nevertheless, today not all teachers are ready to explore their advances. So the official curriculum, e. g., in mathematics today is far from the level of 1980-ies.

In this situation olympiads appeared to be a very strong consolidating factor. Olympiad "curricula" wasn't changed; it was developed further in an essential way. The standards that were elaborated in olympiad movement during many years in some sense became the unofficial standards for advanced education in mathematics. There is a lot of topics that are not included in any official school program but nevertheless are discussed regularly with all students interested in mathematics.

The other positive feature of contests is their stability. Teachers are aware that the olympiads will be held, and they can organize their activities and encourage their students to work additionally for a clear and inspiring aim.

So competitions, which had often been characterized as "elitarian", "discouraging", "far-from-life" etc., appeared to be the strongest support to advanced education at schools in a lot of countries.

All this sets new tasks to olympiads. The competitive factor is still extremely important, but also the educational factor of them has become very significant.From the previous it is clear that math contests today cover broad spectrum of mathematics. It is particularly important also because olympiad and contest problems from previous years are broadly used afterwards in everyday teaching practice.

Modern elementary mathematics

It is a tradition that the words "elementary mathematics" are connected with school only. It's not quite correct. Of course, no definition in the mathematical sense is possible. Trying to list the parts of elementary mathematics we include Euclidean planimetry and stereometry , linear operations with plane and space vectors, scalar, pseydoscalar and vectorial products, the greatest part of combinatorial geometry, elementary number theory, equations and systems solvable in radicals, algebraic inequalities, elementary functions and their properties, the simplest properties of sequences and the combinatorics of finite sets. There are many mathematicians, however, who include also elements of graph theory, simplest combinatorial algorithms, simplest functional equations in integers, etc. There are parts of mathematics which definitely should not be included: we can mention the methods which are effectively used only by a small amount of mathematicians as well as methods which, though used widely, demand a specific and advanced mathematical formalism.

We can give the following approximate description of elementary mathematics. Elementary mathematics consists of:

1) the methods of reasoning recognized by a broad mathematical community as natural, not depending on any specific branch of mathematics and widely used in different parts of it,
2) the problems that can be solved by means of such methods.

Evidently, such a concept of elementary mathematics is historically conditioned. Many new areas of mathematics, especially in the discrete and algorithmic parts of it, are still today exploring elementary methods as the **main tool**. Obviously it can be explained at least partially with the fact that the natural questions there have not yet been exhausted, and natural approaches are therefore effective.

The movement of mathematical contests, especially of mathematical olympiads, has made an important service to elementary mathematics. Becoming a mass activity, the system of math competitions created a large and constant demand for original problems on various levels of difficulty. Clearly school curricula couldn't settle the situation, and the organizers of the competitions turned to their own research fields where they found rich and still unexhausted possibilities.

One of important results that originated from the "olympiad mathematics" was the identification of the so called general combinatorial methods (mean value method, invariant method, extremal element method, interpretation method).[1]

Mean value method

Informally speaking, the mean value method helps to make precise the thousands of years long opinion "to reach considerable results, serious efforts should be concentrated in at least one direction". What does the words "considerable results", "serious effort", "direction" mean depends on each individual problem. In the simplest version of it, the Pigeonhole Principle (or, otherwise, Dirichlet Principle of Box Principle), "considerable result" should bet "$\geq n+1$ elements in n boxes"; "serious effort in at least one direction" in turn should be "≥ 2 elements in some box". From the fact that "considerable result" has been reached we conclude that "serious efforts" have been made.

There are a lot of applications of this simple idea both in mathematics and in computer science, even on the middle and high school level. We mention only few classes here:

a) important classical lemma in number theory: among any n+1 integers you can find two which are congruent modulo n,
b) "overlapping lemma" in combinatorial geometry: if some figures of a common measure exceeding n are situated within a "home" of unit measure, then there is a point in the "home" which is covered by at least n+1 of these figures,
c) Levy's theorem: for any two finite systems of segments A and B, if the sum of projections of elements of A on any axis is greater then the sum of projections of elements of B on the same axis, then the sum of the lengths of elements of A is greater than that of the elements of B,
d) plane Ramsey theory [2],
e) classical Ramsey theorem for graphs [3]:

   for any positive integers m and n there exists such a positive integer R(m,n) that in any complete graph with R(m,n) vertices and each edge coloured either white of black you can find either a clique consisting of m vertices with all edges white of a clique consisting of n vertices with all edges black. (In fact, usually by R(m,n) we denote the smallest of all integers with this property.)
f) analysis of the lower bound of running time of an algorithm [4],
g) impossibility proofs for finite automata in computer science [5],
h) applications to coding theory.

A lot of applications on the school level see, e.g., in [6].

To illustrate the rich possibilities here let's mention only one example – some of more than 50 appearances of the classical result about Ramsey numbers (see above).

It is common knowledge that R(3, 3)=6. As far as we know, for the first time this fact was introduced into math contests in 1953, when it was proposed to the contestants of William Lowell Putnam Mathematical Competition:

"Six points are in general position in space (no three in a line, no four in a plane). The fifteen line segments joining them in pairs are drawn and then painted, some segments red, some blue. Prove that some triangle has all its sides the same colour."

The following list contains only some of theorems and contest problems the proofs/ solutions of which are based significantly on this result.

A. The inductive proof of general Ramsey theorem (see above).

B. There are at least two such “monochromatic” triangles in a two-coloured complete graph K_6.

C. In a two-coloured complete graph K_7 there are at least 4 “monochromatic” triangles.

D. (Goodman’s theorem) In a two-coloured complete graph K_n there are at least $f(n)$ “monochromatic” triangles where

$$f(n)=\begin{cases} 2C_k^3 \text{ for } n=2k, k\in \mathrm{N} \\ C_{2k}^3+C_{2k+1}^3-k \text{ for } n=4k+1,\ k\in \mathrm{N} \\ C_{2k+1}^3+C_{2k+2}^3-k \text{ for } n=4k+3,\ k\in \mathrm{N} \end{cases}$$

and these estimations are the best possible.

E. (6th IMO, 1964). In a group of 17 scientists each scientist sends letters to the others. In their letters only three topics are involved and each couple of scientists makes reference to one topic only. Show that there exists a group of three scientists which send each other letters on the same topic.

F. (20th IMO, 1978). An international society has its members from six different countries. The list of members contains 1978 names, numbered 1*;* 2*;* …*;* 1978*:* Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.

G. (21st IMO, 1979). We are given a prism with pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ as top and bottom faces. Each side of the two pentagons and each of the line segments A_iB_j for all $i; j = 1; ...; 5;$ is coloured either red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been coloured has two sides of a different colour. Show that all 10 sides of the top and bottom faces are the same colour.

H. (33rd IMO, 1992). Consider nine points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either coloured blue or red or left uncoloured. Find the smallest value of n such that whenever exactly n edges are coloured, the set of coloured edges necessarily contains a triangle all of whose edges have the same colour.

I. (Latvian Olympiad). There are $2n$ points in the plane. Among their pairwise distances there are at least n^2+1 which don’t exceed 1. Prove that there are three points which can be cowered by a circle of radius $\frac{1}{\sqrt{3}}$.

J. (Folklore). There are 6 irrational numbers. Prove that there are 3 numbers among them such that all their pairwise sums are irrational too.

K. (Folklore). Six points are given in space such that the pairwise distances between them all are distinct. No 4 of these points are in the same plane. Consider the triangles with vertices at these points. Prove that the longest side in one of these triangles is at the same time the shortest side in another triangle.

This list can be prolonged very far. We see from this example that the same simple idea has served for many years even in such a sensitive area as high-level math competitions. So it is also a suitable tool to explain the idea of mean value method with purely educational purposes.

For more examples of the applications of mean value method on educational level see, e.g., [6].

Conclusions

The mean value method still remains one of the most powerful tools in composing and solving contest problems. It plays also important role in the research in mathematics and theoretical computer science. Acquaintance with it has great educational and aesthetical value. Therefore it must remain in the (official and unofficial) curricula of advanced mathematical education. New applications of it must be carefully collected. A monograph-type teaching aid on this method on college level should be most welcome.

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Conjecturing (and Proving) in Dynamic Geometry after an Introduction of the Dragging Schemes

Anna Baccaglini-Frank
Università di Siena, Tuscany, Italy
& University of New Hampshire, Durham, New Hampshire, USA
abaccaglinifrank@gmail.com

Abstract
This paper describes some results of a research study on conjecturing and proving in a dynamic geometry environment (DGE), and it focuses on particular cognitive processes that seem to be induced by certain uses of tools available in *Cabri* (a particular DGE). Building on the work of Arzarello and Olivero (Arzarello et al., 1998, 2002; Olivero, 2002), we have conceived a model describing some cognitive processes that may occur during the production of conjectures and proofs in a DGE and that seem to be related to the use of specific dragging schemes, in particular to the use of the scheme we refer to as *maintaining dragging*. This paper contains a description of aspects of the theoretical model we have elaborated for describing such cognitive processes, with specific attention towards the role of the dragging schemes, and an example of how the model can be used to analyze students' explorations.

Introduction

Research has shown that the tools provided by dynamic geometry systems impact students' approach to investigating open problems (Silver, 1995) in Euclidean geometry (for example, De Villiers, 1998; Laborde, 2001; Mariotti et al., 2000; Arzarello et al., 1998), and also that such tools can be motivational for students in problem solving (Goldenberg, Cuoco & Mark, 1998; Hadas, Hershkowitz & Schwarz, 2000). The innovative aspect of dynamic geometry software with respect to the traditional paper-and-pencil, is that the figures are "dynamic". That is, points can be dragged along the screen, so that during the process the properties according to which the construction was made are maintained. In a DGE, dragging can be done by the user, through the mouse, which can determine the motion of different objects in two ways: direct motion, and indirect motion. The *direct motion* of a basic element (for instance a point) represents the variation of this element on the plane. The *indirect motion* of an element occurs when a construction has been accomplished. In this case, dragging the base points from which the construction originates will determine the motion of the new elements obtained through the construction (Mariotti, 2006).

The use of dragging allows one to feel "motion dependency", which can be interpreted in terms of logical dependency within the geometrical context. This becomes a key feature in the development of conjectures originating from the investigation of open problems in a DGE. In fact the use of *Cabri* in the generation of conjectures is based on the interpretation of the dragging function in terms of logical control. In other words, the subject has to be capable of transforming perceptual data into a conditional relationship between what will become hypothesis and thesis of a conjecture (Mariotti, 2006), a task which is not at all trivial. The consciousness of the fact that the dragging process may reveal a relationship between geometric properties embedded in the *Cabri* figure directs the way of transforming and observing the screen image (Talmon & Yerushalmy, 2004).

Olivero, Arzarello, Paola, and Robutti (Arzarello, et al., 1998, 2002; Olivero, 2002) presented a theoretical model describing how conjectures are produced by experts and how experts manage the transition from the conjecturing to the proving phase, passing from what they call "ascending control" to "descending control", through abductive processes. Their model shows that abduction plays an essential role in the process of transition from ascending to descending control, that is from exploring-conjecturing to proving. Abduction guides the transition, in that it is the moment in which the conjectures produced are written in a logical 'if…then' form. Once the conjecture is produced through this type of exploration, all the ingredients necessary for the proof are already present, and therefore this model suggests an essential continuity in the process exploring-conjecturing-validating-proving, for experts. Moreover, Arzarello and his team classified subjects' spontaneous development of dragging modalities (Arzarello et al., 2002), which have been referred to as the "dragging schemes".

Building on the studies described above, we (in this paper "we" refers to myself, under the guidance of my advisor Maria Alessandra Mariotti) became interested in integrating the model (or potentially building a new one) in order to describe in as much detail as possible the nature of some cognitive processes that occur when dragging schemes are applied during the conjecturing stage of open problem investigations in a DGE. We further hypothesized that it might be possible to introduce students to certain dragging schemes through in-class activities aimed at fostering their appropriation of the schemes.

Dragging schemes introduced to students

We introduced students to four basic dragging modalities:

- wandering dragging (in Italian: "trascinamento a caso");
- maintaining dragging (in Italian: "trascinamento di mantenimento");
- dragging with trace activated (in Italian: "trascinamento con traccia");
- dragging test (in Italian: "test di trascinamento").

Wandering dragging consists in randomly dragging a base point (draggable point, from which the construction originates) on the screen. Once a particularly interesting potential property of a figure is detected (for example, the possibility that a certain quadrilateral, part of the dynamic figure, might "become" a square), the user can use *maintaining dragging* to try to drag a base point and maintain the interesting property observed. In other words, *maintaining dragging* involves the recognition of a particular configuration as interesting, and the user's attempt to induce the particular property to become an invariant during dragging. Using Laborde's terminology such invariant would be denoted as a *soft* invariant (Laborde, 2005). With *dragging with trace activated* we intend any form of dragging after the trace function has been activated on one or more points of the figure. During the introductory lessons we only activated the trace on the base point that was being dragged. Finally the *dragging test* refers to a test that a figure can be put through in order to verify whether it has been properly constructed or not (Olivero, 2002; Laborde, 2005). During the introductory lessons we used the *dragging test* after having reconstructed the figure we were investigating, adding a new property (by construction) to it that we had hypothesized might induce the original interesting *soft* invariant to become a true (or *robust*) invariant. Thus the dragging test was applied to test whether the originally desired property was actually maintained during dragging. An expert might say we were using the *dragging test* to test a conjecture (even if it might not have been explicit at that point).

Model for the maintaining dragging scheme (MDS)

Let's consider a problem of the following type: "Given a certain step-by-step construction, make conjectures on a certain geometrical figure (that arises from such construction)". The model we constructed (and revised during the study) for the exploration of open problems of the type described above proceeds recursively through levels. In this paper we will concentrate on the phase that originates from experts' use of *maintaining dragging* (MD) to generate conjectures when solving an open problem of the type above. By "experts" we intend subjects for whom *maintaining dragging* (together with the other dragging schemes related with it in this phase) has become an acquired tool with respect to the task of formulating conjectures given an open problem situation. To frame these ideas and give meaning to the terminology used, we may refer to Rabardel and Samurçay (2001) and consider the acquisition of the *maintaining dragging* scheme as an explorative tool that occurs through a process of instrumentation, leading to the formation of a principle that becomes part of the user's knowledge. The principle (or rule), in our case, consists in knowing that while dragging P (the base point in consideration) on some *path* (see next section) the *Intentionally Induced Invariance* (III) will be maintained. Thus, the "creativity," for an expert user, lies in the subroutine related to the "discovery" of a *geometrical description of the path* (GDP).

Let us assume that the solver has encountered an interesting configuration (frequently through *wandering dragging*), and decides to investigate "the conditions under which (or "when") the initial construction falls into this case," using *maintaining dragging*. The general exploration scheme can be described as follows.

1. Choice of an III (*Intentionally Induced Invariance*) to try to maintain during dragging;
2. Application of the maintaining dragging scheme, which presupposes the hypothesis (in the particular case being explored) of the existence of a *path* to be made explicit through the perception of a new regularity or invariance. We refer to such regularity as an IOD (*Invariance Observed during Dragging*).
3. If in fact during the exploration it seems possible to maintain the III (and therefore the solvers believe that a *path* exists and search for a geometric description of it), the solvers propose a GDP in one of two ways:
   a) the solvers interpret the IOD (potentially with help of the scheme *dragging with trace activated*) as a curve they recognize during dragging;
   b) the solvers are not able to visually perceive an IOD (even with help from the scheme *dragging with trace activated*) so they use abductive reasoning (calling into action known geometrical theorems, rules of which the particular figure they are interested in is a case of) to give a geometric description of the *path.*
4. At this point the solvers link the III and the IOD (which now has a geometric description) through a conditional link (CL), passing from a form such as "the figure is a …when…" to a form such as "the figure is a …if (and only if) this point belongs to this curve". In general, this link can occur in two ways:
   a) the solvers link the III and IOD through a conditional link (CL) expressing it as a conjecture (or we might say "hypothesis" in a broad sense) that in their opinion describes the "behavior" of the figure they have explored (the conjecture is not a known theorem).
   b) or they link the two invariants through a conditional link (CL) choosing from their bag of mathematical knowledge (known theorems) a rule of which the situation seems to be a case of.

In either case, the establishment of a CL is key in the transition from "dynamic" to "static": the final outcome of a dynamic exploration is a conjecture – that may be successively refined – which has been crystallized into a "static" statement.

The path and its origin

The application of the *maintaining dragging* scheme, for solvers who have appropriated it, leads to the search of an invariance or regularity in the movement of the base point being dragged. When solvers apply this scheme and verify visually (and manually) that it *is* possible to drag the base point they are interested in and maintain the III they have identified, they already have in mind the idea of a *path* to be found, that is, a set of points on the plane with the following property: *when the dragged base point coincides with any point of the path, the III is visually verified*. Notice that this notion is not associated to a particular geometric shape (or curve), nor does it (necessarily) coincide with the mathematical notion of locus – the set of *all* points of the plane that guarantee verification of the III when the base point is chosen from it (the *path* may, in fact, be a proper subset of such mathematical locus). The solvers then engage in the search of a

geometrical description of such *path* (GDP). During this stage they may choose to activate the *trace* on the dragged base point in order to visualize the movement in a different way. Depending on the student's (the one who is dragging) ability in dragging, the movement (and associated trace if activated) will appear more or less "regular".
In some cases from the movement (and trace if activated) solvers can easily "see" the regularity and they are able to give a *geometrical description* of the *path* they have hypothesized. In other cases the III is difficult for the solver who is dragging to maintain, and therefore regularities are difficult to perceive from the movement (or from the trace, if activated). A *geometrical description* of the conceived *path* is therefore given in a different way. The solvers look at the figure in a "theoretical" way, and through an abduction, according to Peirce's description of abductive inference (Peirce, 1960, p. 372), come to a GDP. The solvers may propose successive more and more refined GDPs leading (ideally) to one that is a *P-invariant*, if P is the base point dragged, that is invariant for dragging of the specific base point P (Baccaglini-Frank et al., in press). Once the solvers have reached a GDP, in order to visually (and manually) test its CL with the III, they construct it as a Cabri object on their figure. There now is a "concrete" geometrical object potentially representing the *path* and that can be used to either drag the base point along "by eye" ("*soft" dragging test*) or to link the base point to and verify (*"robust" dragging test*) visually (and manually) the GDP and the CL.
The *path*, and the GDP in particular, seem to also play a fundamental role in the proving phase. More precisely, the GDP can be seen as a "bridge" to proof: the new relationship(s) between the invariants (that can be translated into geometric properties of the figure), that the GDP has made explicit during the conjecturing phase, become key in the proving phase and solvers at this phase can link back to them in order to successfully construct their proof.

Analysis of two students' exploration

Activity: Draw three points A, B, C. Then construct the line through A and C, and construct the parallel to this line, through B, and call it *l*. Construct the perpendicular line to *l* through C and construct point D as the intersection of this perpendicular line and *l*. Drag the points and make conjectures about ABCD. Then try to prove your conjectures.
Two Students' Response (an episode of their exploration): After making the construction (Fig. 1), through *wandering dragging* of the base point A the students noticed that ABCD could "become" a rectangle, for different (discrete) choices of positions for A. Then they saw that there seemed to actually be "infinitely many" choices of positions for A. This led them to believe in the existence of a *path* (see next section), and to believe that applying the MDS would be possible and insightful. So, with the intention of maintaining the property "ABCD rectangle" (III), the students dragged A and successively continued to drag maintaining the III with the trace activated, as shown in Fig 2.

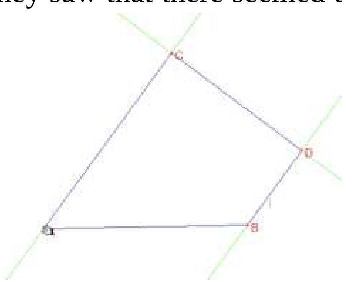

Figure 0: The students decide to drag base point A of the construction.

The students were able to interpret the trace in a geometric way, providing two GDPs, which also described the regularity in the movement of point A (IOD) that they were observing during the *maintaining dragging*. The students then proceeded to construct the *path* according to their second (more refined) geometric description of it (circumference with diameter BC), as shown in Fig 3. Once they constructed this circumference they were not convinced that *all* of it necessarily represented their hypothesized *path* (one student proposed that maybe only dragging A along a part of it would make ABCD a rectangle). To investigate further and reach an answer to their uncertainty, the students performed a robust dragging test, redefining A on the constructed circumference. This led them to believe that the III was (visually)

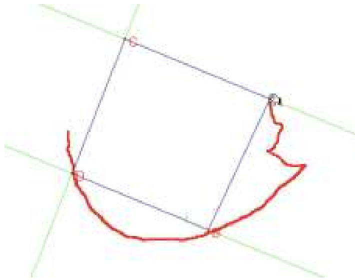

Figure 2: The students are applying the MDS with trace activated.

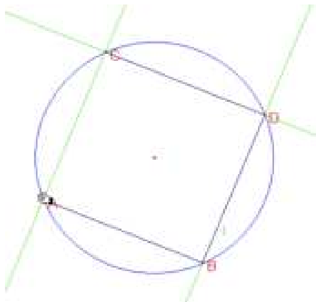

Figure 3: The students perform a robust dragging test.

maintained for the dragging of A along the *whole* circumference, thus confirming their CL between the III and the IOD. They finally formulated and wrote the following conjecture: "If A is on the circumference centered in O (midpoint of CB) then ABCD is a rectangle."

The students then successfully proved their conjecture, making fundamental use of the circumference (their GDP). This is how they proceeded:

1) D, together with A (by their hypothesis), B, and C, have to belong to the circumference, since the angle ∠CDB is right by construction (and so it is inscribed in the semi-circumference centered in O), and thus OB≅OD≅PC≅OA.

2) Triangles AOC and BOD are isosceles and congruent (∠ACO≅∠OBD because they are alternate interior with respect to the parallel lines AC and BD; and therefore also the other angles are respectively congruent), in particular ∠AOC≅∠BOD.

3) C, O, B are aligned by hypothesis; while A, O, D are aligned because ∠AOC≅∠BOD and thus vertically opposite angles (since C, O, B are aligned).

4) Therefore ABCD is a quadrilateral with diagonals that intersect at their midpoints and that divide one another in four congruent segments. Thus ABCD is a rectangle.

Conclusions

Our conclusions here are developed along two lines: considerations upon the model, and importance of the notion we introduce of *path*.

Our model, of which we introduced only a part in this paper, seems to describe cognitive processes that occur in connection with the MDS (and other dragging schemes as described). In fact we were able to interpret data generated by solvers who had appropriated the MDS in terms of phases of the model, as we did for the episode described above. Moreover having a model that seems to describe "experts' use" of the MDS is a very useful tool both for "catching" expert behavior (and thus complete appropriation of the MDS) in students, as well as for diagnosing where and how appropriation has failed. In fact many of the students' cognitive difficulties become describable in terms of "mismatches" between the model and the students' actual behaviors. Therefore the model becomes a powerful tool for designing new activities aimed at overcoming the diagnosed conceptual difficulties and at fostering complete appropriation of the MDS.

The *path* seems to play a main role in the generation of a proof, becoming a part of solvers' "reorganization and transformation" that occurs with abductive reasoning (Cifarelli, 1999). In particular the *path* and the GDP make explicit various new relationships between invariants of the dynamic figure, which can be translated into geometric properties. The *path* and the GDP therefore can become a powerful tool for the solvers to use in their proof, in order to link back to their reasoning and insights from the conjecturing stage, and thus bridge the potential gap between argumentation and proof. In this sense we believe it can foster cognitive unity (Boero et al., 1996; Pedemonte, 2003) and the production of proofs. Given these considerations, teaching students to consciously use certain dragging schemes, and make use of what we describe a *path*, can potentially accelerate and facilitate the entire process of making a conjecture and proving it. This seems to be a significant step towards the achievement of an important goal that the mathematics education community has set for mathematics teaching and learning.

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Mathematics brought to life by the Millennium Mathematics Project (Workshop Summary)

Nadia Baker
Schools Outreach Officer, Millennium Mathematics Project
Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road
Cambridge, Cambridgeshire, UK, CB3 0WA nb344@cam.ac.uk www.mmp.maths.org

Abstract
This workshop aims to share the success of the Millennium Mathematics Project (MMP) in bringing mathematics to life for students and teachers. A range of interactive enrichment programmes and their innovative ideas and resources will be shared. Two MMP outreach projects will be explained in detail as the main focus of the workshop: (1) The Enigma Project, (2) The Risk Roadshow. Both projects travel to primary and secondary schools as well as universities, organisations, science festivals and residential camps, both nationally and internationally.

Introduction to the Millennium Mathematics Project
The Millennium Mathematics Project is a mathematics education initiative for ages 5 to 19 and the general public, based at the University of Cambridge, UK. The aim of the MMP is to support mathematics education and promote the development of mathematical skills and understanding, particularly through enrichment activities. More broadly, we want to help everyone share in the excitement and understand the importance of mathematics.
The project consists of a family of complementary programmes, each of which has a particular focus:

- NRICH website - thousands of free resources designed to develop problem solving skills and subject knowledge.
- Plus website - an online magazine opening a door to the world of mathematics, including a careers library.
- Motivate video-conferencing programme - linking schools to professional mathematicians and scientists to engage in investigative project work.
- Visits to schools all over the UK and abroad by the Hands-On Maths Roadshow, Enigma Project, Risk Roadshow and NRICH staff.
- The Cambridgeshire Further Mathematics Centre - teaching, support and promotion of Further Maths A-level.
- Popular mathematics lectures for schools and the general public, held in Cambridge.
- STIMULUS programme - placing Cambridge student volunteers in local schools to assist with mathematics and science classes.

The MMP's various programmes have won many awards and our resources have been repeatedly commended by the UK Government's Department for Children, Schools and Families (formerly the Department for Education and Skills). Our web-based mathematical resources attract more than 2.3 million visitors worldwide, and around 30,000 pupils and teachers annually are involved in our hands-on activities. In February 2006 the Queen presented the project with the Queen's Anniversary Prize for Higher and Further Education (the counterpart to the Queen's Award for Industry), honouring 'outstanding achievement and excellence' at world-class level.

The Enigma Project
www.mmp.maths.org/enigma
www.enigma.maths.org
The Enigma Project aims to inspire interest in mathematics, science and history through interactive presentations and hands-on workshops focusing on the mathematics behind cryptography – the

science and mathematics of codes and codebreaking. Presentations include a demonstration of a real WWII Enigma cipher machine, loaned to the project by Simon Singh. All delegates attending this workshop will have the unique opportunity to see the Enigma cipher machine in action.

The opening 50-60min interactive presentation introduces students to cryptography. Pupils meet various ciphers used throughout history from Ancient Greece, they see the WWII Enigma cipher machine in action, find out how it worked, and discover why it is one of the most infamous cipher machines of all time.

Students then get the chance to put their problem solving and logical reasoning skills to the test by taking part in a circus of hands-on code breaking activities. The code breaking workshops last for 50-60min with class-sized groups of pupils working in pairs to crack cryptic messages using a variety of traditional and modern methods from Caesar shift ciphers to ISBN numbers. These activities and ideas as well as the Code Book CD-ROM, will be shared during this workshop for future use in the classroom.

The Risk Roadshow
www.mmp.maths.org/risk
www.understandinguncertainty.org
In January 2009, the MMP launched the new Risk Roadshow, in collaboration with Professor David Spiegelhalter, Winton Professor of the Public Understanding of Risk in the Department of Pure Mathematics and Mathematical Statistics, University of Cambridge.

The Risk Roadshow is part of a movement called 'risk literacy', aimed at teaching students the statistical skill they need to make sensible life decisions, which is often ignored in the curriculum. Below are links to the latest media reports of its importance and impact in schools.
'**Probability lessons may teach children how to weigh life's odds and be winners**': *The Times*, 5th January 2009
www.timesonline.co.uk/tol/news/uk/education/article5446920.ece
'**"Risk literacy" for high schoolers gains currency in bid to boost decision making**': *Chicago Tribune*, 1st March 2009
www.chicagotribune.com/news/nationworld/chi-london-risk_goeringmar01,0,631696.story
It bridges the gap between classroom Mathematics and its application in the real world through interactive presentations and workshops.
The opening 50-60min interactive presentation helps students to make sense of the real world through mathematics in situations involving risk, probability, chance and uncertainty. It helps to provide answers to questions such as: What risks do we face in the world? If it sounds too good to be true, what haven't you been told? Is it worth playing the lottery? How can you increase your chances of winning a game? Can you spot a scam before you fall for it?
Following the presentation, students participate in a 50-60min 'Mathionaire Gameshow' workshop, answering multiple choice questions based on the presentation. Each student will use an interactive handset to respond to the questions, receiving instant feedback of results. All delegates attending the workshop will have the opportunity to see and trial this technology.

Conclusion
Teaching and learning mathematics presents challenges. The Millennium Mathematics Project offers an abundance of educational resources, ideas and opportunities to help with these challenges, bringing mathematics to life. The Enigma Project and Risk Roadshow are unique and innovative experiences which challenge, excite and motivate students of all ages and abilities. All delegates attending this workshop will see this for themselves and take away valuable resources and ideas to enhance the teaching and learning of mathematics in their respective countries

Pre-service teachers' mathematics profiles and the influence thereof on their instructional behaviour

Hayley Barnes
Lecturer in Department of Mathematics, Science and Technology, Faculty of Education, University of Pretoria, Pretoria, South Africa hayley.barnes@up.ac.za

Abstract

In this paper the notion of "mathematics profiles" and "instructional behaviour profiles is introduced. A brief explanation of what these profiles are and how they were constructed and represented for pre-service mathematics teachers is provided. An example of one of the participants' profiles is included as an example. The influence of the pre-service teachers' mathematics profiles on their instructional behaviour is then discussed. This is done with regard to using the mathematics profiles as a potential tool to optimise the development of pre-service mathematics teachers' instructional behaviour towards a more reform-oriented approach.

Introduction

How does one mathematically determine whether the gradient of a straight line is positive or negative? I asked this of a mathematics student teacher I was observing and was surprised that he could provide no mathematical explanation. Instead he explained that a positive gradient could be recognised by the fact that if you were walking along the line, it would be like walking up a mountain so you would feel really positive. On the other hand the negative gradient or slope is like coming down a mountain and one usually feels negative coming down a mountain. He confessed that he relied mainly on memorisation to explain mathematical concepts.

This is one of many similar examples where mathematics is endorsed as a process of rote memorisation rather than a discipline requiring understanding. During my role as a mathematics methodologist (or specialisation lecturer), I became increasingly frustrated and concerned at the low level of content knowledge as well as teaching and learning strategies being demonstrated by pre-service mathematics students during such practical teaching periods. Despite the global reform being initiated in mathematics education, the students continued to demonstrate a traditional and rote learning approach to teaching mathematics with only superficial motions towards a more constructivist paradigm. With their own experiences of mathematics teaching at school most likely being limited to a traditional approach, and the lack of deep change occurring in most schools they would end up teaching in, I began to wonder how we can most effectively achieve the reform in pedagogy we are aiming towards. And how much of this may be dependent on the mathematical content knowledge or what I have since come to term the "mathematics profile" of teachers.

Using the literature, I identified important components or indicators of content knowledge (subject matter knowledge, pedagogical content knowledge, conceptions of mathematics and beliefs about the teaching and learning of mathematics) and used data from the final portfolios of seven students[1] in a one-year Post Graduate Certificate in Education (PGCE)[2] programme to compile mathematics profiles for each student and analyse the influence thereof on their resulting instructional behaviour.

Conceptual framework

The research was conducted from within a social constructivist paradigm. Ernest (1991, 1998) suggests social constructivism as a philosophy of mathematics and discusses it also as a philosophy of mathematics education. Through this lens mathematics is viewed as a social construction and knowledge is a result of a process of coming to know including processes leading to the justification of mathematical knowledge

The two main constructs in the study were the mathematics profiles and the instructional behaviour of the participants. The mathematics profile construct was determined with respect to four components, namely, subject matter knowledge, pedagogical content knowledge, conceptions of mathematics and beliefs about the teaching and learning of mathematics. The instructional behaviour construct was studied with regard to participants' use of a traditional versus reform approach to teaching, and whether learners were afforded an authoritarian versus democratic style of learning.

The conceptual framework draws extensively on the work of Ernest (1988, 1991, 1998) in analysing the two main constructs of mathematics profiles and instructional behaviour. However, where there was not sufficient literature in Ernest's work, the conceptual framework was supplemented by other authors such as Ball (1988, 1990, 1991) for the subject matter component, Shulman (1986) and Veal and MaKinster (2001) for the pedagogical content knowledge component, Thompson (1984, 1992) for

[1] These students were not all enrolled for the course in the same year. One of the students completed the course in 2006, two of the students completed the course in 2007 and the other four completed it in 2008.

[2] This is a post graduate certificate that students enrol for once they have obtained an initial Bachelor's degree in order to qualify as teachers.

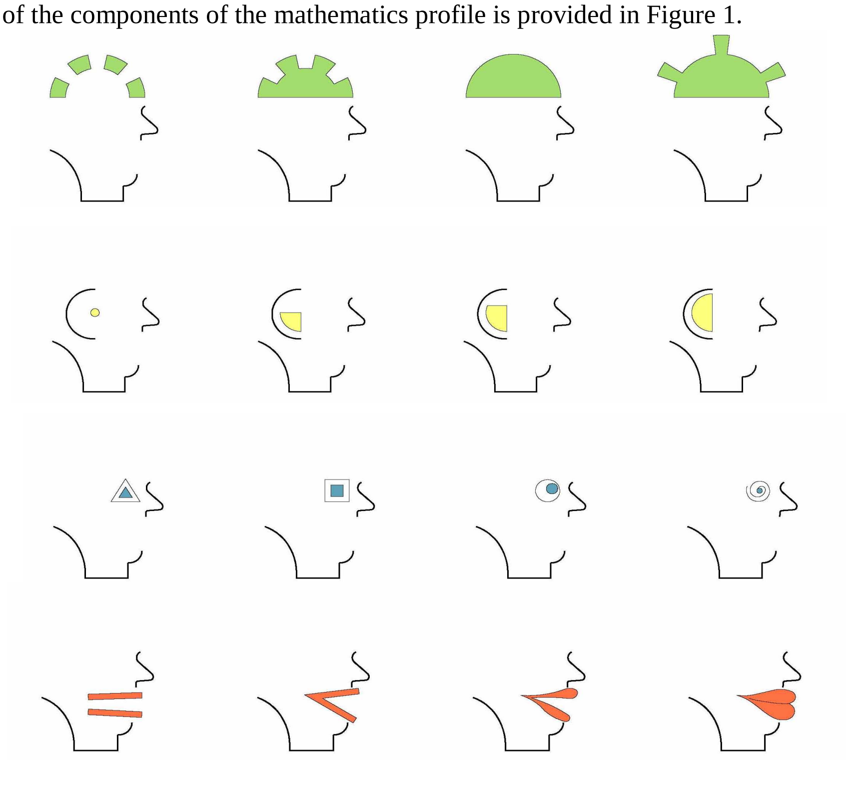
the conceptions component and Goldin (2002), Boaler (2004) and Davis (1997) to inform the instructional behaviour analyses.

Methodology

A qualitative case study design was used as the research methodology for this exploration. The case study was carried out retrospectively or post-hoc, in that the data set was only analysed once the students had completed their PGCE course. A slightly alternative data collection technique was used in this qualitative approach in that interviews were not conducted with any of the participants. The final portfolios that participants handed in were the main source of data. This means that the participants themselves initially selected the “data” they chose to present. I then did the first data reduction in selecting reflections and other entries from participants’ portfolios to compile participant reflections. These were taken directly from the portfolios and written in the voice of each participant. The second data reduction was done in writing the researcher reflections. These reflections were written as a response to the participant reflections based on my experiences and assessments of the participants as their specialisation lecturer. In the third data reduction, the participant and researcher reflections were deductively analysed using the relevant categories discussed in the literature. This analysis was then presented visually displaying an initial and final mathematics profile for each participant and placing each of these in a sub-quadrant on the instructional behaviour Cartesian plane. This plane was made up of the traditional/reform teaching continuum (x-axis) and authoritarian/democratic learning continuum (y-axis). These visual representations facilitated the cross-case comparison.

Presentation of data

As indicated in the methodology, participant reflections, researcher reflections and visual representations of the mathematics profiles and instructional behaviour of participants were used in presenting and analysing the data. For the scope of this paper, a visual representation of the profiles of only one of the participants is provided (Barnes, 2009). A summary and brief explanation of each of the categories of the components of the mathematics profile is provided in Figure 1.



Category 1 2 3 4

Figure 1 Illustration of the four categories of each component of the mathematics profile

The *head* of the face represents the *subject matter knowledge*. In the visual representation, the category on the extreme left indicates obvious and fundamental conceptual gaps in the participant’s subject matter knowledge. In the second category, less fundamental conceptual gaps were evident with some relational coherence of the content. The third category indicates that the subject matter knowledge appeared sufficient with no gaps evident in terms of errors or lack of mathematical understanding observed during the course of the year. The final category on the right depicts subject matter knowledge that is not only relational but also able to extend into other learning areas where necessary.

The *ear* depicts the pedagogical content knowledge. Reasons for this include that much of the pedagogical content knowledge of a student teacher is taken in by what they hear in class at university and what they heard at school. A large part of this in their own teaching practice is their ability to hear the learners, their errors, their thinking and where they are at in their thinking. The category on the far left indicates an incomplete pedagogical content knowledge for a pre-service teacher. The categories towards the right of the continuum show varying levels increased pedagogical content knowledge.
The *eye* illustrates each participant's view or conceptions of mathematics (for obvious reasons). The varying shape of the eye in the four categories indicates a movement from seeing mathematics in its absolutist form as a limited, rigid, structured and rule-bound subject on the far left category to a more dynamic, interrelated and continually evolving subject that is more in line with the constructivist/problem-solving view as expressed by Ernest (1991), in the category on the far right.
Finally, the *mouth* represents the beliefs about the teaching and learning of mathematics that each participant verbalised or expressed. In differentiating between these belief categories, the role of the teacher can be either a transmitter on the far left, instructor, explainer or a facilitator on the far right of the continuum. A transmitter is a device that transmits specific information or signals to "passive receptors" or receivers that receive the signal but do not transmit back. When a transmitter sends out a signal to a transceiver though, the transceiver sends back information. In my view the teacher in the role of the transmitter believes the teacher is an expositor and although they are aware of the learners in the classroom, they talk to them as passive receptors without expecting input. The instructor and the explainer, however, both view the learner as a transceiver that they expect to be more active and communicate with them. The difference though is that the instructor demands a much lower level of input and response from the learner than the explainer, who tends to require responses that demonstrate understanding. Finally, the facilitator has the fuller, closed lips indicating that, similar to the explainer, they also expect learners to communicate their understanding and in my view, they see learners not only as transceivers but as decoders. Facilitators therefore tend to continually demand more high-level mathematical reasoning and facilitate discussions that elicit this. In such cases, the learners are supported to do more of the thinking and construction of knowledge with the facilitator guiding the process (hence the closed mouth in the visual representation).
Similarly to the approach applied to the development of the mathematics profiles, each of the traditional/reform and authoritarian/democratic learning continuums (each forming an axis of the Cartesian plane in Figure 2) was divided into four equal divisions. However, these are not differentiated into categories, but rather form four smaller sub-quadrants in each of the four main quadrants of the Cartesian plane. I purposefully avoided using numbers on the Cartesian plane so that this remains a representation of their changing instructional behaviour, as I see it, without attaching a value or measurement to it. An initial and final quadrant for each participant was derived according to their position on each of the traditional/reform teaching and autocratic/democratic learning continuums. Visual representations such as the example provided above were constructed for each of the seven participants in the study and these facilitated the cross-case comparison. Four main aspects emerged from the comparison.

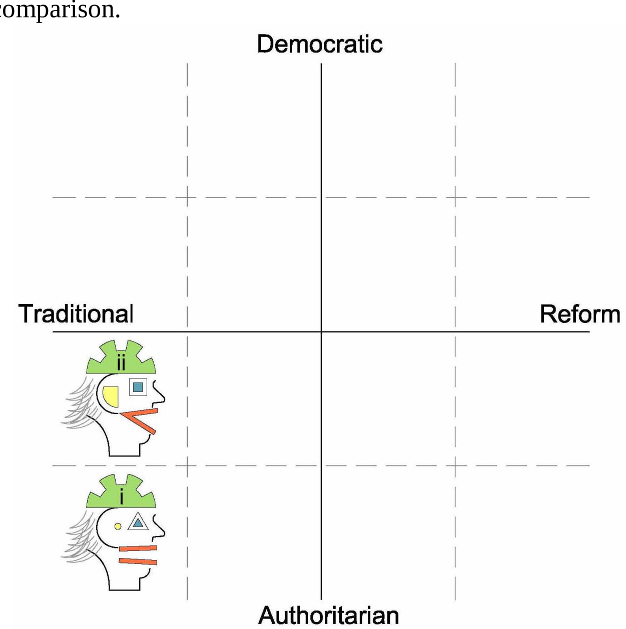


Figure 2 Example of a visual representation of a participants' changes in profiles

Firstly, the component of subject matter knowledge does appear to play an important part in enabling or constraining the changes in pre-service mathematics teachers' instructional behaviour. Secondly, I am suggesting that not just reflecting on one's practice/experiences but that the quality of these reflections may affect the extent of positive change pre-service teachers make in their instructional behaviour. Thirdly, I suspect that encouraging students to access and read more literature in the mathematics and mathematics education domain is something that could be considered developing and improving pre-service teachers' mathematics profiles, with particular reference to their conceptions and beliefs. Finally, it appears that an improvement in pre-service teachers' pedagogical content knowledge does not necessarily have the extent of influence on changing their instructional behaviour that was expected.

These four aspects have important implications for training mathematics teachers in the Further Education and Training Phase. As I reflected on the current intended outcomes and content of the PGCE course that forms the context for this study, I realised that we spend most of the year focusing on improving the pedagogical content knowledge of our students (both general and more domain specific) and on training them to approach teaching and learning in a more reform and democratic-orientated way. Research indicates that this type of approach to teaching and learning is more likely to result in independent and critical-thinking learners. However, the mathematics profile appears to have more of an influence on the instructional behaviour of students than I originally anticipated. As long as we continue trying to focus on training and changing the instructional behaviour of our students without considering their mathematics profiles, we will not be able to achieve our intended outcomes. I am therefore suggesting that evaluating students' initial mathematics profiles and then working to improve and expand the necessary components may be more effective in reforming students' instructional behaviour. The emphasis on improving pedagogical content knowledge without considering students' conceptions of mathematics and their beliefs about the teaching and learning of mathematics does not appear to enable this intended reform. The issue of how best to assist students who exhibit conceptual gaps in their subject matter knowledge also needs to be considered owing to the enabling or constraining impact of this component suggested in this study.

Conclusion

The results of the study indicated that the mathematics profile of a pre-service teacher of mathematics has a considerable influence on their resulting instructional behaviour. The visual representations suggest that the participants who made the most substantial changes in their mathematics profiles also made the most significant changes in their instructional behaviour. I am not trying to indicate a mathematical direct proportion here in that more changes in the mathematics profile imply more changes in the instructional behaviour. Rather I am fore-grounding the trend that the students with final mathematics profiles with components predominantly in the third or fourth category (see Figure 1) demonstrated the most movement in terms of their instructional behaviour. Students' whose final mathematics profiles were predominantly in Category 1 and/or 2 of each component similarly demonstrated the least movement in their instructional behaviour. This suggests that focusing on all of these components of the mathematics profile in teacher training is an important aspect in reforming pre-service teacher's instructional behaviour.

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The Relationship between Didactics and Classroom Management: Towards New Tools for the Training of Math Teachers

Michel Beaudoin, PhD
Université du Québec en Outaouais michel.beaudoin@uqo.ca
Catherine Lanaris, PhD
Université du Québec en Outaouais catherine.lanaris@uqo.ca

Abstract

This paper presents the interest of the "instrumental conflict" concept, developed by Marquet (2005), to understand the relationship between didactics and classroom management in the training programs of math teachers. It also shows some results of a survey, conducted in 2008 among pre-service teachers in the Université du Québec en Outaouais (Canada), revealing a perceived gap between both domains. However, those two domains are closely related during teaching in the classroom. The paper also presents a plan to better understand and improve the situation. Cooperation between classroom management and didactics specialists is highlighted.

Context

Didactics and classroom management are two very distinct domains in the teaching of mathematics. Most research in the education field focuses on one of these aspects, and not on the relationship between the two. In the university setting, activities offered in the didactics classes are very different from the ones proposed in the classroom management classes. The theoretical models on which these two disciplines base themselves are so different that one could, to a certain extent, believe that they are not related. However, in the classroom setting, the reality that teachers face is very different. In fact, when one teaches mathematics, the planning of learning activities, their execution, and the management of classroom atmosphere are closely interrelated. The authors of this paper – respectively mathematics didactics expert and classroom management specialist – are proposing an approach to better understand the situation and find solutions for it in a collaborative perspective.

The importance of the relationship between didactics and classroom management can be illustrated by a situation where one learns through the solution of problems: in this educational context, in tune with the direction of the training program, the mathematical content is studied through situations in which the learners do not know, at first, the method to solve the problem. This approach seeks to foster "situational interest" (Pallascio, 2005; Beaudoin, 1998) in order to motivate students to invest the necessary effort to discover the underlying lessons. What happens if the situation does not generate this interest for the student? In addition to the fact that the lessons are not assimilated, problems related to the learning atmosphere might become more pronounced. The interest in mathematics is actually closely related to the involvement of learners in the activities offered, and consequently, in the classroom management aspect (Beaudoin, 1998), which reinforces the importance of paying attention to the relationship between the didactics of mathematics and classroom management.

Conversely, learning based on mathematical problems can pose certain challenges for the teacher in the management of his or her classroom. These challenges might impel the teacher to avoid this educational approach. In such an educational context, learners must invest efforts in finding a solution to a problem for which they do not possess all the prerequisite knowledge, and the teacher acts more as a guide than an instructor. Some teachers are concerned about the possibility of losing control of their class in this type of setting, where activities are less controlled by the teacher. Some also fear that the time spent on the exploration of problems could detract from the learning of the mathematical concepts and processes that are part of the required curriculum.

Teachers must now work in a unique context with regards to student groups. These include: students with learning and behavioural difficulties that must be integrated into regular classrooms; students of different ages; students in specialized programs (sports-study programs, international school programs, etc.); young adults groups. While the classroom management aspect of a teacher's training takes these parameters into consideration, the discourse of didactics specialists does not address them.

We do not believe that the relationship between subject matter didactics and classroom management has been sufficiently studied, particularly in the existing context in schools. Where, in the training programs for teachers these are presented as two distinct areas, they are closely related in practice. It is through practice and reflection that the teacher (whether teaching or in training) usually succeeds to integrate these two fundamental dimensions of the art of teaching.

Preliminary work conducted at the Université du Québec en Outaouais

During the fall of 2007, a survey was conducted with teaching graduates at the Université du Québec en Outaouais. These students had just completed their fourth (and last) internship. They were enrolled in the preschool and primary school teaching bachelor program, school adjustment bachelor program and secondary school teaching bachelor program (French, Mathematics, Social Studies). A total of 87 students completed the survey. The questionnaire was based on the indicators of the tool for the assessment of professional competencies ("référentiel des compétences professionnelles") used for their internship.

From the bank of indicators, we selected six that were related to didactics, five related to classroom management, and three pertaining to both of these disciplines. Students were invited to select six indicators that appeared to be the most important, and to prioritize them. For each of the six selected indicators, students rated their presumed level of ability.

In all three above-mentioned programs, the indicators that pertained simultaneously to both domains were selected as most important by students. Students also indicated that they felt most proficient in the areas that combined both disciplines. These results should be compelling to those who train teachers, both in the fields of initial training and continuing education. Even if the theoretical university training does not prepare them explicitly, students consider that the intersection where didactics and classroom management meet is a priority. The role of practical training (internships) in the combined application of these two disciplines, which pertains to teacher's contact with his or her students, is therefore very important.

On the other hand, graduating students considered less important such elements as interpretation of the subject matter and mastering of basic techniques in classroom management. They also feel less proficient in these fields. This situation does not seem ideal, to the extent that new teachers could lack important competencies related to the interpretation of programs and to the use of basic techniques in classroom management. Lefstein (2005) has underlined the distance that seems to exist between classroom management and what pertains to didactics in the material used to train teachers. An interpretation of the disconnection between these two dimensions would allow to better understand this challenge and to remedy it.

Goals

We would like to better define the issue regarding the relationship between classroom management and didactics to validate a training model for teachers that takes it into account. Our goals are as follows:

- Document the potential conflicts between didactics and classroom management with teachers in training, particularly in the field of mathematics.
- Consider the potential impact of these conflicts.
- Observe the manifestation of these conflicts with the teaching interns.
- Develop and validate a training model that could solve these conflicts.

Conceptual framework

The mathematics curriculum of the Province of Québec favours the acquisition of knowledge through problem solving (Pallascio, 2005) and the use of democratic models for classroom management. The theory of didactical situation (Brousseau and Balacheff, 2004) serves as an important base for this curriculum and for the training program for teachers. This model covers the relationship between knowledge, the learner and the agents responsible to guide the learning. However, it barely takes into account the behaviours of the student or of the student group that may help or hinder the learning process. In the field of classroom management, many documents (Charles, 1999; Nault; 1998) define classroom management as the ability of the teacher to consider the complexity of the environment, the needs of the students and the objectives of the program.

Although the link between didactics of mathematics and classroom management has been acknowledged by a few authors (DeBlois, 2009), the study of this relationship has not been thoroughly documented and the intersection of didactics-classroom management does not explicitly urge the renewal of practices in the teaching of mathematics.

Moreover, the concept of "instrumental conflict" developed by Marquet (2005; 2003; 2004) in the context of computer environments for human learning ("Environnements Informatiques pour l'Apprentissage Humain") presents a great deal of interest for the study of the relationship between didactics and classroom management. Marquet defines instrumental conflict as the consequences of an interference that could arise between one of several tools at play in an instrumental situation. The educational intervention of the teacher might be considered to be a situation where he or she must simultaneously use and apply several tools in order to sustain the learning of students. According to Rabardel and Pastré (2005), this constitutes an instrumental situation, one in which the educational and didactics approaches can be considered to be artefacts that are transformed into tools by teachers.

Rabardel (1995) has underlined two simultaneous processes in the development of a tool: the *instrumentation* and the *instrumentalization.* The instrumentation involves the emergence of a utilization approach on the part of the subject, whereas instrumentalization involves the adaptation of the tool to the subject. Obstétar (2008) and Marquet (2005) expanded on the concept of the instrument to include objects in a learning situation and gave rise to the potential conflicts between didactics and educational approaches. The study of these potential interferences in the instrumentation and instrumentalization processes in the didactics and educational approaches amongst teachers in training form the basis of our contextual framework.

Measures

By basing ourselves on our respective expertise in the field of didactics of mathematics and of classroom management, we will put the following measures forward.

<u>Conflict documentation</u>

The preliminary study conducted at the Université du Québec en Outaouais has allowed us to identify, through the input of students, some potential areas of conflict between the didactics and educational aspects. We will further document these conflicts through a questionnaire to be completed by internship supervisors and associate teachers in schools.

<u>Study on interns' conflicts</u>

We will observe all intern mathematics teachers. These subjects will be divided between internships 2, 3, and 4 of their teaching training program. We have excluded the first internship, as it is based on observation. These interns will be observed by a classroom management specialist and a didactics expert during two classes where they will be asked to accompany students in their learning. The conflicts pertaining to the didactics and educational aspects will be identified, and the student will be required to provide comments on the subject. The analysis of the comments will allow for a better comprehension of the perception of the conflict from the student, and will allow him or her to find a solution.

<u>Model development</u>

The documentation and study of these conflicts with the interns will allow us to develop training activities that will integrate both dimensions, particularly in the context of learning through problem solving situations. At this stage of our reflection, we can predict the following elements:

- Interventions by the mathematics didactics expert in the classes given by the classroom management specialist, and vice-versa, particularly for the management of learning through problem solving situations;
- Integration of elements related to classroom management in the planning of activities for the teaching of mathematics during internships;
- Development of training sessions for the continuing education of practising teachers, under the joint responsibility of a mathematics didactics expert and a classroom management specialist.

Impact in the teaching of mathematics

1. Learning through problem-solving situations

This education approach, which is a priority for the mathematics curriculum, is undergoing implementation difficulties in Québec. Classroom management could be one of the causes. Our research will contribute to better implement this educational approach, and to foster greater success for students.

2. Use of ICTs in the teaching of mathematics

The integration of ICTs in the learning of mathematics also seems limited by problems associated to classroom management. The results of our model will be able to be reinterpreted in this context.

3. Differentiated instruction

Mathematics teachers are now asked to work in environments where differences between groups and learners within a same group are particularly important. The management of these differences is difficult because of the sequential aspect of several mathematical learning processes. Our work will help to better equip teachers in this field.

4. Initial training and continuing education of teaching staff

Our work will have impact on the initial training and continuing education of mathematics teachers. The preliminary results we have obtained have already elicited thoughts on the compartmentalization of the training fields of teachers. Our project will allow for the implementation of continuing education for practising mathematics teachers, done jointly by a didactics expert and a classroom management specialist.

Impact in the research field

In terms of knowledge, we will be able to develop a niche that has been, up until now, neglected by research. While some authors write about the importance of good classroom management in order to encourage the optimal functioning of didactics and vice-versa, the approach that we are proposing will enrich knowledge by studying it through instrumental conflicts. The project will also allow for a broadening of the scope of the instrumental approach, developed through the appropriation of technologies, to another field. Our work will also allow us to document the importance of studying the impact of the conflict between the different approaches used by the teacher (Trouche, 2007) on the students.

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Learning Mathematics through Scientific Contents and Methods

Astrid Beckmann
University of Education Schwäbisch Gmünd, Germany astrid.beckmann@ph-gmuend.de

Abstract

The basic idea of this paper is to outline a cross-curricular approach between mathematics and science. The aim is to close the often perceived gap between formal maths and authentic experience and to increase the students' versatility in the use of mathematical terms. Students are to experience maths as logical, interesting and relevant through extra-mathematical references. Concrete physical or biological correlations may initiate mathematical activities, and mathematical terms are to be understood in logical contexts. Examples: physical experiments can lead to a comprehensive understanding of the concept of functions and of the intersection of medians in triangles. Biological topics can lead to the concepts of similarity and proportion as well as to the construction of pie charts. In the European ScienceMath Project a variety of teaching modules was developed and tested in secondary schools.

Background

The European ScienceMath Project is a co-operative project between universities and schools in Germany, Denmark, Finland and Slovenia (coordination: University of Education Schwäbisch Gmünd, Germany; www.sciencemath.ph-gmuend.de). Its central objective is the development and testing of teaching modules for the promotion of mathematical literacy. A cross-curricular approach is used in the natural sciences, in particular in physics, but also in biology and chemistry. Through extra-mathematical references, students are to experience maths as appropriate, meaningful and interesting. The learning in meaningful contexts is designed to contribute to an intuitive mathematical understanding. The idea is, on the one hand, to close the gap between formal maths and authentic experience through the use of contexts and methods taken from the natural sciences (Kaput 1994), and, on the other hand, to allow students to experience the versatility of mathematical terms (Michelsen & Beckmann 2007). That way, mathematical content can be learned in realistic and meaningful contexts and the students' sense of reality can be increased through mathematical insight. In the following, four teaching modules will be presented that were developed at the University of Education Schwaebisch Gmuend. These are concerned with promoting the acquisition of mathematical terms through physical experiments in the fields of the function and the intersection of the median in triangles (centre of gravity) as well as promoting students' competences in the fields of similarity/geometrical proportions and in the construction of pie charts in biological contexts.

Promoting the acquisition of the Term Function through Physical Experiments

Experiments are well known as methods used in the natural sciences. In maths teaching, they can become part of a new form of instruction. Due to my own extensive experiences and trials in teaching, they are recommendable here, too. There is a lot to be said for experiments in connection with functions because

- the activities in an experiment correspond to the respective aspects of the functional concept: Collecting the individual data equals the aspect of correspondence; collecting the data of a complete series of measurements corresponds to the discrete co-variation, respectively the process level, and transferring the data to a graph leads to continuous co-variation and the object level (Malle 2000, Vollrath 1978, Dubinsky & Harel 1992, Swan 1982, cf. also House of Functional Thinking, Hoefer 2008).
- in carrying out experiments, essential aspects of functional terms are actively experienced. Example: the idea of continual co-variation is experienced, when observing an experimental car that is in motion and continually increases the distance travelled (with stop watch). The object idea, such as anti-proportionality, can be experienced by pumping up a tire with a closed bicycle pump: in reducing the air volume inside the pump, one simultaneously notices an increase in the air pressure.
- experimenting addresses various objectives simultaneously (data collection, functional context, modelling) and allows students to gain experience in various relations to reality (cross-curricular context, every-day experience and concrete quantities, instead of x and y).

Using physical experiments, to promote the understanding of functions, was once again motivated by the results of numerous tests and international comparative studies in the past years and decades, all showing that many students have only a limited understanding of the term function (i.e. Vinner & Dreyfus 1989, PISA-consortium 2004, Beckmann 1999). A special deficit here concerns the inability to interpret graphs correctly and to recognize the functional connection between two variables. It is here that experiments offer a particular opportunity to demonstrate, experience and talk about this functional connection in a concrete way (Beckmann 2006).

Using physical experiments, to promote the understanding of functions (and also of the term variables), was meanwhile tried out with 400 high school students between the ages of 13 and 17 at various school levels (cf. Hoefer & Beckmann 2009, Beckmann 2006, 2007, Zell 2008, Beckmann & Litz 2009). The emphasis here was on experiments, made up of simple materials, to ensure an uncomplicated implementation in maths teaching. However, there were, in addition, cross-curricular tests, in which the reference to maths as well as to physics was discussed (Lukoscheck 2005, Haas & Beckmann 2008). In class, the students carried out the experiments by themselves in groups. They were either observed directly or via video and audio. In addition, the worksheets, that were prepared for each experiment, were evaluated (Beckmann 2006).

The important results, made in these trial runs, were, that the reflection on the data sheets and the discussion of the graphs stimulated the co-variation aspect as well as the formulation of hypotheses in relation to reality, on the background of the co-variation aspect. Furthermore, presentational activities, such as the data's relation to a co-ordinate system, respectively, modelling activities in general, were stimulated. In their descriptions, the students partly used the link with reality and the model simultaneously: "In an additionally plotted graph we were able to see exactly how much light there was at the beginning of the tunnel and how the light intensity was gradually reduced". After carrying out the experiments, many students were able to interpret the graphs. The following graph (figure 1), for example, was no longer interpreted as "The car is rolling downhill" but as "The car is stopping at a crossroads.", "The car had to brake because of an obstacle." "It is in a traffic jam.".

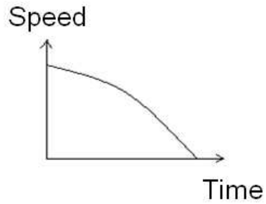

Figure 1

Functional correlation between speed and time

Point of Intersection of the Medians in a Triangle and the Term Physical Centre of Gravity

In connection with the topic "Special Points in a Triangle", experiments enable a more global view and more global links, in that the point of intersection of the medians in a triangle appears as only one of the numerous examples for the centre of gravity in bodies. The basic idea of the realisation in class is to enable students to experience the term centre of gravity by placing it in real situations, respectively, in natural science contexts and through experimental activities. All known methods for determining the centre of gravity, such as the hanging or weighing methods, (Heine & Prommersberger 1999) result from the definition of the point of gravity as the point of intersection of all lines of gravity. In the hanging method, the given bodies are hung up in at least two different positions, the lines of gravity are marked and their point of intersection is determined (graph 2). The rule to be learned from this is, that the point of intersection of the medians in a triangle is its centre of gravity and that all axes of symmetry are lines of gravity.

In station work in secondary I, sheets of cardboard and bodies with fixtures for hanging (from the geometry collection or at random composed of Duplo blocs) are made ready for the hanging and weighing methods. The stations stimulate students on the one hand to find the points of gravity experimentally and on the other hand to establish them graphically and check them experimentally. There is also a dynamic geometry system, available for the latter, with the help of which the quality of the median's intersection point in a triangle can be established. Printing the triangles and sticking them onto cardboard reveals the connection with the point of gravity.

In the station work at secondary II level, the analytical description of the points of gravity is made additionally. As an example, the following applies for the point of gravity S in a triangle ABC with the position vectors of S: $\vec{s} = \frac{1}{3}(\vec{a} + \vec{b} + \vec{c})$, which leads us to the interesting question whether that correspondingly applies to the quadrangle $\vec{s} = \frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d})$. The result of the hanging methods is, that the validity of the equation depends on the distribution of the mass in the quadrangle. In a full trapeze, e.g., this results in a different value from that of a “hollow trapeze with an uneven mass distribution at the corners” (figure 2).

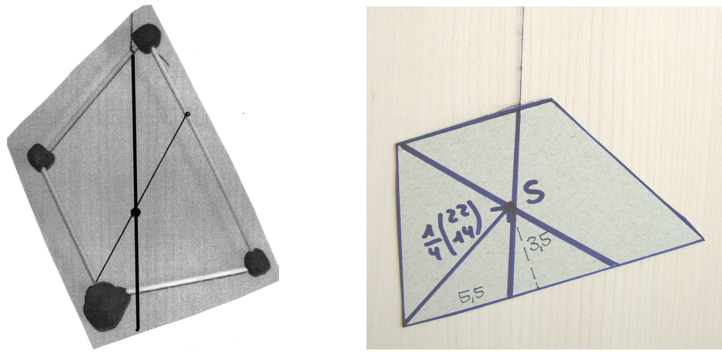

Figure 2
Definition of the point of gravity with hanging method,
Left: hollow trapeze with “heavy corner”,
co-ordinates of the point of gravity S (4.8/4.2).
Right: cardboard trapeze, same size, point of gravity (5.5/3.5).

The difference in the position of the point of gravity in quadrangles of similar size but a different distribution of the weight, can be confirmed mathematically. For the trapeze in figure 2, with corners weighted by plasticine of the masses $m_1=m_2=m_3=0.1$gr. and $m_4=0.2$ gr., the point of gravity works out as:

$$\vec{s} = \frac{1}{0{,}1+0{,}1+0{,}1+0{,}2}\left(0{,}1\begin{pmatrix}0\\0\end{pmatrix}+0{,}1\begin{pmatrix}12\\0\end{pmatrix}+0{,}1\begin{pmatrix}8\\7\end{pmatrix}+0{,}2\begin{pmatrix}2\\7\end{pmatrix}\right)$$

$$=\frac{1}{0{,}5}\left(\begin{pmatrix}1{,}2\\0\end{pmatrix}+\begin{pmatrix}0{,}8\\0{,}7\end{pmatrix}+\begin{pmatrix}0{,}4\\1{,}4\end{pmatrix}\right)=2\cdot\begin{pmatrix}2{,}4\\2{,}1\end{pmatrix}=\begin{pmatrix}4{,}8\\4{,}2\end{pmatrix}$$

In further tasks, corresponding observations of bodies, such as pyramids, are stimulated. A linked procedure is characteristic of this approach, in that the point of gravity is worked out experimentally with the help of the hanging method of real bodies and also mathematically using analytical geometry and integral calculus. A three-dimensional representation and an examination of the point of gravity on the computer, e.g. with the help of Mathematica, enrich the overall experience of the learning process (Cf. learning material at www.sciencemath.ph-gmuend.de).

Similarity and geometrical Proportions in biological Contexts

The topic “similarity” plays a central role in maths instruction. A definition can, e. g., be formulated with the help of centric dilation, in that by the term similar figures those figures are understood, that mutually overlap through centric dilation. The original figure and the centrically dilated figure correlate in length, surface and volume, and this can be expressed by the relevant dilation factor k, respectively k^2, k^3. In biology, there is no similarity in a mathematical sense. Biologists rather speak of allometry, which describes the differences in the proportions (e.g. of organs and limbs) of similar looking animals. By comparing the volume m or the measurements of live animals or their photographs, allometric features can be ascertained by assessing the proportions of externally similar animals, such as domestic cats and wild cats, white and black rhinos, pups and fully grown animals. Example:

$$\frac{m_{tiger}}{m_{domestic\,cat}}=\frac{100kg}{5kg}=20$$

The dilation factor is therefore: $k=\sqrt[3]{20}=2{,}7$. Their shoulder heights (acromion) relate as 1: 2.7, while their surfaces relate as 1: 7.3. Allometries have their biological reasons, such as the different functions of internal organs or the “puppy scheme”, according to which young animals have a bigger head, in order to look more appealing and thereby stimulate a protective behaviour in the parents.

The relationship between surface and volume is also of particular importance. In maths teaching, an important objective is to realize that bodies of the same volume may have totally different surfaces. This can, e. g., be shown with the help of a cube of 1m edge length. If one changes its original dimensions of 1:1:1 edge length to 1:1:2 and then to 1:1:4 etc., retaining the same volume, this leads to a continuous elongation of the cuboids. The biological consequences of the relationship between volume and surface show in the specific body shapes and forms of behaviour of the animals. Dragonflies, e. g., have a slim long body to be able to fly fast, while a beetle of the same volume is rather round and compact. Small endotherms such as field mice and hummingbirds must constantly consume high energy food, in order to be able to equalize the unfavourable ratio between volume and surface (Glaeser 2004).

In the teaching module, these relationships are worked out individually with the help of worksheets (Cf. worksheets at www.sciencemath.ph-gmuend.de). From a mathematical point of view, this is all about deepening the understanding of the topic "similarity - centric elongation" and the "relationships between surface and volume". It is overall about proportions in geometry and the animal kingdom, at which the networking between biology and mathematics takes centre stage: On the one hand, the mathematical relationships are extended in an important way through the biological perspective, and on the other hand, mathematics helps to discover biological phenomena and to understand the biological consequences.

Pie Chart, Percentage Calculation and Nutrition Circle

The special characteristic of this teaching module on "healthy eating" is the cross- curricular approach between mathematics and biology, and the teaching about a highly topical issue: more and more children eat the wrong things and are overweight (KIGGS, Kurth & Schaffrath 2007). They often don't even know what is meant by healthy eating. The nutrition circle of the German Nutrition Association (Deutsche Gesellschaft für Ernährung e.V.) (figure 3) can be of support here and forms the starting point and the basis for the teaching module to be introduced (Grube 2008). It informs about healthy eating in a clearly laid out circular diagram, in which the individual segments represent food categories, such as cereals, fruit, milk products and meats. The size of each segment gives an indication of the respective salutary amount.

Figure 3 Nutrition Circle of the German Nutrition Association (DGE 2005)

From a mathematical point of view, the teaching module serves the purpose of introducing circular diagrams (including the application of percentage calculation); from a biological point of view it serves to stimulate healthy eating. The module starts with a homework task that the students have to do before the start of the sequence: They are asked to weigh all the foods that they eat in a (if possible) normal day and to note down and order the quantities according to the categories of the nutrition circle. They enter these in a chart. In the first lesson, the nutrition circle is introduced. It serves as an impulse and to initiate a discussion, based on the question: Are my eating habits in accordance with the nutrition plan of the nutrition circle? For comparison, one's own eating habits are entered into a circular diagram. This is achieved by calculating the percentages and transferring them to angular measures. A further task is establishing one's own perfect nutrition plan that takes into account the parameters of the DGE. This requires a form of thinking in which percentage calculation, circular diagrams and nutritional recommendations are directly linked.

This teaching module was tried out in a German secondary I school in the school year 2007/2008 with students aged 14 – 16 (Grube 2008). The students were highly motivated and were even stimulated to talk about nutrition into the breaks. The topic and the tasks constantly led them to linked thinking between biology and mathematics. The need to reflect on the procedure of entering data in the circular diagram, in order to compare their own eating habits with those suggested by the DGE, proved to achieve excellent motivation. The comparison of nutrition plans among class mates led to exited discussions, and students worked out independently, that in spite of a higher percentage value a lower total percentage can be achieved, as the basic value is the decisive factor.

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Rescuing Statistics from the Mathematicians.

Mike Bedwell, MA, MSc
Michael_Bedwell@hotmail.com

Abstract : Drawing on some 30 years' experience in the UK and Central Europe, the author offers four assertions, three about education generally and the fourth that of the title. There the case is argued that statistics is a branch of logic, and therefore should be taught by experts in such subjects as philosophy and law and not exclusively by mathematicians. Education in both Statistics and these other subjects would profit in consequence.

Introduction

I am old, I have been working as a jobbing instructor in former Warsaw Pact countries since 1998, and while my main interest has been in statistics, I have in my time taught French, English literature and other subjects in between. With this idiosyncratic introduction, I ask you to indulge my non-academic style, with its shameless use of the first person singular.

I hope to provoke you with four assertions, the most important of which, inspired by Paton (1990) and implied in the title, I shall leave till last. The others are:

Assertion 1: Keep Technology out of the classroom.

In my UK university, I took for granted that I could turn up to my classroom safe in the knowledge it would be unlocked, warm and well lit, that kind administrators would have informed the students of the where and the when, would have provided sufficient seats, while equally long-suffering technicians would have put in place any equipment I had asked for. But even so, I witnessed too many colleagues going spare when their high-tech aids – over the years ranging from 8mm film to PowerPoint – fail at the critical moment; a particularly embarrassing case was when the material being taught included the reliability of systems comprising components in series.

So in Central Europe, where classroom technicians don't exist and administrators see their roles differently than in the West, I have learned to approach the classroom prepared for the worst; I don't carry candles (but if I did, one of the many smokers among the students would volunteer a light), but I do equip myself with my own chalk and topcoat. For statistics, I may add a few dice, and a few things the traveller naturally has to hand, such as coins and an opaque bag of variously coloured socks; in their alternative use these are rather more comfortable than the urns and billiard balls beloved of textbook writers. But nothing more.

Lest I be accused of Luddism, I rejoice if outside class hours students can access IT tools. These are invaluable in freeing classroom time for discussion with the students, thereby striving for those higher goals that educationalists wax lyrical about. Indeed, I have gone to the lengths of offering cash to students claiming they cannot afford to access Google, safe in the knowledge that they will be shouted down by their colleagues who know more about available facilities than I do; I have considered a similar approach to embarrass my employers into installing IT tools like a bulletin board.

Assertion 2: Look West, young teacher.

After my own Oxbridge education, with its frequent essay-writing and infrequent examinations, it came as a shock to teach in Central Europe. For there I have worked in US-sponsored institutions, where the tradition is to require essays only in the liberal arts, but to test frequently, usually with multi-choice questions.

While I still hold reservations about a system that requires, for instance, a biologist to teach a class without having any say in a putatively prerequisite statistics course, and while I continue to object to the coy practice of calling tests 'quizzes' and essays 'research papers', I have nonetheless become a zealous convert to multi-choice tests. So these I now aim to set every class; I hasten to add that the students are encouraged to work in teams of any size and constitution of their choosing. For me this makes the marking (or 'grading' in American) tractable, even with a class size of over 50. And while it also makes life easy for the class 'passengers', these find themselves shunned over time by the 'workers', and in any case get caught out in the exams, which are sat individually. In Appendix II you will find some sample questions, where I (a) give an example from a non-quantitative subject area, and (b) demonstrate a technique which owes something to Socrates in that it teaches as well as tests. This is essential as after the opening class I aim never to lecture.

Anyone with doubts about American-style testing should tackle one of the Princeton-based Goliaths like the GRE or GMAT; with no other purpose in mind, I submitted myself recently to the latter. I emerged humbled and mentally exhausted, but with increased respect for Princeton's reputation.

Assertion 3: Exams should be open-book and pre-published.

No-one would dream of assessing an electrician by asking him/her to write a description of a screwdriver. And most of us are delighted when our doctor consults some authority -- on-line or otherwise – before prescribing our medicine. So I believe the case is overwhelming for open-resource examinations in Statistics and, I suspect, in many other subjects.

Of course, students will use every opportunity to cheat that such examinations seem to present -- and weren't we all students once? This is especially true in Central Europe, where a student whom I would have trusted with my wallet cheerfully boasted to me that academic cheating 'was a national sport'. But my classroom rules in Appendix I and sample questions in Appendix II show ways of meeting this problem; in particular, the theory of Latin squares shows us how to construct an examination comprising a prime number ***n*** questions, each of which has at least ***n*** variants, in such a way that no two out of $\boldsymbol{n}^2$ students will share the same variant in more than one question.

The arguments for pre-publishing exams are less strong, but in my experience the students who have proved leaders in team-testing also do well in their exams, encouraged by my published practice of grading on a relative scale with 100% going to the best. At the same time, the willing donkeys often work out in advance the answers to all variants of all the questions, and also do well, as they surely deserve to.

Assertion 4: Stand back, Mathematicians.

Lastly, I advance arguments to support the assertion of my title:

i) As Moore (2002) has extensively illustrated, the most serious problems with statistics in the social sciences, from economics to medicine, have arisen through using an inadequately representative sample, and in particular in coping with non-response. Thus the flawed predictions before the US Presidential elections in 1935 and 1948 have both passed into folklore. If there is a systematic answer to this problem, mathematics is unlikely to provide it.

ii)Mathematics has been described as a branch of logic (Encyclopaedia Britannica, 1998). I hold the same to be true of statistics, so in any taxonomy it should stand side-by-side with maths but not beneath it.

To justify, see what reaction you get the next time you tell someone how you earn your living. Two typical replies are 'there are lies, …', and 'oh yes, I did a bit of that on my ___ course'. In the second case, ask the following 'acid test' question:*Does the p-value derived from statistical testing give the probability of the truth given the evidence, or that of the evidence given the truth?*

The sophisticated response is 'Truth is a difficult philosophical construct; I can reply only if I may substitute "null hypothesis" in its stead'. But, with this modification, how often do you get a straight answer?

In my experience, all too rarely. People who can't answer must have studied from The equivalent of the Biblical book of Proverbs and not of Revelation, yet to see the light they need apply only a little arithmetic, as developing the following table demonstrates:

<table>
<tr><td colspan="4">What can we learn from the single toss of a coin?</td></tr>
<tr><td>*Null Hypothesis= Prior belief: coin is↓</td><td>*Outcome = evidence↓</td><td>What does the evidence tell us?</td><td>Bidirectional p value</td></tr>
<tr><td rowspan="2">..conventional – the two sides differ</td><td>H (Heads)</td><td rowspan="5">Nothing</td><td rowspan="2">100%</td></tr>
<tr><td>T(Tails)</td></tr>
<tr><td rowspan="2">.. a forgery – both sides the same</td><td>H</td><td rowspan="3">100%</td></tr>
<tr><td>T</td></tr>
<tr><td rowspan="2">.. a forgery – both sides H</td><td>H</td></tr>
<tr><td>T</td><td>Everything</td><td>0 %</td></tr>
</table>

In this development, we need first to discuss the row and column headings, and the equivalence of the terminology shown *. We could then go on to explore the students' own prior beliefs by asking what they know of Karl Popper's *potential falsification*. And, again before answering, we could explain a *null hypothesis* with the example from criminal law that the accused is innocent until "proved" guilty, but what do we mean by "proof"? In any standard text on statistics, by comparison with one on maths, how often do we encounter this word? None of this discussion needs maths.

But to return to the 'acid test', in deriving the *p* values we are effectively defining 'significance' as the probability of the evidence given our prior belief. We might heartily wish it were the other way round, but once recognized we have the cornerstone of any subsequent discussion of Bayesianism. This topic

Matthews (2005) has not only placed among the 25 ‘Big Ideas’ of current science, but with little maths has explained to the layman.
A further explanation from the table is of an apparent paradox: that the case for abandoning a null hypothesis is stronger when the alternative is unidirectional (the bottom line).
These points can be reinforced by considering two tosses of a coin, as exemplified by test Question 1 in Appendix II. Further, in the context of this question, it is helpful in any more advanced lessons for the teacher to appreciate that, in a sample of two, t is equal to

a) The ratio Sum/Difference, precisely , and
b) tan (p% π), where p is the unidirectional p value.

These can be helpful in any later teaching; for example we can readily check the tabulated value of t = 1.00 for the 25% unidirectional significance level, and then relate this to the probability of both members of the sample having a median greater than that of the null hypothesis. I would like to continue arguing for thus teaching non-parametric before conventional significance tests, but space precludes. With a final increase to three tosses, we can introduce the concept of *confidence*:

What can we learn from an outcomes other than HHH and TTT from tossing a coin three times?					
Outcome		HHH	Any other	TTT	Sum
Statistic→	Number of Heads	3	1or 2	0	
Object is to calculate ↓	Assumption that the coin is: ↓	Probabilities			
Significance	Fair	1/8	¾	1/8	1
Confidence	Biased as in the sample, i.e. Heads:Tails = 1:2	1/27	1- (1/27+8/27) =2/3	8/27	1

In developing this table I first introduce the terms *exclusive* and *exhaustive*, so avoiding the need for any further knowledge of probability. I then argue that the two outcomes of either one or two Heads constitute the critical values for a bidirectional significance test at the 1/8 + 1/8 = ¼ probability level, but for a confidence interval of only 2/3 probability, not ¾ .I make this point as many authorities, (eg Upton & Cook, 2001 and Wood, 2003) imply that that sum of the bidirectional *p*-value and confidence interval probability must be one, a generalization explored for more realistic and continuous distributions in Exam Question 3 of Appendix II. Professors Upton and Wood have both been kind enough to debate this issue with me on email, yet have so far left me unrepentant. But I have a hair shirt at the ready!

Conclusion We teachers could do better in (1) exploiting new technology, (2) assessing how we assess, and most of all, (3) addressing the question ‘*Logic*’, *said the Professor....’why don’t they teach logic at these schools*?’ from C. S. Lewis’s *The Lion, the Witch and the Wardrobe.*

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Appendix I Instructions to Students, issued in first class.

Welcome. My aim in this course is to minimize the class time we spend in administration and acquiring information, and so maximise the time we spend, often in one-to-one discussion, on the fascinating subject that is Inferential Statistics. While you may not become an expert, you will develop the confidence to hold your own with a professional statistician. And you will learn enough to enlist the help of mathematicians and IT specialists who themselves know no statistics. To these ends, the following will apply:

1. I will make all the documentation you need available on-line on… at least two weeks before you need it. This includes a copy of this Introduction, as well as your weekly tests and Exam. You should print out copies before the class in which you will need them; thus before we next meet, you should print Test 1. You should see me privately if you experience difficulties, financial or otherwise, in achieving this.
2. You should hand in your answers to tests at the end of the appropriate class, and at no other time. You will be encouraged to work on these tests in teams of whatever size or constitution you choose, and to hand in a team answer.

3. You should borrow from any library an introductory statistics text which, on browsing through, you feel comfortable with; this can be and in any language you understand.
4. Your grades-to-date will be published on-line from time to time unless you object that this makes an unwarranted intrusion on your privacy.
5. No late work will be accepted, and there will be no opportunity to sit examinations at other than the prescribed time. This applies no matter how valid your reasons or often you have to be absent from any class, including the first one. However, at the end of term individual oral examinations will be given to students who plead special consideration. At such orals you should present any relevant written medical or other evidence, and be prepared to be interrogated on any of the tests and exams, whether or not you have submitted earlier answers. Other teachers apart from me may be present.
6. 100% in all the tests will earn you a bare pass, but nothing more. But I am happy to report that in the last class the best student got 100 % in the exams, and 99% overall.

Appendix II Specimen Test Questions The first time you meet any terms in Italics, check the meaning in a statistics text or on-line equivalent, since in Statistics words often have a more specialized sense than in a general dictionary.

1.Three different cooks A, B and C each measure the diameters of two of their pizzas, and find they are all over-size by the following, in millimetres: A, 0 &12; B, 5 & 15 and C, 4 & 6. All the following are correct, but which two cannot be deduced from this information alone?:

i) A has the biggest *range* ii) B has the biggest *mean*

iii) The range and mean are the most commonly monitored in *Statistical Process Control*

iv) C has the biggest ratio mean/ range, and so has the strongest evidence for taking some corrective action, such as using a smaller pan.

v) The ratio (difference in mean from that of *null hypothesis*): (some measure of *dispersion*) is fundamental to *hypothesis testing*

2.Circle the Null hypotheses in the following:

i) My beliefs are right; show me evidence otherwise.

ii) Your beliefs are wrong; show me evidence otherwise.

iii) In a school there were 10 false fire alarms last term, but the first time the alarm sounds this term the teacher orders “this might be for real –evacuate!”

iv) Elsewhere in the school an examiner orders “this is obviously another false alarm – stay and finish your exam!”

v) In criminal law, the accused is deemed innocent unless strong evidence is found otherwise.

Specimen Questions for an open-book, pre-published Exam

This exam comprises 7 questions, each with at least 7 variants. On the day of the exam, answer only the variant indicated in pen. Should you accidentally see your neighbour’s paper, you will find he/she shares with you the variant to one question at most.

1. The following five individuals were prominent during the 20th century: Pope John-Paul II, Joseph Stalin, Adolph Hitler, Margaret Thatcher, and John Lennon. On the day of your exam, you will find one pen-annotated ‘A’, and a second ‘B’. Write an imaginary script for a television interview where A critically interrogates B. [This yields 20 variants]

2. For the sample indicated in the following table, check the calculation of *s* (the best estimate of the population standard deviation when this is otherwise unknown) and calculate the range. The first column gives an example

Sample	1 2 3	0 1 5	1 2 6	1 3 8	1 2 9	0 7 8	2 7 9	4 8 9
s	2	√28	√28	√52	√76	√76	√52	√28
range	2							

3. Compute the values of the empty cells in the column indicated.

	Examples*			Etc.
Sample size n	40	100	100	
μ	0	0	0	
Σ x	41.56	-143.5	0	
Σ (x²)	107.4673	1204		
∴ s²	1.6484	10.0816		
Test statistic z	5.12	-4.52	1.96	
95% CI	200.64 to 201.44	-2.06 to 0.81		
Bidirectional *p*- value	<<1%	<<1%	5%	

* based on Upton & Cook (*op cit*) p409-410 & 419, after coding. .

Presentation of the Digital School Journal: Revista Escolar de la Olimpiada Iberoamericana de Matemática, Sponsored by the O.E.I. Organización de Estados Iberoamericanos para la Educación, la Ciencia y la Cultura

Francisco Bellot Rosado, Editor

Dos de Mayo street, number 16, 8th floor, Valladolid, Spain franciscobellot@gmail.com

Abstract

The purpose of this paper is to present all participants of the Dresden Conference the digital School Journal *Revista Escolar de la Olimpiada Iberoamericana de Matemática,* online since May-June 2002, with almost 16,000 subscribers at current issue number 32. Subscribers are based all over the world, but mostly in Spain, Portugal and Latin America.

The Digital School Journal: Revista Escolar de la O.I.M.

Leading, mathematically speaking countries, have school journals with the purpose of popularising mathematical science , its teaching and learning. As examples we can quote Canada, with *Crux Mathematicorum*; Hungary, with *Kömal (Közepiskolai Matematikai Lápok);* Romania, with *Gazeta Matematica.* The two journals last quoted have more than 100 years of life, starting in 1895. The situation in Spain and several countries of Central and South America was very different. Some small countries are not capable of publishing and distributing a mathematical journal that emphasises problem solving – showing different exercises and papers suitable for both students and teachers ranging from secondary school to university level.

Taking advantage of my experience of more than 10 years, from 1988 till 1997, in the International and Iberoamerican Mathematical Olympiads I presented the O.E.I. in 2002 the project of a digital journal, only distributed electronically, to cover the gap before mentioned. The project was accepted and the first issue of the journal appeared in May-June 2002. The URL is http://www.oei.es/oim/revistaoim.

All the papers and problems are written in one of the two official languages of the O.E.I., that is, Spanish or Portugeze. Materials submitted to the editor in other languages, such as English, French, German, Romanian, Bulgarian or Russian, are translated into Spanish. The materials must be sent to the editor to the emails fbellot@hotmail.com, franciscobellot@gmail.com , or the official email of the journal, revistaoim@oei.es

All issues of the journal are in .pdf format and they can be downloaded from the journal website for free (each article separated, or the full issue). Subscription is also free.

The journal is structured in the following sections: Articles, Notes and lessons of Olympiad preparation; Problems for the youngest; Problems of medium level and Olympiads; Problems; Mathematical Miscellany; Comments of web pages; Comments on published books. This list is open within limits , and therefore we are always ready to include new items in the journal. I would like to explain with some more detail these sections.

Articles, Notes and lessons of Olympiad preparation is a sort of Pandora box where all the mathematical papers can be included. Of course the idea is not to include research papers, which have specific journals to be published in; we can quote some of the papers we had published there : *Maxims and minimals without derivatives,* by Abderrahim Ouardini (issue number 3); *Some theorems and its proofs,* by the late Juan Carlos Salazar from Venezuela , (issue number 13) ; *The Mathematical Open,* by Antonio Ledesma López, from Requena, Valencia, Spain (issue number 11); *Couples of heronian triangles with the same perimeter and the same area: a description,* by K.R.S.Sastry, from Bangalore, India (issue number 16); *A methodological proposal of early introduction of the concept of local approximation in its manifestation of tangent line, via the mathematical assistant,* by Pedro Vicente Esteban Duarte, from Medellín, Colombia, and Pedro Pérez Carreras, from Valencia, Spain, etc, etc.

Problems for the youngest is the section devoted to present problems for students from Primary Education up to Lower Secondary School. For example, in the last issue (number 32) we had presented the problems of the Kangaroo Mathematical contest 2007 for students 13-14 years old; and in other issues we published the problems from the Primary Education Mathematical Olympiad from Costa Rica (OMCEP), kindly sent to us by Dr. Víctor Buján Delgado, the Coordinator of that contest.

Problems of Medium level and Olympiads is a section which includes problems for students of Higher Secondary level; the problems from the National or International Mathematical Olympiad, or the Mediterranean Mathematics Competition, or the Balkaniada belongs to this section.

Problems is the section which includes problems up to the third year of University. Usually, five problems are proposed in each issue; from all received solutions, the editor chooses that the one that, in his opinion, is the best (or eventually the two best) for publication, with authorship design, also giving a list of solvers (name, institution – if any – and location). Up to issue number 32, we had published 160 problems. Anyway, the motto of this section is the same as appears in *Crux Mathematicorum:* No problem is definitely closed. The editor is always pleased to consider new insights or new solutions of an old problem.

Mathematics Miscellany (in Spanish the title of the section is *Divertimentos Matemáticos*) is a very versatile section. We had included in this songs like *El tango del algebrista*, by Carlos Domingo, with the tune of the tango *Mano a mano;* the Spanish translation from German of *Die Ballade von armen Epsilon*, included in a book titled *Carmina Mathematica* (by Hubert Cremer, 1965); parodies of the bourbakist style; etc, but also short biographies of Iberoamerican mathematicians, prepared by the editor for another event and never used.

Comments of web pages and Comments on published books are self descriptive and require, in my opinion, no further explanation.

I invite everybody to access the journal web page , read some of the published issues or papers, and send me any collaborations which you may think suitable to be published. I promise to read all the material with extreme attention, and answer the author as promptly as possible. Thank you very much for your attention.

Valladolid, December 2008. Prof. Francisco Bellot Rosado

In what case is it possible to speak about Mathematical capability among pre-school children?

Anna V. Beloshistaya
Professor, Faculty of pre-school and early school education, Murmansk State Pedagogical University, Murmansk, Russia, E-mail: beloshistayaAV@email.su

Abstract:
Most of people have fatal attitude to Mathematics: some of them are capable to learn it form nature, but the others are not. So is their fate – to suffer from it for the whole of life… But it is a rude though natural mistake, as it results from means of mathematical education and its content. Most of parents and teachers are directed on these aspects both in kindergarten and at primary school. Of course, parents are different. Nevertheless so many parents can't possibly but speak about achievements of their children. Some start making their own children learn better by the example of success of the others. They make their children learn long chains of figures with no understanding. It is even more sad to see how a mom asks her 4-year old son: "How much is two plus three?..' But he replies just because he learned the answer but not calculated.
Not only parents but also kindergarten tutors don't want to understand that drilling for arithmetic has no sense. For a specialist it would take two days only…But teach him how to think logically – is a goal demanding from him, reached by different means.

Introduction

About Mathematical Abilities

It is clear, that capability to any subject or other activity are determined by individual psycho features, genetic predisposition. Although nowadays there is no evidence to stipulation of abilities by neural tissues of any kind. Moreover, it is possible to compensate even unfavorable abilities. Task-oriented approach will lead to personal growth, formation of clear-cut abilities, which is proved by certain experience.

Mathematical abilities are from a group of so called *special abilities* (e.g. musical, painting etc.). To reveal their existence certain knowledge is needed, together with certain skills, namely skill to use knowledge in mental activity.

Mental activity – the key type of mathematical activity. Realization of its results is one of the strongest stimulations for current development of the civilization.

The problem of knowledge digestion and accumulation is traditionally connected with natural figures' apprehension and operations with them: counting, adding on, arithmetic operations and comparing, changing the scalar quantities, as well as quantities with nonnegative results of change.

Many educational programs create the mathematical content with the focus on "natural numbers and operations with it". The process of mathematical … formation is aimed at content (knowledge) and operational (skills) elements of curriculum. In other words, "certain knowledge base" is associated with knowing the natural numbers, whereas "collection of certain skills" can be understood as practical operations with numbers – counting, adding on and use of symbols (operational figures and signs), typical mathematical problem solutions etc.

Both Russian and foreign researchers associate formation and development of mathematical abilities among school children with mental processes (not with subject knowledge and skills).

Talented children usually have number of specific characteristics, namely, flexibility of mind, i.e. fresh thinking and ability to various cognitive problem solving, easy transfer from one problem to another, ability to come out of usual activity and find new solutions under changing conditions. Such peculiarities of mind are directly depending on specific memory organization as well as on imagination and perception.

Researchers point out also such characteristics as *deepness of thinking*. By this they mean ability to penetrate into essence of each fact and event, observe their interconnections with other facts and events, uncover specific, implicit characteristics of the learned material.

Among major characteristics in mathematical thinking there is *task-oriented thinking* in combination with its *breadth*, i.e. ability to formulate general ways of thinking, skills of team vision of a problem.

Prior to all other categories mentioned above, specific or natural aptitude to structural approach to a problem and maximum stability, concentration and amount of attention.

Mathematical abilities are closely connected with cognitive abilities, including *sensitive* (perception and observation of subjects and events) and *intellectual abilities* (out-coming information processing). Consequently, task-oriented development of all mental characteristics as well as sensitive and intellectual abilities (thinking as operational process, i.e. independent analysis making, synthesis,

comparison and other mental operations) on the mathematical material will favor general development of mathematical abilities among children.
Why do some challenges appear?
Special or subject knowledge allow us “speak the language of Science” – operate with sign systems? Peculiar to a particular system, reveal and describe logics of conclusions with the help of familiar symbols (in our case – figures, letters, signs). Knowledge recorded in such a way becomes clear to an onlooker (a teacher, a tutor, parents), seeing and estimating cognitive results. Although the most important part of a mathematical process is left outside.
Initial mathematical visions of a child are formed on work with numbers and operations with them (i.e. counting and arithmetic operations). Great variety of symbols allows to make the process “transparent” and controlled. On the other hand, such process can not serve development neither of mathematical thinking nor mathematical abilities.
The main way of pre-school children development is empiric generalization, i.e. generalization of their sense experience. Accumulation of such experience is based on sensory capabilities of a child (vision, hearing, sense of touch) and its “processing” is realized through intellectual capabilities. It is necessary to provide a child with conditions for investigation and experimenting, in order to start the “engine” of this process. In other words, educational content should be both acceptable by senses and favor his experimental needs. Such experimenting may result in development of a child on the way of the World perception and understanding.
You, probably, have mentioned that there is a kind of contradiction: a figure as a mathematical issue is a highly general abstraction with … from basis of its construction. Despite of the way chosen for “natural number’s” construction – on the meaning of “set” or on scalar quantities’ measurement – Number as the key issue of mathematics is abstract, impossible to be directly perceptible for senses. Any “object snap” of a Number (e.g. use of trees, rabbits for counting) is a double loss of abstraction and consequently loss of the essence’s generality. We should speak about “double” loss because in this case we deal not with graphic image (number of pixels) but variety of trees or rabbits, etc. This image is directly percepted by a child, acts in experiments. Results are fixed in empiric generalization. This can e proved by the fact that primary school children often loose results of such generalization when the teacher change rabbits for trees, for example. They see this change as a new situation and repeat the whole process from the very beginning.
Theoretically we may conclude about importance of numerous experiments with different objects for the sake of right empiric generalization. But in practice it is not true for many cases. Reasons are different and vary from individual perception abilities up to lack of descriptive materials. In this sense, traditional substitution of independent work with observation of the teacher’s activity can not be adequate. So, contradictions mentioned contain reasons for high level of unpredictability, if we speak about creation of mathematical abilities.
Early introduction into numeric and sign symbols (i.e. early symbolization) is not widely recognized. Pre-school children learn it very easy as it is usual way of coding for their plays. Nevertheless, symbols get separate meaning due to absence of ready symbols configuration. Herewith its external manipulation replaces implicit operating with mathematical notions and relations.
There is a great variety of examples from teaching practice. They prove independence of symbols if we speak about the children’s mind. At the same time, its link with real sense of notions and relations is quite peculiar. Judging from experience and examples given above it is evident to say that children can easy remember order of presentation as well as symbols themselves. On the one hand, examples show lack of flexibility and deepness of child’s thinking; on the other hand, they reveal tendency to formalization (it is easier to learn strictly shaped images).
Arithmetic? Algebra?? Geometry!!!
There are several components in the mathematical content: arithmetic material, algebraic and geometrical materials. The first and the second ones are incorporated into quantitative characteristics of subjects and their groups (arithmetic is based on notion “number”), are connected with generalization process of their qualitative characteristics (letters are used in algebra for qualitative characteristics) and operations (algebra is based on notion “operation” equal to more general notion “actions” from arithmetic).
Even slight analysis of mathematical notions mentioned above proves that we deal with abstractions of high-level difficulty and generality. In particular, counting of apples in a set or rabbits on a meadow need a child to be disembodied from all perceived objects’ qualities (color, size, shape, taste etc.). At the same time a child should concentrate on such characteristics as “quantities of variety ”. As for

algebraic symbols, it needs disembodiment not only from qualities and characteristics of objects but also from their quantity: x of rabbits, y of carrots.

Learning of Geometry has its specific character, too. Its major components are figures and bodies on two- and three-dimensional space. As it is possible to create models of all geometric objects, investigate and operate with them, initially and in pre-school period we usually use sensor abilities of children.

Analysis of mathematical programs and manuals for school children reveal an interesting tendency: mathematical educative material mainly consists of arithmetical material. Among typical exercises for 1st-year pupils there are counting, numbers and natural numbers' qualities accompanied with arithmetical exercises, addition and subtraction tables, arithmetical problems, multiplying and division tasks, double figures etc.

It is a kind of paradox because all notions mentioned are highly abstract and demand not on "imagination" but abilities of abstraction without sensory support, which turn to be impossible for a 5-6 years old child.

Use of geometrical content in work with pre-school children helps to omit all these methodological challenges. A model of any geometrical notion can be directly perceived by a child. Besides there are other advantages of geometrical material use:

It helps in work with the "Zone of proximal development" with reference to experience and knowledge of children. More difficult task motivates a child for new activities in mathematics. First, children copy models and way of work with them assisted by a teacher; then try to construct according to a picture etc.

2. It helps in creation of evolutive environment with the help of new material use (but not by speeding educational process).

For example, a 2-3 year child makes easy compositions operating with geometrical figures. In fact, he learns their features and qualities (sides' length, parts positions etc.) In 3-4 years age a child analyses their similarity and differences in their size, sides' length, their number etc. When a child is 6-7 years old, he compares different objects, formulates comparison's and generalization's results, makes an assessment of qualitative characteristics, describes separate space and qualitative aspects of the objects etc.

In this case, we don't need annually insert new figures, enlarge list of notions, borrow new issues from school program. The only thing needed is set of new exercises, reveal of new features in familiar notions and new relations between them

3. Geometrical material helps in resting on children's interest in experiments – natural means for learning the material by children of a certain age.

4. It stimulates the process of mental development, necessary for any cognitive problems solution.

5. It gives an opportunity for building the educational process based on plays. They are interesting for children as constructive activity itself is perceived by children as a play, makes them interesting and does not acquire additional plots.

6. It promotes graduate and more stable learning the material. Initially, on the stage of an adequate mental act's formation an external base is needed. It will be used by a child as a model later on. Working with arithmetical material we may face some problems with such external basis' creation.

7. By means of mathematical activity it helps in development of such qualities of a child as observancy, assiduity and ability to plan succession of operations etc. So, structures of any kind need child's ability to work with notions and relations.

Effective development of mathematical way of thinking on the geometry material is connected with formation and development of cognitive abilities (both sensory and intellectual). In this context, education is based not on qualitative but space objects' characteristics. It means that first forms and motions are perceived, and then come qualities. It makes all children equal in ability to learn mathematics. Consequently, we may state that reasons for "mathematical abilities" being a rare case lie in educational system as such. System of introduction into the world o mathematics does not coincide with children's way of understanding it.

It is well known that not all abilities of children are seen on the surface, so a teacher needs to find, reveal them. Unfortunately, this pedagogical axiom does not work if we speak about methods of teaching mathematics. Teaching the subject is aimed at content but diminishes the key objective of any kind of education – personal development of a pupils resulting in abilities creation, mathematical abilities including.

Possibilities and Challenges of Mathematical Modeling in Teacher's Formation

Maria Salett Biembengut
Post-Graduation Program in Education, Universidade Regional de Blumenau – FURB
Blumenau, Santa Catarina, Brazil salett@furb.br

Abstract

In this article are the results of research of empirical data from two pedagogical experiences using Mathematical Modeling with two groups: one with 28 students from the last period of a course of mathematics teachers, and another with 21 teachers of a course of continuing education. The objectives of the course were: teach Mathematical Modeling, and in sequence, modeling as a method of teaching. The data about the interest for the proposal and the need of the two groups in learning modeling for use in practice was raised from interviews and issues raised and works done by them. Even though the importance of Mathematical Modeling as a method of teaching is not underestimated, some aspects exemplify the difficulties for the participants in changing the concept of teaching and learning: formation of the participants and the need for formation.

Key-words: Mathematical Modeling, possibilities and challenges.

Presentation

In the last three decades, the growing interest for mathematical modeling in Brazilian education has generated curricula reformulation, and new pedagogical proposals, research and ways of mathematical modeling. Motivated by these new reformulations and proposals, many states and cities have promoted courses of continuous education for mathematics teachers, with the proposal of improving the quality of teaching and the interaction between teachers and students. At the same time, many courses of mathematics teachers' formation have tried to insert into the curricular schedule subjects about lines of research of Mathematics Education, in particular, about mathematical modeling. For example, according to government data, there are 413 courses of mathematics teachers' formation in Brazil; from these, the author of this research identified (until March/2009) that about 30% have in the curricula schedule the subject of modeling.

In spite of the law and critics, in the greater part of the courses of mathematics teachers, the curricula is still subdivided into disciplines, without any connection with each other, composed of strict plans, teaching methodologies and evaluation in the traditional way. Except for isolated instances, the specific disciplines are treated without any connection to the matters that should be dealt with by future teachers in Basic Education; and only in the pedagogical disciplines the task of showing to the future teachers the actual tendencies of the methodological teaching proposals. (BIEMBENGUT, 2004). Generally, the classes are no more than passing content, exercises and techniques or the exposition of theorems and proper demonstrations without meaningful objectives.

The concern with the preparation of this future teacher for the field in which he/she will act belongs under the responsibility of some disciplines, such as: Mathematical Education, Tendencies of Teaching or Mathematical Modeling. However, the number of hours available for these disciplines (between 45 to 90 class /hour) is not enough to prepare a future teacher to act differently from what he/she has experienced during his/her school years. In the same way, the teachers who come to participate in a course of continuing education, of mathematical modeling for example, (between 30 to 90 class/hour), have the motivation and interest for another proposal, although there are few that are able to break with the teaching practice they have been using, (BIEMBENGUT, 2009; BASSANEZI, 2002). Thus, despite the results of research and recommendations in official documents of the Brazilian Education Department, few changes have occurred in classroom practices. For the most part, mathematics teachers in all levels are still stuck to the textbook and reproducing the same teaching they experienced since the beginning of their school life. These perceptions have initiated a search for an answer to the question: what factors make the mathematics teacher not to alter his/her practice despite the difficulties presented by many students?

To answer this question two pedagogical experiences were carried out with two groups: one with 28 students from the last period of a course of mathematics teachers and another, with 21 teachers in a course of continuing education; the objectives were: to teach the art of modeling – mathematical modeling in research, and in sequence, the art of teaching – modeling in teaching. To reach these objectives, this researcher elaborated on material of didactic support that is organized in four phases, denominated: mathematical language; experiences based on classical models, modeling as a research method and modeling as a teaching method. The data relating to the interest for the proposal and the need of the two groups in learning modeling to use in their practice were raised from the interviews and issues raised by and works done by them. To better understand the happenings during the pedagogical experiences philosophical literature concepts and definitions were searched about interest and necessity. For Dewey (1922), interest means an internal achievement or feeling of value given that each person has; it is dynamic and pushes one to action; necessity, for Claparède (1958) is what stimulates the person to move or to act. “The interest implies a necessity, or then the interest produces a necessity” (Habermas, 1987, p.220).

Modeling and the classroom: two pedagogical experiences

The course given for each of the groups lasted 60 class/hours. For the students of the last period of the teachers’ formation course these 60 class/hours were developed during the semester, four class/hours per week and for the teachers of the continuing course, in six weeks, 10 class/hours per week. For both groups the course was divided into four steps: in the first, two problem questions, not formulated for them to present the solutions; in the second, twp classical mathematical models to check their validity through experiments; in the third, to elect a theme from an area of knowledge and do mathematical modeling and in the fourth step, adapt the work of Modeling for the teaching of Mathematics in Basic Education. The participants gathered in teams of two or three members to develop the activities. It is important to mention that both groups were motivated and interested in learning modeling in the first class. In the following there is a synthesis of the common occurrences of the two groups in the four steps proposed by the course: main difficulties and possibilities.

The first two steps had four activities with the proposal to inspire in the participants the spirit of a researcher: raise data, decide the mathematical language to be used in the formulation and resolution and analyze the validity of the solution. None of the two proposals demanded mathematics besides the one of the Basic Education. In the first, when they faced the two non-formulated proposals, which data they should find and then formulate to later solve students had difficulties to read, attend the prescribed orientations and recognize in from their previous formation the mathematical knowledge demanded for the solution. In the second, they should have done simple experiments to better understand two classical mathematical models (restricted growing and cooling of a liquid). These models are part of the program of the discipline Integral Differential Calculus. Despite having a meaningful timetable of integral differential calculus, algebra, geometry, analysis, among others, they did not know how to use this knowledge. The lack of understanding showed how these participants are conducted with the way of teaching that lived along their school life.

In the following two steps, it demanded that each team (two or three members) elected a theme according to their interest and empathy, and elaborated on a mathematical model, following the orientation proposed and after such elaboration, adapted this model for the teaching of mathematics for some school level. To do modeling it is supposed to study and to interpret a theme, a topic from any area of knowledge, and after, raise issues whose answers or solutions are not explicit or without need of formulation and resolution. To use mathematical modeling in classrooms the teacher needs to know how to do modeling and moreover, how to adapt one or more mathematical models that allow him/her to develop the content, while at the same time, calls the attention of the students to do and learn mathematics. They know that they improve with experience, so this step should complement this formation (to know how to model and to know how to adapt models) for them to use in the school practice.

This condition – to do modeling to be able to teach mathematics through modeling – contributed the most to resistance in doing the proposal. In the course of continuing education, for the most part the teachers affirmed not needing to know how to create their own material of didactic support, saying that a great number of schools already have a textbook. According to Claparède (1958) to make a person act it is necessary that he/she is in good condition for the appearance of a need and that it calls his/her interest in satisfying that need. Thus, it is supposed that the need to ‘learn to teach’ was remote by the value that is attributed to the teacher, dismissing the initial interest of these three participants.

Modeling in the classroom: possibilities and challenges

Even though the importance of the mathematical modeling is not underestimated as a method of teaching and learning, some aspects should be verified not to underline with too much emphasis, forgetting the limitations that the educational structure produces for the teacher as well as for the students. The educational structure with the curricula broken into many disciplines, each discipline under the responsibility of a teacher, and schedules and periods to accomplish each school phase, with no doubt, is the main difficulty to turn mathematical modeling into a method of teaching and learning in the classroom. (Saeki, Ujiie and Kuroki, 2007; Biembengut, 2007). In the proposal described above, some happenings were indicators of difficulties for the participants (teachers and future teachers); happenings that were gathered in two categories: formation of the participants and need of formation.

Formation of the participants: one of the main problems of the Brazilian school formation, from the elementary to the superior, is that rarely does the student learn to do research, or is taken to be responsible for his/her own learning, except for isolated experiences. The structure used makes only the teacher responsible for the student's learning. In this scenario, many students assume their physical space, transfer the content somewhere, the questions or exercises that the teacher presents and answer these questions or exercises, if and only if, they are asked to take a test for a specific evaluation and giving a grade. Rarely, he/she is called upon to search for an answer, in a kind of 'to do to know' and 'to know to do' as suggested by Maturana and Varela (2001).

Thus, for the most part, the participants of these courses, even being teachers or students at the end of their graduation course, assumed the same position as the students: wait for the teacher to tell them what to do, how and which results he/she 'would like' to receive from them. As affirmed by Maturana and Varela (2001), between the person and the environment there is a necessary structural congruence. The interaction between the person and the environment may promote changes in the biological structure. This suggests that if the teacher wants to adopt a different position from the student he/she was, it will be necessary to be aware of the effort that is demanded from him/her in the change of structure, living another way of learning.

As all the participants from this course were promoted, worked in at least two periods, they had little time to study and do the activities involved in the course. Thus, prevented by time constraints and by their own school life, the same interest that took them to wanting to participate in this experience forced them to think again about their needs in learning. The interest is a kind of feeling that prevents the action. It comes from the vicissitude that one person faces and that incites them to find an answer; developed by the observation and is associated with concepts, to its contrasts and interconnections. HABERMAS said (1987, p. 73) "the interest only transcends the simple perception, by the fact that in it the thing observed conquers preferably the spirit and if imposes certain causality among other representations."

Need for formation: To learn depends on the interest and the need if that person and moreover, demands from the person: diligence, discipline and perseverance. According to Habermas (1987), knowledge is found at the top of climbing done during a person's life and is part of his/her process of human formation. The activities done come from his/her interests and needs. "The interest implies a need, or then the interest generates a need." (HABERMAS, 1987, p. 220). According to Claparède (1958), every human being tends to keep intact until something disturbs his/her interior balance and promotes doing necessary acts to his/her own reconstruction. It is about "a continuous readjustment of a balance perpetually broken" (…); a search to reach "an objective and not to disappear the needs that show up" (CLAPARÈDE, 1958, p.40). In the functional perspective, of overall relevance, defended by Claparède, it is the need that makes the human beings move, it is that which vibrates the interior stimulus for doing the activities.

The personal life, particularly in this virtual time, is found to be multifaceted in occupations of multiple interests and needs. The participants, when faced with the proposals of the modeling course, and that to do them demanded enough experience or understanding to be able to describe and refine this description, were required to think again about the needs of the multiple occupations with which they were involved. From one side, their difficulties in being self-taught, to know how to read and interpret different contexts of the questions of the textbooks and, on the other side, the many demands or factors coming from the educational system used.

Despite the official Brazilian documents emphasizing the importance of turning mathematics into something meaningful for the students, in promoting learning, skills and their critical senses in using it, and the research indicating how modeling contributes to that, the educational politics present certain contradictions between the propositions and actions. From one side, they prescribe a pedagogical orientation that respects the socio-cultural differences among the students; from another side, they keep the same curriculum schedule and do not value the professional enough.

The educational structure in all the levels (from Basic Education to Superior) with the curriculum ruled in many subjects, not enough time for teachers to prepare thoroughly and having each one of these subjects under the responsibility of one teacher, makes it difficult for meaningful changes to be done in the students' formation. Besides that, this teacher has little availability to gather with other teachers of similar subjects to organize a proposal that brings an academic formation efficiently. And because of that, each time more students without interest and without realizing any need in getting this academic knowledge, keep showing results that are each time worse on the exams and in the job market, when they start to act. And the teacher in this context keeps using his/her techniques and strategies, sometimes taken to make new attempts just because of his/her virtuousness.

The pattern adopted by the governmental organs that (un)shape the objectives of the education when alluding to the assistance to the student strongly, even that it is sealed. The student knows that he/she does not need to make an effort, because he/she will be asked to present a minimum understanding of a concept. Therefore demands little of themselves or what he/she produces. The teacher in this scenario hardly dares to change his/her practice, especially if this practice demands more time from the teacher to prepare and to orient the students.

Facing these conditions, the educational proposals with results that tend to improve the Mathematics Education are tenuous. In particular, if guided by these signs it is possible to move on to other educational areas: from the needs to the interests of survival and from the interests to the needs of survival, in a continuous circle, without carrying improvements in the academic formation of the people. In the attempt to give words to the problems coming from these contradictions from the public educational politics, it is prudent that there is an analysis of the objective manifestations of the students' and teachers' reality. It is not less meaningful that "to take the consciousness of your reach, of the doctrinal autonomy, and the extension of your success to a popular auditorium, it is precisely situated" nowadays. (GRANGER, 1969, p. 38).

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How to increase the understanding of differentials by using the Casio-calculator model 9860 G I/II to solve differential equations.

Bjørn Bjørneng, lecturer in Mathematics and Physics at Dokka Secondary School, Norway
bbjornen@bbnett.no

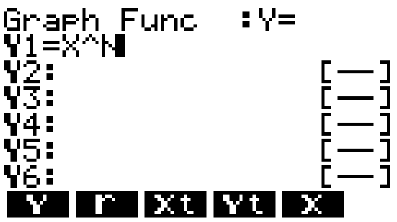
Abstract

The major aims of this paper are to present how we can improve the students understanding and involvement in mathematics by using a programming/graphic calculator. I will use differentials as examples such as differentiation ,integrals and differential equations, creating lines of slopes for differential equation of the type y'= f(x,y) . Find the solution of some differential equations by using regression and create the graph connected to the differential equation. As we have different approaches to solving a problem, it is a hope the students interest in mathematics will improve. The tools used will be programming, graphic commands as plot, f-line, etc. One goal is also to show how we can create small programs solving problems in mathematics. For many students this will be a stepping stone for further work with programming. The programs used can be copied using the program FA 124 that can be downloaded from Casios homepages. On request I can send you the programs.

Summary of workshop

An ordinary graphic calculator can be a very helpful tool to strengthen the understanding in mathematics. In this paper I will use differentials as examples.

Example 1: Differentiation of the function $f(x) = x^n$
The students are challenged to find a formula for derivative for the function finding local slopes for x = a numerically $\dfrac{\Delta y}{\Delta x} = \dfrac{f(a+0.01) - f(a-0.01)}{0.02}$



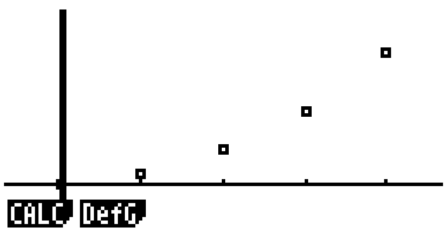
We start by making the graph

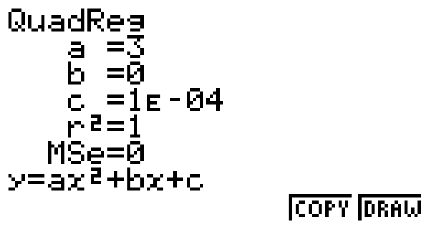
y1 = x^n and allow the students to make the program derive .

The program is short and put x-values from 0 to 4 in list 1

and the local slopes in list 2.

List 1 : shows 0. 1.2.3.4 List 2 shows 0, 2, 12,,27, 48





Making the graph combined with regression we find $x^{3\prime} = 3x^2$

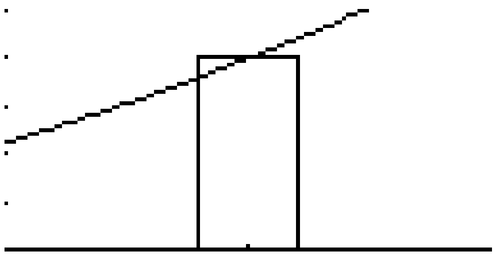
With trying n = 1,2,3,4 and 5, most of my students find the formula $x^{n\prime} = n\,x^{n-1}$

Later giving the proper proof, they easily recognize the result.

The next challenge is to find out if the formula is correct for n being negative or a fraction.

I find that many students have big problems with functions like $y = \sqrt[3]{x^2}$ and $y = \dfrac{1}{\sqrt[2]{x^3}}$ It becomes easier if they can translate the expressions to $x^{\frac{3}{2}}$ and $x^{-\frac{2}{3}}$ when they shall find the derivative.

EXAMPLE 2: Integral of the function $y = x^n$



We find the integral by sum up the areas of narrow columns with height h= f(x) and width 0.02 going from x-0.01 to x +0.01 that gives a good approximation for the real area.

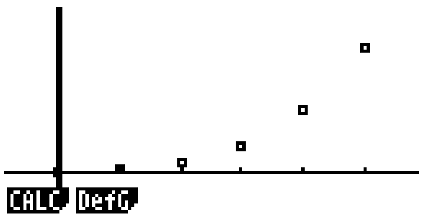
Again a small program integral:

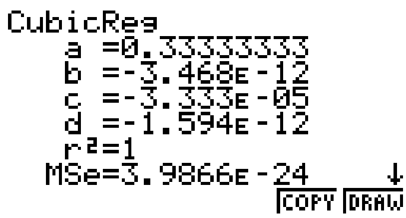
The program give starting values for x , S= 0 and for the total area A = 0. For every x the area is added up and for every whole number of x, the x-value goes to list 1 and the antiderivative of f(x) goes to list 2

list 1 : 0,1,2,3 List 2 : 0, 0.3333, 2.666 , 9,

In this example we choose n = 2

We make the graph and use regression:





By trying the students find regression of power 3 to fit; $\int x^2\,dx = \frac{1}{3}x^3 + C$ and trying n = 0, 1,2 and 3 find the general formula : $\int x^n dx = \dfrac{1}{n+1}x^{n+1} + C$

What now when n is negative or a fraction?
The big question is also what happens when n = -1 ?

It is good to have some questions unanswered before the proper proofs are given.

EXAMPLE 3

We shall find slopes of line for equations of type y' = f(x,y).

To make this we need the command F-line which creates the line between two given points.

F-line 1,1,3,2 makes a line between (1,1) and (3,2)

Diagram with lines of slopes for points (x,y) with slope D.

The angle A between tangent and x-axis is given by: $A = \tan^{-1} D$ (SHIFT tan D)

A short line with length 0,6 will then join the point (x-0,3cosA,y-0,3sinA)

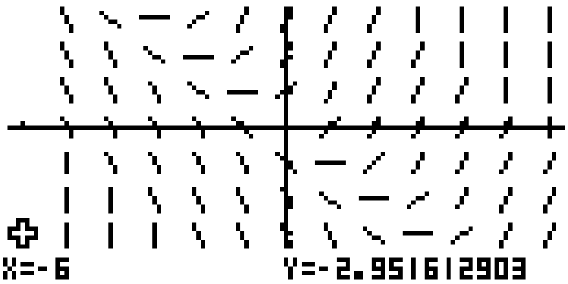
with point (x+0,3cosA,y+0,3sinA) If A = 90^o The line is between (x,y-0,3) and (x ,y+0,3)

The example : y' = x + y ; x+y -> D
You can easily change to other equations.

The third program is also a short one.

The loops are governed by Lbl and Goto and conditions for breaking out of the loop is for

example y>5 => Goto The result is fascinating :



The line given by x + y = -1 is special. If we start with a point on this line,

y' will be -1 all the time and we will follow this line.

y = -x – 1 is one solution of the differential equation y' = x+ y

EXAMPLE 4 :

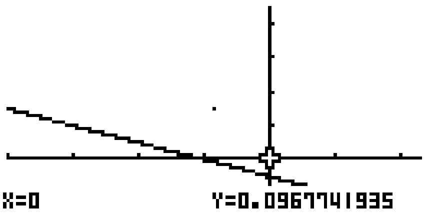
A GRAPHIC SOLUTION FOR THE EQUATION y' = x + y USING THE PLOT COMMAND.

In order to plot points following the graph we need a starting point, "startpoint", (A,B) and small increasements in x and y ; Δx and $\Delta y \approx y'\Delta x$ I choose $\Delta x=0.01$ in this program.

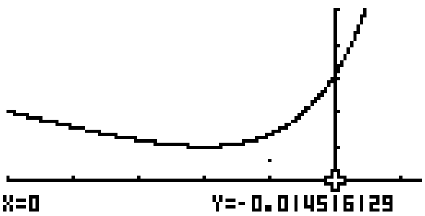
The last program diffgr:

Using the program we get following graphs.

Startingpoint (-4,3) is on the line x+y = -1 and the graph follows the line.

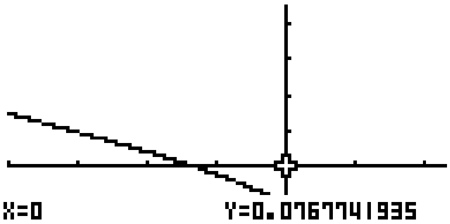


Startingpoint just a bow the line (-5, 4.05) gives :



With a minimum at (-2, 2)

and just below the line x+y = -1



We can solve the equation y' = x+y

y' – 1 . y = x ; y' + f(x) y = g(x)

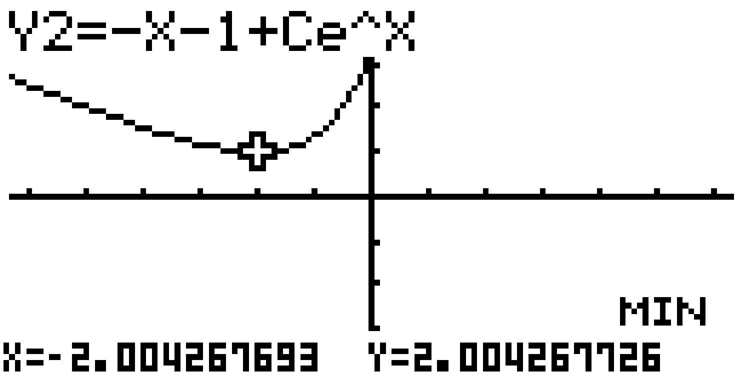
The antiderivative to f is -x , that gives an integrating factor e^{-x}

The solution is: $y = e^{x} \int e^{-x} \cdot x\,dx + C\,e^{x} \;=\; e^{x}\left(-e^{-x}x - e^{-x}\right) + Ce^{x} = -x - 1 + C\,e^{x}$

with $C = \dfrac{x_s + y_s + 1}{e^{x}}$ (x_s, y_s) is the startingpoint.

For x+y = -1 C becomes zero so one solution is y = -x -1

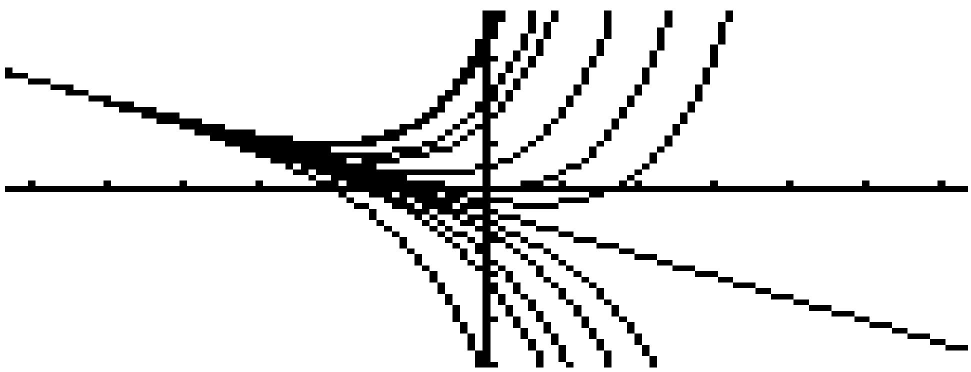
Startpoint (-5, 4.05) gives C = 0.05 e5 ≈ 7.42 with following graph.



We can make a lot of graphs having C vary from – 7 to + 7 .

If we choose the mode simultaneous graph

we get this picture:



Let this remind us that Bloxberg is not far away from Dresden.

The programs used:

Computer(DERIV)

```
Filename:DERIV
Fix 2↲
5→Dim List 1↲
5→Dim List 2↲
?→N↲
0→X↲
Lbl 1↲
X→List 1[X+1]↲
X+0.01→X↲
Y1→B↲
X−0.02→X↲
Y1→A↲
X+0.01→X↲
(B−A)÷0.02→List 2[X+1]↲
X+1→X↲
X=5⇒Goto 2↲
Goto 1↲
Lbl 2↲
"END"
```

Computer(INTEGR)

```
Filename:INTEGR
Fix 3↲
"N =":?→N↲
5→Dim List 1↲
5→Dim List 2↲
0→A↲
0→S↲
Lbl 1↲
S→List 1[S+1]↲
A→List 2[S+1]↲
S→X↲
Goto 2↲
Lbl 2↲
X+0.01→X↲
Y1×0.02+A→A↲
X+0.01→X↲
X=S+1⇒Goto 3↲
Goto 2↲
Lbl 3↲
S+1→S↲
S=6⇒Goto 4↲
Goto 1↲
Lbl 4↲
"END"
```

CASIO FA-124 - [Computer(RETNDIAG)]

File Edit Link View MENU PRGM VARS OPTN KEY SET UP SHIFT CATALOG Characte

```
Filename:LINSLOPE
−4→Y↲
ViewWindow -6.3,6.3,1,-3.3,3.3,1↲
Lbl 1↲
1+Y→Y↲
Y>5⇒Goto 3↲
−6→X↲
Lbl 2↲
X+1→X↲
X>6⇒Goto 1↲
X+Y→D↲
tan⁻¹ D→A↲
F-Line X−0.3cos A,Y−0.3sin A,X+0.3cos A,Y+0.3sin A↲
Goto 2↲
Lbl 3↲
Plot 0,0↲
```

CASIO FA-124 - [Computer(DIFFGRAF)]

File Edit Link View MENU PRGM VARS OPTN KEY

```
Filename:DIFFGRAPH
"STARTPOINT (A,B)"↲
"A ="?→A:"B ="?→B↲
ViewWindow A,A+6.3,1,B−8,B+8,2↲
A→X:B→Y↲
Lbl 1↲
Plot X,Y↲
X>A+6⇒Goto 2↲
X+Y→D↲
X+0.01→X↲
Y+0.01D→Y↲
Goto 1↲
Lbl 2↲
Plot A,B↲
```

Origami-Mathematics Lessons: Researching its Impact and Influence on Mathematical Knowledge and Spatial Ability of Students

Norma Boakes, Ed. D.
Associate Professor of Education,
School of Education, Richard Stockton College of New Jersey,
Pomona, New Jersey, United States Norma.Boakes@stockton.edu

Abstract

"Origami-mathematics lessons" (Boakes, 2006) blend the ancient art of paper folding with the teaching of mathematics. Though a plethora of publications can be easily found advocating the benefits of Origami in the teaching of mathematics, little research exist to quantify the impact Origami has on the learning and building of mathematical skills. The research presented in this paper targets this common claim focusing on how Origami-mathematics lessons taught over an extended period of time impact students' knowledge of geometry and their spatial visualization abilities. The paper begins with a brief overview of Origami as it relates to teaching mathematics followed by a summary of research done with two age groups: middle school children and college students. Gathered data in these two studies suggest that Origami-mathematics lessons are as beneficial as traditional instructional methods in teaching mathematics.

Introduction

Mathematics stands as an essential part of a child's education. Beyond the obvious every day applications, mathematics is seen as a needed element in developing student readiness for the workforce demands of the 21st century (National Council of Teachers of Mathematics, 2000). Though this is widely recognized and accepted the National Council of Teachers of Mathematics (NCTM), the largest mathematics association in the United States, calls for continued improvement in methods of teaching mathematics that engage, excite, and develop mathematical thinkers (2000). A recent report put out by the National Mathematics Advisory Panel (2008), commissioned by United States Department of Education (USDOE), concurs citing students lagging scores on international assessments like the Trends in Mathematics and Science Study (TIMSS) and Program for International Student Assessment (PISA). When compared to our international economic competitors, US students rank 8th and 9th out of 12 countries reviewed. Of the topic areas reported, the study of geometry was identified as a significant area of weakness on international assessments.

The word geometry broke down into parts stands for "geo"-earth and "metry"-measure. Though this topic has an obvious tie to our natural world and is a core element of mathematics, it stands as one that US students struggle to grasp. NCTM's standards describe geometry as the study and analysis of shapes and structures (2000). Part of this study includes the development of spatial visualization defined as "[the] building and manipulating [of] mental representations of two- and three-dimensional objects and perceiving an object from different perspectives" (NCTM, 2000, p.41). It is this skill that children use to describe, interpret, and understand their natural surroundings. To develop these skills, NCTM calls for teachers to actively engage students through a variety of hands-on, engaging tasks. Of these experiences, once such tactic noted is "paper folding".

The use of paper folding as a way to engage students in mathematical thought is far from a new concept. Publications in the US have featured its benefits since the 1960s with the printing of *Geometry Exercises in Paper Folding* (Sundara Rao, 1966). As of 2007, 27 books relate Origami to teaching mathematics (Tubis, 2004). NCTM alone has published 8 articles in their national teaching magazines featuring the benefits of Origami as an instructional tool. Internationally, paper folding in the classroom dates back to the 1800s when the founder of kindergarten, Froebel, included the art in his curriculum as a way of promoting children's mental growth and grasp of basic geometry. In all cases, Origami is seen as a powerful tool to teach mathematical concepts, particularly in geometry (Boakes, 2006). Writers speak of a variety of benefits including familiarization with geometric figures & principles (Pearl, 2008), developing spatial sense (Robichauz & Rodrigue, 2003), and engaging children in the discourse of mathematics (Cipoletti & Wilson, 2004).

Beyond the tie of Origami to mathematics, the art of paperfolding also has links to learning theories (Boakes, 2009). For instance, within Piaget's work on cognitive development Piaget discusses the development of logical-mathematical intelligence. Emphasized is the need for children to construct and develop their own meanings of mathematics through physical manipulation and play. Learning modalities and preferences also relate well. Modalities deal with how we prefer to learn information when first presented. There are three major categories within modalities that are universally accepted including auditory, kinesthetic, and visual-spatial. Auditory learners make sense of knowledge through what they hear. Kinesthetic learners, as the name suggestions, use touch and movement as a way to grasp presented concepts. The final modality, visual-spatial, relates to those that need visual stimulus and images to help understand material presented. By addressing all modalities in instruction, teachers are more likely to have success in reaching all children.

Learning preferences (also known as learning styles), slightly different from modalities, deal with the way in which a student processes information once it is presented. Known best in this field is Martin Gardner. His theory speaks of a set of multiple intelligences that all individuals possess. These intelligences include linguistic, logical-mathematical, spatial, bodily kinesthetic, musical, naturalistic, interpersonal, intrapersonal, and existential. Students taught in ways that allow them to explore these various intelligences tend to be more motivated, engaged, and retain more of what they are taught. Consider how Origami relates to these theories. The practice engages children physically requiring listening and visual stimulation. The act of folding involves spatial skills and geometric shapes. It is for this reason that Origami seems to have captured the interest of those that teach mathematics and has been accepted as a beneficial practice in the classroom.

One nagging issue remains among all that has been presented thus far. It's clear that there is room for growth in the area of geometry instruction. The national mathematics standards of NCTM support the use of such hands-on, physical tactics for learning geometry. Origami is an accepted method for teaching mathematics and relates easily to learning theories. However, how can one be sure that Origami truly is a beneficial experience in the mathematics classroom? To answer this question the next logical thing to do is to seek out research that quantifies the impact Origami has when instructed in a mathematic classroom. Interestingly, there is very limited research to substantiate what we think is true about Origami and mathematics (Boakes, 2006). It is this fact that prompted the research presented in this paper on the effect Origami instruction has on an individual's mathematical knowledge and geometry skills.

Anatomy of an Origami Mathematics Lesson

In both sets of research conducted, the term "Origami-Mathematics Lesson" was coined for the treatment method in the studies. This term "refers to a mathematical lesson taught using an origami activity linking students' mathematics knowledge and skill during the folding process and with the resultant Origami figure" (Boakes, 2006). In a normal folding session, an instructor of Origami would verbally and visually show each folding step. Origami-mathematics lessons simply blend mathematical terminology and discussions within this process. An article found in the online journal Mathitudes details this process and provides an example of what an Origami-mathematics lesson might include (Boakes, 2008).

Research with Middle School Students

The first research study was conducted in a suburban middle school in southern New Jersey. The purpose of the study was to "compare the spatial visualization abilities and mathematical achievement of seventh-grade students taught by two different methods of instruction" (Boakes, 2006, p.82). To do so, a basic quasi-experimental design was used. A control group of 31 students received traditional instruction over the course of a month long geometry unit. Meanwhile, a treatment group of 25 students received traditional instruction along with the infusion of a collection of 12 Origami-mathematics lessons. Both sets of students were taught by the same classroom teacher. The researcher served as the instructor for all Origami-mathematics lessons. During sessions where Origami-mathematics lessons were taught, about 20 minutes was taken from the normal 80 minute class held daily.

The groups were pre- and post-tested using an excerpt from a national mathematics assessment and a set of spatial tests. The mathematics achievement test used contained 27 multiple choice questions within the geometry/spatial skill strand of the National Assessment of Educational Progress (NAEP) assessments better known as the "Nation's Report Card" (National Center for Educational Statistics, 2004). Three subtests measured students' spatial abilities through a card rotation, paper folding, and surface development test taken from the Kit of Factor-Referenced Cognitive Tests (Ekstrom, French, Harman, & Derman, 1976). The card rotation test measured students' ability to mentally manipulate and analyze 2-D figures. The paper folding test requires the student to imagine folding and unfolding a square sheet of paper. The final of the three spatial tests, the surface development test, has readers try to match parts of a 3-D geometric figure with its 2-D net.

Data on the quantitative assessments was analyzed using an analysis of covariance (ANCOVA). The ANCOVA allowed the researcher to control for initial differences in spatial skills and mathematics achievement levels. A summary of results by group are presented in Table 1. The analysis completed using statistics software was a 2x2 between groups ANCOVA to determine the differences in mean scores among groups and gender. Looking at variables individually, no statistically significant differences were found among any of the 4 tests. However, a significant interaction effect [$F(1,51)=3.59$, $p=0.64$] was found on the Card Rotation Test between gender and group where males in the experimental group and females in the control group earned higher gains in scores than their counterparts in the same groups.

Table 1

Descriptive statistics for all instruments with middle school age students

Instrument	**Group**	**N**	**Pre-test Mean**	**SD**	**Post-test Mean**	**SD**
Card Rotation Test	Control	31	52.29	14.85	60.00	13.52
	Treatment	25	56.56	17.46	60.04	16.22
Paper Folding Test	Control	31	3.61	1.56	4.32	1.89
	Treatment	25	4.04	2.34	5.00	2.14
Surface Development Test	Control	31	10.16	7.13	13.84	8.04
	Treatment	25	10.00	6.98	14.28	8.30
Mathematical Achievement Test	Control	31	14.97	3.86	15.68	3.96
	Treatment	25	14.00	4.38	16.60	3.91

Without delving into gender differences and instead looking more generally at the overall results, data were not strong enough to warrant statistical significance. While this is true, if one examines Table 1 it can be said that both groups improved their mean average score on all tests. Further, the treatment groups had a similar or higher mean score than that of the control group. There are many reasons for this occurrence, only one of which may be the infusion of Origami-mathematics lessons. It is fair to conclude though that this instruction method was as beneficial as traditional instruction in the mathematics classroom and stands as an acceptable tool for improving children's spatial skills and geometry knowledge.

Research with College Level Students

A second study was conducted during the spring terms of 2008 and 2009 at the Richard Stockton College of NJ to further investigate the impact of Origami-mathematics lessons on students' abilities. In this case, a college course called The *Art and Math of Origami* was utilized. The course is similar in purpose and structure to that of Alan Russell's (2007) Origami course, *Mathematical Origami*, discussed at the Ninth International Conference of the Mathematics Education in the 21st Century Project. Students studied the art of paper folding while also learning about the history, culture, and mathematics that relate to it. The researcher, who served as the instructor of this course, infused Origami-mathematics lessons into the 27 two-hour sessions. A total of about 10 sessions were dedicated specifically to the tie between geometry and the folding process.

Students in this study ranged in age from 20 to 45 years old and were mainly junior & senior level based on the number of credits earned. Data gathered was the culmination of two groups of students, 24 during spring 2008 and 23 during spring 2009. Due to the difference in age of this group versus the previous study containing seventh graders, the assessments used to determine change in mathematical abilities was limited to the three spatial tests. (These tests are rated as appropriate for both age ranges.)

Pre- and post-test data were gathered on all three spatial tests. Since all students experienced the Origami-mathematics lessons in this study the data analysis method used was a paired-sample t-test. Results are shown below in Table 2. In all cases there was a significant increase

Table 2

Paired sample t-test statistics for the college age groups

	Pre-test	SD	Post-test	SD	Mean	SD	*t*	df	Sig.
Card Rotation Test	87.23	51.51	90.98	48.64	-3.75	12.21	-2.10	46	.04*
Paper Folding Test	12.13	3.32	13.43	3.80	-1.30	2.23	-4.00	46	.00**
Surface Development Test	39.60	13.09	45.87	11.13	-6.28	7.23	-5.95	46	.00**

*p<.05, **p<.005

seen in students' mean scores on all three spatial tests. Eta-squared values calculated indicate moderate (.09 for card rotation) to large effect sizes (.26 for paper folding & .44 for surface development). These results seem to support the conclusion that Origami does indeed have an impact on students' spatial skills. However, because this is a group of diverse individuals with varied academic backgrounds it is difficult to say with certainty that Origami was the sole cause of such change. Based on this being done over a short period of time however it is likely to be a contributing factor.

Discussion & Conclusion

The attempt in both studies was to examine the claim that Origami is an effective teaching tool capable of strengthening students' mathematical and spatial abilities. This paper provides an abbreviated review of each study conducted on very different groups of individuals. In the case of the middle school students, increases in both knowledge and spatial skills were seen with treatment students doing as well or better than their control student counterparts. While this is true, these differences were not pronounced enough to find statistical significance. There are a variety of reasons this may have occurred including the fact that this study was run over a very short period of time with a limited number of students. A fact worth noting though is that while 20 minutes of a daily lesson was taken away in order to fit in the

Origami-mathematics lesson, treatments students performed as well overall on 3 of the 4 tests used including the mathematics achievement test. For the group of college students, results were all significant. A large increase in the mean score received was found for all three spatial skills tested. Though again it cannot be said for sure what caused this increase with great certainty, it stands to reason that the 4 hours of Origami instruction received per week (plus work outside of course hours) may very well have contributed.

Origami, mathematics, and teaching have had a long standing relationship. In this paper, you've learned why this is the case. It was the purpose of the research conducted to shed light on whether this coupling of ideas is beneficial for students. Based on what's been presented, there is definitely potential shown for Origami-mathematics lessons. While there is still a great need for research in this field to be certain of this claim, these results should warrant inclusion of such practices in the mathematics classroom.

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Linking Geometry, Algebra and Calculus with GeoGebra

Josef Böhm
Austrian Centre of Didactics of Computer Algebra & Technical University of Vienna
nojo.boehm@pgv.at

Abstract

GeoGebra is a free, open-source, and multi-platform software that combines dynamic geometry, algebra and calculus in one easy-to-use package. Students from middle-school to university can use it in classrooms and at home. In this workshop, we will introduce the features of GeoGebra with a special focus on not very common applications of a dynamic geometry program. We will inform about plans for developing training and research networks connected to GeoGebra.
We can expect that at the time of the conference a spreadsheet will be integrated into GeoGebra which offers new ways teaching mathematics using the interplay between the features of a spreadsheet and the objects of dynamic geometry.

Workshop Examples

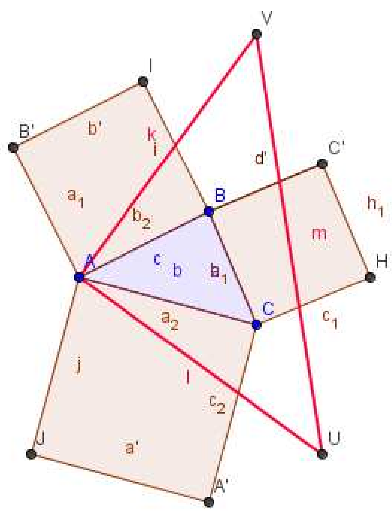
We will adjust the examples to a possible previous knowledge of the participants and make a selection of the following problems according to the wishes of the audience.

1 Another Property of a Triangle

Given is an arbitrary triangle ABC. Construct squares over all sides of ABC. Take two vertices of the triangle – say B and C – and complete two adjacent sides of the respective squares to parallelograms. The resulting fourth vertices U and V of the parallelograms are connected with the remaining vertex of the triangle – A.

Try to use macros for your GeoGebra construction!

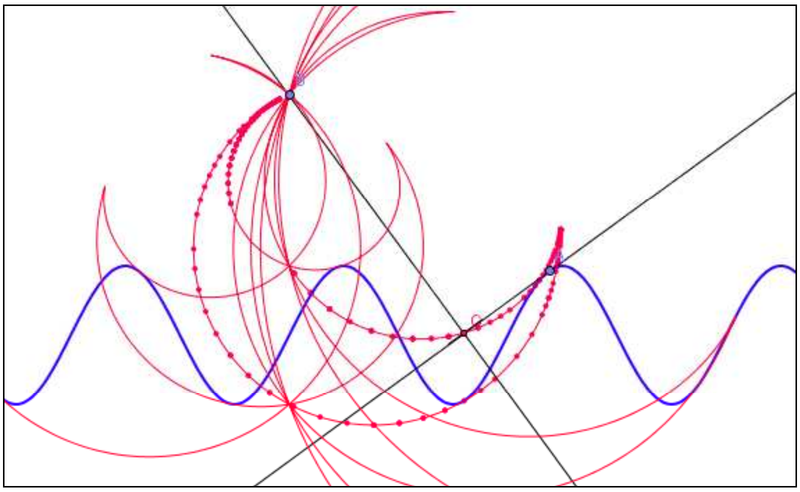
Do you have any conjecture about the properties of the generated triangle AUV?
Can you prove your conjecture?
What happens if the squares are erected in the other directions?
Can you find a formula for the area of the triangle?
We will show a generalized result given by Geometry Expressions.



Credit goes to Peter Lücke-Rosendal. He found this problem in an US-journal.

2 Some Calculus – Finding Pedal Curves and Relatives

1. Given is a curve and a fixed point B (the pole).
2. Draw a tangent at any point A on the curve.
3. Mark a point Q on this tangent so that BQ and AQ are perpendicular
4. Find the locus of all points Q when A is tracing the curve. This locus is the pedal curve of the given curve with respect to the pole.

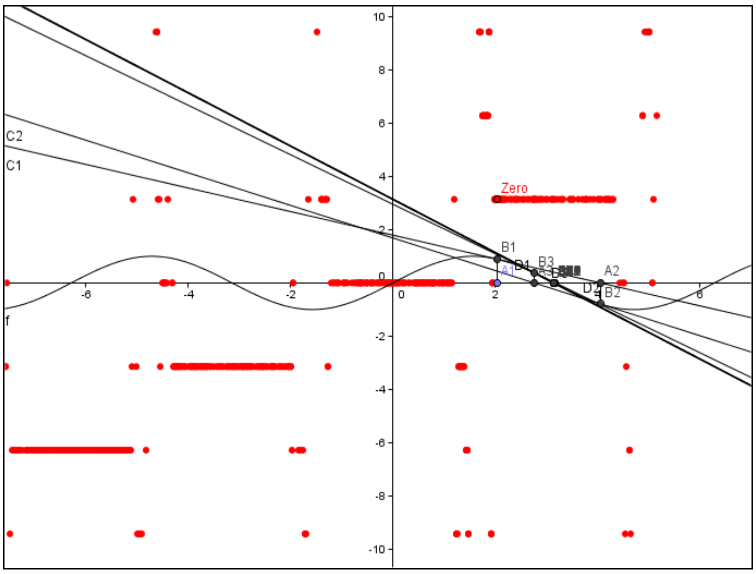


The graph shows a pedal curve of a sine wave.

Further investigations: Find the Negative Pedal Curve, find the Contrapedal.
Find the equations of these curves using a CAS and compare with the loci produced supported by GeoGebra.

3 Using the Spreadsheet for the Newton-Raphson-Algorithm

We can assume that everybody is familiar with the Newton-Raphson Algorithm for finding roots of equations numerically. We will show in a dynamic way the dependency of the root on the chosen initial value.

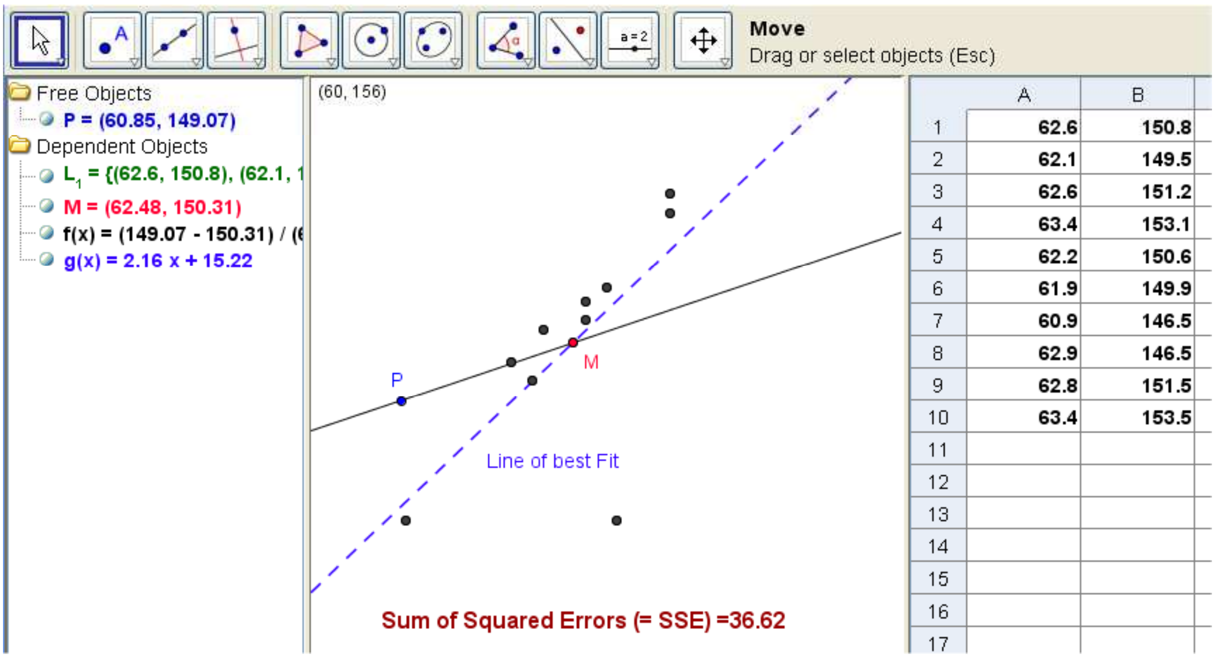


The red points - ini_val,zero(ini_val) - represent the result of the algorithm as a function value of the chosen initial value. A chaotic pattern appears. Zooming in will give interesting insights.

4 Some Statistics with GeoGebra

This is an example which can be found in an Austrian textbook ("Mathe mit Gewinn" = "Maths with Profit"). Given are the height, the length of the stride and the shoe size of 10 students. The problem is to find a relation between the height and the stride (the height and the shoe size).

Name	Größe cm)	Schrittw.(cm)	US-Schuhgröße
Josephine	150.8	62.6	4
Carl	149.5	62.1	5.5
Stanley	151.2	62.6	6.5
Terence	153.1	63.4	7.5
Larry	150.6	62.2	7.5
Walter	149.9	61.9	5
Patricia	146.5	60.9	4.5
Eleonor	146.5	62.9	6
George	151.5	62.8	8.5
William	153.5	63.4	6.5



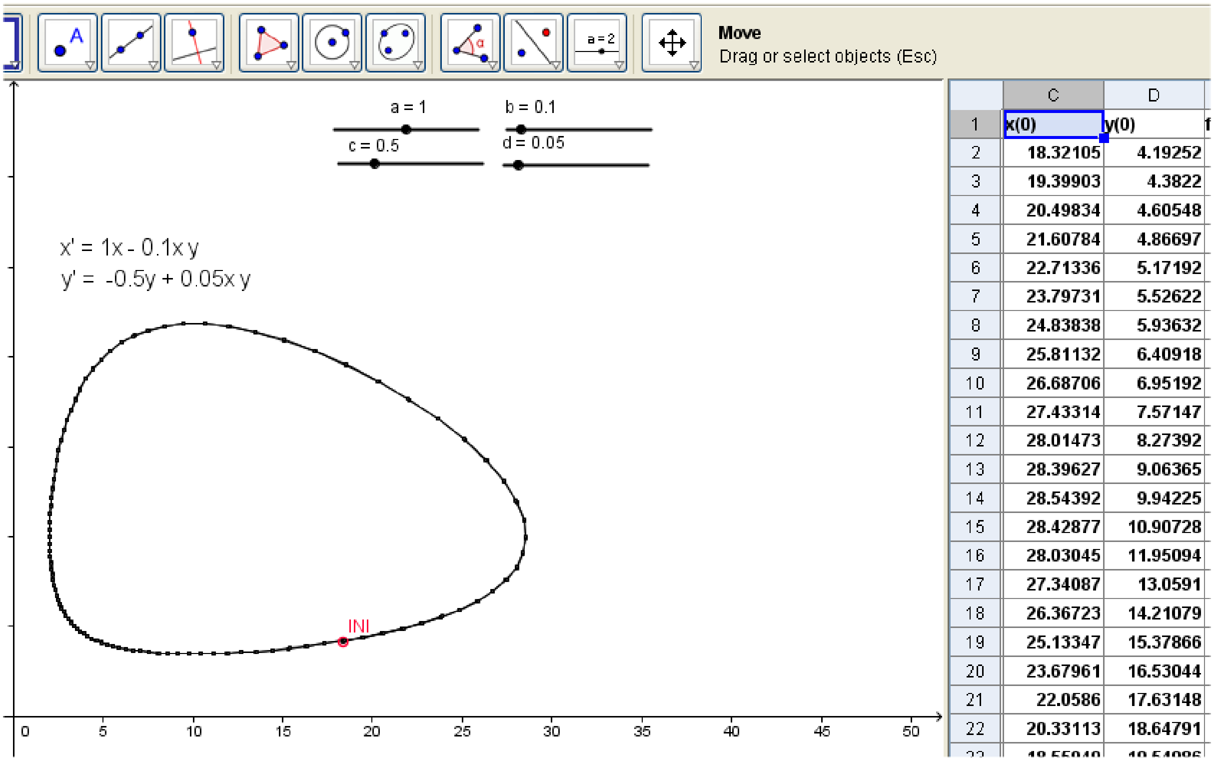
The students enter the data and represent them in form of a scatter diagram. They are instructed to find the "mean point" and any line passing this point. Then they shall find the position of the line giving the minimum of the sum of the squared errors.

Finally we add the linear regression line and compare the results.

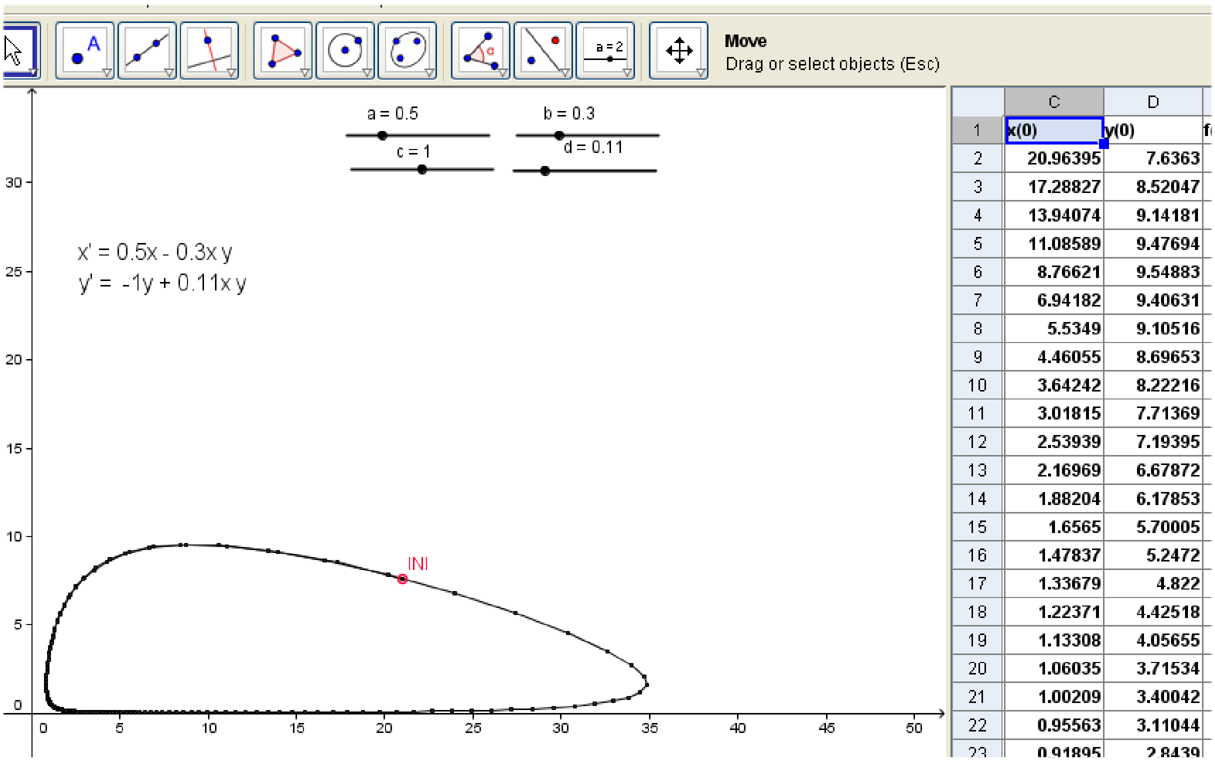
Do the same for shoe size vs height and discuss the outcome.

5 Prey-Predator with the Improved Euler Method and Sliders

This is one of the classic problems in dynamic systems. We will use the spreadsheet together with sliders to make the model as flexible as possible. Students might be inspired to realize better numeric routines like Runge-Kutta-method (and worse routines like Euler-method).



You can see the mathematical model (in form of a dynamic text). The sliders control the parameters. The initial point can be moved by the mouse. The phase diagram changes accordingly. We can add the plots of both populations vs time.



Innovations in Educational Research and Teaching of Experimental Calculus

Horacio E. Bosch, Claudia Guzner, Mercedes S. Bergero, Mario A. Di Blasi,
Adriana Schilardi and Leonor Carvajal
Inter-academic Network of Research and Teaching of Experimental Mathematics
National Technological University, Argentina hbosch@funprecit.org.ar

Abstract

For several decades, there have been a varying number of books on Calculus following the classic line of mathematical thought, where Mathematics is taught for everybody by means of rigorous definitions, theorems, and carefully detailed and extensive demonstrations. For mathematical education into the XXI Century the students need to achieve ability in handling of present mathematical tools and concepts from the beginning of their courses. These needs can be achieved today by means of a paradigmatic change in the focus of mathematics teaching: to learn to develop ideas and to experiment and test those ideas in such way that students can verify their own inferences. In this paper we report an educational research in teaching and learning functions models according to a new paradigm in hands-on experimental mathematics, with applications in the real world, i.e. sciences and engineering by using Computer Algebra Systems. The study of functions is presented, focused into the framing of Exploratory Learning Systems, where students learn by means of the action and their participation in it. It is designed for teachers working together with students in a computer laboratory like hands-on workshops-type activities on other sciences. In this way students have a more "alive", "realistic" and "accessible" touch in Calculus.

Introduction

For several decades, there have been a varying number of books on Calculus following the classic line of mathematical thought. Without minimizing the quality of the books, and their authors' authority, this line of thought has nonetheless had serious difficulties in being understood and learned by most students of natural sciences, engineering, economic sciences and so on [(1)].

The Mathematical Education into the XXI Century

Students need to achieve ability in handling of present mathematical tools and concepts from the beginning of their courses without getting lost in the labyrinth of demonstrations and tests. These needs can be achieved today by means of a paradigmatic change in the focus of mathematics teaching, particularly of Calculus. Students should assimilate the methodologies of experimentation, simulation, and graphical interpretation in problem solving. For several years now, the interactive graphing and calculation facilities in "Computer Algebra Systems" (CAS) such as "Maple®", "Mathematica®" "MATLAB®", and online software releases like "GeoGebra", have been available. The use of such computer assistance in Mathematics is equivalent to the use of telescopes in astronomy or of microscopes in biology. These tools do not explain the facts, but show new possibilities. This mathematical approach facilitated the discovery of new celestial bodies, and of the effects of mass, gravity and acceleration in space-time. The experimental mathematical approach is maturing and promises extraordinary beneficial effects [(2, 3)].

New focus vision concerning mathematical education at the laboratory

In order to prepare students for the *Mathematical Education into the XXI Century*, an educational research group on experimental mathematics has been created at the Argentine National Technological University. Since many years ago, a complete set of presentations to international congresses in Latin America have been performed [(4-9)]. Based on this new paradigm, we have elaborated a ***collection on topics of Calculus.*** In particular, in VOLUME 1, a study of functions is presented, focused into the framing of Exploratory Learning Systems. We have selected a few of the varied number of tools available in the market for use in mathematical applications.

A group of model functions is selected: lineal, quadratic, exponential, logarithmic, harmonic, and periodic non harmonic. The structure of each description is common:

1. Experimental Study of the model (Assisted by "Mathematica")
2. Complementary Experimental Activities
3. Experimental Study of the model (Assisted by "GeoGebra")
4. Integrating problems
5. Numerical and graphical applications

A short series of activities are described in the following paragraph.

Results

A complete book on Functions has been designed and is ready for publication. Some activities contained in this book are shown below, derived from specific stated problems.

Linear model. Representation with Mathematica

Activity 1. Relationship between Fahrenheit and Celsius scales

The water freezing point is 0° Celsius scale, and 32° Fahrenheit scale. The water boiling point is 100° Celsius scale, and 212° Fahrenheit scale.

Which are the increments ΔC and ΔF between the water bowling point and freezing point in both scales?

Which is the relationship between both increments $\frac{\Delta F}{\Delta C}$ *?*

Find the relationships between both scales ºC y ºF, being ºC the independent variable. Draw a graphics of this relation. Complete the following Table.

ºF	***32***	***42***	***150***	*¿?*	*¿?*	*¿?*
ºC	*¿?*	*¿?*	*¿?*	***24***	***37***	***-5***

Table 1. To be completed after experimentation.

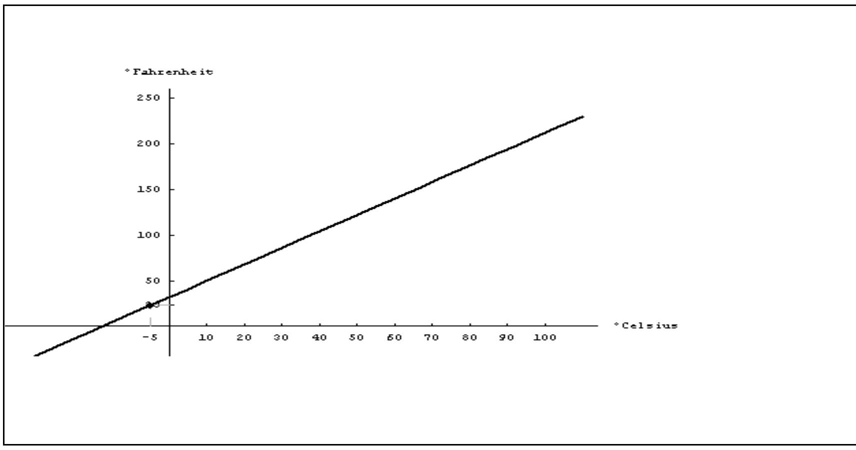


Figure 1. Experimental relationship between Fahrenheit and Celsius scales.

F (**º*C***) = 1,8 **º*C*** + 32

Quadratic model. Representations with GeoGebra and Mathematica

Activity 2. Representation of the "Security Curve"

Find the equation for the "security curve". For a fixed initial projectile velocity, by changing the angle with respect to the horizontal line, the projectile attains different altitudes. These are "limited" by the "security curve".

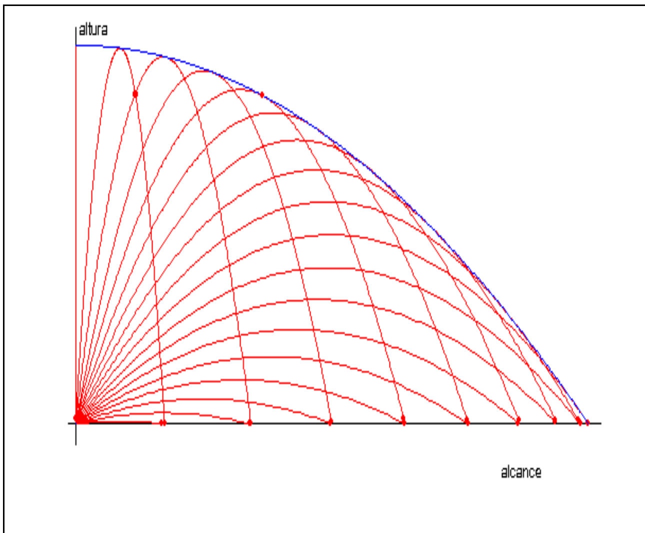


Figure 2. "Security curve" representation.

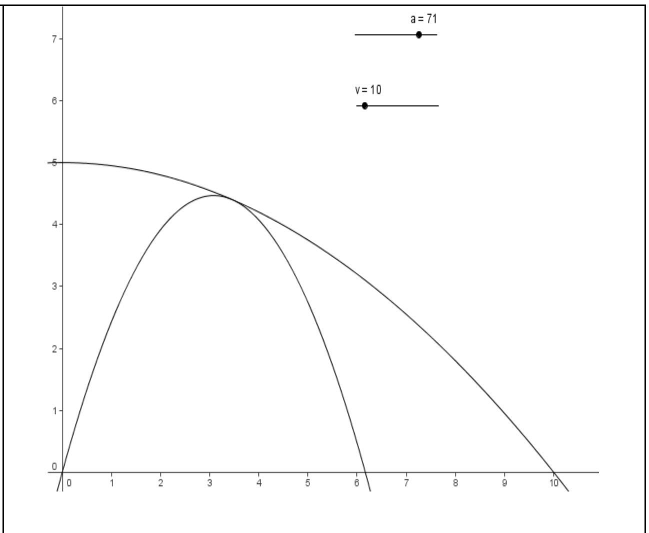
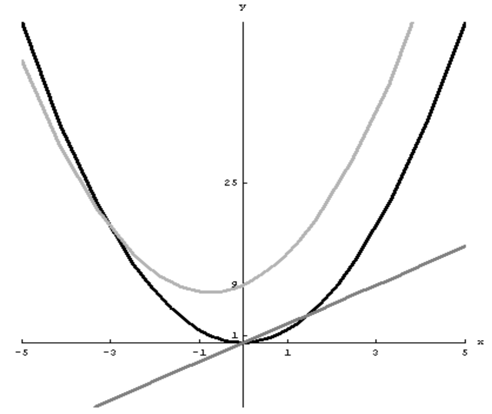


Figure 3. Alpha is the projectile angle with respect to the horizontal line. As alpha varies, the security curve is drawn.

*Activity 3. Let are three functions $f_1(x) = 2x^2$, $f_2(x) = 3x$, $f_3(x) = 9$. Find the sum of them and represent them in a "Mathematica" graph. Let be the function $f(x) = mx^2 + bx + c$. Experiment with different values of **m**, **b** and **c,** draw the graphics and write the surmises.*



m = 2, b= 3, c=9

Figure 4. Graphics with Mathematica

Exponential model. Representations with Mathematica

Activity 4. Voltage decay of a capacitor through a resistance as a function of time

Let are a capacitor and a resistor connected in series and a power supply with a certain voltage difference V. As one measures V (t) as function of time one observes an exponential - type decay

$$V(t) = V_0 \cdot e^{-t/\iota}, \qquad (4.1)$$

Where ι = R·C. *Apply the definition of period T and find the relation between it and the time constant ι*

$$T = \ln 2 \cdot \iota = \ln 2 \cdot R\cdot C = 0{,}693 \cdot R\cdot C \qquad (4.2)$$

Represent the function (4.1) for V_0 = 12 V and R = 2000 Ω y C = 5 μF for 5 periods T of time.

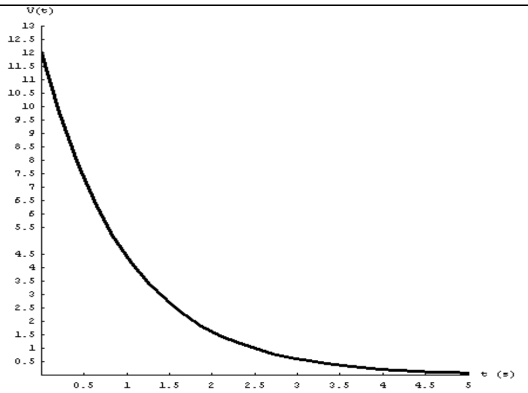


Figure 5. Voltage decay as function of time

Harmonic motion. Representations with Geogebra and Mathematica

Activity 5. Coupling of harmonic motions with GeoGebra. Analyze the two harmonic motions coupling represented by the following conditions. Projection on the x- axis of two vectors of different amplitudes that rotate counterclockwise with the same frequency with a phase difference α = π/3 *(Fig. 6).*

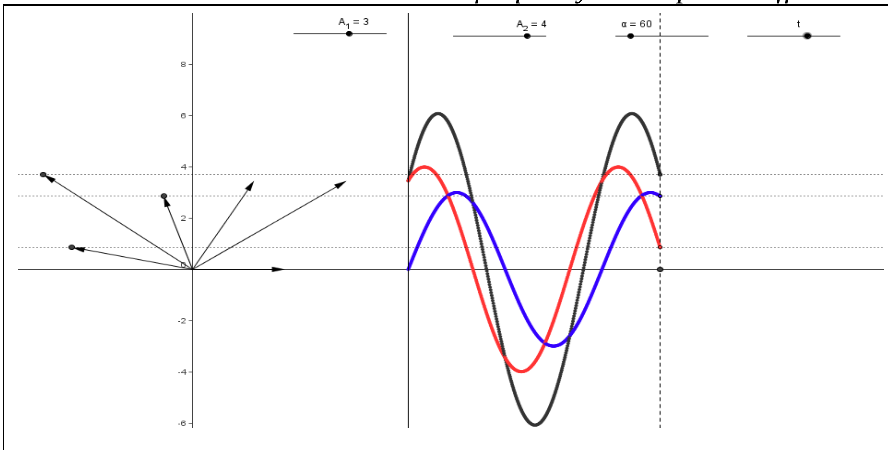


Figure 6. Projection on horizontal axis of two vectors which differ in magnitude and phase π/3, with same frequency. They are coupled to the sum vector, which rotates at the same frequency. The projections on the horizontal axis of the three vectors are represented as function of time.

Activity 6. Mathematica representations

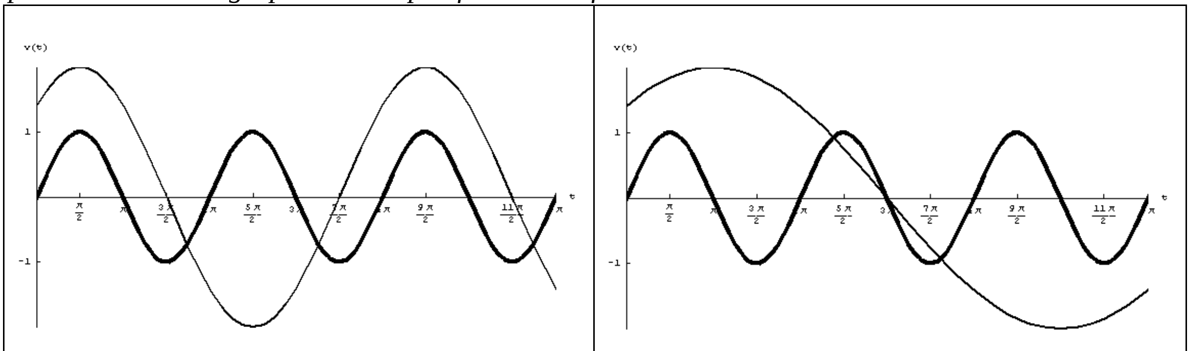
A person is at a Park Wheel. The projection on the y – axis of its position is represented by a function of the type y(t) = sin(t).

Find the time elapsed between two successive identical positions. Define the function period T.

Let are the functions $y(t) = 2.\sin(0.5t + \pi/4)$ *and* $y = y(t) = 2.\sin(0.25t - \pi/4)$*.*

Which is the period for each function? How are they related with respect to the former function $y(t) = \sin(t)$*?.*

Find a general harmonic function comprising amplitude A, period T, the relation between ω *and T and initial phase* α*. Find its graphics with specific values of each variable.*



y(t) = sin (t); y(t) = 2·sin(0.5t + $\pi/4$)   y(t) = sin (t); y(t) = 2·sin(0.25t - $\pi/4$)

Figure 7. Graphics.

Periodic non harmonic function. Representations with Geogebra

Activity 8. "Tangentoid"

A laser beam rotates vertically and impinges a wall. An angle made by the laser beam and a horizontal line is defined as alpha .Changing the angle, segments on the wall with different "heights" are obtained. If one represents these segments as a function of alpha, one gets the "tangentoid" (Fig. 8).

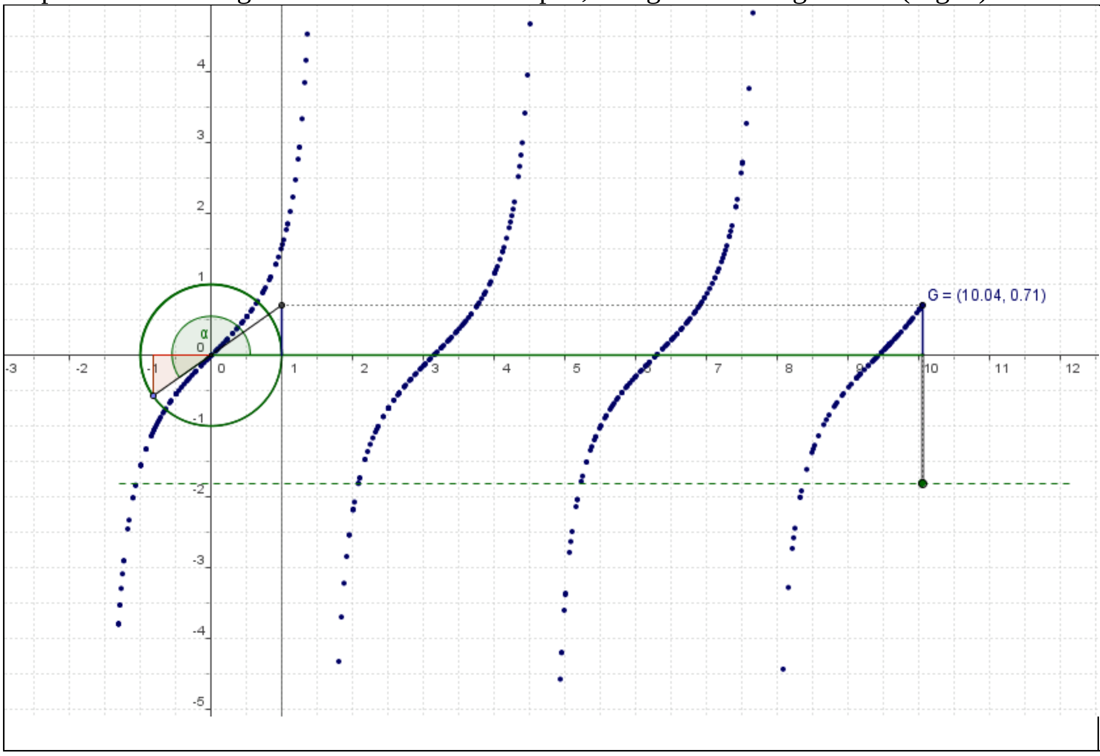


Figure 8. "Tangentoid" Graphics of the projections obtained by changing the α angle.

Transfer of the model to different environments

One of the main concerns the research group has is to disseminate the results in such a way that these can be grasped by teachers, students, graduate schools, and teachers´ careers.

A great effort in this direction has been done organizing local seminars, workshops, and conferences. On the other hand, typical "hands-on" workshops for mathematics teachers at the Technological University have been organized in the last years and for secondary school teachers as well. Many conferences and workshops in Latin American Countries have been arranged [4-9].

Special publications in Spanish, named "Experimental Mathematics Notebooks" have been distributed among several secondary school teachers.

Conclusions

We report some results of an educational research in teaching and learning mathematics carried out for several years, according to a new paradigm in hands-on experimental mathematics, with applications in the real world, i.e. sciences and engineering by using Computer Algebra Systems. A complete book on Function models has been developed according to the above mentioned paradigm. The experimentation with university level professors and students, as well as with secondary school mathematics teachers has been carried out in the last years. A feedback from those experiences has been incorporated by our research group, and we can state that this "new" paradigm has been assimilated and welcome from both, teachers and students.

In this way, a contribution to the actions on mathematical education into the XXI Century is presented highlighting the importance that students need to achieve ability in handling of present mathematical tools. We claim that with this contribution students have a more "alive", "realistic" and "accessible" touch in Calculus.

Nevertheless, a real reconversion of mathematical teaching is still very distant. Many books of the kind we present here have to be written and disseminated in a bigger scale. The fundamental action must be concentrated in teacher's professional careers. As an example, we highlight the thorough efforts done by U.S. National Science Foundation and the U.S. National Academy of Sciences on STEM (Science, Technology, Engineering and Mathematics) [10].

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The Best of Both Worlds: Teaching Middle School and College Mathematics

Daniel J. Brahier, Ph.D.
Professor of Mathematics Education, School of Teaching and Learning, Bowling Green State University
Bowling Green, Ohio, USA brahier@bgsu.edu

Abstract
As a full-time Professor of Mathematics Education, as well as a part-time eighth grade (13 and 14 year olds) mathematics teacher, I have the opportunity to experience the teaching profession from "both sides of the fence." My university courses are enhanced by my work in the field, while my eighth graders' learning is strengthened by educational principles studied at the university. In this paper (and presentation), I will explain this partnership and the benefits to both audiences.

Introduction

For the past 15 years, I have taught mathematics and mathematics education courses at Bowling Green State University (BGSU) in Bowling Green, Ohio (USA) as my full-time responsibility. However, I have also taught one middle school (Grade 8) mathematics course at a local school each day during the same time period. I teach at St. Rose School in Perrysburg, Ohio (USA) – a Catholic school with approximately 400 students. My typical class size has been about 20 students per year.

I have found that my experiences in the middle school classroom feed and enhance my teaching at the college level, as well as my ability to make presentations and author books and journal articles. Additionally, my eighth graders have benefited from my sharing with them about the nature of college mathematics – what they will be expected to know, which of the "big ideas" in our class will be expanded upon later, and so forth. When making presentations to teachers and administrators, I always share *real* examples of teaching strategies I use with my own secondary school students, including challenges that arise and opportunities that present themselves.

Dividing Responsibilities

When I was hired at our university (BGSU) in 1994, there was an agreement that I would be permitted to continue teaching eighth grade mathematics, which I had been doing for five years prior to beginning my college teaching career. My eighth grade class started at 7:15 a.m. each day, Monday through Thursday (ending at 8:15 a.m.), and the university agreed to allow my teaching, office, and committee responsibilities to begin late enough that I could get to campus in time. Consequently, I generally never teach a class that begins before 10:00 a.m. and schedule all meetings and office hours for 9:00 a.m. or later. This agreement has held for 15 years and makes it possible for me to teach in both settings.

St. Rose School pays me to teach the class, although the salary is minimal. A colleague of mine who lives in another state teaches in a local school in a similar manner. However, in her case, rather than the school paying her to teach, the school district transfers money to the university, who uses the funds to buy her out of a course-teaching load. Therefore, her "in load" of teaching includes two university courses and one middle school class, as opposed to teaching three classes on campus. This arrangement has not been feasible at my institution, so I teach three courses on campus each semester, in addition to the eighth grade class. I estimate that I spend approximately 10 hours per week on planning, teaching, and grading papers at St. Rose. I accomplish this work by rising early in the morning and sometimes work well into the evening. I do not allow my eighth grade teaching time to interfere with my productivity at the university and try to keep the two separated. In reality, both positions feed and enhance one another, and I am more productive in both settings with this arrangement.

Seven years ago, I took a sabbatical from BGSU and was hired as a mathematics teacher at a local high school. For that academic year (2001-2002), I taught high school mathematics on a full-time basis, including teaching five classes across the grade levels (Grades 10-12), co-chairing the Mathematics Department, and doing other school-related work, such as chaperoning or supervising student events. My intent was to immerse myself into the school setting for a full year to enhance my ability to teach education courses upon returning to the university. Similarly, I am on leave for the autumn of 2009 and am again teaching in schools. I am teaching my eighth grade class, along with teaching a statistics course in a local high school.

Issues Raised in Classes

One day, in the fall of 2008, I was teaching my eighth grade class how to solve linear equations. The equation on the board was: $5-2(x+4)=x+6$. One of my students went to the board, subtracted 2 from 5 (ignoring the correct order of operations), distributed the "3" through the parentheses, and obtained an answer of $x=-3$. Another student raised her hand and said, "That can't be right. He forgot to distribute the -2 through the parentheses." She went to the board, followed the "correct" procedure, and also found a solution of $x=-3$. Of course, a third student raised his hand and asked, "So, does it matter whether you distribute first or not? It looks like you get the same answer either way!" Suddenly, I found myself having to defend why the proper use of order of operations would *always* result in a correct answer and why this particular problem was an exception, rather than a rule.

The next morning, I went into my mathematics teaching methods course on campus, put the problem on the board, explained what had happened, and asked my students two questions:

1. How would *you* have handled the eighth grader's question?
2. In what cases would we expect the answer to be the same, whether the distributive property is used or not?

The first question got us talking about how students learn mathematics and the pedagogy that can be used to meet students' needs, while the second question allowed us to delve more deeply into the mathematics of the problem. Eventually, we graphed the functions $y_1=5-2(x+4)$ and $y_2=3(x+4)$, as well as $y_3=x+6$ and showed that, by coincidence, all three of these linear functions intersected at the point $(-3,3)$. Since all three of the lines share the same intersection point, both ways of working the problem will result in the same answer. A counter-example had to be used to prove that this is not the case for a similar equation, such as $5-2(x+4)=x+5$. My students went on to explore the conditions under which both the "right" and the "wrong" processes would always result in the same correct answer.

This is just one example of a situation that arose in my eighth grade class that I could take back to my university students. On another occasion, an eighth grader asked me, "I know that a line has a slope, but does a parabola have a slope too?" Again, I took this question back to my methods students at the university, and we discussed how to get a secondary school student to understand the basic idea of "slope of a tangent line," which eventually develops a fundamental idea of calculus. Similarly, one day in my university class, a student revealed a misconception about what it means for a weather forecaster to say that there is a "20% chance of rain." Several of my college students thought it meant that 20% of the area would receive rain today, while the other 80% would not. I went back to my eighth graders and asked them what *they* believed the prediction meant. It made for an interesting contrast of thinking, as it turns out that my 14-year-olds understood the notion of probability better than many of my university students! The situation helped me to understand the thinking of both levels of students, which in turn makes me a better instructor.

Problems Used with Students

One of my favorite problems I use with my students at both levels is called the Orange Grove Problem. Students are presented with a scenario in which an orange farmer has a grove of 120 trees in which each tree averages 650 oranges per season. But if he plants additional trees, each tree takes up enough nutrients from the soil that the average production of every tree decreases by 5 oranges. The question is how many trees to plant to maximize the orange production. What I find interesting about the problem is how students at different levels solve it. For example, most eighth graders will make a table and then look for a pattern. So, if one tree is planted, the grove has 121 trees, but each produces an average of 645 oranges. The second tree results in a total of 122 trees with 640 oranges apiece, and so on. By multiplying the number of trees by the average number of oranges per tree, students can see the pattern and determine that planting five trees maximizes production.

At the college level, I find students more likely to find and solve equations. Typically, students will write an equation such as $y=(120+x)(650-5x)$, where *x* represents the number of new trees planted, and *y* stands for the total orange production. They will either graph the function on a calculator and identify the vertex or multiply the binomials and take a first derivative to determine its maximum value. In the end, they also determine that planting five trees maximizes the production, but the approach is much more technical and makes use of higher mathematical skills.

In each case, once students have finished solving the problem, I show them how the other group typically deals with it. The eighth graders look at the parabola and discuss the visual approach to seeking a maximum, while college students are asked to "step back" and look at the problem as someone would view it with no calculus and very little exposure to quadratics. The experience of sharing the problem across grade levels makes the process of solving the problem much richer for everyone involved. I frequently try to pose problems that can be solved both in a secondary and in a university setting to deepen my understanding of how students handle the same situation with different backgrounds and problem-solving tools.

Lesson Learned

Probably the most important lesson that I have learned through this teaching arrangement is how important it is for a university professor to remain up-to-date with what is happening in the field. So often, university students complain that their education professors "have never been a 'real' school teacher" or that it has "been so long since he/she was a teacher, that he/she is out of touch with schools." I believe it is essential that university education faculty maintain a presence in school settings. While on sabbatical, for example, I had to learn the system for taking attendance on the computer and uploading grades and attendance to a school Web site so that parents and students could check their grades every day. These tools were not available when I was a full-time teacher, so it was important for me to put myself into the situation to learn what teachers are up against.

There are several avenues that can lead to a high level of exposure to teaching schools, such as the following:

1. The situation described in this paper is one way to remain in a school setting. If a local school is looking for a part-time teacher, an arrangement with a university faculty member may be possible.
2. The sabbatical experience also discussed here is another way for a university professor to be immersed in a school setting. Some institutions actually *require* education faculty to get out into schools to teach every few years.
3. Grants or other collaborative ventures can allow university faculty to interact directly with students in a school setting. In some cases, a grant project allows for the faculty member to teach model lessons, to tutor small groups of students, or to team teach lessons with mathematics teachers.
4. Another initiative that I have taken is to simply call a local teacher and ask him/her if you can visit his/her classroom for a couple of days and teach a sample lesson or unit. In the process of writing a journal article a few years back, I called a fifth grade teacher and arranged to spend three days teaching her class, during which time I was able to gather data, take photographs, and, most importantly, gain experience. The teacher and children were enriched by my visit, and I learned much about the day-to-day struggles (and the excitement) of being a fifth grade teacher.

Conclusions

I am often asked, "Why would a full-time university professor choose to spend 4 hours a week in a classroom with 24 adolescents when he doesn't have to do it?" The answer is simple: Because I *enjoy* it. I chose education as my profession because I want to help children appreciate mathematics and become successful problem solvers. I believe that having an impact on these students can happen from both ends of the spectrum – as a professor who teaches others how to teach, as well as working directly with the children themselves. While I conduct most of my work at the college level, I have never lost sight of the importance of working with school-age students as well.

By working in both environments, I have found my credibility level among professionals to be very high. At St. Rose School, students, parents, and administrators know that I prepare mathematics teachers, so my opinions and work are highly valued. At the university level, my students and colleagues frequently seek my opinion on issues, knowing that I am "in the trenches" every day. I was recently conducting an in-service workshop for teachers and sharing a teaching idea. One participant raised his hand and asked, "This sounds like an interesting problem in theory, but I'm not sure 'real' students can handle it. I wonder how it would play out in a classroom." I replied that it was interesting he would ask that, given that I just solved that problem with my own eighth grade class within the past couple of weeks. I proceeded to show the teachers samples of how my own students had approached the problem. I try to share only those ideas that I have actually attempted myself, with my own students. As a result, I can give a presentation, write an article or book, or teach a class based on experience, rather than relying entirely on theory. I am the author of one textbook and co-author of another, and both of these were written out of my experiences in the classroom.

One major drawback of my situation is that it can be exhausting. When my eighth graders are on vacation or a class trip, I suddenly "find" 10 working hours that I didn't have the previous week, and I find myself feeling less stressed and getting more sleep. Another difficulty I experience is with adjusting to different age groups. There are days when I teach eighth graders in the morning, college students in the afternoon, and experienced teachers in a workshop in the evening. Knowing the audience and adjusting the pace and content for the situation can be challenging. In general, however, I would not trade this experience for anything. I look forward to both components of my work and have managed to put them together in my teaching, writing, and presenting. I look forward to continuing my relationship with a local school until I retire and encourage colleagues to consider any type of arrangement that puts them in a school environment on a routine basis – not just *observing* teachers and children but *interacting* with them.

Language and Number Values: The Influence of the Explicitness of Number Names on Children's Understanding of Place Value

Sandra Browning, MS PhD

Assistant Professor of Mathematics Education, School of Education, University of Houston-Clear Lake, Houston, Texas, USA

Abstract :In recent years, the idea of language influencing the cognitive development of an understanding of place value has received increasing attention. This study explored the influence of using explicit number names on prekindergarten and kindergarten students' ability to rote count, read two-digit numerals, model two-digit numbers, and identify the place value of individual digits in two-digit numerals. Through individual student interviews, pre- and post-assessments were administered to evaluate rote counting, reading five two-digit numerals, modeling five two-digit numbers, and identifying place value in two two-digit numerals. Chi-square tests for independence showed two significant relations: (1) the relationship between the control and treatment group membership on the post-assessment of modeling two-digit numbers and (2) the relationship between place value identifications and group membership. Analysis of the children's performance and error patterns revealed interesting differences between children taught with explicit number names and children taught with traditional number names. The improvement of the treatment group overall exceeded the improvement of the control group. This study indicates that teaching children to use explicit number names can, indeed, have a positive influence on their understanding of place value.

Introduction The superior performance of Chinese speaking countries in international tests in mathematics and science raises the question of what advantages there might be in the language itself. While many reasons for national differences in these tests can, should, and have been posited, the question of language advantage remains unanswered. A quantification of the number of words needed to name the numbers from one to one hundred was used to investigate correlation between this and position in national test rankings (Beauford, 2003). Languages considered were English, German, French, Spanish, Chinese, Japanese, and Korean. When learning to count to 100 in Mandarin Chinese, children use a total of 11 words. In English, the task requires 26 different words or word parts. Correlations were negative, strong, and significant. Almost without exception, the fewer words or word parts needed to name numbers to 100, the better was the position of the country in international comparison.

The question of whether language influences mathematics performance has prompted a progression of studies. Miura (1987), Miura and Okamoto (1989), Ginsburg (1989), Miura, Okamoto, Kim, Steere, and Fayol (1993), MacLean and Whitburn (1996), and Alsawaie (2004) compared children from different language heritages. These studies involved a single assessment of the mathematics performance of intact groups of students without intervention. Most of these studies were conducted in the United States comparing native English-speaking students with students whose native language was not English and who were attending classes taught in their native language. Fuson and Briars (1990), Fuson, Smith, and Cicero (1997), Fuson, et al. (1997), and Cotter (2000) conducted studies that applied and analyzed the results of using explicit number names for several months with first grade students.

Beauford (2003) extended the research further in an investigative study using explicit number names as an intervention with four- and five year-old students beginning their first formal introduction to number. The four-year-old students were taught using only explicit number names for the entire year. The five-year-old students were taught using explicit number names during the first semester and both explicit and traditional number names during the second semester of the school year. Beauford's study involved a small sample with no control group.

This research extended the study of language differences and the understanding of place value to a quasi-experimental study of a larger sample that included both an experimental and a control group when interventions were utilized. To investigate whether number names affect students' cognitive understanding of place value, this research involved the use of explicit and traditional names of numbers with young children. In examining the influence of language on the cognitive understanding of place value, the following questions involving the components of place-value understanding were examined:

1. Is there a difference in rote counting between children taught with explicit number names and children taught with traditional number names?
2. Is performance in reading two-digit numerals independent of group membership of children taught with explicit number names and children taught with traditional number names?
3. Is performance in modeling two-digit numerals independent of group membership of children taught with explicit number names and children taught with traditional number names?
4. Is accuracy of identifying the place value of the digits of two-digit numerals independent of group membership of children taught with explicit number names and children taught with traditional number names?

Definitions *Place value.* The term place value refers to the value assigned to a digit due to the position of the digit in a numeral. The three elements of place value understanding are: (a) grouping by tens, (b) spoken names of numbers, and (c) written names of numbers (Van de Walle & Lovin, 2006).

Explicit number name. In this study, students were taught to say numbers explicitly. In other words, instead of saying “forty two”, the students were taught to say “four tens two”. This method of naming numbers accurately indicates the base-10 place value of numbers; therefore this is called the “explicit” method for naming numbers. This is also called base-ten language.

Modeling number. Students were also asked to represent two-digit numbers with straws. Straws bundled into groups of tens to represent the numeral in the tens place were called ten-bundles. Single straws represented the numeral in the ones place.

Canonical representation. A representation of a two-digit number using the correct number of ten-bundles and single straws was defined as a canonical representation (see Figure 1).

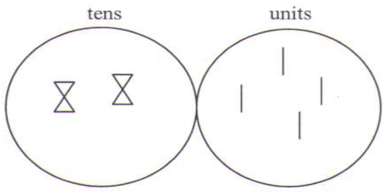

Figure 1. Canonical representations of “24”. ⧖ = ten-bundle, | = unit.

Noncanonical representation. A correct representation of a two-digit number that was not canonical was defined as a noncanonical representation. Three possible noncanonical representations were noted (see Figure 2).

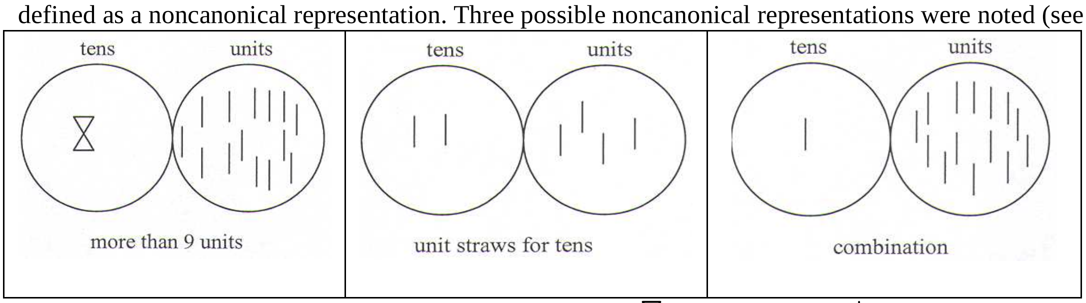

Figure 2. Noncanonical representations of “24”. ⧖ = ten-bundle, | = unit.

One-to-one representation. A representation of a two-digit number in which the student uses only single straws with no ten-bundles was defined as a one-to-one (1-1) representation. In a 1-1 representation, the student counted each straw only once as the student represented the given number (see Figure 3).

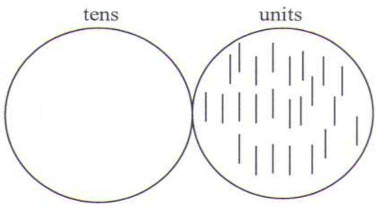

Figure 3. 1-1 representations of “24”. ⧖ = ten-bundle, | = unit.

Incorrect representation. A representation of a two-digit number that is completely incorrect due to any other reason was simply called an incorrect representation (see Figure 4).

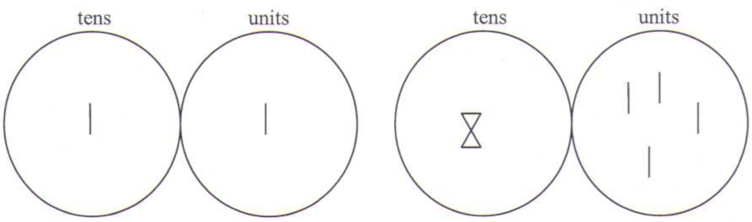

Figure 4. Examples of an incorrect representation of “24”. ⧖ = ten-bundle, | = unit.

Research Design This study was a quasi-experiment comparing the performance of students using explicit number names and the performance of students using traditional number names in demonstrating a cognitive understanding of place value. Three groups of participants were involved in this study. The first group consisted of prekindergarten and kindergarten students. These students were taught to use explicit number names such as "one-ten two" for the quantity "12" and were the treatment group. Spanish-dominant students in the bilingual classes were taught to say "uno dies dos" rather than "doce" for the quantity "12". The second group of prekindergarten and kindergarten students was taught to use traditional number names, such as "twelve" for the quantity "12". This group was the control group. The third group consisted of the teachers of the students in both the control group and the treatment group. Explicit number names were used during the full year of kindergarten and prekindergarten. A pre- and post-assessment of rote counting, reading five two-digit numerals, modeling five two-digit numbers, and identifying the place value of individual digits of two-digit numerals were conducted. The teachers of both the control and treatment groups were interviewed.

Participants. The participants were selected from students enrolled in a small public school district in a south central city. The school district has an enrollment of approximately 500 students per grade level reflecting the diversity of area with 68% Hispanic, 24% Anglo, and 6% African-American. The study involved the three prekindergarten classes, one of which was a bilingual class comprised of students whose first language was Spanish.

Two kindergarten classes from two elementary campuses participated in the study. One elementary campus served predominantly affluent students while the other elementary campus served predominantly low socio-economic students. One of the kindergarten classes on the more affluent campus was a bilingual class comprised of Spanish-dominant students.

A total of 115 students participated in the study with 53 (46%) prekindergarten students and 62 (54%) kindergarten students. The study was comprised of 49% male participants and 51% female participants. The treatment group comprised 57% of the participants, and the control group comprised 43% of the participants.

Data collection. Students in both the treatment group and the control group were individually assessed two times during an eight-month period. Using a script to standardize the interviews, the interviewer asked students to perform four tasks: (a) count as far as they could, (b) read five cards with two-digit numerals on them, (c) model with bundles of ten straws and single straws a different set of two-digit numbers on five cards, and (d) identify the place value of the digits of two-digit numerals for two numerals. Data was collected during each interview using a standardized recording document. In addition to interviewing the students, teacher observations specific to lessons involving the teaching of place value were conducted. The class visits allowed the researcher to: (a) become a familiar face to the children, (b) observe children in the class setting, (c) observe teaching practices, and (d) serve as a resource for teachers.

At least two observations of each teacher were conducted during the study. District personnel requested that the researcher use the district-wide observation form. The teachers were familiar with this form and were, therefore, very comfortable with observations being recorded with this form.

Results

Rote counting. Research question one asked whether children taught to rote count with explicit number names would perform differently from children using traditional number names. From September to May, the treatment group and the control group for both prekindergarten and kindergarten students improved in their ability to rote count. No significant differences were found in the highest number reached when rote counting for prekindergarten students or kindergarten students when taught to use explicit number names rather than traditional number names. However, notable differences in counting errors were found between the kindergarten control and treatment groups. The percentage of minor errors in the treatment group decreased considerably (pre-assessment 20%, post-assessment 3%) while the control group's percentage of minor errors remained constant at 19%.

Reading two-digit numerals. Research question two asked if performance in reading two-digit numbers was independent of group membership for children using explicit number names or children using traditional number names. Performance in reading two-digit numerals was independent of group membership for prekindergarten students on the post-assessment. However, the prekindergarten treatment group read about 30% fewer of the numerals incorrectly than did the control group. On the post-assessment, a significant relationship was found for kindergarten students (χ^2 (3) = 13.99, p < .001) when taught to use explicit number names rather than traditional number names. Approximately 70% of the numerals were read correctly by the treatment group, and approximately 40% of the numerals were read correctly by the control group.

Modeling two-digit numbers. Research question three asked if performance in modeling two-digit numbers was independent of group membership for children using explicit number names or children using traditional number names. For prekindergarten students, performance in modeling two-digit numerals was dependent on group membership (χ^2 (2) = 12.76, p < .01) with the number of canonical representations increasing in the treatment group.

For kindergarten students, performance in modeling two-digit numerals was also dependent on group membership (χ^2 (3) = 43.90, p < .001). The percentage of numbers modeled correctly by the treatment group was about twice that of the control group. Also, the treatment group was able to model 44% of the numbers canonically compared to about 10% of the control group. Notably, the kindergarten control group placed the correct number of ten-bundles and unit straws into the appropriate side of the two-sided container for 7 numbers, and the kindergarten treatment group placed the correct number of ten-bundles and unit straws into the appropriate side of the two-sided container for 62 numbers.

Identifying place value. Research question four asked if performance in identifying the place value of individual digits of two-digit numerals was independent of group membership for children using explicit number names or children using traditional number names. The largest difference in reading numerals occurred when the kindergarten students were asked to point to the tens and ones place of a numeral. A significant dependence was found between group membership and identifying place value (χ^2 (1) = 16.36, p < .001). The percentage of correct responses in the treatment group (53%) was approximately twice the percentage of correct responses in the control group (28%). The kindergarten students in the treatment group who correctly identified the place value of a numeral on a card did so with confidence.

Discussion

This study included prekindergarten and kindergarten students since language acquisition is a primary goal for these students. Learning to count, whether using explicit or traditional number names, is part of the process of acquiring language and building vocabulary. The results of this study correspond with Sousa's (2008) observation that American children have trouble counting past ten at age four while most Chinese children can count to 40 by age four. According to Sousa, this difference is due to the simplicity of explicit number names as well as the syntax reinforcing the decimal system resulting in "Chinese speakers [processing] arithmetic manipulation in areas of the brain different from those of native English speakers" (p. 19).

Kindergarten students are still developing an understanding of place value, and the use of explicit number names seems to have had a positive effect on the students' ability to read two-digit numerals as evidenced by the larger percentage of numerals read correctly by the treatment group. According to teacher reports, the students taught explicit number names had fewer digit reversals than did students using traditional number names. The reinforcement of place value in the explicit number names may have allowed students to realize that the position of the digits in a numeral determines the value of the digit.

Unlike reading abstract numerals, modeling numbers with manipulatives is a concrete activity in that the students can see, touch, and physically move the manipulatives. The use of explicit number names, reinforcing place value, coupled with ten-bundles of straws that represent numbers in the tens place and single straws that represent numbers in the ones place may have provided students with the support necessary to connect the position of a digit with the number of ten-bundles or single straws needed to represent the value of the digit.

These findings supported the findings by Alsawaie (2004). Assessing students in first grade, Alsawaie found that 51.2% of the students represented numbers canonically when prompted while 12% of the students represented numbers canonically when not prompted. However, Alsawaie's study assessed the students only once rather than in a pre- and post situation.

Cotter (2000) also studied the use of explicit number names with first grade students. No pre-assessment was administered, but an intervention was utilized with a treatment group for eight months. A control group received no intervention. Cotter's findings that the students' performance in modeling two-digit numbers canonically improved when taught explicit number names were supported in this study.

While conclusions in this study must be tempered by the lack of longitudinal data on the students involved concerning their success in mathematics in subsequent grades, the findings suggest that the use of explicit number names does increase students' performance in reading, modeling, and identifying place value in two-digit numerals. The improvement of the treatment group overall exceeded the improvement of the control group. Pending a longitudinal study, the cautious conclusion is that using explicit number names can increase the understanding of place value of prekindergarten and kindergarten students.

Possibilities for Transfer to Different Environments

Since the beginning of this project in 2003, we have developed strategies and techniques to improve the validity and reliability of assessment of these young children. While the sites involved has proven to be valuable learning laboratories for research strategy, the relatively small size of the study dictates that it can only be considered a beginning for this research. To achieve generalizability will require similar projects in many classes using a variety of teaching styles and curricula so as to negate the effects of any one environment. We are continuing this research in bilingual elementary schools in Saltillo, Mexico, and also in Santa Cruz and Cochabamba, Bolivia.

Integrating Technology into the Mathematics Classroom: Instructional Design and Lesson Conversion

Marcia M. Burrell, Ph.D.
Associate Professor burrell@oswego.edu
Clayton Cohn, B.A,
Master of Arts in Teaching Student ccohn@oswego.edu
Curriculum and Instruction, State University of New York College at Oswego, Oswego, New York 13126

Abstract

The use of technology in Kindergarten to grade 12 classrooms provides opportunities for teachers to employ mathematical rigor, to integrate problem solving strategies and to extend mathematical ways of knowing (Drier, Dawson, & Garofalo, 1999). The presentation consists of two parts. One investigation maps secondary mathematics technology lessons and materials to the elementary school mathematics standards and converts the mathematics concepts to manageable elementary school lessons. The other investigation analyzes pre-service teacher lessons written using ASSURE instructional design format. The major aims of this paper are to present two teacher preparation practices, one for secondary mathematics pre-service teachers (converting secondary materials to elementary materials) and the other for elementary mathematics pre-service teachers (writing lessons using the ASSURE model).

Introduction

The researchers intended to increase the use of technology integration in pre-service mathematics methods courses. Two processes were undertaken. The first process involved a graduate student adapting materials and lessons for the secondary classroom to the elementary classroom setting. This process required that the graduate student review a variety of software and hardware such as Autograph, Geometer's Sketchpad, National Library of Mathematics manipulatives. The investigation included looking at the materials, mapping the materials to the New York State mathematics standards and then writing objectives for the elementary classroom.

The second process involved elementary teachers writing a lesson for the grade level classroom that integrated technology. The integration used the ASSURE instructional design model to ensure the proper use of technology. The design model was used by every student in the elementary mathematics methods class, and each of the lessons was presented in methods class for critique using the Lesson Study model (Takahashi, Watanabe, and Yoshida, 2006). The ASSURE model stands for analyze the audience, state the objectives, select the methods, materials, and media, utilize the methods, materials and media, require participation, and evaluate what happened upon instruction.

Adapting Materials for Integrating Technology

After one semester of reviewing lessons written by pre-service teachers taking a mathematics methods course, the researcher realized that assigned technology based lessons lacked the characteristics expected. The researcher had expected as with Chaney-Cullen and Duffy (2001) that integrating technology into mathematics lessons would provide pre-service teachers with opportunities to investigate problem based constructivist mathematics lessons. Given adequate time, teacher attitudes about technology often begin to adapt towards a more constructivist perspective (Derry & DuRussel, 2000; Richardson, 2003; Thomas et al., 1994), where "constructivist goals focus on students ability to solve real-life practical problems, and its methods call for students to construct knowledge themselves rather than simply receiving it from knowledgeable teachers" (Roblyer & Edwards, 2000, p. 67). It was hoped that the pre-service teachers would exhibit qualities of constructivism in their lesson delivery and write ups. While the pre-service teachers delivered their lessons to their peers, there were gaps in their constructivist thinking, gaps in the types of objectives they were attempting to assess, and gaps in the types of activities that seemed to fit under a problem based methodology.

As a response to the concerns about the lessons mounted, the researchers investigated using the Teaching Secondary Mathematics (TSM) resources and other secondary technology based resources as a means to more problem based (more constructivist) set of materials that were mathematically rigorous and integrated technology. It was hoped that by investigating the objectives, standards and possible assessment goals using the TSM materials that elementary mathematics methods students could improve the types of lessons created. In response to these educational imperatives, the U.K.-based technology for secondary /college mathematics (TSM) group serves as clearinghouse for software-based technological enablers for math instruction at the secondary and college levels in order to encourage the integration of technology with mathematics instruction.

The researchers provide elementary mathematics teachers with variants of TSM activities, modified for use with elementary students that aid in the introduction to probability in the first and fifth grades. The technology-based activities, in conjunction with the sample lessons plans, have the potential to cover a wide range of New York State performance indicators for each grade level. The graduate student researcher investigated technology-enhanced lessons in the first grade and fifth grade in order to represent both the lower and upper elementary grades, choosing fifth grade as the benchmark for the upper elementary grades in the probability module because it is in this grade that the New York State performance indicators require experiments using the formal language and methodology of probability & statistics.

Model Lessons

In integrating technological resources into elementary probability lessons, the graduate student researcher began with a spinner activity, an activity not included in the TSM resources but nevertheless recommended by education researcher Marilyn Burns as a "good beginning probability and statistics experience for children of all ages," an opinion that was supported anecdotally by several sample lessons available online (Burns, 1992, p. 61). The elementary teacher can introduce the spinner activity in the context of a simple board game or an arts and crafts project in order to avoid introducing the mathematical experience of the spinner activity as an arbitrary and irrelevant activity divorced from "real life." By constructing their own spinners, as described by Burns, grade level students can explore performance indicator 1.G.2 (Recognize, name, describe, create, sort, and compare two-dimensional and three-dimensional shapes) while addressing possible Individual Evaluation Plans (IEP) goals for students with modified curricula such as improving manual dexterity and following directions (Burns, 1992, p. 62).

Once the classroom students have constructed the spinners, the teacher should ask students which number is more likely to come up more than any other number and why, addressing the reasoning and proof, communication, and problem-solving performance indicators (Burns, 1992). After explaining and demonstrating how to record the results of the spinner experiments on a pictogram or bar graph, addressing performance indicators 1.S.2 – 1.S.8, the teacher should ask students which number will reach the top of the graph first and what the graph will look like once one number reaches the top of the graph (Burns, 1992). The teacher can address the number sense and operations performance indicators by asking well-placed questions about the graphs, requiring students to count and make judgments of greater than, less than, or equal to. By using this questioning methodology, the teacher allows students to formulate conjectures, using their intuition to sense what should happen, and subsequently conducting experiments to collect data to support or refute those conjectures, allowing for the "continuous interaction between a child's mind and concrete experiences with mathematics" which forms the basis of a constructivist approach to mathematics (Burns, 1992, p. 24).

After the completion of the conjecture-based activity using the cardboard spinner, the introduction of technology is useful in extending the spinner activity by enabling students to explore different spinner configurations and large numbers of spins without necessitating the use of the additional time required to manually construct additional spinners and graph large numbers of trials. Although complete generality and theoretical probability are beyond the grasp of elementary age students still in the concrete operational stage of cognitive development, encouraging students to begin constructing these higher-order concepts using concrete experiences as a conduit can serve as a valuable introduction upon which additional knowledge can be constructed as students mature (Owens, 2002).

Linking Activities to Curriculum Objectives

The Microsoft excel document (exercise) is undertaken to give pre-service teachers (students in the mathematics methods course) an opportunity to analyze problems and activities that are of interest in the classroom. The Excel chart which serves as a single place for information regarding the activity includes title, grade level, mathematical processes and content, and a place to document specific objectives that can be taught based on the original problem. Teachers may use the excel file to analyze information and to discover gaps in teaching standards within grade level, and across particular mathematical processes and content.

Pre-service Teacher Lessons using the ASSURE Model

Organizations such as NCTM (2000) and TPACK (2008) speak of designing digital age learning environments and experiences for students in mathematics. These organizations provide information for pre-service and in-service teachers to gather exemplars of these lessons, but the process of developing and delivering these lessons around student higher order cognitive thinking described by TIMMS (Keene, 2008) is a difficult and dynamic process. When pre-service teachers are asked to create these lessons or move these created lessons into a design template for presentation, there are difficulties. One of these lessons will be analyzed in order to discuss the intricacies of the technology integrated lesson and to provide some feedback for how to better integrate these lessons into a typical childhood mathematics lesson.

Pre-service teachers are asked to create a problem based lesson that integrates technology. The lesson creation activity provides an opportunity for students to translate the theory about good lessons, as described earlier, into a workable, usable lesson that could be used in any classroom. All of the lessons should have evidence of how students are collecting evidence around mathematical ways of knowing. The acronym ARRCC is used to help remind pre-service teachers that mathematical ways of knowing should include Accuracy, Reasoning, Representation, Communication and Connections. Pre-service teachers are asked to create lessons and think about the mathematical ways of knowing that are connected to ARRCC as they integrate technology.

Convincing teachers to change their practice has much to do with systematic activities with technology and the interactions with a variety of vehicles that move teachers along a continuum for making changes in their thinking (Barnett, Walsh, Orletsky, and Sattes, 1995), and that change should take place via experiences in the college classroom first. These technology based lessons are used to help teachers create learning environments for teachers to help negotiate individual needs for learning and teaching (Hung, 2001; Ross et al., 2002; Flake 2001).

Discussion

These lesson plan samples provide a glimpse to pre-service thinking about technology based lessons in the mathematics classroom. Each of the students was shown the materials created by the graduate student, technology integrated materials, and then students were asked to create lessons that use of a problem based methodology. The lessons cannot fully provide evidence related to the criteria as defined earlier, but the lessons reinforce that the process for creating the lessons is a difficult process that requires much research.

The lessons demonstrate enormous effort around interaction, but the interaction and the objectives appear to be disconnected. For example the objectives from lesson one are from a variety of learning levels, but the evaluation activity, does not map very well to the learning objectives listed. The second lesson also fails in that area. The activities are dynamic and make use of the smartboard, but the objectives, assessment, and activity are often disjointed. For example in the assessment with theoretical probability and the like, the objectives speak to reflecting on the thinking, but the assessment examples do not demonstrate any assessment. It is also difficult to read the lesson and understand how the smartboard moved the objectives forward, but the difficulty in creating lessons and then observing behaviors that bring out the appropriate mathematical thinking is not easy to document.

The comments made by the students in their evaluation phase are of interest too. Their comments reveal a sense of the difficult nature of the task and the learning process required of the teacher as they develop lessons and reflect on their teaching. There is a lot of good information in the words of the students as they grow beyond just delivering information and reflect on their role in the classroom. As teachers rethink their roles to meet the diverse needs of their changing, inclusive classrooms, technology provides a concrete instrument for teachers to step back and observe classroom interactions. As the students use the technology the teacher is able to observe the understandings via the student interaction with the computer technology, the interaction with peers and of course the interaction with the actual content of interest.

Observations about pre-service teacher understandings must be carefully parsed so that what the teachers observes is based on teacher perceptions about what student behaviors are supposed to look like when students understand. So, pre-service teacher perceptions, teacher observations opportunities and skills must be carefully developed in order to ensure teacher interpretations that are accurate and reliable over time. The assumption here is that during a constructivist organized lesson, the teacher is making observations about student understandings based on a variety of sources of evidence, but those observations are connected to teacher perceptions of him or herself and are further clouded by issues of race, socio-economic status, administrator expectations, disability and ability status, curriculum constraints and other factors that affect learning and the assessment structure in schools. These factors play a part in how the teacher will make inferences in the classroom about student thinking, behaviors, successes and failures, etc. The teacher observations which are conducted during classroom interactions and are part of the teaching process can be facilitated via the use of technology in the classroom. The technology provides a space for students to interact with the material, whatever that content may be and for the student interaction with their peer, so that the teacher can spend more time observing and for assessment purposes so that the instruction and assessment cycle (formative assessment) is seamless.

Paradigm Shifts in Mathematics Education and Their Implications on Mathematics Education

The researchers believe that teachers, when given the opportunity to convert technology based materials for the elementary classroom, both the secondary and elementary pre-service teachers gain. One, the pre-service teacher learns about the technology deeply enough to think about how he or she might use that technology in a future secondary setting. Two, the conversion process of mapping the technology materials to the elementary mathematics standards provides useable materials for perspective elementary teachers who may (or may not) have the mathematical background to create and write materials for use in their own classroom. The assumption is that the work of adapting the technology for the elementary classroom is done by those who may be more mathematically confident, and therefore better able to map the mathematics to usable lessons in the classroom (Math Tools).

We assumed that requiring pre-service teachers to integrate technology would be straightforward. It was assumed that using technology could move forward the teaching of diverse learners (Math Tools). However, the preliminary results of the study show that elementary pre-teachers mostly see the technology integration as a superfluous add-on and not a potentially integral part of the instructional process. Students work hard at using the technology to create their lessons, but often the qualities of a constructivist lesson are lost in the process. Evidence of proper questioning was lacking in both lessons presented in this paper. The assessment was connected to the objectives, but the original objectives rarely map to higher order thinking skills required or expected in a constructivist lesson (Burrell, 2008). The lessons use technology but the technology may not enhance the instructional process.

The lessons were created by the graduate student as an instructional tool to prepare pre-service teachers to implement technology into Kindergarten to Grade 6 classrooms. Unfortunately the researchers learned that the process of introducing technology into the classroom requires more time. The time would be used to observe the lessons in the real K-6 classroom, as opposed to only observing the lesson in the higher education classroom with adult peers. The evaluations (from the E part of the ASSURE model) reveal pre-service teacher perceptions about their instructional process in a good way, but also reveal to all who observed the lesson the inadequacies of measuring mathematical gains in the instructional process. The work by the graduate student was useful and profitable, in that pre-service teachers viewed a variety of elementary lessons with rigorous levels of mathematics and technology. The final step in the process was for pre-service teachers to use the sample lessons that the graduate student created, to create their own, based on the mappings in Microsoft Excel, and based on a topic of interest. The instructional design template does not ensure, in itself, that teachers will integrate technology properly, but provides a lens by which technology based lessons may mechanically include the parts of the lesson that demonstrate some constructivist thinking.

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The Role of Dynamic Interactive Technology in Teaching and Learning Statistics

Gail Burrill, Michigan State University, USA   burrill@msu.edu

Abstract

Dynamic interactive technology brings new opportunities for helping students learn central statistical concepts. Research and classroom experience can be help identify concepts with which students struggle, and an "action-consequence" pre-made technology document can engage students in exploring these concepts. With the right questions, students can begin to make connections among their background in mathematics, foundational ideas that undergrid statistics and the relationship these ideas. The ultimate goal is to have students think deeply about simple and basic statistical ideas so they can see how they lead to reasoning and sense making about data and about making decisions about characteristics of a population from a sample.Technology has a critical role in teaching and learning statistics, enabling students to use real data in investigations, to model complex situations based on data, to visualize relationships using different representations, to move beyond calculations to interpreting statistical processes such as confidence intervals and correlation, and to generate simulations to investigate a variety of problems including laying a foundation for inference. Thus, graphing calculators, spreadsheets, and interactive dynamic software can all be thought of as tools for statistical sense making in the service of developing understanding.

NEW OPPORTUNITIES

Dynamic interactive technology has the potential to extend this tool to help students understand central statistical concepts. The ability to link representations, where changes in one representation are reflected in the others, enables students to take an action, immediately see the consequences and reflect on the meaning of these consequences to make sense of the statistics- an action-consequence principle (Dick & Burrill, 2006). To maximize this potential and allow students to explore statistical concepts in deeper ways, it is possible to impose constraints on what they can do, in essence creating action-consequence "microworlds" in which students can play with a statistical concept in a variety of ways but where the opportunity to go astray, both mathematically and operationally, is limited. An action-consequence document is similar to an applet (e.g. Rice University, www.bbnschool.org/us/math/ap_stats/applets/applets.html) but can be modified or adapted by the user in ways not always possible in applets.

An action-consequence document is a technology document or file that

- requires very little knowledge on the part of the user of the device itself and how it operates;
- focuses on a fundamental statistical concept in a simple and straightforward way with both mathematical and statistical fidelity (is mathematically and statistically sound and accurate); and maintains pedagogical fidelity (does not present obstacles such as cluttered screens or too many decimal places that interfere with learning) (Dick & Burrill, 2006);
- is based on the action-consequence principle;
- have one object such as a point or graph serving as a driver for the interaction with little or no use of menus.

The design of technology-based activities for learning statistics needs careful consideration, however. There is a real danger that such materials will fall into categories absent any emphasis on what statistical learning they will enable. For example, materials such as the following are likely to appear (adapted from Belfort & Guimaraes, 2004): 1) the author's interest is on mastering the use of the technology where the statistics is secondary; 2) the activity is merely a demonstration of an idea where students are treated as spectators; 3) the activity revisits a topic to show how it can be done in a simple way with the new technology where the students' role is verification; 4) the activity replicates activities from the point of current instructional materials, underestimating the technology's potential, where the ideas are fragmented and obtaining a formula is often the objective. Materials developed for microworld environments can suffer from these same pitfalls: construction of the mathematical or statistical objects involved in the problem becomes the focus along with all of the details needed to master the technology; elaborate constructions that demonstrate a relationship but allow no interaction on the part of students, use of the technology to perform relatively meaningless operations (i.e., entering 100 numbers to find the summary statistics); replicating what can be done on a graphing calculator with no attention to the opportunities afforded by the new capabilities.

To exploit a dynamic environment, just as with handhelds and dynamic statistical software (Doerr & Zangor, 2000; Schwarz & Hershkowitz, 1999) students need to have adequate opportunities to conjecture, reflect, explain, and justify. Thus, technology's influence on students' statistical learning is either amplified or limited through the kinds of statistical tasks teachers provide and the questions they ask. Thinking and conjecturing on the part of the students can be enhanced or inhibited depending on the kind of answers the questions elicit. Questions such as "What did you get?" or "What's the next step?" will not prompt student thinking as much as "What does this mean?" and "How is this different?" - questions that push students past passive recording to active sense making.

The discussion below examines how these dynamic interactive worlds together with questions and tasks focused on reasoning and making sense of the statistics might address issues related to understanding central statistical concepts using several examples from analyzing single variable data and building understanding of inference.

VARIATION

In general, the research makes clear that students struggle with the concept of distributions, failing to abstract from individual data points to view the collection as a whole with its own characteristics. Students should be able to integrate both centers and spreads when they deal with data, yet Cooper and Sloan suggest that student understanding of variability is "tenuous" (2008). Students often describe distributions using language such as "the dots are close together here and spread out there", which Bakker (2004) calls "local views" on spread. In his study, none of the students viewed spread as a dispersion from a mean or median value. Many texts introduce the concept of mean and median as algorithmic processes, without first exploring the notion of center, and very few connect either of these to the concept of spread.

Standard Deviation.

Using a dynamic interactive plot, students can explore distributions and, in particular, the concept of standard deviation as a measure of variation. Allowing students to experiment with a distribution, noting how the data cluster around the mean can precede the formal definition of standard deviation and can help them reason about when such a measure would be sense making and useful. When the user grabs and drags a point or set of points (figure 1), the distribution changes, as do the corresponding values of the mean, and the plus/minus interval (one standard deviation from the mean).

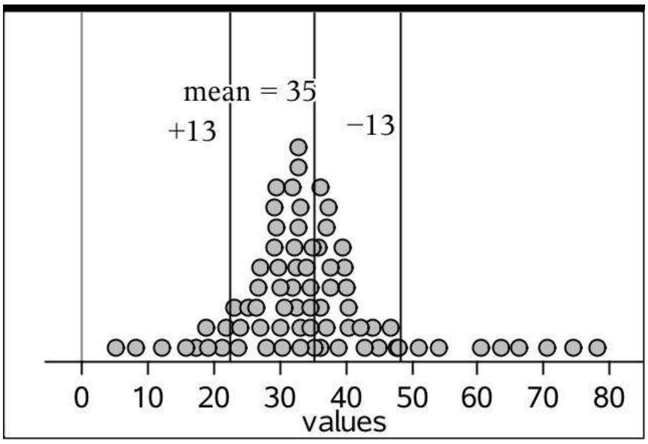


Figure 1: Mean and standard deviation

Research suggests interventions that support the development of aggregate reasoning in younger students are more successful if they are rich in student discussion of qualitative aspects of distribution such as shape, including identifications of the location of hills or deviation bumps, gaps, and spread-out-ness (Konold & Kazak, 2008). To promote such discussions, students can be asked to move points to find a distribution with certain characteristics, such as mean of 35 and a larger interval than 19 or find a distribution where the plus/minus interval would not convey much meaningful information about how the points were distributed. Conversations that encourage students to describe their distributions can help them develop a sense of what constitutes a distribution. Students can be given a mean and an interval and asked to create distributions with those characteristics, comparing their work and observing what the distributions have in common. Questions such as "What if…" or "Is it possible to…. Why or why not?" can help students focus on the relationship between the mean and the interval as a characteristic of the distribution. Asking students to describe possible contexts for distributions they create and to interpret the mean and interval in terms of those contexts can help them make the connections that lead to understanding. The goal of the activities should be to help students develop a sense of an expected variability that would seem to be reasonable around the expected value, almost like an intuitive confidence interval (Shaughnessy, 2006).

Box Plots Box plots are a powerful tool for comparing distributions, but research has shown that some of their features make them particularly difficult for students to interpret. These difficulties include: individual cases are usually hidden in box plots; box plots operate differently than other displays (the area of a histogram represents the frequency, for example); the median is not intuitive to students who often think of it as a cut point not as a measure of center and thus as a characteristic of the data; quartiles divide the data into groups in ways that are difficult to understand (Bakker, Biehler, Konold, 2004). Action-consequence documents are promising resources that might be used to address some of these difficulties and to better develop students' ability to interpret box plots.

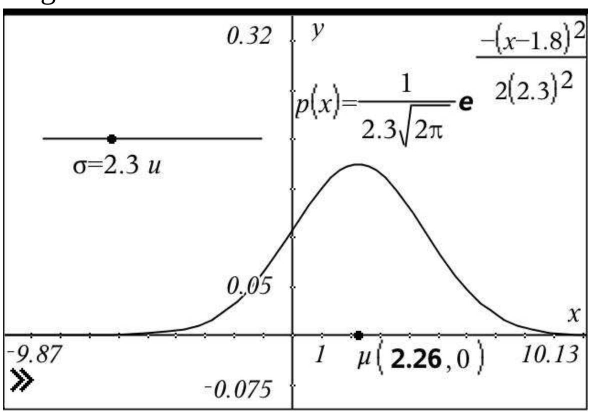
Consider a set of possible times it might to take for students to get to school displayed in a document where details such as the same scales for the dot plot and the box plot and the presence of the mean in the dot plot are not left to chance, and students can focus on the statistical concept rather than on creating the document. Selecting a data point (or points) highlights the position of that point in the box plot. Grabbing and moving a point(s) changes the value of the point(s) and thus, the distribution.

Students can highlight points inside the box plot and relate them to the corresponding points on the dot plot; teachers can ask students to describe the relationship between the group in that region of the box plot and the distribution. Students can change the distribution by moving the points, exploring questions such as, "What would the distribution look like if the median is at the lower quartile?" or "Can a distribution have a box plot with no whiskers? What would this suggest about the data?" If you drag one of the segments of the box plot, the distribution will also change, providing the opportunity for teachers to ask students about connections among the distribution of the data and the box plot, the role of the median and its relation to the data.

Inference Students enter and often leave statistics education courses with misconceptions. For example, Lunsford and colleagues (2006) found that students at the end of a course still confused variability with frequency, had problems with the "averaging reduces variation" concept and did not fully understand that for a fixed sample size, the sample mean was a random variable and thus had a distribution with a shape, center, and spread. Technology can address some of these issues, and, in addition, some studies suggest that technology greatly facilitates a "predict-and-test" strategy that can establish the cognitive dissonance necessary for students to change their ways of thinking about a concept (e.g., Posner et al, 1982).

Normal Curve How many normal curves are there? This question can produce surprising answers. Students often think there is only one normal curve, with mean 0 and standard deviation1. They have trouble understanding that just as other functions have a basic structure with the characteristics determined by varying parameters, a family of normal curves is determined by the mean and standard deviation. Investigating the relationship between the graph, the mean and the standard deviation can help students anchor and generalize their concept of a normal curve (figure 2).

Figure 2. A normal curve



To understand how a normal curve behaves, students can respond to questions such as "How is p(x) the same or different from f(x) = ax^2+bx+c?"; "What will happen to the distribution when the mean is changed?"; "the standard deviation?"; or "Compare changing the mean to changing the standard deviation." Answering these questions will force students to think about how the distribution connects to other mathematics they know and, if the curve is plotted on a grid, can set the stage for understanding that the area under the curve is the same for any combination of standard deviations and means.

Simulations

Research suggests that simulations can develop students' understanding by having them carry out many repetitions (i.e., comparing different samples of the same size from a population), controlling parameters (such as sample size), describe their observations and reflect on what these mean statistically rather than on concentrating on theoretical probability discussions, which can often be counterintuitive (delMas, Garfield, & Chance 1999). Interactive dynamic technology enables students to generate a sample from a given population, calculate a statistic for that sample, and look at the distribution of the statistic, all on a screen simultaneously. The repetition of many simulations for distributions using the same parameters can help address the concern students' understanding of statistical inference seems to be hindered by a limited understanding of related concepts such as distribution and variability (Chance et al, 2004). By pressing a control key, students can be asked to compare *t* distributions to *z* distributions from the same population, noting the consistent difference in shape for a given sample size and what changing the sample size does to this difference. They can investigate the central limit theorem for distributions that are normal and those that are not and how changing the sample size affects the distribution of the sample means (figures 3 and 4).

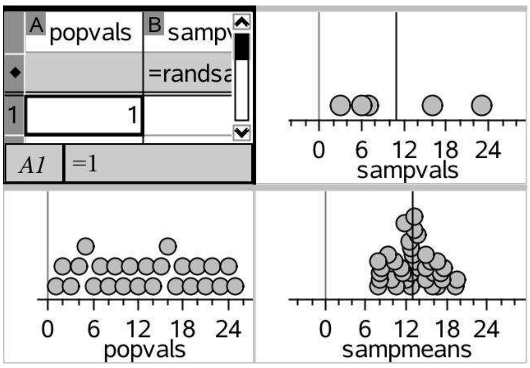


Figure 3: Sample means sample size 5

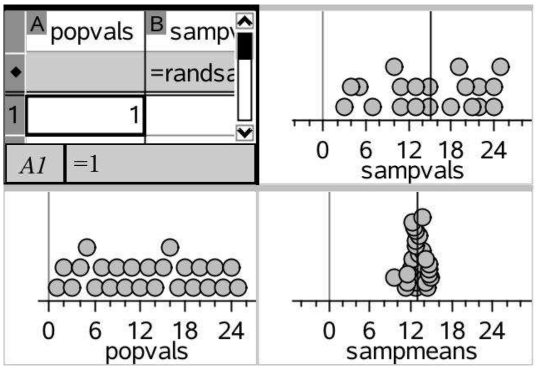


Figure 4: Sample means sample size 20

These "action consequence" documents as well as others that allow students to investigate confidence intervals, probability distributions such as the chi square or geometric, or what it means to have a significant observation can engage students in reasoning about the conditions, assumptions and theoretical tools involved in using statistics and random sampling to make inferences about some unknown aspect of a population. The questions posed can help students make connections to mathematics such as interpreting graphs they have encountered in other mathematical settings. A caution here, however, is in order to maximize the benefit of these interactive environments, the use of computer simulation for demonstration purposes only is not sufficient for developing real understanding of the concepts, in particular sampling distributions and the central limit theorem (Lunsford, Rowell & Goodson-Espy, 2006).

CONCLUSION

Technology has clearly been instrumental in the practice of statistics where statistical tables are no long needed as calculators easily provide accurate values, "resampling statistics" (Good 2006) offers an intuitive alternative to model-based inferential models, and new methods for visualizing and exploring data are emerging as powerful tools for analysis (Chance et al, 2007)

As technology advances and more students have access to dynamic interactive technologies, more opportunities become available for helping students learn. The discussion above has been on the use of this technology to support student learning in a targeted way, where the focus begins with core concepts with which students struggle. Dynamic interactive technology allows the creation of action consequence documents that make possible not only a re-examination of "*what* [statistics] students should learn as well as *how* they can best learn it" (NCTM, 2000).

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Presentations Using Autograph

Douglas Butler

(1) The fun of localizing dynamic software

I will describe some of the highs and lows of getting Autograph translated into various languages around the world, including those that use very different alphabets and writing direction. The biggest challenge had to be Arabic, where even the standard Unicode fonts set needed to be expanded to include the likes of a backwards (to us) facing square root!

Arabic apart, most countries use similar written mathematics to the Western conventions, and that surprisingly includes Russian and Chinese! The translation process will be discussed, especially the need to checking by local teachers. Just think how many ways the English use the word "normal" … This presentation will also look at how other software titles have coped with all this, including Geogebra and Cabri, and discuss the implications of the comma-decimal point issue.

(2) Those magic moments when you realize you could not have taught it that way with chalk

I will run through a number of topics using dynamic software and amazing web resources. These will be based on those magic moments in class when you realize that there is no way you could have explained it so effectively when it was just you and a piece of chalk. Topics will include: Frequency density; introducing 'e'; areas and volumes; the many and various uses of the parabola, hypothesis testing and some lively statistics.

There will be discussion of the pedagogical advantages of using a graphics tablet in the classroom (for total mobility round the room by the teacher). The effective use of slowing down or stopping the graphing will be discussed, together with judicious use of an interactive pen tool to intervene, predict and, importantly, to raise expectation from the students.

(3) Autograph Workshop for ages 11-16

I will give delegates the opportunity to work through a number of lessons plans covering topics as diverse as transformations of shapes and functions (in 2D and 3D), conic sections (in 2D and 3D), applications of the parabola through world-wide images, and there will be some lively data to analyse. There will be an emphasis on the pedagogical advantage of using autograph tools, especially the slow plot and scribble tool, but also the animation and constant controllers.

(4) Autograph Workshop for ages 16-19

I will give delegates the opportunity to work through a number of lessons plans covering topics as diverse as vectors (in 2D and 3D), differential equations (1st and 2nd order), areas and volumes, and a range of topics from probability and statistics. There will be an emphasis on the pedagogical advantage of using autograph tools, especially the slow plot and scribble tool, but also the animation and constant controllers.

Elementary Teacher Candidates' Understanding of Rational Numbers: An International Perspective

Rose Elaine Carbone, Ed. D., Professor,
Mathematics Department, College of Arts and Sciences, Clarion University
Clarion, Pennsylvania, USA ecarbone@clarion.edu

Abstract

This paper combines data from two different international research studies that used problem posing in analyzing elementary teacher candidates' understanding of rational numbers. In 2007, a mathematics educator from the United States and a mathematician from Northern Ireland collaborated to investigate their respective elementary teacher candidates' understanding of addition and division of fractions. A year later, the same US mathematics educator collaborated with a mathematics educator from South Africa on a similar research project that focused solely on the addition of fractions. The results of both studies show that elementary teacher candidates from the three different continents share similar misconceptions regarding the addition of fractions. The misconceptions that emerged were analyzed and used in designing teaching strategies intended to improve elementary teacher candidates' understanding of rational numbers. The research also suggests that problem posing may improve their understanding of addition of fractions.

Introduction

Research conducted over the past three decades provides evidence of the difficulties that many elementary teacher candidates experience with rational numbers [1], [2], [3]. The international research project described in this paper concurs with past researchers about the importance of identifying the level of mathematical knowledge that should be required of future teachers [4], [5]. Besides acquiring procedural knowledge, elementary teacher candidates must also exhibit a deep conceptual understanding of rational numbers to adequately teach their future students. Identifying understandings as well as analyzing student errors to determine their misunderstandings is critical in preparing candidates for teaching [6], [7]. Additional rational number research also shows that teaching number sense helps learners to understand mathematical symbols by relating them to referents that are meaningful to them. Connecting instructional examples to real life experiences, thus, is extremely important in overcoming difficulties and in developing conceptual understanding [8], [9]. It is also suggested that future teachers should be able to pose a valid problem for their future elementary students that can be solved by using a particular arithmetic operation. By doing so, they will exhibit a deeper understanding and be more likely to help their future students develop their own understanding [10]. In the two studies described in this paper, mathematics educators from Northern Ireland, South Africa, and the United States were involved in similar research projects where the mathematics educator from each country asked their respective elementary teacher candidates to write problems for their future students regarding operations with fractions. The findings of these studies demonstrate the understandings as well as misunderstandings regarding addition of fractions that were exhibited by the elementary teacher candidates from each country.

Fractions Research by United States, Northern Ireland, and South Africa

During the spring of 2007, two mathematics educators from Northern Ireland (NI) and the United States (US) asked their respective elementary teacher candidates to pose problems involving addition (and division) of fractions [11]. Six elementary teacher candidates from NI and 34 US elementary teacher candidates were included in this initial study.

In the spring of 2008, the same US mathematics educator collaborated with a mathematics educator from South Africa on a similar research project that focused solely on the addition of fractions. There were 26 elementary teacher candidates from SA and 18 additional US subjects in this second study.

The research that was used is grounded on previous research that also involved problem posing with the addition and multiplication of fractions [12], [13]. In these two new international studies, elementary teacher candidates were asked to relate the problems that they posed to real life experiences that would be appropriate for their future elementary students.

The following problem was completed by 84 subjects in both studies.

Write a story problem where students in the elementary grades would add ¾ + ½ to complete the problem.

Description of participants

In the three countries involved in the studies, all subjects were current elementary teacher candidates at the universities of their respective researcher. During the 2008 study, the SA and US teacher candidates were simultaneously taking courses that included similar mathematical concepts: whole numbers, fractions, decimals, percents, ratios, and proportions. The NI subjects did not have a similar course that reviewed these concepts in their curriculum. All NI subjects, however, had a concentration in elementary mathematics and were in the senior year of their education program, where the US and SA subjects were both beginning their mathematical studies, and none were concentrating in mathematics. All three countries used the English language in instructing their courses, but the SA subjects were unique in that 73% of their subjects did not use English as their first language (19 out of 26).

Cultural Differences

Mathematics is a universal language, and the procedures for adding fractions are the same, however, in the analysis of the problems, some differences in the cultures of the US and SA emerged in the researchers' conclusions as to whether several of the problems posed were acceptable. Each researcher had first analyzed the problems from their own country, and then analyzed the problems from the other country. When comparing the results, there were several problems in which the researchers initially disagreed. For example, in SA, margarine is usually sold in 250 g, 500 g, and 1 kg blocks, but in the US, margarine may be found in ½ cup sticks. The SA researcher rejected the example below because she was not familiar with "sticks" of margarine, but "sticks" are appropriate for the US culture.

Mrs. Newton is baking cookies. The recipe calls for 1 cup of margarine. Her husband, Mr. Newton, will sometimes open a new stick of margarine before finishing the old one. So Mrs. Newton has 2 used sticks of margarine. One is ½ c long and the other is ¾ c long. Does Mrs. Newton have enough margarine to make her cookies.

The US researcher rejected the problem below regarding loaves of bread, as loaves are not a standard size in the US. The SA researcher explained that loaves of bread are a standard size and can be considered a nonstandard, but informal unit of measurement in SA.

In this story is very poor family, with sick mother, absent father, and 3 children. The mother was worried because there only ½ loaf bread left, but she manage 2 get some bread, it was ¾ of a loaf. So how much bread do they have now

After cultural differences were explained, researchers were in agreement that problems regarding the differences in the cultures should be considered acceptable.

Analysis of the Results from the Three Countries

In the first analysis of all of the problems that were posed, it was immediately noticeable that food was mentioned in a high percentage of the problems. Problems from all countries included categories of pizza, sweets, fruit, and, uniquely from South Africa, loaves of bread.

In a deeper analysis of the results of the addition problems, the researchers found that those who wrote acceptable problems had two basic similarities.

The first similarity in posing acceptable problems was that the teacher candidates from all three countries exhibited number sense that was sufficient to realize that the sum of the two fractions (1/2 + ¾) is greater than one and that two wholes were necessary, and must be included in their problems.

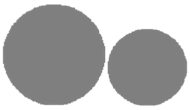
The second similarity in the acceptable problems from SA and US was that the teacher candidates displayed an understanding that referring to two similar unit wholes for both fractions was required. None of the NI subjects made reference to equivalent wholes, but all of the problems written by the NI teacher candidates included two wholes, which showed that all NI subjects exhibit sufficient number sense that the sum of the fractions is greater than one.

Findings from Unacceptable Problems

Difficulties in writing the problem in all three countries were similar in regard to the referent whole. Two themes emerged in analyzing the misconceptions of the unacceptable problems.

First, teacher candidates did not stipulate that the wholes being added are the same size and shape.

Kim has half a cake and Jamie has ¾ of a cake. How much cake is there altogether?



Students did not indicate that half of the left cake and ¾ of the unknown size of the right cake should not be added since they are not the same size. Since a cake is not a standard unit of measurement, the researchers agreed that for the problem to be considered acceptable, the writers should include that the cakes are of a uniform size. This writer, however, indicates an understanding of number sense that the sum of the fractions is greater than one, since two cakes are identified in the problem.

Jack and Jill had just gone trick-or-treating. They realized that if they added their candy together they would have more. Jack's bag was ½ filled and Jill's was ¾ filled. When they combine them together, how many bags of candy will they have?

Here we are unsure if Jack and Jill have the same size bags for their candy, thus adding the candy together may not be appropriate.

The second misconception that emerged in the analysis of the candidates' problems was that SA and US teacher candidates had not acquired sufficient number sense to realize that the sum of the given fractions is greater than one whole.

In Mrs. C's class, ½ of her students got A's on the test, and ¾ of her students got B's. How many students got A's and B's?

My class teacher bought us Pizza, she then gave ½ to boys and ¾ to girls. After we have eaten she told that the other pieces will be shared the following day. How many pieces were there altogether?

Each of these problems could not be modeled in a real life situation since the sum is greater than one in each situation. Acquiring such number sense is necessary to pose a realistic problem.

Findings from Acceptable Problems

Examples of acceptable problems that show understanding of number sense are presented below.

You have 2 pizzas that are exactly the same size. You give one pizza to your friend and you eat one half of your pizza. Your friend eats ¾ of his pizza. How much pizza did you both eat together?

Brian Habanna goes to Debonaires and orders one large Hawaiian pizza and one large chicken pizza. He eats half of his Hawain pizza and his girlfriend eats a quarter of her chicken pizza. How much pizza do they have left altogether?

Both problems also indicate "the same size" in the first problem, and an indication in the second problem that both pizzas are the "large" size from the same vendor, and thus equivalent.

Examples of acceptable problems that show understanding in employing a referent whole are presented below. When students included a referent whole in their problem, many problems included standard units of measurement in the problem so that the sum, which is greater than one, was appropriate for a real life situation.

You are helping your grandma make cupcakes. She tells you to add ½ cup of flour, then add ¾ cup more. How much flour did you put in total.

John ran ½ mile in the race for life on Saturday. Sue ran ¾ mile, how many miles did the run all?

If your mother tells you that you can spend ¾ of an hour at your friends house. Then you need to come home and get ready for school which will take you ½ an hour. In fraction form, how long does it take you to visit your friend and get ready for school?

Standard units of *cups*, *miles,* and *hours* were used in the problems above and easily make use of a familiar standard referent whole. In using these types of units of measurement, however, it was not evident if the students considered if the sum of the two fractions is greater than one.

Acceptable problems were also written that include non-standard units of measurement, as the following example from South Africa.

Dad gave John ½ of a peppermint chocolate and Susan ¾ of a peppermint chocolate too. How much chocolate(s) did dad have in total?

Response to Findings

In the first study none of the problems posed by the Northern Ireland elementary teacher candidates and only 20% of the US problems were categorized as acceptable prior to instruction but all NI candidates showed knowledge of number sense. In the second research study, four (15%) of the South African problems and seven (39%) of the US problems were categorized as acceptable prior to instruction. Of the 84 subjects in the two studies, only 21% were able to write an acceptable problem prior to instruction.

Analyzing student work has been shown to help mathematics educators reflect on their teaching and improve the teaching and learning of mathematics [14], [15]. After analyzing the elementary teacher candidates' problem posing work with addition of fractions, the US researcher shared some of the US sample research problems with her classes. A short discussion ensued concerning whether several of the problems were acceptable. The problems were carefully selected so that problems discussed were not written by any students in that class, but were from another class. Problems were chosen so that each of the identified themes of misconceptions was discussed. The classes also practiced estimating answers regarding the sum of fractions to the nearest whole.

At the end of the rational number unit, a few weeks after the problem posing discussions, when the US students were evaluated on fractions, decimals, and percent, the same problem posing item of the addition of two fractions was included on their unit exam. An analysis of the post test results in the first study shows that there was an increase in understanding such that 82% were able to write acceptable problems after instruction as compared to 20% on the pretest. After identifying the difficulties of the US teacher candidates and addressing them through a different type of instruction than was normally employed, the teacher candidates showed an improvement in their ability to pose a relevant problem.

Conclusions

Groups of elementary teacher candidates from three different countries, representing three different continents, wrote most of their problems about a most basic human need – food. Whether it was a pizza or a loaf of bread, similar understandings and misconceptions emerged from the three countries. The two salient themes that emerged regarding their understanding of addition of fractions and also their misconceptions are number sense regarding the sum of the two fractions being greater than one, and reference to a uniform unit whole. In future discussions, the international examples provided in this research can be used to raise the level of the classroom discussion to a global discussion by presenting problems from different countries. As the identified difficulties are addressed, a deeper understanding of addition of fractions may result. An improvement in the ability to pose problems was shown on the US post evaluations, after the misconceptions were addressed. The similarity in misconceptions held by all groups indicate a possible universal need to address the misconceptions that emerged in this research when teaching addition of fractions. The findings suggest factors to consider when teaching addition of fractions with elementary teacher candidates. This study contributes to the research that supports the use of problem posing to improve conceptual understanding. The findings also alert international researchers that cultural differences should be considered in analyzing qualitative data from different countries.Two questions naturally arise from the results of this study. Does the use of problem posing with elementary teacher candidates deepen their understanding of addition of fractions? Can the use of similar problem posing studies in other areas of mathematics help to clarify other misunderstandings for future teachers?

The author wishes to acknowledge Patricia Eaton and Pamela Austin for their contributions to this paper.

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Use and misuse of quantitative and graphical Information in Statistics
An Approach in Teaching

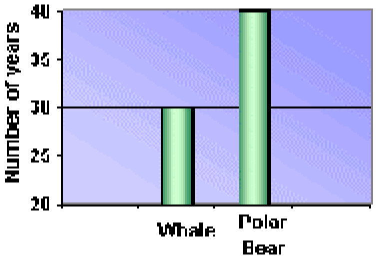
Lucette Carter* and Cécile Hardouin**
* University Paris X, lucette.carter@gmail.com
** SAMOS, CES, University Paris I, and University Paris X, hardouin@univ-paris1.fr

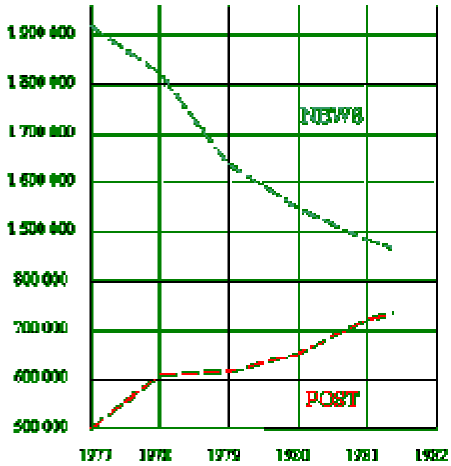
Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write. – H.G. Wells

Abstract
Miscellaneous examples of misleading statistical data or interpretation are presented in a form suitable for students in mathematics or Social Sciences during a first course of statistics. The aim is to promote critical thinking when confronted (mainly by the media or scientific papers) by information that is biased, incomplete, poorly defined, or deliberately oriented towards a preconceived target. Starting with the simple manipulation of Simpson paradox, the emphasis is put on the need for counfounding in the analysis of relationship between variables.

Introduction
Data and statistical analysis are often presented (particularly in the media or in politics) without the background information necessary for its proper interpretation (omission often due to carelessness, but sometimes with a deliberate intention to mislead). Several well known historical cases of misleading statistics are presented in statistical textbooks to stimulate students' critical thinking. The most famous example occured in the early days of opinion polling in United States ([12]). A survey based on a sample of over two millions, collected mainly from telephone directories (thus biased in favour of relatively rich voters), predicted that Alf Landon would beat the incumbent democratic President Franklin Roosevelt. (It must be noted that, at the same time, a Gallup poll, based upon a representative sample, predicted rightly that Roosevelt would win).
Another example, presented in many textbooks ([6]), is the bias towards expectation in Gregor Mendel's pea experiment, designed to confirm his hypothesis on genetical inheritance laws. More than 50 years later, the Mendel's experiment could be subjected to statistical analyses (applying mainly the chi-square test) which led to the conclusion that his data were suspisciously close to expectation. The controversy was instigated in 1936 by R. Fisher ([5]) who concluded that the "data of most, if not all, of the experiments have been falsified so as to agree closely with Mendel's expectation " . Fischer attributed the falsification to an unknown assistant. The controversy continued with other hypotheses such as an unconscious data manipulation by Mendel himself. This example of confirmation bias, frequently encountered, encourages the students to apply critical scrutiny to evidence supporting a preconceived idea.
These two classical examples of biased sampling are often presented to the students at the stage of inferential statistics. But at the level of the first introduction to descriptive statistics, other kinds of examples should be presented such as numerical information given without error margin, lack of a precise definition of the variables involved in the study, omission of confounder variables to assess the relationship between two variables, confusion between causality and correlation....
A few miscellaneous examples of such misuses are presented in this paper. They have been analysed by mathematics students, as well as students in Ethnology, in the course of Descriptive Statistics at University of Paris X ([2], [3]). It is hoped that these might lead to usefull discussions in teaching applications, and also to a better scrutiny of statistical informations in various media. Simple examples could already be discussed at the secondary school level. Some of these examples are derived from Penombre ([10]), a web site founded in 1993 by jurists and statisticians and extended to other academic fields. The authors attention focusses on the quality of quantitative and statistical information. Papers published by Penombre try to discern between methods and presentation (particularly in the media).

Misleading graphic Information.
The start-at-zero-rule has not been followed in the two graphs below. Moreover, in the second one, the bias in the vertical scale purposedly emphasizes the progression of the New Yorker Post (from New Yorker Post, 1981, see [12]) The pupils could be asked to reconstruct and comment the second graph with the right origin and the correct vertical scale.





The right construction of a histogram is unknown and misleading on the graphics hereafter.

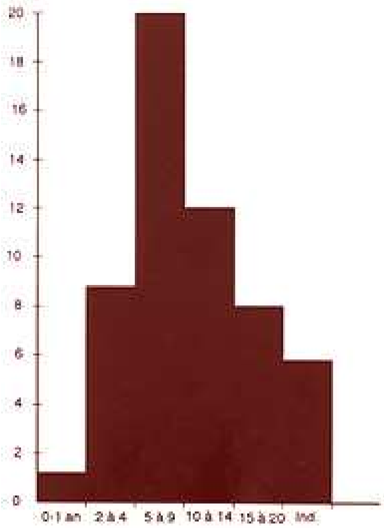


Fig. 1 : courbe de mortalité enfantine.

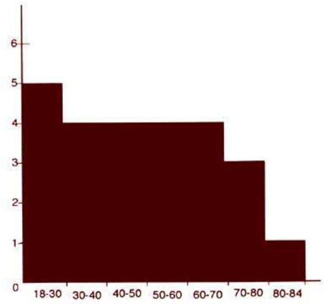


Fig. 2 : courbe démographique des individus masculins de la première période d'occupation de la nécropole (effectif réel).

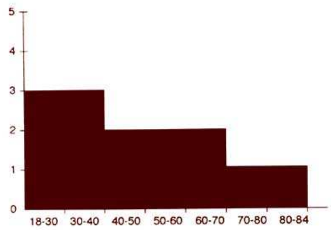


Fig. 3 : courbe démographique des individus féminins de la première période d'occupation de la nécropole (effectif réel).

These graphics are extracted from an archeological study ([9]); the one on the left hand side represents the age of the children found in a necropolis; the graphics on the right hand side represents the age of adult bodies for men (high) and women (below). The author comments the graphics about modal classes, number of deaths, ignoring the lengths of the classes. Therefore, there are some mistakes which could be easily avoided on the basis of a first course in statistics.

The role of confounding variables: Simpson Paradox

A common problem, particularly important in analysing clinical data, is that of confounding. This occurs when the association between exposure and outcome is investigated but the exposure and outcome are strongly associated with a third variable, called confounder.

The desirability of taking as many covariates as possible into account for assessing the association between two qualitative factors is illustrated by the Simpson's paradox which occurs when the direction of the relationship is reversed when a third factor confounder, sometimes also called effect modifier factor, is taken into account.

A classical case illustrating the Simpson's Paradox took place in 1975 ([1]), when UC Berkeley University was investigated for sex bias in graduate admission. Overall data showed a higher rate of admissions among male applicants, but broken down by departments, data showed a slight bias in favour of female applicants. The explanation is that the female applicants tended to apply to more competitive departments than males, and in these departments the rate of admission was low for both sexes.To analyse this paradox, we present to the students the following simplified version (with only 2 departments) similar to the Berkeley admission data The table below shows the repartition of 400 students candidates in two schools (of Economics and Engineering) depending on the result of the candidacy and the sex of the candidate:

Result	*Admission*	*Non Admission*
Female	66	134
Male	96	104

The frequency of admissions for female candidates (66/200 = 0,33) is smaller than the frequency of admissions for male candidates (96/200 = 0,48)
Considering this table only, a question might arise: is the rate of admissions influenced by sex?
But another interpretation emerges when each school is taken separately!
For the school of Economics:

Result	*Admission*	*Non Admission*
Female	40	120
Male	12	48

In this case, the girls (40/160 = 0,25 or 25 % admissions) are doing better than the boys (12/60 = 0,20 or 20% admissions).)
For the school of Engineering:

Result	*Admission*	*Non Admission*
Female	26	14
Male	84	56

In this case, again, the girls (26/40= 0,65 so 65% admissions) are doing better than the boys (84/160 = 0,60 or 60% admissions).
So when the data takes into account the type of School, the direction of the relationship between sex and result of admission is reversed!
This is an example of the Simpson paradox whereby the introduction of a third factor (the confounder or effect modifier factor) reverses the relationship between two factors.
To explain the paradox on this example, one can first compare the total percentages of admissions:
for the school of Economics, the percentage is equal to [(40+12)/220] x 100 = 24%;
for the school of Engeneering, the percentage is equal to [(26+84)/180] x 100 = 61%.
The next step is to compare the percentages of the candidacies for both sexes for each school:
for the girls, the percentage of the choice of the school of Economics is equal to [(40+120)/200]x100 = 80 % and for the school of Engeneering [(24+14)/200] x 100 = 20%;
for the boys, the percentage of the choice of the school of Economics is equal to [(12+48)/200] x 100 = 30 % and for the school of Engeneering [(84+56)/200] x 100 = 70 %.
The discrepancies between these various percentages can be summarized by observing that the girls tended to apply to the more competitive department with a low rate of admissions while men tended to apply to the less competitive department with a high rate of admission.
Examples of the Simpson's paradox occur in Epidemiology and clinical trials, particularly in the comparison of two treatments. For example, in a comparison of treatments for kidney stones ([7]), treatment B is more effective. But when a lurking variable, the size of the stone, is introduced in the study, the conclusion on the two treatments effectiveness is reversed
The students could be asked to reconstruct the related contingency tables and try to explain the source of the paradox.

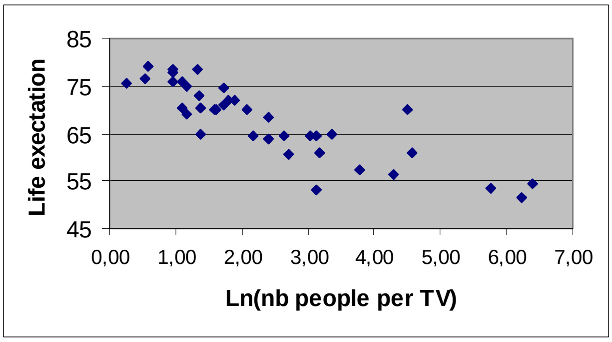
Confusion between causation and correlation. An example.

One of the most common errors in the press is the confusion between correlation and causation in scientific and health related studies. When the stakes are high, people are much more likely to jump to causal conclusions. This seems to be doubly true when it comes to public suspicion about chemicals and environmental pollution. There has been a lot of publicity in former years over the purported relationship between autism and vaccinations, for example. As vaccination rates went up across the United States, so did autism. However, this correlation (which led many to conclude that vaccination causes autism) has been widely dismissed by public health experts. The rise in autism rates is most probably to do with increased awareness and diagnosis, or one of many other possible factors that have changed over the past 50 years.

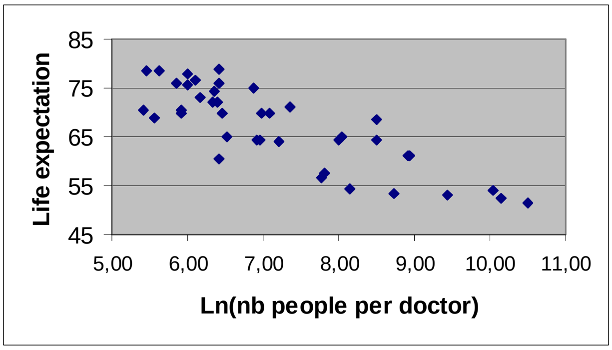
A very usefull dataset (drawn from *TheWorld Almanac and Book of Facts 1993)* helps the students to grasp that a strong association between two quantitative variables does not imply causation. The data for 40 countries (with populations of more than 20 millions as of 1990) gives life expectation at birth, the number of people per television set, the number of people per doctor ([11]).

The strong negative association between life expectation and each of the two other variables is observed by inspecting the raw data, and confirmed by the correlation coefficients (equal to -0,606 and -0,666 respectively). To avoid the graphic clustering of the richest countries (Japan, USA, France..;), the two graphs below are constructed after a logarithmic transformation of the 2 variables "nb of people per tv" and "nb of people per doctor". Discussing the difference between the two observed associations, the students are led to conclude that in the case of the variable "nb of people per tv", the association with life expectancy is spurious, entirely explained by the effect of economic confounders (such as GNP). On the other hand, the association between the variables " nb of people per doctor " and life expectation is partly direct, partly explained by economic confounders.

This example could be presented at various levels of statistical teaching. A. Rossman ([11]) presents the analysis of these data for advanced students, including regression, role of outliers, and so on..



Correlation coefficient = -0,855



Correlation coefficient r = -0,832

Another example, which is as simple as easily understandable by students, is to compare the increasing evolution of foot size with mental development for children. Obviously, the correlation is high, but explained by the fact that both variables are strongly dependent on age.
The following assertions, published in various Media, could be presented to the students to encourage them to judge whether the implicit or explicit interpretations are valid, and suggest some possible confounders:
* Eating a vegetarian diet is positively correlated with intelligence (USA media December 2006)
* Children evacuated to Texas during Hurricane Katrina do worse in school than children from Texas (USA media 2006)
* Watching a lot of television for a teen (A) is correlated negatively with SAT verbal score (B). Could reading ability be one confounder?
* 3 car accidents out of 4 take place at proximity of the drivers's home, implying that that the cause is a relaxation of vigilance (French TV news, 2006)

Unexpected or untractable confounders.

How, then, does one ever establish causality? This is one of the most daunting challenges for public health professionals and pharmaceutical companies. The most effective way of doing this is through a controlled study. In a controlled study, two groups of people who are comparable in almost every way are given two different sets of experiences (or treatments) and the outcome is compared. If the two groups have substantially different outcomes, then the different experiences may have "caused" the different outcome.

It is useful to present some examples where the confounders are unexpected or difficult to detect. For example, in a criticism of a study on the risk of passive smoking ([8]), focused on the risk of the health of the non-smoking partner of a smoker, it has been noticed that a confounding missing factor is the inclination of smokers to have a rich diet (which is naturally shared by the non-smoking partner).

Exagerated precision of ill defined data.

From an article published by Populations et Sociétés de l'INED (1996): At the global level, the FAO estimation for the average of food availability for 1992 is 2718 calories per day and person. A figure given with such precision (and without error margin) denotes a failure to appreciate the notion of confidence in estimation.

A even most flagrant example, occured during the heath wave of 2003 in France, with daily swarm of death numbers highjacked by the media (such as 42 deaths (July 26); 65 deaths (July 27); 10416 additional deaths for the first 3 weeks of August,...). A complete critical scrutiny of these figures is given in [4].

In both cases, the over-impressive figures lend an aura of precision to the inexact.

Information in terms of percentages.

In an era of increased quantitative information, such examples as the following could be analyzed.

Does the announcement that there is 10% extra free of a certain product implies that the discount of the price is also of 10%?

To increase the impact of certain statistics informations (for example, concerning the improvement of security, or of a medical treatment), relative risk (which could be impressive) is often preferred by politicians or the media, since it may produce a spurious impression of great improvement. For example, if the chance of something happening is increased from 1 per thousand to 2 per thousand, the relative risk is 50%, which seems a lot. If it concerns the chance of survival after a treatment, the absolute chance of survival is increased from 99.98% to 99.99% which is relatively small.

Conclusion

The introduction of examples of misleading “statistics” encourages the students to become more discriminating consumers of political, economic or medical statistics. It seems particularly usefull at the time when one is assaulted through various media by the hegemony of numbers, statistical data, and often incomplete or biased interpretation (often in sensitive fields such as medicine, ecology, social sciences...).

Suspicion should not be systematic but one should be ready to ask relevant questions whether confronted by a media declaration or an “expert” article (see the critic on the passive smoking study [8]). It is also a way to enhance rigor in the students statistical applications or memoirs and meticulous presentation of their work: original aim of the study, definition of the variables, of the sample, possible biases induced by questionnaires....

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Basic knowledge and Basic Ability: A Model in Mathematics Teaching in China

CHENG Chun Chor-Litwin
The Hong Kong Institute of Education, Hong Kong   cccheng@ied.edu.hk

Abstract

This paper aims to present a model of teaching and learning mathematics in China. The model is "Two Basic", basic knowledge and basic ability. Also, the paper will analyze some of the background of the model and why it is efficient in mathematics education. The model is described by a framework of "slab" and based on a model of learning cycle, allow students to work with mathematical thinking. Though the model looks of demonstration and practice looks very traditional, the explanation behind allows us to understand why Chinese students achieved well in many international studies in mathematics. The innovation of the model is the teacher intervention during the learning process. Such interventions include repeated practice, and working on group of selected related questions so that abstraction of the learning process is possible and student can link up mathematical expression and process. Examples used in class are included and the model can be applied in teaching advanced mathematics, which is usually not the case in some many other existing theories or framework.

Introduction

China started reviewing her mathematics curriculum in the early eighties. Chinese learn mathematics in a very traditional way, mostly by imitation and practice. Many thought that such delivery of mathematics education is not good for the students. However, it is also a fact that many Chinese students achieve well in mathematics standard and they did not lose their interest in mathematics. Most of the time, discussion of learning theory in China focus on western scholar such as Dienes, Freudenthal etc. Very few of the Eastern scholars and theory are mentioned.

There are a lot of developments in the near 100 years in learning in the West. For example, the nonassociation theory of thinking, explanation of learning through behaviourism and Gestalt, and the development of cognitive science by Piaget. The following is a list of theories of learning mathematics in the West that is related to the learning of mathematics.

Dienes and multiple embodiments

Dienes maintained that structured teaching aids can help students to establish their concepts. And play is important for learning. Dienes proposed that learning of mathematics can be more efficient with multiple embodiments. The design of instruction is described by the grids in the box representing the increase in depth in mathematics and the increase in abstraction in perceptions.

RME in Holland, Van Hiele and Freudenthal

The development of cognitive science resulted in RME (Realistic Mathematics Education) brought by Freudenthal in his process of mathematization in mathematics education. Van Hiele proposed that the stages learning go through "visual, analysis (part and whole), abstraction, informal deduction, formal deduction". The characteristic of RME is related to Van Hiele's levels of learning mathematics.

Problem Solving by Polya

Polya proposed three principles of mathematics learning, which is active learning, best motivation, and stages of progression (exploratory phase, formalizing phase, assimilation phase).

APOS by Dubinsky

Dubinsky proposed that learning of mathematics is through APOS (Action□Process□Object□ Schema)

Richard Skemp, Schema and principles in learning concepts

Skemp introduced his idea of schema, instructional knowledge and relational knowledge. Skemp consider that concepts are formed from abstraction of context, which came through from abstraction of operation. Mathematical knowledge is a special kind of abstraction.

David Tall – Procepts and proceptual divide

Tall proposed that students are troubled by procedure and concept during their learning. Hence they could not realize the real meaning of the concept. Tall proposed a term of Pro-cept, which is a combination of procedure and concept. Procepts compressed Process, and if students can cross the perceptual divide, then they are able to compress their knowledge.

Origin of the Two Basic models in learning mathematics

In China, learning is related to "comparison of object and emerge into knowledge" has a long history dating more than 1000 years in China. Teachers have a mentor role, demonstrate and intervene. Students imitate and internalized. Most of the knowledge we acquired are not from direct personal experience, but indirect experience, such as schooling. And schooling provides experience in learning mathematics by discovery approach and re-invention.

Two Basic Models in Learning Mathematics (Zhang Dianzhou)

Two basic means "Basic Knowledge" and "Basic Skills". According to Zhang, high-level knowledge and ability are intervened. Skills can be developed into knowledge and knowledge can induce skills. This also echoes the old Chinese saying, knowing and practicing. In fact, skills and knowledge overlap at some instances.

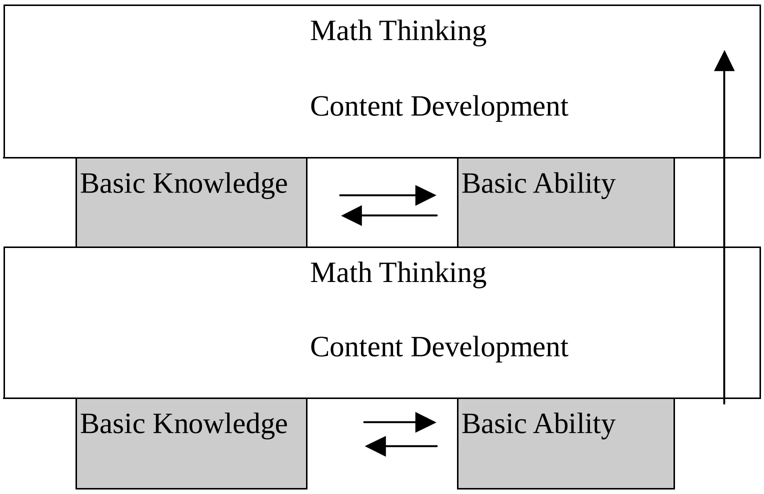
Metaphor of Knowledge formation in Two Basic, building of slabs

Zhang has proposed a metaphor of knowledge formation in the "two basic model", building of slabs. Basic knowledge is similar to reinforcement in the slabs. There are two kinds of knowledge, direct knowledge through exploration and investigation, and these reinforcements are thick. The second kind of knowledge is indirect knowledge. They came from imitation, there reinforcement are many but thin. Basic ability is the concrete in the slab, with the suitability reinforcement, the slab formed is strong.

Two basic are the slabs, these slabs connected together by ability. The successful of solving the problem depends on whether such slabs are connected. Connection of slabs means connection of knowledge.

The quality of foundation depends on number of reinforcement and also the quality of the concrete, and also how the slabs are joined together. As knowledge structure is not conceived in the same format for everyone, structure of knowledge is differently arrayed in every learner.

Slab Model of the "Two Basic"



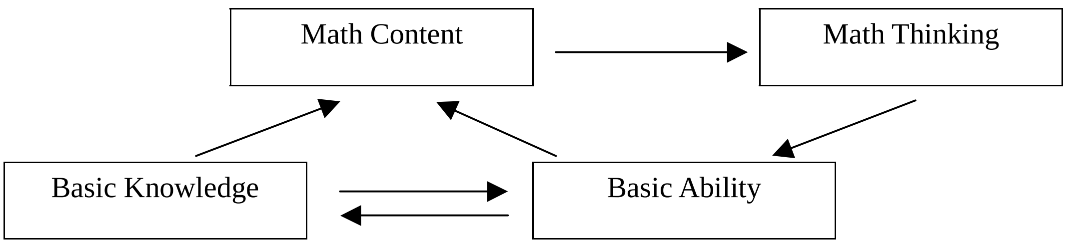
The three steps of Two Basic Model-(1) Imitation and Representation, (2) Intervention, and (3) Abstraction and Internalization

The first step is imitation, with teacher's guidance and demonstration. Demonstration is the premiere of imitation. Imitation means students need to observe and internalize. The second step is the intervention by teachers. This includes criticism, correction of concepts, and using more strong examples for the concepts, and the summary of knowledge learned in a trunk and practice. These will raise the level of understanding of the students. The third step is abstraction process by students, which is a kind of internalization and self monitoring. By internalization, students can connect different knowledge, and deduce new knowledge.

Step 1		Step 2		Step 3
Imitation		Teacher's Intervention		Internalization and Abstraction
Observation and representation Internalization (level 1)	→	Correction and reinforcement of concept	→	Connect to different domain of mathematics Deduce mathematics

And we can present the model of the learning cycles as following.

The Model of Learning cycles in the "Two Basic"



Imitation and intervention before construction

The Chinese theory attained that construction is the process of imitation and intervention. It is achieved through two levels. The first level is imitation of what is taught by teachers, and the second level is construction of knowledge through internal abstraction. And practice has an important role. How to avoid having practice becomes mechanical repetition? The two basic provide the following principles:

(1) memory leads to recognition and become intuition,

(2) good speed of operation provide grounds for efficient thinking,

(3) using deduction and reasoning to sustain precise logical thinking,

(4) rising of standard through variation of problems and learning process.

Zhang maintained that teacher's direct demonstration has a strong role in establishing learning objective, and provide guidance during learning. Learning of a language is an obvious example. Such imitation and internalization can also found in the domain of mathematics learning. And construction is based on intuitive of concepts so that formal concept is built. He attained that western theories stressed the importance of construction, but have skipped the important role of imitation. Knowledge that can be constructed in a short while is only very surface knowledge.

How classroom teaching process leads to mathematical thinking

There are two levels of teaching. The first one is using daily lives context, which aims to arouse the interest of students. The second level is the learning process through abstraction. Teachers teach mathematics based on the process of "correspondence, induction and deduction". Correspondence means to map the concept of mathematics problem to another problem; it also involves correspondence of expression, and structure. For example, learning of fraction division correspondence to integers division. The induction process involves the process pattern of calculations and then generalizes the pattern. And by deduction, students deduce related results of the generalized pattern. For example, $4 \times 3 = 12$, $4 \times 4 = 16$, $4 \times 3.5 = ?$ Through deduction, the answer is half of $(12+16) = 14$. The process can continue to 4×3.25, which is half of $(4 \times 3 + 4 \times 3.5)$.

Example: Integers divided by fractions

Question:
A walk 1 km in $\frac{1}{4}$ hours. How much distance can he covered in an hours?

The answer is $1 \div \frac{1}{4}$. The answer is not demonstrated by teachers. The answer is obtained through interaction between teachers and students. The process is by using analogy and related questions. Using intuition and multiple, the answer is 4 km.

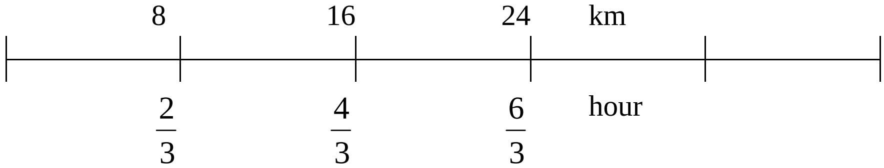
hour	km
$\frac{1}{4}$	1
$\frac{1}{2}$	2
1	4

After such process of "inquiry imitation", students started to analyse the question and proceed to group of related question as practice.

> Question:
>
> A dove flies a distance of 8 km in $\frac{2}{3}$ hours. How much it can cover in one hour?

Teacher intervenes by building correspondence among questions. For example, $\frac{2}{3}$ correspondence to 8 km, $\frac{4}{3}$ corresponds to 16 km, and $\frac{6}{3}$ corresponds to 24 km. Hence, 2 hours corresponds to 24 km. That is, 1 hour corresponds to 12 km. The process is the understanding of the context and the problem through ratio. Finally, students need to internalize that the answer is $8 \div \frac{2}{3}$, and the process is 8×3÷2, and hence $8 \div \frac{2}{3}$ is equivalent to $8 \times \frac{3}{2}$.

Another intervention is through diagram. Through the following diagram, teachers explain the process of the above process.



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How Involving Secondary Students in the Assessment Process Transforms a Culture of Failure in Mathematics to a Culture of Accountability, Self-Efficacy and Success in Mathematics: *Student Action Plans, Assessment, and Cultural Shift*

Katharine W. Clemmer, M.A.T., Mathematics
Director and Founder, Center for Math and Science Teaching, Loyola Marymount University, Los Angeles, California, United States of America kclemmer@lmu.edu

Abstract
Learn how to realize a measurable increase in student engagement and achievement in mathematics through a guided, collaborative, and active process grounded in mathematics. Students and teachers collaboratively devise a data-driven plan of action that moves learning forward for all students and effectively supports at-risk secondary students in urban environments. Learn how teachers in the LMU Math and Science Teaching Program effectively implement assessments as motivations for student achievement and develop opportunities for students to demonstrate comprehension and retention of essential content over time. Students become active participants in the assessment process in an environment where learning is an individual progression and risk-taking is valued and encouraged. Find out how students, guided by teacher-provided descriptive feedback, make decisions in a process of self-reflection in which they critically analyze and compare their learning outcomes to expectations of content mastery. By comparing mastery to current performance, students utilize failure and engage in error analysis to deconstruct prior shortcomings and devise a plan of action that will move learning forward thereby overcoming failure.

Introduction
The Loyola Marymount University (LMU) Math and Science Teaching (MAST) program inspires and supports cultural transformation through collaborative, innovative, research-based, and transformative practices that focus on students as the impetus for improving teaching and learning in mathematics and science and merge pedagogy with academic content. The MAST program serves as the link between the theories developed through educational action research specific to math and science education with practical applications designed to increase the number of students who are engaged and achieving proficiency in mathematics and science, and connect to the academic discipline of mathematics and science. The MAST program teaches how to analyze, organize, present, and assess information in innovative and active ways that stimulate students' critical thinking processes, embrace failure as part of the learning process, balance collaborative and independent learning, and instill the value of mathematical and scientific thinking as a solution for moving teaching and learning forward [1]. The goal is to transform classrooms so that they effectively implement and analyze assessments as motivations for student achievement, as well as develop opportunities for each student to demonstrate comprehension and retention of essential content over time [7]. Within this structure students become active participants in the learning process, influence the teaching process, and discover the joy and value of mathematical and scientific thinking.

Collaborative Learning Portfolios
In order to involve students as active participants in the learning and assessment process, a framework that supports them in successfully interacting with the process and mathematics is required. An effective structure for engaging students in this process is the MAST Collaborative Learning Portfolio. This portfolio organizes data about what students know and need to know with supporting evidence and goal-setting structures. A key component of the Collaborative Learning Portfolio is the MAST Learning Target Logs. Through use of the log students self-monitor their progress toward academic learning targets by using a rubric to evaluate understanding of the content. The Learning Target Log groups learning targets with levels and shows a progression of these levels throughout the learning process [2]. The learning targets detail what students are expected to know, understand, and do in order to achieve mathematical literacy and competency. They provide a balance between conceptual knowledge and skills [8]. The levels assigned to these targets on the log are based on an

analysis of academic performance on assessments and understanding of the content aligned to learning targets.

Since Learning Target Logs are part of the Collaborative Learning Portfolio, these logs support students in making choices about their individual learning needs. With the logs, students assess and track their own progress in meeting their goals and meeting proficiency while also collaborating with the teacher in analyzing progress and next steps. When students receive returned assessments, scored by the teacher, or participate in an activity, which is self-assessed, the student identifies which learning target is being assessed, records the appropriate level of understanding based on the rubric, and then determines areas of strength and areas of growth. A primary vehicle for teaching students how to appropriately self-assess is the provision of descriptive feedback and communication about learning. Communication is a continuous process, in which teachers and students dialogue back and forth during active practice and individual communication about learning goals, progress, and questions [10]. This may be written, as feedback in the Collaborative Learning Portfolio, or verbal, shared in class or during a conference. The focus is supporting progress to achieve the end goal of mathematical literacy.

Feedback is given on a variety of assessments. Students can write questions which teachers answer, or teachers write questions for students. The modes of communication are exit slips, reflective slips, quick checks, quizzes, tests, and the collaborative learning portfolio. As the teacher and student consistently and effectively engage in this process, students internalize the appropriate and high-level usage of content-related and academic vocabulary and the ability to critically analyze their learning outcomes for errors [9]. Collaboration becomes the norm within the classroom and a transformative culture around learning emerges. While students need critical thinking in applying rubrics to learning target-aligned outcomes to record learning, the power behind the learning log lies not in the sorting of levels of understanding but in the analysis and action plans that result from this datum and other samples of outcomes.

Learning Target-Aligned Instruction and Assessment

Having discussed the framework in which the students interact with assessments and various other learning outcomes and the interaction between students and teachers throughout the assessment process, the relationship between the assessment process and instruction is addressed. The goal of instruction is not to teach for short-term recall, but instead to support a depth of understanding and long-term retention. The MAST spiral cycle supports the internalization of learning through continued accountability and practice. This cycle is unlike the traditional focus on spiraling content that includes the continual re-teaching to students who remain unsuccessful in mastering the content at the initial time of instruction to support learning. The MAST spiral cycle supports continued, as opposed to initial, mastery of the content. It operates under the assumption that all students need to be able to maintain mastery of content throughout the duration of a course and into future courses. Spiraling aligns with the flow: students master a specific learning target, learning moves forward, and the previously mastered learning target is incorporated in teaching of future content to support long term retention and strengthening of learning through mathematical connections [12].

As the content is consistently incorporated and connected following the MAST spiral cycle, differentiation aids all students in accessing the content. In order to support retention, assessment is embedded within each component of daily instruction. A MAST-sequenced lesson includes a Hook to engage students at the beginning of a lesson, an Activity Before Concept (ABC) to transition from the hook to the delivery of new content, an Interactive Mini-Lecture to deliver new content and engage students in critically analyzing material as they learn, Active Practice to challenge students as they interact with the material in class, and an end of lesson assessment such as an exit slip [4]. Each of these lesson components facilitates multiple opportunities for assessment throughout the lesson [3]. The Hook is used to assess the students' level of interest in the topic to be delivered. The ABC measures student background knowledge connected to the concept; this information is used by the teacher to determine the amount of re-teaching and review that is necessary for a given lesson. The Interactive Mini-Lecture incorporates multiple components; such as brain bubbles, think boxes, and think pair share; to evaluate comprehension of the content as it is delivered as well as students' ability to connect the new content to what they have already learned. The Active Practice component of a lesson measures students' ability to communicate understanding through written and spoken word. The aforementioned assessments measure understanding throughout the flow of the lesson but provide no deliverable for analysis. Throughout a unit of instruction, assessments that are marked or graded to support future learning include exit slips, quick checks for retention, spiraled summative assessments, and assessments that encourage content connections. Exit Slips are utilized to formally measure the learning of new content at the end of the lesson. Quick checks are used to formally assess retention of past content over extended periods of time. MAST spiraled assessments incorporate all learning targets from the course in the same assessment; these topics can be measured in isolation or can be combined to encourage content connections and provide a more challenging problem set that requires careful critical thinking. As students interact with these formal assessments they can engage in a process of intentional and purposeful reflection and error analysis [12].

Reflection and Error Analysis

Learning is an interactive process by which learners try to make sense of new information and integrate it into what they already know. Students are always cognitively processing information and they are either challenging or reinforcing their thinking on a moment-by-moment basis [4]. Before teachers can plan for targeted teaching and before students can engage in practice and continued learning that is tailored to their individual learning needs all parties need to know about students' thinking and understanding for a given learning target. This knowledge goes beyond the simplicity of considering whether students have a question right or wrong, but instead connects to the constructs and patterns that students have internalized about the content. To effectively deconstruct patterns that are incorrect students need to critically analyze their learning outcomes to not only obtain the right answer, but to rectify their misunderstandings. To do this students engage in the MAST Assessment Protocol. In the framework of this protocol, students analyze their work, scores, and descriptive feedback to assess their own work and identify errors, complete an analysis of their errors - identifying the specific place where the error occurred and explaining what went wrong, and self-reflect. As students reflect they respond to questions that engage them in exploring what next steps are needed to achieve mathematical literacy and proficiency. Students analyze the learning targets for which they were ready to demonstrate mastery, targets for which they were not prepared, and the reasons for strengths and gaps in learning. Learning is enhanced when students are encouraged to think about their own learning, to review their experiences of learning, and to apply what they have learned to their future learning. Making these connections and then articulating next steps strengthens learning [5]. The teacher's role is to provide the structure and space to facilitate this process in connection to assessments and learning goals. After analyzing assessments, students analyze their progress of the learning targets and devise study plans based on their individual learning needs. As students engage in reflection they use the contents of their Collaborative Learning Portfolios to support the identification of learning targets that are strengths and those that require more review. Teachers must provide descriptive feedback in order to ensure a feedback loop for this process [11]. When students and teachers become comfortable with a continuous cycle of feedback and adjustment, learning becomes more efficient, students begin to internalize the process of standing outside their own learning and considering it against a range of criteria, rather than the teacher's judgment about quality or accuracy, and begin to take ownership of their learning. When combined with a focus analysis of assessment these activities reveal to the students their individual understanding of and gaps in the material. Using this analysis, and descriptive feedback from the teacher, students then set goals for and devise and implement a plan to support achievement of learning targets.

Goal Setting and Action Plans

Connecting learning targets to academic expectations allows for written, reflective responses to meaningful questions posed about the learning process. It extends the meta-cognitive analysis of learning the content to include the creation of individual growth targets and the process of creating goals. Goals are not simply stated and revisited at some obscure future date, but instead aligned to a specific action plan that is generated in collaboration between the teacher and student and visited frequently [6]. The development of goals relies on accessing information from the Collaborative Learning Portfolio, including the Learning Target Log, that align to learning goals in order to align them with three categories: strengths, topics for review, and highest priority for studying. Review topics are connected to learning goals for which a basic level of understanding has been demonstrated whereas the highest priority for studying are learning goals for which the lowest levels of understanding have been demonstrated.

Once learning targets have been separated into categories based on evidence, students can generate goals that include demonstrating an increase of the level of understanding for a selection of learning targets that are not currently categorized as strengths. These goals are then aligned to an action plan that is developed in collaboration with the teacher to ensure that the goals are measurable and that steps are taken to realize these goals. As students work toward their established goals in alignment with the action plan, they continue to engage in the process of reflection and error analysis. Students reflect continuously on their work and growth to determine what is supporting their understanding and what aspects of their action plan need to be modified to support their learning and proficiency in mastering the learning goals. Student responses and products throughout this process give an indication of the students' understanding of and gaps in the material as they reflect on learning, as mathematicians and scientists, and of their growth in meeting their goals. As they reflect the Collaborative Learning Portfolio and Learning Target Logs are updated to help students connect their efforts to growth in meeting the learning goals. Students continue to monitor their performance of these learning targets and change the plan as needed. This process empowers students as they take ownership of their learning and begin to view the teacher as a support and facilitator in the process of learning.

Cultural Shift

As a result of implementing the Collaborative Learning Portfolios, Learning Logs, Learning Goal-Aligned Instruction and Assessment, Reflection, Error Analysis, Goal Setting, and Action Plans a shift of culture occurs for the student. In this framework students are able to take ownership of their learning and take intentional actions to realize measurable goals. Teachers become partners with students in the process of learning. Effectively implemented assessments motivate student achievement and develop opportunities for students to demonstrate comprehension and retention of essential content over time. Consequently classrooms in which this paradigm is implemented realize a measurable increase in student engagement and achievement.

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An Alternate Route to Urban Mathematics Teaching: The NYC Teaching Fellows Program

Dr. Laurel A. Cooley, Associate Professor
Mathematics Department, Brooklyn College, City University of New York
LCooley@brooklyn.cuny.edu

Abstract

The NYC Teaching Fellows (NYCTF) program, as the nation's largest alternative certification program, aims to provide high-needs NYC public schools with highly qualified teachers in such hard-to-staff areas as math, science, and special education. Reports of NYCTF teacher retention are mixed; The New Teacher Project (TNTP) claims high retention rates, but other research indicates that fellow recruits have lower retention rates than other teachers in similar NYC schools – only Teach for America (TFA) exhibits higher attrition (Boyd *et al.*, 2006).

After scrutinizing these contrary claims, this paper examines the retention of a recent cohort of approximately 300 Mathematics Teaching Fellows (MTFs) in the NYCTF program, examining MTF's early attrition, movements from school to school in the NYC system, and professional plans for the future. We also include findings on teacher induction, school leadership, and school context that affect MTF retention.

NYCTF Retention

The NYCTF program was instituted to help the NYCDoE replace uncertified teachers with certified teachers. Though Fellows initially replaced uncertified teachers, they now replace each other, as MTFs leave and new ones enter. Hence, while NYCTF has helped NYCDoE meet NCLB's "highly qualified" teacher guidelines, it has not solved the problem of math teacher turnover in NYC public schools. The transitional teaching license (Transitional B), created in 2000 as part of alternative route legislation, counts Fellows and other alternatively certified teachers (ACTs) as "certified" after completing a short summer preservice program. In September, they become teachers of record and are expected to, within 2 to 3 years, complete a Master's degree at 1 of 4 cooperating universities in or near NYC. In regard to retention, TNTP claims the following:

> Today, 87% of Fellows begin a second year of teaching, a higher rate than the national average, and nearly 75% teach a third year. These retention rates are noteworthy since Fellows teach in some of the hardest-to-staff schools in the city. Nearly half of all Fellows who start their first year continue into at least a fifth year in the classroom.

Such comparisons can be deceptive though; after fulfilling their 2-year commitment to NYC public schools, many Fellows move to schools outside NYC or private schools, so many of those who "continue into at least a fifth year in the classroom" are teaching outside the NYC public school system at that point. The portability of the DoE-subsidized Master's degrees all but encourages Fellows to seek jobs in better-resourced districts or private schools.

Second, the comparison of Fellows to "the national average" of teachers who "begin a second year of teaching" is problematic. Unlike many new teachers, Fellows receive financial support upwards of $30,000 in exchange for signing a 2-year commitment, which includes $2,500 for completing a summer preservice program. In addition, the Master's coursework required by NYS is heavily subsidized by the NYCDoE; Fellows who leave teaching before fulfilling their 2-year commitment have to repay their funding. Unlike some states, NYS requires new teachers to receive mentoring and other induction supports. In sum, sizable incentives and induction supports help Fellows fulfill their 2-year commitment to NYC schools. Given that other beginning teachers generally are not provided similar support, it is unsurprising that Fellows' initial retention rates beat the national average. Given the resources spent on each Fellow, it is actually surprising that the first-year attrition rate is as high as 13%, but after a second year, Fellows no longer beat the national average for teacher retention, as extra incentives and supports drop away.

Third, calculating NYCTF retention is not straightforward. TNTP does not count recruits who complete the NYCTF preservice program but fail the Content Specialty Test (CST) in math on their first try, although some of these do pass the CST on subsequent tries. Nor do they count those few recruits who drop out of preservice before becoming teachers of record. A less selective count could increase TNTP's attrition rates by at least 5%. Retention calculations that include this currently uncounted information are reasonable; the NYCTF spends $4,000-7,000 on each Fellow, and NYS requires university-based teacher education programs to include any student who takes teacher education courses in state reports of program-wide failure rates on the CST.

Fourth, TNTP retention reports gloss over Fellows' school-to-school movement in NYC. While TNTP implies that Fellows stay in the hardest-to-staff schools, evidence runs contrary. Boyd *et al.* (2005) find that "high-achieving" teacher candidates, like those recruited by NYCTF, are less likely to stay in these NYC schools. Rather than helping stop the "revolving door" of teachers in high-needs schools, Fellows may actually worsen the problem.

Fifth, contrary to the implication that Fellows' retention rates beat "the national average", extant research finds that they (and TFA recruits) have lower retention than other teachers in similar NYC schools. Adjusting for differences in school context, Boyd *et al.* (2006) find that, while first-year TFs have higher retention than traditionally certified teachers, it falls in later years.

At times, Boyd *et al.* (2006) may favor alternative route programs, downplaying the fact that students of novice Fellows under-perform on math exams relative to students of other novice teachers. While they factor school context into their analysis of teacher retention, they neglect the specific support systems in place for Fellows but not for other novices in NYC. After describing our methodology, we examine retention rates for first-year MTFs, school-to-school movement in NYC, and early indicators of long-term retention. We then examine some induction supports and school context factors shown to be important in new teacher retention and end with a brief discussion.

Methodology

The primary data source for this paper is a large-scale survey study of an annual cohort of MTFs who began the NYCTF program in June 2007. At the end of the 7-week NYCTF preservice program, 269 of approximately 300 MTFs were surveyed. In the summer and fall of 2008, approximately 90 members of the same cohort completed a second survey at their partner universities. Completion rates for the first-year survey were between 85-90%. The inservice survey response rate was 70-80%. The lower number of inservice surveys and lower completion rates reflect attrition from the program (some quit, some repeatedly fail the CST, some are excessed), slippage between cohorts (some fall behind in coursework and take classes with later cohorts), and greater difficulty in gaining access to the MTFs as they progress through the Master's programs (one university partner would not allow us to give surveys on campus).

The preservice and inservice surveys were both in-depth, including Likert-type, forced-choice items and open-ended questions, and took respondents about 35 minutes to complete. Survey items dealt with the content of Master's coursework, in-school supports (e.g., mentoring, planning periods, administration), their perceptions of their students and school administrators, and their professional plans.

Analyzing preservice and inservice surveys, we adopted a mixed-method approach, including classical statistics, exploratory data analysis, and coding of expository responses. In the first phase of analysis, we have begun to examine connections and correlations between item responses. Because many MTFs completed both surveys, we can assess the development of individual MTFs and have yet to begin analyses of individual change.

The inservice survey did not fully address MTF retention; it only included data from MTFs who remained in the program after their first year. So, to further examine first-year retention, we supplied key personnel at the 4 NYCTF university partners with lists of Fellows who had completed the preservice but not the inservice survey. They told us who was "active" and "inactive" in the 2008-09 academic year. We stress that this manuscript is exploratory and the findings tentative.

Math Fellows and Teacher Turnover

Reports by personnel at the 4 NYCTF university partners for math indicate that, of the 269 MTFs who took the preservice survey, 55 were inactive at the start of the 2008-09 school year; some left teaching in their first year and some at year's end. This translates into an attrition rate of at least 20%, which would be higher if the dozen or so recruits who left the preservice program prematurely (i.e., before the preservice survey) were included. It does include a small number of preservice candidates who were never officially counted as "Fellows" because they failed the mathematics CST test – even though they may have passed it later and become a teacher of record. Even without this latter group, the first-year attrition rate for MTFs is over 16%. Whether closer to 20% or 16%, MTF retention is higher than the 13% reported by TNTP for the NYCTF population.

Of the inservice respondents who began a 2nd year of teaching in NYC, approximately 20% report moving from school to school by the start of their second year (4% moved in their first year, while 16% found a new school for their second). While most moved of their own volition, a few report being "excessed" and having to find a new school. A cursory analysis indicates that the highest-needs schools are most impacted by MTF movement in the NYC system. This resonates with extant research on teacher turnover in NYC schools (Boyd *et al.*, 2005; Loeb *et al.*, 2005). Combining results on MTF turnover, we find that more than 35% of these MTFs had either left teaching or moved to new schools by the start of the second year. The highest-needs schools – those that NYCTF was designed to help – appear to be impacted most. Further, retention rates of MTFs are significantly lower than for general Fellows reported by TNTP.

Short-Term and Long-Term Career Plans: Implications for the Revolving Door

A majority of inservice respondents report that they planned to teach for an additional 6 years or more; however, more than 1 in 3 planned to be out of the NYC system within 3 years. Combining these results with the 16+% of MTF teacher leavers, it appears that only about half of MTFs will be teaching four years after they began the program. Of course, these are reports of future plans and some MTFs who plan to leave may have a change of heart. Also, some respondents may inflate the amount of time they plan to stay in the classroom. In terms of teacher turnover, almost a third of inservice MTFs report that they would leave their current school by their second year's end. Combining these results with the 35% first-year turnover rate, it appears that less than half of MTFs will continue to teach in their first school after their 2-year commitment is up, and only about 30% for four or more years.

A year or less of teaching in NYC schools appears to have convinced many MTFs to teach elsewhere or not in any school at all. That said, our data suggest many MTFs viewed teaching as a short-term commitment from the start. On the preservice survey, only 38.5% of preservice respondents claimed they planned to teach in NYC for more than 5 years. An additional 14.2% claimed they would still be teaching at that point, just not in NYC public schools. An additional 12.3% of preservice MTFs said plans were uncertain. Some MTFs report that NYCTF officials stated or implied in admissions interviews that it was acceptable for them to view teaching as a short-term job. Many Fellows were recent college graduates – about 4 in 10 MTFs were 23 years old or younger – and seemed particularly prone to view teaching as a resumé builder.

A second forced-choice item asked preservice MTFs about their longer-range career plans, which provides further insights into MTFs' professional plans to teach. In particular, while just more than a third of MTFs viewed teaching as a long-term career, large numbers reported being unsure. In terms of long-range plans, close to one-fifth reported planning to work in "another education position." When prompted for clarification, many wrote that they aspired to be school administrators. Smaller numbers planned to teach in public schools outside NYC, teach in a private school, or attend graduate school. Another approximate fifth reported being unsure about their long-term plans. It would seem, then, for the majority of MTFs, that teaching in NYC schools was seen as a short-term way to help advance their careers.

Yet another follow-up item on the inservice survey was on open-ended item that asked MTFs “who planned to leave NYC schools within the next three years” about the incentives and changes that would “keep them in the profession for longer”. The most common response amongst those who planned to leave within 3 years had to do with raising salaries and the high cost of living in NYC. A smaller number complained about “unsupportive” administrators, schools that lacked structure or discipline, and the lack of planning time. A few critiqued NYCTF for not providing enough “support.” A small number yearned for “more intellectual” students, “better kids” who were “more respectful”, and students with “parents who cared” about education. A few disliked NYC, wanted to move back to their hometowns, or desired a shorter commute to work. These are familiar refrains in the literature on teacher retention (see Boyd *et al.*, 2005; Guarino *et al.*, 2006).

A companion item on the inservice survey asked respondents planning to stay in NYC schools “for more than three years” what “motivated them to stay in the profession”. Many wrote that they “love” NYC students and want to help them succeed in academics and life. A few of those planning to stay discussed job stability and benefits (e.g., summers off), while a few mentioned career advancement (e.g., moving into school administration).

First-Year Induction for Math Fellows

Hypothetically, NYCTF induction goes beyond basic induction; Fellows are assigned to a NYCDoE mentor who visits weekly, if not more. A NYS law passed in 2005 (still unfunded) requires school districts to give all new teachers mentoring and induction. Alternative route legislation, passed 5 years earlier, requires ACT programs to provide mentoring. In addition to a DoE mentor, Fellows are assigned a university mentor who visits once a month, in part, for evaluative purposes. In terms of supportive administrators, after the 2000-01 school year, when many Fellows were placed in unwelcoming schools, NYCTF began advising Fellows to take jobs in schools with administrators that supported the program (Goodnough, 2004). More generally, the NYCDoE expects principals and other administrators (e.g., math or literacy coaches) to support all new teachers, including Fellows. In terms of teacher networks, once Fellows secure a school position, NYCTF connects them with more experienced Fellows who work at that school. NYCTF uses a cohort model; Fellows are encouraged to network with others in the cohort; while they teach in different schools, they see each other weekly at 1 of 4 universities where they complete Master’s coursework in math and education. Finally, Fellows receive something that approaches a new teacher seminar in their preservice programs; specifically, in these NYCTF-provided professional development “sessions”, Fellows are oriented to district policy and NYC schools and discuss lesson planning, classroom management, and their clinical experiences in summer school classrooms. The inservice survey included items that addressed most induction components outlined above. However, since DoE mentoring is the centerpiece of NYCTF induction, we focus on it here. First and foremost, 3 of 10 inservice respondents report never being assigned a NYCDoE mentor in their first year. Again, this does not include data on the over 20% of MTFs who quit teaching prior to the second year. Hence, 90 or more MTFs lacked a key component of new teacher induction in the 2007-08 school year. Further, the 7 out of 10 MTFs who were assigned a DoE mentor report significant variation in the amount of mentoring they received. Over 60% of these MTFs report that their mentors visited at least twice a month during the 2007-08 school year, while the remainder (37.9%) received 1 visit per month or less. Combining results, less than half of MTFs (43.4%) had a mentor that visited them at least twice a month.

While many mentors visited regularly, they often did not observe whole lessons. Indeed, just over a third (35.7%) of MTFs who received mentoring report having an entire lesson observed twice a month or more. Hence, when mentors came, they generally only observed part of a lesson. Further, while the majority of the follow-up meetings lasted from 20 to 45 minutes, about 1 in 5 lasted 10 minutes or less.

There was considerable variation in what DoE mentors and MTFs discussed in meetings. Some MTFs claimed such things as, “my mentor was candid with me and gave helpful classroom management tips and lesson planning tips.” Others reported that mentoring was more about moral support than practical advice; e.g., 2 respondents described their mentors as, “just a person to talk to who had been through a similar situation” and “a sympathetic ear”. Still other MTFs claimed that their DoE mentors provided little if any support. One wrote, “None! He did nothing but sit and watch and leave.” We also note here that a small number report being mentored by an out-of-field mentor who may have been unable to provide assistance in the area of math.

A Likert-type item examined the extent to which DoE mentoring helped MTFs become proficient in specific areas of teaching (e.g., instructional strategies, management) and provided more general types of support (e.g., moral, psychological). While most of the 70% of MTFs who received mentoring saw some benefit in it, reports of what was most “helpful” about it varied considerably. On average, DoE mentors provided more general encouragement than specific help in any one area. An exploratory analysis suggests that MTF turnover in the first year correlates with a lack of mentoring. MTFs who received little or no mentoring were more likely to move from school to school than others. That said, MTFs were also more likely to leave the most troubled schools – schools that likely have the most difficulty getting DoE mentors to work with their teachers. Importantly, more than 80% of inservice respondents reported being mentored by someone other than a DoE mentor. The most commonly named persons providing support were: a) math coaches (officially not mentors), b) university mentors, and c) fellow teachers.

Discussion

We find substantial teacher turnover amongst Cohort 14 MTFs and a first-year attrition rate that appears higher than that of other Fellows. The majority of MTFs who entered NYC schools in 2007-08 plan to stay a total of four years or less. At least in the case of math, rather than stopping the "revolving door," NYCTF may actually be exacerbating the problem.

While salary and organizational factors contribute to rapid and massive teacher turnover, the backgrounds of MTFs are not unimportant. More than 1 in 3 (35%) of MTFs report that they entered NYCTF program having completed minimal college coursework (i.e., 1 to 3 courses) in math (Donoghue *et al.*, 2008). Teaching math was not their first-choice discipline and becoming a math teacher may not have been in their future plans at all when they applied to NYCTF. In interviews, a number of MTFs claim that NYCTF administrators convinced them to become math teachers during the admissions process. Some MTFs report that they were told that teaching math would increase their competitive advantage in getting into NYCTF. The fact that many MTFs have relatively weak backgrounds in math could contribute to lower rates of teacher satisfaction and comparatively high rates of turnover amongst the MTFs. Many of these MTF are not particularly passionate about math and may even feel underprepared to teach it. Further, in interviews and open-ended survey responses, many MTFs discuss feeling considerable social distance from their students. As adolescents, most Fellows were in selective school programs (i.e., honors tracks, private schools). Over 70% report coming from middle- and upper-class backgrounds. More problematic is that many preservice MTFs articulate deficit views of NYC students and their guardians after having only minimal contact with them in summer fieldwork classes.

> [The neighborhood I grew up in was] really obsessed with class and prestige and wealth . . . 99% of the students go to college. So it was just like [a] school culture where kids paid attention in class . . . But from my experience with summer school, these [NYC public school] kids won't do their homework at all. And half the time they don't care if they fail. So I think I'm going to be a lot more aggressive in my classroom than maybe some of my teachers were at home by necessity (Interview, Summer 2006).

As this excerpt shows, there is an implicit contrast between normal (privileged) students who care about school and urban students who ostensibly "don't care". While some MTFs certainly do develop into effective and caring NYC teachers, initially at least, privileged-class teachers may not be particularly good "fits" for high-needs urban schools (Brantlinger, Cooley, & Brantlinger, in press). As our data indicate, the lack of fit is likely exacerbated by unsupportive administrators and ineffective mentoring.

The idea that the *type* of ACT recruit matters as much as the teacher education program itself is one that is gaining currency amongst scholars who study alternative routes to teaching (Humphrey & Weschler, 2008; Johnson, 2004). They argue that "early entry" ACT programs, such as NYCTF, work for some recruits. To a large extent, the success of these programs depends on what individual recruits bring to teaching and the schools they teach in. As our paper suggests, there is considerable diversity to ACTs and their first-year experiences in NYC schools. While large numbers leave their initial school or teaching altogether within the first year, others are committed and survive. One wonders, however, what could be done to reduce massive math teacher turnover in the highest-needs urban schools and, given the high rates of attrition, whether the money spent per Teaching Fellow is worth the cost.

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Analyzing the effects of a linguistic approach to the teaching of algebra: students tell “stories of development” revealing new competencies and conceptions

Annalisa Cusi
University of Modena and Reggio Emilia annalisa.cusi@unimore.it

Abstract
This work is part of a wide-ranging long-term project aimed at fostering students’ acquisition of symbol sense through teaching experiments on proof in elementary number theory (ENT). In this paper, in particular, we highlight the positive effects of our approach analysing the written reflections that the students involved have produced at the end of the project. These reflections testify an increased level of awareness, developed by students, about the role played by algebraic language as a tool for thinking and a positive evolution in their vision of algebra.

A focus on meaning in the teaching of algebra: introducing our approach

A central aspect of a teaching aimed at fostering a mature conception of mathematics is helping students “conquering” the meanings which are connected to the learning of concepts and to the use of mathematical instruments during school activities. The main problems related to a “warped” conception of mathematics have been highlighted by research in mathematics education. Skemp and Sfard, who have respectively described the opposite approaches to mathematics through the instrumental-relational dichotomy (Skemp 1976) and the operational-structural dichotomy (Sfard 1991), have singled out what kind of processes students should activate in order to be able to develop an appropriate vision of mathematics and to pass from an instrumental-operational approach to the learning of mathematics to a more meaningful relational-structural approach. Skemp (1976), in particular, stress that, although many teachers intentionally chose an instrumental approach to the teaching of mathematics, the relational vision of mathematics and of its teaching should be considered the desirable one. According to the author, the reason of a widespread disaffection toward mathematics, also among those students who have studied mathematics at higher levels, must be found in years of a teaching which turned out to be unable to help them develop a real relational understanding.

Sfard e Linchevski (1994) have highlighted the centrality of the problem of the “conquest” of meanings also in the context of the teaching and learning of algebra because students could develop “warped” conceptions like those that the authors define pseudo-structural, which are typical of students who consider algebraic formulas as mere strings of symbols to which they apply, without control, routine procedures. These students identify mathematical objects with their representations, therefore they consider formal manipulations as the only kind of meaning which can be associated to algebraic formulas. In order to “fight” against this pseudo-structural conception of algebra, teachers must work in order to help their students becoming “active sense-seakers”, who always look for the meaning of symbols.

The centrality of a teaching of algebra focused on meanings is also stressed by Arcavi (1994), who claims that, in addition to stimulating students’ abilities in the manipulation of algebraic expressions, teachers should make them see the value of algebra as an instrument for understanding, expressing and communicating generalizations, the establishment of connections, or the production of argumentation and proof. Also Bell (1996) states that it is necessary to favour the use of algebraic language as a tool for representing relationships, and to explore aspects of these relationships by developing those manipulative abilities that could help in the transformation of symbolic expressions into different forms. Similar observations are found in Wheeler (1996), who asserts the importance of ensuring that students acquire the fundamental awareness that algebraic tools “open the way” to the discovery and (sometimes) creation of new objects.

Our research enabled us to highlight a widespread pseudo-structural vision of algebra also in school-contexts in which mathematics plays a central role: the analysis of a questionnaire on the vision of algebra, proposed to a group of 53 students attending the first year of a liceo scientifico (grade 9) helped us verify that also those students who do not face difficulties in performing syntactical algebraic manipulation actually display a partial vision of the meanings associated to the activities they face during their school experience (in the following we will refer to this group as group B). The following is one of the questions we posed to students through the questionnaire: *What is your idea of algebra? Try to describe it through 5 key-words/key-sentences.* We analyzed written reflections produced by students who tried to answer to this question. Because of space limitations, we chose to propose only a sentence, written by one of these students, R, which concisely expresses the widespread distorted vision of algebra that we were able to highlight: *“If I have to be rational, algebra is somehow useless because in ordinary life we do not need to sum or multiply letters. But I am in a liceo scientifico, so probably I like it a little”*. The vision of algebra that R’s reflection conveys (like many other reflections proposed by group B students) is that of a useless ‘game’, too abstract and too difficult to be played. We believe that this idea could be considered the result of a lack in students’ understanding of the deep meanings which are subtended to algebraic activities. In order to foster a recovery of these meanings, our research project is aimed at promoting a different, more appropriate vision of algebra (we refer to the idea of symbol sense proposed by Arcavi, 1994) through the planning and implementation of innovative experimental paths to be proposed to upper secondary school students. For students aged 14 and 15, in particular, we planned and experimented a path for the introductions of proofs in elementary number theory (ENT). We, in fact, believe that activities of proof in ENT would help students appreciate the value of algebraic language as a tool for the representation and solving of situations that are difficult to manage through natural language only and we agree with Wheeler (1996), who states that activities of proof construction could constitute “a counterbalance to all the automating and routinizing that tends to dominate the scene”. The path, proposed to 84 students of 4 different classes (in the following we will refer to this group as group A) of upper secondary school (grades 9 and 10), is articulated in six different gradual phases of work, characterized by the following activities: (1) Translations from verbal to algebraic language and vice-versa; (2) Study of the relationship between properties of a given formula and properties of the variables it contains; (3) Analysis of the truthfulness/falseness of statements concerning natural numbers and justification of the given answers; (4) Exploration of numerical situations, formulation of conjectures and related proofs; (5) Construction of proofs of given theorems. The work with students (about 20 hours) was articulated through small-groups activities (some groups were audio-recorded), followed by collective discussions (audio-recorded) on the results of the small-group activities. Before and after the activities of the didactical path, students were asked to face two written tests: the initial one was aimed at monitoring students’ competences in translating form verbal to algebraic language and in interpreting algebraic formulas; the final test was aimed at highlighting students’ achievement of those competences useful in the development of thought processes through algebraic language. After the final test, we also proposed to students a questionnaire on their vision of algebra, similar to the one that group B students had to face, in order to highlight similarities and/or differences between the two groups of students.

Hypothesis and aims of this work

Our hypothesis is that the approach we propose, if adopted by a teacher who pose him/herself in an aware and effective way in the class (we analysed the role of the teacher in a previous paper, Cusi 2009B), could produce positive effects in students, not only in terms of acquisition of competences and development of a new awareness about the meaning of algebraic activities, but also in terms of construction and consolidation of a relational vision of algebra.

The aim associated to this hypothesis is to highlight the effects of this approach on those students who participated in the activities of the didactical path in terms of both competences acquired and vision of algebra developed by students. In order to highlight these effects, at the end of the experimentation, we carried out an in depth-analysis of: the written protocols produced by students during the different activities of the path (we also analyzed audio-recordings of group discussions and collective discussions referring to these protocols) and the answers students gave to the two written tests and to the final questionnaire on their vision of algebra.

Methodology of research and theoretical references for the analysis of students’ written reflections

In order to introduce the results of our analysis, we chose (Cusi 2009A): (1) to tell some “stories of success”, which testify how students’ participation in the activities of the didactical path helped them (a) develop those competencies that are fundamental in the construction of reasoning through algebraic language, (b) reach an high level of awareness about the role played by algebraic language as a tool for thinking, (c) modify their vision of algebra; and (2) to show, through the analysis of the written reflections produced by students of group A, the clear gap between these reflections and those produced by students of group B (who have learned algebra through a “traditional” teaching approach, focussed on syntactical aspects) in order to highlight how students’ participation in the project helped them realize a recovery of their attitude toward the meanings which are associated to algebraic activities. Because of space limitations, we devote this paper to the presentation of the results of part (2) of the analysis we carried out.

Our main reference for the analysis of students written reflections about the new vision of algebra they developed thanks to their participation in the activities of the innovative path is the work by Di Martino and Zan (2003, 2007), who introduce a new way of conceiving the concept of attitude toward mathematics. The authors stress the need to overcome the positive/negative dichotomy (usually associated to the concept of attitude) through an approach based on an analysis of students’ attitudes done by referring to the different components which constitute them, the complexity of their interactions and the mathematical context of reference. Thanks to an analysis of students' written essays on their relationship with mathematics, Di Martino and Zan were able to identify three different key-aspects around which the essays are developed: the emotional disposition toward mathematics (expressed with terms like "I like / I do not like"); the perception of being / not being able to succeed in mathematics (expressed with terms like "I can do it / I can't do it"); the vision of mathematics (expressed with terms like "mathematics is…"). These key-aspects are connected each other in different ways. The analysis of these connections helped the authors conclude that a deep description of students' attitude toward mathematics require to highlight not only their emotional dispositions, but also their vision of mathematics and the beliefs on the sense of self-efficacy associated to it.

A recovery of the attitude towards mathematics: students tell their personal stories through their reflections

We stimulated students' written reflections through two questions which follow the one proposed to group B students. The two questions are formulated in order to help students reflect on the changes occurred in their vision of algebra thanks to their participation in the didactical project: (a) *What was your idea of algebra before your participation in the didactical project? Try to describe it through 5 key-words/key-sentences.* (b) *Did your participation in the didactical project modify your previous idea of algebra? If yes, what are the changes occurred? Try to describe your present vision of algebra through 5 key-words/key-sentences*. Our analysis of students' written reflections helped us highlight how the participation in the didactical project, also for those students who display scarce aptitudes and poor interest toward mathematics, could positively influence their way of facing and perceiving the discipline. Group A students' reflections, in fact, clash with those produced by group B students.

Most of group B students refer to key-words which reflect a warped or, at least, partial vision of algebra and testify an occurred development of a pseudo-structural conception of this subject. Very few students of this group, instead, propose sentences which reflect a structural vision of algebra or a relational understanding of the discipline. In students' reflections it is possible to identify many references to a sense of uselessness of algebra, strictly related to the field of the emotional disposition toward the learning of this subject. This idea is often associated to the one of a discipline which is considered too abstract and difficult: *"If I have to be rational, algebra is somehow useless"* (R); *"It is abstract, difficult, useless"* (S); *"It will be useless in the future"* (A_1). These ideas of uselessness and senseless complexity are often associated to warped or partial visions of algebra, conceived as a mere set of rules to be respected and often relegated to the sphere of the simplification of algebraic expressions: *"Algebra is somehow useless because in ordinary life we do not need to sum or multiply letters"* (R); *"I think that arithmetic is more useful than algebra because in every day life we do not need to solve equations or to perform factorizations"* (S_1); *"I think that algebra is not useless in every day life, unless you choose a future of numbers and letters"* (V). These students identify the only key to success in the practice of memory and concentration and in the development of their abilities in performing syntactical manipulations: *"You need to be really careful. Algebra is very unpleasant because, as soon as you miscalculate, also if it is a simple careless mistake, all the expression changes. It irritates me, especially if the expression is very long!"* (A_1); *"According to me, algebra is a set of rules. It requires a lot of memory and accuracy. Moreover you cannot be inattentive while you perform calculations because the smallest mistake could change the text of the expression. Algebra is difficult if you do not know the calculation rules, while it is easy if you know them"* (A_M).

We can find similar ideas also among those students who declare to love algebra and to think that it is not too complex. The only difference between these students and those who show their disaffections toward algebra is that the first ones identify the reason of their love for this subject exactly in the possibility to be successful thanks to a mere application of rules and a constant training: *"I think that algebra is not difficult because in order to be successful it is enough to apply some rules. I like it very much, especially when I succeed in solving expressions "* (F); *"I like algebra, differently from geometry, also because you do not need to learn many rules and the most important thing to succeed is a constant training in perform calculations"* (A_3). Also when students refer to algebra as a useful discipline, this idea is however associated to the possibility of developing abilities in performing calculations and mental flexibility. The few who assert that algebra is a tool for reasoning and understanding, actually associate the term reasoning to ideas like "training for mind", "tool do develop the brain", "stimulus to concentration and to precision", "capability to apply rules".

Our analysis of group B students' written reflections let us highlight how, for these students, both the negative emotional dispositions ("I do not like algebra") and the positive ones ("I like algebra") are almost exclusively associated to an instrumental-procedural vision of algebra, focussed on the role played by memory and by precision in the application of rules. Very few reflections are focussed on the need to understand what students are doing when they deal with algebra. In any case, because a predominance of an idea of algebra as a mere exercise for mind rather than a language for producing reasoning and for communicating, these ideas could be considered signs of a relational-structural conception of the discipline which is still far from a complete development. The fact that also students who are good and motivated in the study of mathematics do not display a relational vision of algebra represents a strong signal for a need of a restructuring in the approach to the teaching of this subject, which must be focussed on the processes of construction of meaning.

In order to highlight the contrast between the attitude displayed by the students of the two groups, in the following we propose some reflections produced by four group A students, who express the awareness of having developed an idea of algebra which is completely different from the one they had before their participation in the project:

"*Before the project I used to like algebra, but I looked at it as a set of rules to be applied. Only thanks to the project I discovered that this topic offers difficulties that I have never met before. It does not mean that now I do not like algebra. On the contrary, it fascinates me more than before. Now I know that algebra is something more than the subject I learned at school. Algebra can open your mind and show you things in a different and deeper way*" (G_2);

"*Before the project I used to have an idea of algebra as 'following rules', that is I used to think that using logic and reasoning was not so essential in order to solve the exercises. Now, instead, I have understood that reasoning is necessary and that mathematics involves arguing*" (A_1);

"*I have never liked algebra and I have always thought that only those who were able to appreciate it and to dedicate a lot of time to its study could be successful. I have always looked at it as something difficult and abstract because I used to think that it could not be used in every day life. Now I do not look at algebra as something unreachable and incomprehensible, but I feel that if I study algebra with dedication I can be able to really understand, learn and apply it*" (I);

"*Before my participation in the project I used to think that algebra was a complicated subject and that it would have been useless for my future life. I thought it was simply a set of rules to be learned by heart and difficult to be applied. Now I have changed my mind. Thanks to algebra you can prove something which is difficult or sometimes impossible to be proved by means of words. When you work with algebra you must reason and reasoning is a way to open your mind to many different perspectives. Algebra is useful to understand why something is like this and not like that*" (S_1).

Written reflections produced by group A students, in tune with those of G, A_1, I and S_1, testify how their vision of algebra have been changed. In fact:

(1) while group A students asserts that before the project they used to look at algebra as a mere set of rules that can be learned through a constant training and a lot of precision and that must be respected also if they are often incomprehensible (*instrumental* vision of algebra, which is similar to the one expressed by group B students):

> "*Before the project I used to have an idea of algebra as 'following rules'*" (A_1);
>
> "*My idea of algebra before the project was that of a set of numbers to be calculated trough procedures*" (A_3);
>
> "*I used to think that algebra is a set of rules to be learned by heart and difficult to be applied and that it is a subject which requires a lot of precision, otherwise you cannot understand anything you have written*" (S_2);
>
> "*I have always thought that only those who were able to appreciate it and to dedicate a lot of time to its study could be successful*" (I);

(2) the same students declare that, thanks to their participation in the project, they have developed a different conception, according to which the use of algebraic language can support reasoning processes and, therefore, it allows to analyze reality from a different perspective and to 'catch' aspects that cannot be studied through natural language only:

> "*Algebra is not only something mechanical but also something logic*" (M);
>
> "*Algebra, thanks to the use of letters, can help you reach wider concepts*" (G_1);
>
> "*Algebra expresses concepts and it is more simple if you use reasoning*" (G_3);
>
> "*Algebra opens your mind and allows you to see things in a different and deeper way*" (E);
>
> "*Now I think that algebra is useful to help us to carry out a deep analysis of things*" (A_3);

"Thanks to algebra you can prove something which is difficult or sometimes impossible to be proved by means of words. When you work with algebra you must reason and reasoning is a way to open your mind to many different perspectives. Algebra is useful to understand why something is like this and not like that." (S_1).

Students show to be aware that this approach to algebra involves greater difficulties (because the activities of the didactical path require to construct reasoning by means of algebraic language), but they look at these new difficulties as ways toward a deeper comprehension and a thorough analysis of problems:

"Only thanks to the project I discovered that this topic offers difficulties that I have never met before. It does not mean that now I do not like algebra. On the contrary, it fascinates me more than before. Now I know that algebra is something more than the subject I learned at school." (G_2);

"I used to think that using logic and reasoning was not so essential in order to solve the exercises. Now, instead, I have understood that reasoning is necessary" (A_1).

Finally, it is interesting to highlight the idea expressed by I's written reflection. The student, in fact, stresses that, starting from a vision of algebra as an incomprehensible subject, thanks to the project she was able to develop a new vision of algebra as a subject that can be learned only through a constant dedication aimed at a real comprehension: *"Now I do not look at algebra as something unreachable and incomprehensible, but I feel that if I study algebra with dedication I can be able to really understand, learn and apply it"*.

The occurred recovery of the attitude toward algebra that our analysis of groups A students' reflections was able to highlight (although it is only partially documented because of space limitations) testifies how the participation in the didactical project allowed students to perceive algebraic language from a richer perspective and to grasp the deep meanings of its learning. Their participation during both the activities of the didactical path and the moments devoted to the sharing of their learning, has allowed students to tell us, as well as to themselves, *different stories of their vision of algebra*.

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Localization of Learning Objects in Mathematics

Prof. dr. Valentina Dagiene,
Head of Informatics Methodology Department, Institute of Mathematics and Informatics
Vilnius, Lithuania dagiene@ktl.mii.lt
Inga Zilinskiene
PhD student. Institute of Mathematics and Informatics, Vilnius, Lithuania
ingasolyte@ktl.mii.lt

Abstract Mathematics learning seems to be a demanding and time-consuming task for many learners. Information and communication technology (ICT) is an attractive tool of learning for students at any level and it can provide an effective atmosphere for understanding mathematics. The question is how to combine mathematics teaching contents, approaches, curricula, and syllabus with new media. The key issue in European educational policy (and other countries as well) is exchange and sharing digital learning resources (learning objects) among countries. In order to accumulate the practice of various countries and use the best digital resources created by different countries, it is necessary to localize learning objects (LO). The paper deals with some problems connected with localization of LO, developed for mathematics education, and presents some solution. Software localization is mainly referred to as language translation (e.g., translation of user interface texts and help documents). However, there are many other important elements depending on the country and people who will use the localized software. In this paper, the main attention is paid to localization of learning objects used for teaching and learning mathematics.

Keywords: teaching mathematics, learning objects, mathematics software, mathematical notation, localization

Introduction A constant development of ICT has unleashed new challenges in education. Its use is a strong element for knowledge construction support. In the Lithuanian community of mathematics educators, teaching mathematics takes a strong position in the education policy and, particularly, in the school community. Many students and parents consider mathematics knowledge as the key success for future life. However, the understanding of teaching mathematics is mostly based on an academic approach that is good enough mostly for motivated students. The majority of students especially teenagers are not interested in academic knowledge and are not able to develop mathematical literacy skills at all.

How could we make mathematics studies easier for both students and teachers? There are lots of suggestions that fall between deep "rethinking of mathematics" by S. Papert (1980) and gaming (Kahn, 2006). Using learning objects (LO's) for teaching mathematics can be one of the best ways. LO can be described as any digital resource that can be reused to support learning. To develop good LO's for teaching mathematics is great and time consuming work. So it is necessary to exchange LO's among countries. Therefore the European digital learning content implementation is based on the exchange of learning resources (European Schoolnet, 2006).

Therefore it is necessary to run the localization and adaptation of high quality content – digital learning resources. In addition, while developing new learning content it is necessary to take into account its future adaptation to other locales and other countries' educational systems i.e. to pay attention to internationalization of LO.

Overview of the tasks of localization

Countries and different speaking peoples use ever more and more diverse software. One of the main problems in software adaptation to local users is localization. Localization can not be interpreted as an action of translation (Grigas, 2000). Despite the fact that localization of software is estimated by translated resource lines, translation makes up only a small part of software localization (Esselink, 1998).

Discussions on software localization usually point out three main parts of localization (Fig 1): 1) software adaptation to target locale, 2) translation and adaptation of user interface, 3) translation and adaptation of software documentation.

Software adaptation to particular locale norms serves as the basis of a localization process. According to the international standard ISO/IEC 15897 (ISO/IEC 1999), locale is “the definition of the subset of a user’s information technology environment that depends on the language, territory, or other cultural customs”. Usually, three main components are attributed to locale: 1) language (which can be understood by the user and which must be handled by the software), 2) culture (non-verbal aspects of the product’s functionality (Hall, 1990, Schäler, 2002)), 3) local practices and conventions (aspects such as legal requirements, notation, measurement units, etc.) (Hall, 1997).

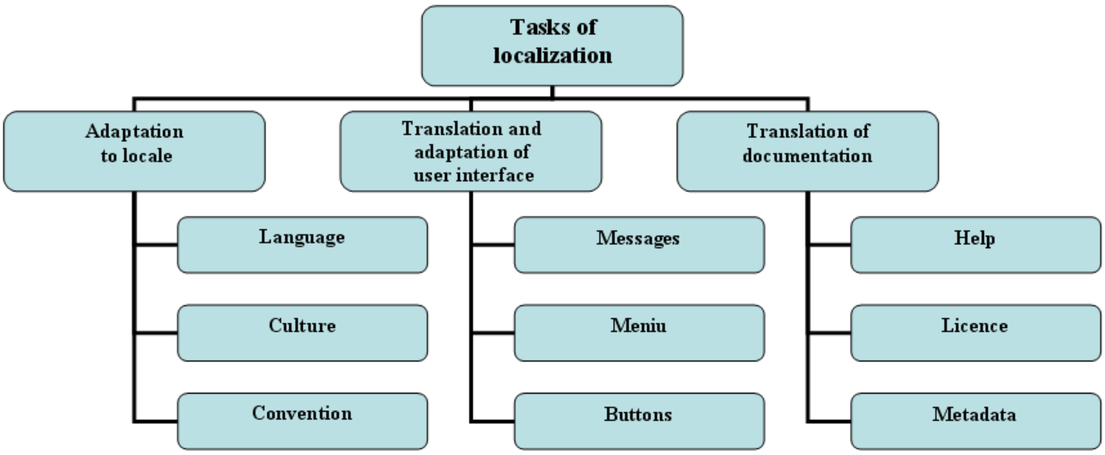


Fig. 1. Main components of software localization

Locale is usually identified by the language, using a two-letter language code (ISO 639-1), and by territory, using a two-letter territory code (ISO 3166-1). Locale depends not only on the language (for instance, locales of Great Britain and the USA are different, although these countries use the same language) or only on the country (for example, in Canada there are two official languages, English and French, each of these combinations of language and country usually have their own ways of expressing dates, times, numbers and other elements).

POSIX (Portable Operating System Interface for Computer Environments) standard was one of the first to define basic locale data. POSIX locale model (ISO/IEC 9945-2) has six main categories, that define (Jevsikova, 2006): 1) Character classification and case conversion. 2) Collation order. 3) Monetary formatting. 4) Numeric, non-monetary formatting. 5) Date and time formats. 6) Formats of informative and diagnostic messages and interactive responses.

This is a minimum set of locale elements for any software including LO as well. However that is not enough for high quality localization. Some new elements were added in later locale models.

Adaptation of user interface texts is the second component of localization. User interface texts are records and messages (text strings) within software’s dialogs and their elements (buttons, captions, boxes, menu bars, etc.).

Translation of help and documentation files (printed or online) is the last component. It is important to point out that translation of user interface texts and help files usually is the most time and effort consuming task because of a large amount of such texts and a requirement to preserve the consistency with the adapted user interface.

The first step of localization is to adjust the software to the norms of locale. Only after that the adaptation of resources and the translation of help files may be performed.

Localization of mathematical notation

The software used in comprehensive schools of many countries is: operational systems, programs of file managing, text processors, spreadsheets, database programs, internet browsers, e-mail programs, antivirus programs, and presentation programs. They are almost exempt from mathematical elements (except the use of the usual mathematical symbols in the keyboard, other minimal tools for writing equations, fractions, etc.)

Localization problem of mathematical programs have been noticed since the interest in learning objects (LO‘s).

When localizing the parts of mathematical software, as usual, the main attention was paid to mathematical characters. They are very different in different countries. The notation of even the main arithmetic operations (multiplication and division) is distinct in various countries. (Table 1).

Different countries use distinct measuring units and their notation. For example, the notation of length, weight, temperature, currency, etc. is quite different in Great Britain. The decimal separator is also a point, while in Lithuania it is a comma. It is easy to change that in one program, however, if the program uses another one that can be not localized, then we have to envisage the consequences of a possible conflict. An analogous situation is with other mathematical notation.

It is not so complicated to replace (localize) separate characters, however, it is much more complicated to localize if the order of writing or notation is changed (e.g., in Japan postfix notation is used to designate functions).

Localization of mathematical teaching strategies

When localizing computer programs of teaching (learning) mathematics especially, we have found some methodological differences between Lithuanian mathematics didactics ant that of foreign countries, e.g., while performing some arithmetic operations: not only the notation, but also the ways of calculation are different (LO “Rechenheft“, http://www.rechenheft.com, “Rainforest“, http://www.rainforestmaths.com, etc.). Thus,

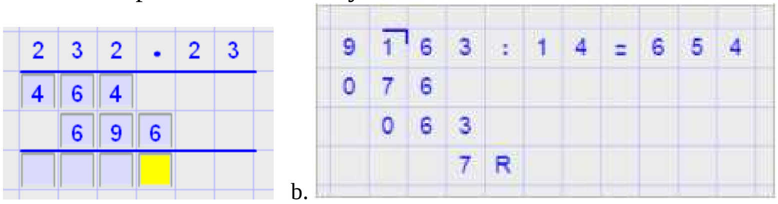
Table 1. Mathematical notation in different countries

		Mathematical notation					
Mathematical operation	Multiplication	**Austria** 1014·163 6084 3042 165282					
	Division	**Austria** 9163:14=654 076 063 7R	**Denmark** 5/375\75 35 25	**Israel** 75 375| 5 35 25 25 00	**Japan** 75 5) 376 35 26 25 1 Answer: 75…1	**Croatia** 376:5=75,2 35 _ 26 25 10	**Australia** 174 r1 5) 8^37^21

localization of software, especially that of LO‘s embraces not only technical, but also methodological problems, after exposing of which there is a possibility to present a qualitative product to the society.

Some LO‘s have been essentially changed, reprogrammed in the process of localization. An example of this kind is “Rechenheft“(created by Christian Nosko, a teacher from Austria). At present the program is operating in the Lithuanian language, though it was rather difficult to localize it (despite that there is not so much of the text). It has turned out that in Austrian schools, arithmetic operations are performed quite in other methods than in Lithuania. For instance, multiplication is performed not from left to right, as is the habit with us, but from right to left.

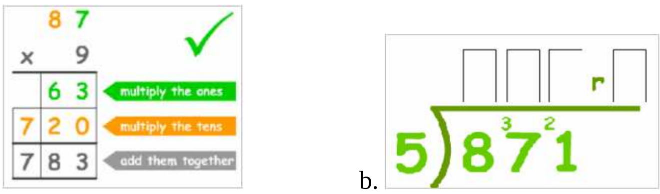
Division is also performed in an extremely complicated way. Intermediate operations of multiplication and subtraction are omitted by writing below the difference obtained only: all the intermediate operations are performed mentally.



a. b.

Fig. 2. Methods of multiplication and division in Austria

One more aspect for teaching multiplication is displayed in the program developed by Jenny Eather in Australia „Rainforest Math“ (Fig. 3) Multiplication is performed in the way usual to us, however, while multiplying units by tens (hundreds, thousands, etc.), the exact number is written that is obtained multiplying units by tens, i.e., in this example number 720 but not 72, but transferred one position to the left. The multiplication operation is performed in Australia just in the same way as in our country; only it is written in another form.



a. b.

Fig. 3 Methods of multiplication (a) and division (b) in Australia

Software Internationalization

The software designed for international market should be internationalized, i. e. the provision for its adaptability in any locale should be ensured. It’s just several years ago the wider attention to the issues of program internationalization was paid. (Reinecke, 2007) That’s why these issues are still poorly analyzed from theoretical point of view and still there are no solid standards or specifications for the internationalization of software. The contemporary tools for the software development are also still not adopted in the production of the proper internationalized programs. Namely because of this the adaptation of the software localization very often turns to be quite complicated.

Localization was initially approached as an "add-on"; i.e., after the original program was fully functional in English, localizers had to work on a Spanish, French, Japanese, etc. versions. However, programmers soon had to realize that such ex post facto solutions were inadequate; in many cases they required the re-writing of source code, which was a costly step that could have been avoided, if only the future internationalization would have been a part of the initial programming plan. (Uren, 1993) Many other authors, for example, Tuoc, David, and Driscoll (Tuoc, 1995), argue that internationalization must be part of the earliest design stages of any program, which must be written so that localization would be possible without rewriting the program's source code.

Internationalization involves isolation of the culturally and linguistically-dependent parts of software. Software internationalization is a framework for software localization; it is the process of designing and developing products with sets of features, functions and options to facilitate the adaptation of the product to various international markets (Hall, 1997).

Conclusions

Software localization is the process of adapting a software product to the linguistic, cultural and technical requirements of a target country or language. This process often requires a significant amount of time from the development teams.

Localization process can be divided into three main components: 1) software adaptation to target locale, 2) translation and adaptation of user interface, 3) translation and adaptation of help and other documentation. All components are related to each other.

Learning objects are the core concept in an approach to learning content in which content is broken down into "bite size" pieces. These pieces can be reused, independently created and maintained, also pulled apart and stuck together like lego bricks. That means that a learning object could be a piece of software as well as a text document, a movie, a presentation, an mp3, a picture or even a website.

When localizing mathematical learning objects all three above mentioned components should be considered: adaptation to the locale, translation and adaptation of user interface as well as translation of documentation. Additionally, attention should be paid to the mathematical notation which might be distinct in various countries. But it is not enough.

Analysis of many localized mathematical learning objects has shown that different countries have been using different teaching approaches and strategies. So these teaching approaches and strategies should be recognized and localized as well. Sometimes it is very hard work and requires substantial reconstruction of the learning object. Consequently, before starting localization of a mathematical learning object, the LO should be thoroughly analyzed not only in technical and notational aspects but also in point of the educational approaches and strategies. Localizers of mathematical learning objects have to be acquainted with various teaching strategies and be able to adopt them.

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Large-Scale Assessment as a Tool for Monitoring Learning and Teaching: The Case of Flanders, Belgium

Erik De Corte*, Rianne Janssen**, & Lieven Verschaffel*
* Center for Instructional Psychology and Technology (CIP&T)
** Center for Educational Effectiveness and Evaluation
University of Leuven, Belgium

Abstract

Traditional tests for large-scale assessment of mathematics learning have been criticized for several reasons, such as their mismatch between the vision of mathematical competence and the content covered by the test, and their failure to provide relevant information for guiding further learning and instruction. To achieve that large-scale assessments can function as tools for monitoring and improving learning and teaching, one has to move away from the rationale, the constraints, and the practices of traditional tests. As an illustration this paper presents an alternative approach to large-scale assessment of elementary school mathematics developed in Flanders, Belgium

Using models of item response theory, 14 measurement scales were constructed, each representing a cluster of curriculum standards and covering as a whole the mathematics curriculum relating to numbers, measurement and geometry. A representative sample of 5,763 sixth-graders (12-year-olds) belonging to 184 schools participated in the study. Based on expert judgments a cut-off score was set that determines the minimum level that students must achieve on each scale to master the standards. Overall, the more innovative curriculum standards were mastered less well than the more traditional ones. Few gender differences in performance were observed. The advantages of this approach and its further development are discussed

Introduction

Assessment is concerned with the design, construction, and use of instruments for determining how powerful learning environments are in facilitating in students the acquisition of the different aspects of competence in a domain, e.g., mathematics. Assessments of learning can either be internal or external. Internal assessments are organized by the teacher in the classroom, formally or more informally; to the contrary, external, usually large-scale assessments come from outside, organized at the district, state, national, or even international level using standardized tests or surveys. As argued by the National Research Council (2001) in the US, assessments in both the classroom or a large-scale context can be set up for three broad purposes: to assist learning and teaching, to measure achievement of individual pupils, or to evaluate school programs. I like to argue that a major purpose should be to use assessment *for* learning which means that it should provide useful information for students and teachers in view of fostering and optimizing further learning. Sloane and Kelly (2003) contrast assessment *for* learning or formative assessment with assessment *of* learning, the goal of the latter being to determine what students can, and whether they attain a certain achievement or proficiency level. In this paper I will focus on large-scale assessment of mathematics education in Flemish primary school.

Large-scale assessment of learning: A critical discussion

The massive use of standardized tests in education has always been more customary in the United States as compared, for instance, to Europe. But especially since the beginning of the 1990s the traditional tests have been criticized.

Analyses of widely used standardized tests show that there is a mismatch between the new vision of competence in different domains, on the one hand, and the content covered by those tests, on the other hand. Due to the excessive use of multiple-choice item format, the tests focus on the assessment of memorized facts, rote knowledge, and lower-level procedural skills. On the other hand, they do not sufficiently yield relevant and useful information on pupils' abilities in problem solving, in modeling complex situations, in communicating ideas, and in other higher-order thinking skills. A related criticism points to the one-sided orientation of the tests toward the products of pupils' mathematics work, and the neglect of the processes underlying those products.

An important consequence of this state-of-the-art is that assessment often has a negative impact on the implemented curriculum, the classroom climate, and instructional practices, dubbed the WYTIWYG ("What You Test Is What You Get") principle (Bell, Burkhardt, & Swan, 1992). Indeed, the tests as characterized above convey an implicit message to students and teachers that only facts, standard procedures, and lower-level skills are important and valued in mathematics education. As a result teachers tend to ‘teach to the test’, i.e. they adapt and narrow their instruction in the sense that they give a disproportionate amount of attention to the teaching of the low-level knowledge and skills addressed by the test at the expense of teaching for understanding, reasoning, and problem solving.

An additional major disadvantage of the majority of traditional evaluation instruments is that they are disconnected from learning and teaching. Indeed, also due to their static and product-oriented nature most achievement measures do not provide feedback about students’ understanding of basic concepts, nor about their thinking and problem-solving processes. Hence, they fail to provide relevant information that is helpful for students and teachers in view of guiding further learning and instruction.

Apart from the previous intrinsic criticisms on traditional standardized achievement tests, a major issue of debate is their accountability use as high-stake tests, i.e. their mandatory administration for collecting data on the attainment of students as a basis for highly consequential decisions about students (e.g., graduation), teachers (e.g., financial rewards), and schools and school districts (e.g., accreditation). According to the No Child Left Behind Act in the US this accountability use should result in the progressive acquisition by all students of a proficiency level in reading and mathematics. However, a crucial question is whether current testing programs really foster and improve learning and instruction; and, there are serious doubts in this regard. In a study by Amrein and Berliner (2002) involving 18 US states it was shown that there is no compelling evidence at all for increased student learning, the intended outcome of those states high-stake testing programs. Moreover, there are many reports of unintended but unfavorable consequences, such as increased drop-out rates, negative impact on minority and special education children, cheating on examinations by teachers and students, teachers leaving the profession, etc. In addition students tend to focus on learning for the test at the expense of the broader scope of the standards.

The Flemish approach to large-scale assessment of mathematics at the primary school

To achieve that large-scale assessments do indeed foster and improve student learning, one will have to move away from the rationale, the constraints, and the practices of high-stake testing programs As one example I will briefly review here an alternative approach to large-scale testing developed in the Flemish part of Belgium (for a more detailed discussion see Janssen, De Corte, Verschaffel, Knoors, & Colémont, 2002).

In a project commissioned by the Department of Education of the Flemish Ministry, we developed an instrument for the national assessment of the new standards of the entire new mathematics curriculum. These standards represent the basic competencies that students should master at the end of the primary school (which consists of 6 grades starting at the age of 6). The instrument was used to obtain a first, large-scale baseline assessment of the attainment of those new curriculum standards. The aim was thus not to evaluate individual children or schools as a basis for taking high-stake decisions, but to get an overall picture of the state-of-the-art of achievement in mathematics across Flanders at the end of primary education. The instrument consists of 14 measurement scales, each representing a cluster of standards and covering as a whole the entire mathematics curriculum relating to numbers, measurement, and geometry.

Using a stratified sampling design, a representative sample of 5763 sixth-graders (12-year-olds) belonging to 184 schools participated in the investigation. Taking into account the aim of the assessment it was not necessary to have individual scores of all students, and a population sampling approach could be used

“whereby different students take different portions of a much larger assessment, and the results are combined to obtain an aggregate picture of student achievement” (Chudowsky & Pellegrino, 2003, p. 80).

This approach also allows to really cover the total breadth of the curriculum standards. More specifically, the instrument involved 10 booklets, each containing about 40 items belonging to two or three of the 14 measurement scales. To get booklets that were somewhat varied the scales in each booklet represented distinct mathematical contents (for instance, the items in booklet 2 related to’percentages’ and ‘problem solving’). Each booklet was administered to a sample of over 500 sixth graders. Four different item formats were used: short-answer (67%), short-answer with several subquestions (14%), multiple-choice (11%), and product and process questions (8%). Especially the latter type addressed higher-order skills by asking for a motivation or an explanation for the given answer; an example is shown in Figure 1.

Chantal wants to buy a pair of Tiger sneakers and saw these ads in the local paper.

Family Shoe Center	Van Dierens shoe shop
Bottom prices every day	This week only
Tiger sneakers only 1200 BF	Sales: Tiger sneakers 1100 BF

The Family Shoe Center is within walking distance.
To go to Van Dierens shoe shop Chantal has to take the bus. That would cost 80 BF for a one-way ticket.
If Chantal wants to spend as little money as possible, at which shop should she buy her sneakers?
Answer: ____________________
Explain why.
Answer: ____________________

Figure 1. Example of a product and process question

Models from item response theory were used for the construction of the measurement scales. Starting from the test results scales were constructed for the set of items relating to each of the 14 clusters of standards. On each scale the items as well as the pupils are represented, as shown in Figure 2.

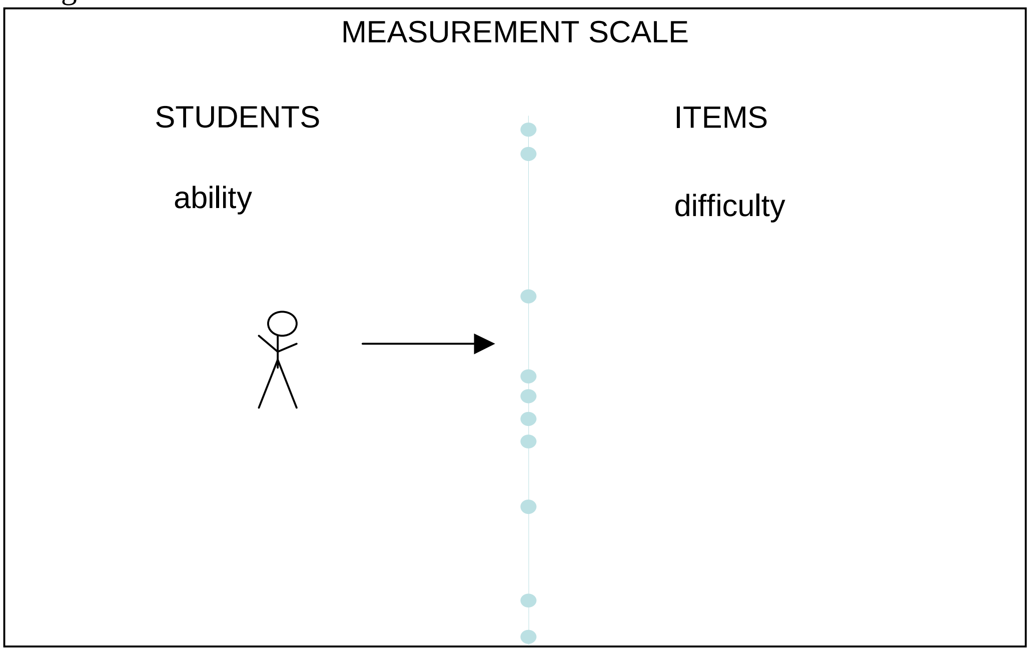

Figure 2. The principle of a measurement scale: the bullets on the vertical line represent the items in order of difficulty; the arrow indicates the position of a student on the scale

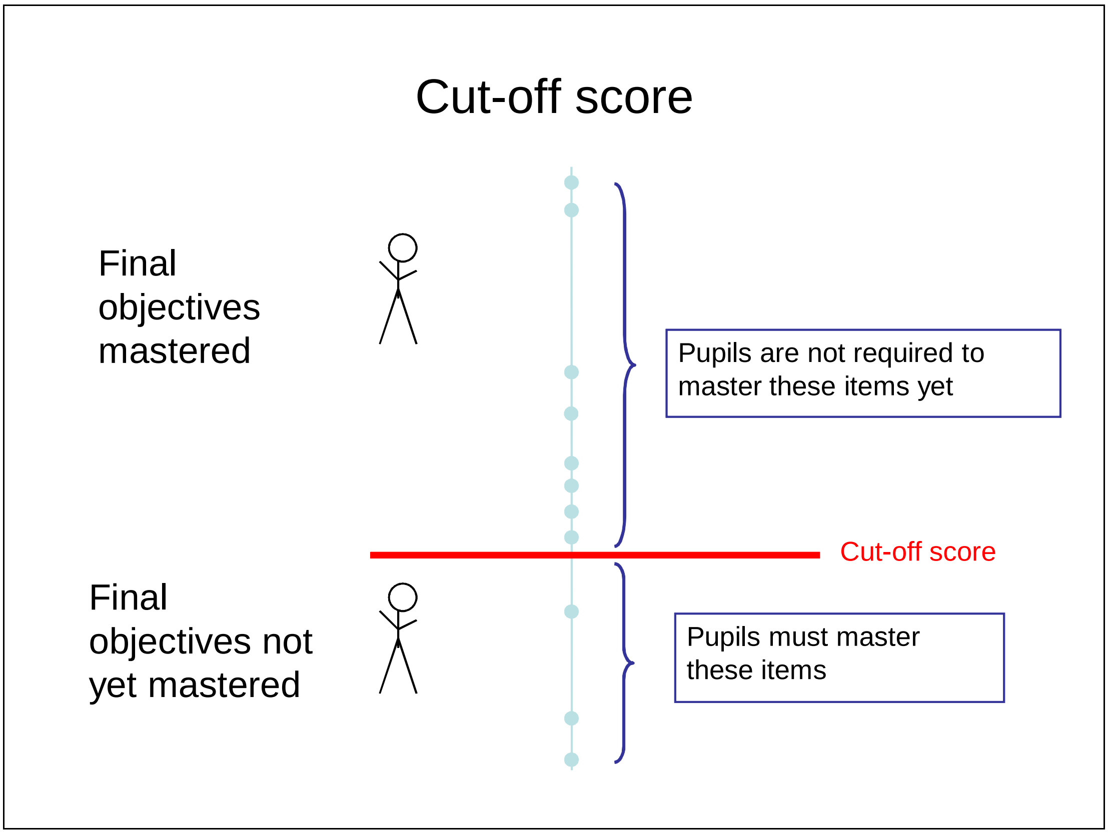

Figure 3. Cut-off score showing the divide of the items as well as the students

The next step consisted in determining on the measurement scales the minimum level that students must achieve in terms of the test items. Indeed, the standards describe that basic competencies in general terms, but because for each standard items of very different difficulty levels can be developed, there is a need to set a cut-off score that defines the minimum level of competence for each scale. This was done by consulting a group of expert judges who were asked to set the cut-off score for each scale based on a careful analysis of the content of the items. The cut-off score distinguishes the items the students must master well to attain the standards or basic competencies and the items that go beyond the minimum level (see Figure 3).

The results of this assessment shown in Table 1 can be summarized as follows. Scales about declarative knowledge and those involving lower-order mathematical procedures were mastered best. The scales relating to more complex procedures (e.g., calculating percentages; calculating perimeter, area, volume), and those that address higher-order thinking skills (problem solving; estimation and approximation) were not so well mastered. The latter finding is not so surprising as those scales relate to standards that are relatively new in the Flemish mathematics curriculum. It is also interesting to mention that few gender difference in performance were observed.

It is the intention of the Department of Education of the Flemish Ministry to organize such a large-scale assessment of mathematics education periodically in the future. The next assessment will take place in May 2009. The advantages and the potential of this approach to large-scale assessment are obvious. First, because this assessment covers the entire curriculum, its findings are a good starting point for continued discussion and reflection on the standards in and among all education stakeholders (policy makers, teachers, supervisors and educational counsellors, parents, pupils). Second, due to the breadth of such an assessment approach, it uncovers those (sets of) standards that are insufficiently mastered. In doing so the assessment provides relevant feedback to practitioners (curriculum makers, teachers, counsellors) by identifying those aspect of the curriculum that need special attention in learning and instruction; and researchers could focus intervention research on those weaknesses in pupils' competence. Third, due to the alignment of the assessment and the curriculum the often heard complaint about 'teaching and learning to the test' can largely be avoided, especially if appropriate counselling and follow-up care is provided after the results are published. Moreover, because the Ministry does not at all intend to use the results for the evaluation of individual teachers or schools, and because scores of individual children, classes, or schools are not published, the negative consequences of high-stake testing referred to above are also avoided.

Table 1. Overview of the assessment results for the 14 measurement scales

Measures in meaningful context	**88%**
Units of measure	**88%**
Geometry: concepts	**87%**
Space and spatial orientation	**86%**
Number values and equivalence	**86%**
Ratios	**74%**
Reference points	**72%**
Problem solving: measurement and geometry	**68%**
Problem solving: numbers and operations	**68%**
Fractions and decimal numbers	**64%**
Rounding off, estimation, approximation	**63%**
Meaningful reductions	**56%**
Perimeter, surface area, volume	**53%**
Calculating percentages	**42%**

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Connections between Mathematics and Arts & Culture: An exploratory Study with Teachers in a South African school

Joseph Dhlamini
Mathematics Education, PhD student, Tshwane University of Technology, South Africa
jjthemba@yahoo.com

Abstract This paper presents results of a two year study, at Master's level, which was undertaken to investigate how two Grade 9 Arts and Culture teachers incorporated mathematics in their Arts and Culture lessons in their classrooms in South Africa. Data from concept mapping activities and subsequent interviews with both teachers were collected and analysed using typological methods of analysis. Data collected from the study revealed that teachers still continue to grapple with the notion of integration. Lack of proper training and insufficient teacher knowledge seem to be the challenging factors for teachers to navigate successfully through the notion of integrated teaching and learning. Drawing from the theory of situated learning, this paper argues that although integration between mathematics and Arts and Culture is desirable in teaching and learning, it is problematic in practice. The analysis from this study raises important pedagogical issues about the link between 'integrated teaching' and 'teacher training-and-content knowledge'.

Introduction and contextual background

Curriculum reforms are taking place in many countries across the world. In a South African context, the se reforms have meant that schooling shift from following a structured schedule of study characterized by a strong separation between bodies of knowledge (Snyder, 2000), to an interdisciplinary approach of instruction. The introduction of Outcome Based Education (OBE, Department of Education, 2003a) has placed demands on teachers to adopt new styles of teaching, with integrated approaches being central. The notion of integration has permeated and become synonymous with the new curriculum such that teachers are encouraged to organise their teaching in ways that promote integration of one learning area with another. There is a belief that integration can foster stronger working relationships not only amongst educators, but also amongst learning areas that are being taught. So intergration seems to be key in understanding learning systems being framed in the new ways of working in the South African curriculum. When teachers collaborate across disciplines, as Jennifer Stepanek (2002) argues, they gain new insights and new ways of approaching familiar and often complex subject matter. However, implementing the new curriculum has been marked with complexities and criticisms, with others raising concerns that teachers are not adequately trained to handle new curriculum demands. It is realised that this could be due to the fact historically, they are educators who are certified in specific disciplines. Therefore, it is both ambitious and unrealistic to expect them to posses knowledge that can enable them to navigate integratively effectively across subjects (Czerniak, Webber, Sundmann and Ahern, 1999).

These concerns, in addition to claims that integration might mean different things to different teachers (Davison, Miller and Methey, 1995), make integration a seemingly complex teaching innovation to comprehend and also implement in practice. Regarding integration, Adler, Graven and Pournara (2000) have noted that 'the teacher is expected to possess a broad general knowledge of his or her subject matter and possibly also to be an expert in other subject areas. This is clearly seldom possible and might leave the teacher feeling powerless to cope with the new demands' (p. 6). The challenge still remains for teachers to incorporate and articulate new pedagogies that emphasize integration within and across disciplines. From this background, it becomes clear that there is a challenge for Arts and Culture teachers to incorporate new pedagogical approaches that emphasize the need to integrate mathematics and Arts and Culture.

This paper reports on a study that involved two Grade 9 Arts and Culture teachers in South Africa. The aim of the study was to establish how these teachers deal with situations in which they are called upon to reflect on their mathematical knowledge while teaching their Arts and Culture curriculum (DoE, 2003b). This paper reports on aspects of the following research question of the study:

> Within the context of the new South African curriculum, what connections do Grade 9 Arts & Culture teachers make between mathematical concepts and concepts in an Arts & Culture topic?

It was important for this study to focus on links between mathematics and Arts and Culture because most of the research reviewed (for example, Davison, et al., 1995; Huntley, 1999; Lyublinskaya, 2006) have placed more emphasis on exploring integration between mathematics and science, than on mathematics with other learning areas.

Literature review and theoretical framework

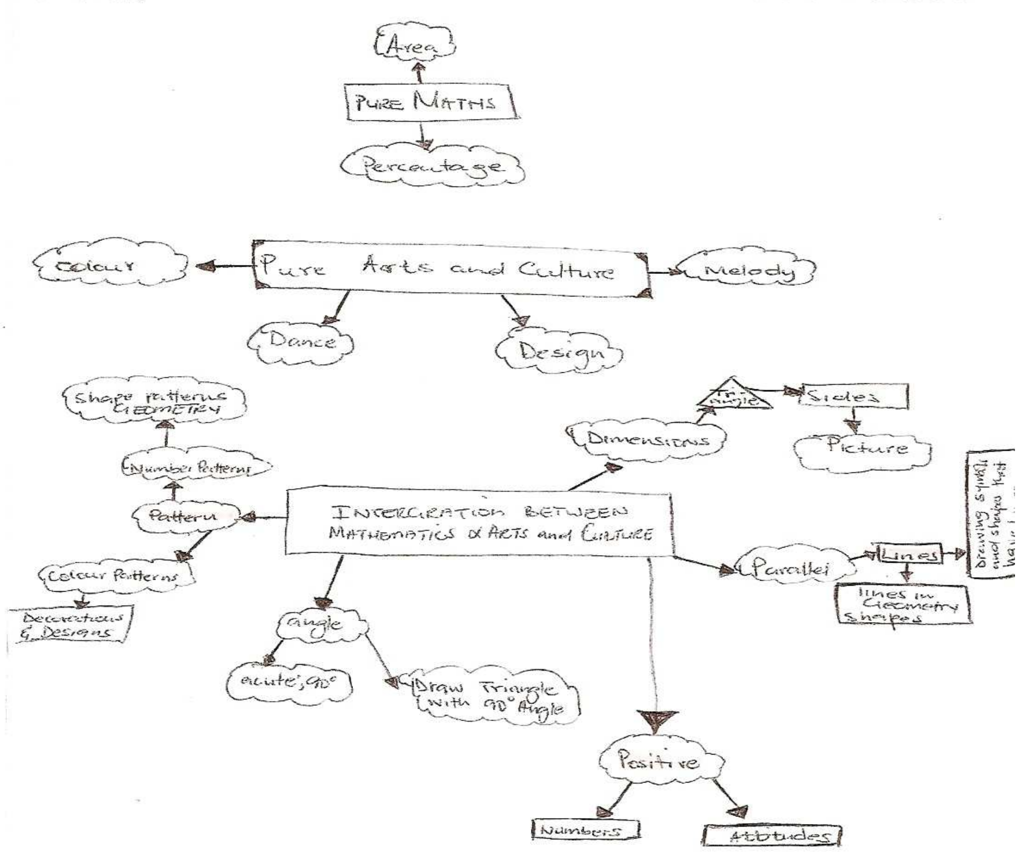
Interest in this study was initiated by the realization that integration is placed as a fundamental aspect of the new progressive South African curriculum (Adler, et al., 2000). In South Africa, when the traditional curriculum was replaced by Curriculum 2005, the OBE-oriented approach, it became clear that teachers had to familiarize themselves with new pedagogical approaches to teaching and learning. A wide range of new concepts was introduced to the system of education, with integration being the most popularised. Few years after its inception, Curriculum 2005 was reviewed and most of its ‘design features’ (Adler, et al., 2000) were removed. However, integration continued to remain the key feature of the subsequent revised version. Teachers, as they enact the new curriculum, are expected to implement integrated teaching in their lessons. However, as already alluded, there is a concern that teachers are not well oriented to deal with the demands of integrating across disciplines (Czerniak, et al., 1999; Adler, et al., 2000; Huntley, 1999). This raises serious pedagogical concerns, particularly in view of the fact that substantive teacher content knowledge is a prerequisite in order to facilitate connections between disciplines (Huntley, 1999). This study was therefore designed to provide space to investigate how teachers worked with these new pedagogical demands for integrating across subjects. The focus here was on how Grade 9 Arts and Culture teachers are able to make connections between concepts that are embedded in the two subjects, mathematics and Arts and Culture.

It was believed that the findings of this study would be critical in helping curriculum designers to heighten the awareness of the need to train teachers and also encourage them to integrate across subjects. Some of the literature on integration has claimed that there is a ‘historical lineage of connections between mathematics and Arts and Culture’ (Beckmann, Michelsen and Sriraman, 2005). The two learning fields are historically inter-connected, with Arts and Culture providing possibilities to visualize mathematical thinking and expressing mathematical thoughts that are possibly complex to comprehend theoretically. Mathematics, on the other hand, can contribute to the solution of significant unresolved cultural problems, for instance, global birth control and epidemic control (Sriraman, 2005). It is therefore significant that these issues are brought to the awareness of the Arts and Culture teachers with the view that their awareness might stimulate them to make important and relevant connections. Teachers need to realize that there are various opportunities that provide possibilities for connecting mathematics and Arts and Culture, and that they can explore these opportunities in order to enhance their professional experiences of integrated teaching and learning. Arts and Culture is a critical learning area within the South African school setting. Learning Arts and Culture can assist in liberating students’ potentials to do well in subjects such as mathematics. According to the Department of Education (2001), ‘liberating the imagination, [is] a first step in the creative process, and the expression of culture, is a skill, a goal in itself which ranks in importance with mastery of numbers and natural laws in the school setting’ (p. 9).

The theory of situated learning (Lave and Wanger, 1991) was used to frame this study. This theory is founded on the premise that knowledge is situated, and is a product of the activity, context and culture in which it is developed and used (Brown, Colllins and Duguid, 1989). This theory presupposes that one learns differently in different situations, so learning is situated within a context. The key issue here is that the ‘development of knowledge, and how it is later applied, is situated within a context’, that is, the development and formation of identity are both tied to the setting in which they are acquired (Adler, et al., 2000). Subsequent to this view, Adler, et al. argue that the transference of knowledge from one setting (context) to another is always problematic because ‘knowledge and skills cannot be neatly lifted out of one setting and imported ready-to-use into a new setting’ (p. 11). The researcher has used this theoretical framework to highlight the fact that implementing the notion of integration, in teaching and learning settings, is potentially problematic as this process involves the sharing and transfer of knowledge from one context to the other. This view is strengthened by the realization that mathematics and Arts and Culture present two different learning contexts that are constituted with different learning activities and culture. Furthermore, how one conceives and embeds concepts from one discipline to another depends on one’s understanding of the possibilities of connections that are available. Such a conception is likely to be subjective and highly dependent on the nature of the contexts involved.

Study design and data collection

A qualitative descriptive research methodology was used, and because qualitative research is inherently multi-method in focus (Naidoo and Parker, 2005), the researcher was able to collect data through a concept mapping activity and subsequent interviews which were administered on the participating teachers. A concept mapping activity was fore-grounded because concept maps are quicker, more direct and considerably less verbal (White and Dunstone, 1992). The concept mapping task which was administered on teachers involved the following concepts, which were drawn from a Grade 9 Arts and Culture textbook, Millennium Arts and Culture Grade 9 Learner’s Book: angle; area; colour; dance; design; dimension; melody; parallel; pattern; percentage and positive. Teachers were asked to identify concepts which they perceived as ‘pure mathematics’, ‘pure Arts and Culture’ and those they considered to be integrating between mathematics and Arts and Culture (parallel concepts). After this activity both teachers were interviewed based on the results of their concept mapping activities.



The Arts & Culture teacher's Concept map

Results and observations

The observations in this paper are based on an analysis of the concept map constructed by one Arts & Culture teacher. The teacher was teaching Arts and Culture on a fulltime basis and had teaching experience of approximately 15 years. The teacher indicated that he was quite familiar with both versions of the curricula, the traditional and the new revised curriculum. He acknowledged having attended OBE training workshops arranged by the Department of Education and, at other times, by non-governmental organizations. The figure below shows the concept map drawn by the Arts and Culture teacher.

As can be seen from figure above, there are three components in the map: it shows concepts that seem to belong to the top part "Pure Maths"; secondly, those that are "pure Arts and Culture" concepts, and those that "Integrate between mathematics and Arts & Culture". The teacher felt that the concepts *colour*, *melody*, *dance* and *design* were pure Arts and Culture concepts, and so could not be linked to mathematics. This is interesting, particularly in light of Graumann's (2005) comments below in relation to the connection between mathematics and music:

Rhythm and notation is a relatively simple mathematical field, which provides a good opportunity for application-oriented practice of fractions. The determination of pitches and scales respectively tunes is a big chapter in which the development in the theory music from Pythagoras to twelve-tone music can be opened up by mathematics. In this context, the ancient theory of music can serve as a field of application for fractions as well.

The absence of connections between these concepts and mathematics was further confirmed in the interview with the Arts and Culture teacher when he repeatedly stressed that the concepts *colour* and *music* could not be linked to mathematics. However he was passionate about integration and acknowledged that "teachers should be trained on integration" because some of her colleagues were "still struggling with integration". He noted that teachers sometimes relied on other teachers as well as learners for assistance with integration. According to him, "the learners will definitely help you, you can give them a problem, they will give you answers that you did not expect, they will integrate". He identified 'area' and 'percentage' as the only concepts that can be linked to mathematics. During the interview, he insisted that 'percentage' is a mathematics concept. He stated: "In Arts I don't talk about percentage".

The teacher noted that integration is encouraged in curriculum documents through the statement of Learning Outcomes (LO). He acknowledged that some of the LOs encouraged the integration of mathematics with Arts and Culture. He particularly quoted Assessment Standard 9 in the Grade 9 Arts and Culture Curriculum in which the concepts 'positive' and 'negative' are mentioned, namely that there are "positive and negative effects of television, radio, documentaries or films on our lives" (DoE, 2003). He acknowledged that such topics are likely to stimulate mathematical discussions, thus opening up opportunities for connections between the two subjects. He noted that learners can "learn two things at the same time" when such topics emerge during lessons.

Overall, this activity revealed interesting observations. Teachers acknowledged that integration had replaced the traditional style of teaching. This was made evident by the fact that both teachers had clustered concepts on the integration category of their map, thus revealing that they regarded most concepts as integrating. However, their actual teaching practices were not in line with this acknowledgement, as was revealed later in the interviews with the teachers. There were areas where it became evident that teachers do not foster a spirit of non-coercive collaborative work. For instance, teachers' views differed on certain concepts such as 'percentage', 'area' and 'pattern'. In some cases teachers were not sure in which category they needed to place a concept, i.e. whether the concept belonged to mathematics or to Arts & Culture. Interviews with the teachers also revealed that teachers acknowledged links between mathematics and Arts and Culture, they provided numerous examples to support this claim. However, both teachers acknowledged that they had little knowledge of mathematics, and consequently struggled to incorporate mathematics in their instruction. As a result of this lack of mathematical knowledge they either conduct their teaching in traditional ways or rely on students to assist them with integration. There was a strong emphasis on the fact that teachers relied on their students for integration to be feasible. In fact one of the teachers insisted: "these kids help us many times, they know more maths than us. We prefer them than teachers". From this view it was evident that teachers regarded themselves as incapable of implementing the notion of integration properly, they regarded students as being more knowledgeable on integration than them. They strongly insisted that they preferred working with students than working with other teacher colleagues.

Conclusion

This qualitative case study has provided us with knowledge regarding challenges that teachers face as they attempt to integrate across learning fields. Teacher-knowledge is a critical area that requires immediate attention. The findings from this study concur with Huntley's (1999) observations whereby it is noted that:

> Teachers are positive about integration, however, they do not posses the necessary knowledge and expertise to properly enact this teaching innovative in their classerooms. These gaps in their knowledge limit the extent in which teachers explore inter-disciplinary connections. Teacher knowledge is a prerequisite to making connections between subject fields.

The fact that most of the teachers were trained and socialized to be subject specialists (Czerniak, et al., 1999), and in the process not possessing general knowledge to other subjects, also emerged in this study.

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Mathematical Competitions for University Students

Alexander Domoshnitsky, Dr.
Professor of Mathematics,
Dean of Faculty of Natural Science
Ariel University Center, Ariel, Israel
adom@ariel.ac.il

Roman Yavich, Dr.
PhD on Education,
Ariel University Center, Ariel, Israel
romany@ariel.ac.il

Abstract

We present several possible forms of mathematical competitions for University students. One of them is Blitz Mathematical Olympiad. It is a team competition, when all teams receive the same problem and are allotted 10-15 minutes to come up with a solution. This cycle is repeated 6-8 times with different problems. Modern Internet technologies allow us to organize Blitz Mathematical Olympiads for the teams which are in different cities and even countries.

Introduction

Lecturers teaching mathematical courses for students, whose main specialization is not mathematics, can confirm that almost all their efforts and time (except lectures of course) are invested only on the weakest students in order to help them survive in academic institutes. Hardly anything is done for average or even relatively strong students, whose level cannot be defined as an excellent one, but they understand the basic things in mathematics and, in corresponding sense, they even like mathematics. In the system of high education there are almost no developed forms of the work with good students, although namely these students will become the kernel of specialists working in industry, education, hi-tech and even in science. Many enthusiastic lecturers tried to find such forms. Scientific circles for students and the institute Math Olympiads present only a short list of ideas in this area, but all of these forms require a lot of time and efforts of their organizers. However, only a very small part of students was involved in this activity. The scientific circles were very similar to additional courses and were usually stopped when the real exams approached. Only a few students participate in various Mathematical Olympiads.

Several Notes about Classical Mathematical Olympiads

Mathematical competitions among students have two main goals. The first goal is just like any other competition - to discover the strongest competitors. The second is to enhance interest in mathematics. The first aim is quite achievable, however the other goal, is far less attainable. Mathematical competitions in the classical form of the exam is not, in our opinion, the best way to incite the students' interest. Although we invest a lot of time and efforts in choosing suitable competitors, we find that students, who lose in the competition, lose their confidence. As a result, these students are reluctant to participate in future competitions and are left out of our organizing efforts instead of getting additional motivation in studying mathematics, which leads to serious psychological problems. If we want a mass of students to participate in mathematical competitions we have to choose their forms in a way that the psychological problems will lessen and students will mainly enjoy this competitions.

How can we achieve it? Let us start with the recognition that the classical mathematical Olympiad in the form of an exam is a game! This fact is not accepted even by many specialists, but that is a game! As a result of this recognition, we make the following conclusion: to make the mathematical competitions more attractive for a mass of students, we have to strengthen their game component.

Notes about Games in the Education Process

Games play an important role in child's development: through games, children obtain information on the world at large. In kindergarten and, to some extent, in elementary school, games are used to teach languages and science. However, teaching methods that use games have disappeared from use in high schools and in institutions of higher education, despite the fact that even at these ages, games can help

learners learn rapidly and with ease.
Our goal is to construct models, which could be used to involve school children and students in educational games. In all games, learners solve mathematical problems in their free time. This is clear to the teacher, but the activities are presented to students as a game.

The Blitz Mathematical Olympiad
One of the advantages of team competition is that nobody takes a full responsibility for the team's loss. This essentially solves negative psychological problems, which appeared in the classical individual competition.
The scenario of Blitz Mathematical Olympiad can be described as follows. All teams receive the same problem and have to come up with a solution. In order to score points, teams must submit the solution to the problem within the allotted time (10-15 minutes). The solutions are immediately checked by the panel of judges, who immediately announce the results to the participants and to the audience. All of this is repeated with the next problems. Dynamics of Blitz Mathematical Olympiad is one of the main principles. We inform participants about their results after solving each problem, rather than after solving all problems, which allows everyone to track the teams' score in real time.

The principle of scoring can be, for example, the following. The number of points for solved problem is inversely proportional to the number of teams who solved this problem correct. Consider, for example, the case of 4 teams. Suppose every problem is worth 12 points. In the situation that all of the teams solved correct, each team gets 3 points. If three teams solved correct, then each of them gets 4 points and the fours team get 0 points. If two teams solved correct, each of them gets 6 points, and the other two teams - 0 points. In the case of only one right solution, this team gets 12 points and the other three - 0 points. It is important that the order of problems is from simple to more complicated problems. This usually allows us to hold non evidence of the final result till the last problems.

Another option is to propose an alternative scoring method. Every team starts with the same number of participants, for example, five. If a team solved a problem correct, one of its members can go out from the game. If after solving several problems correctly, a team has "disappeared" (all members went out of game), it means that this team won Blitz Mathematical Olympiad.
Modern Internet technologies allow us to organize Blitz Mathematical Olympiads when the teams stay at their Universities in different cities and even different countries.

Our Experience
Blitz Mathematical Olympiad was first organized in Perm Polytechnic Institute, Perm, Soviet Union in the 1980s. Blitz Mathematical Olympiads were regularly organized for high school students in the Department of Youth Activities in the Technion, Haifa, Israel in 1993-1999. In November 2008 the First International Blitz Mathematical Olympiads for University students from Russia, Romania and Israel was successfully organized in Ariel University Center, Israel.
As an example, how the problems of Blitz Mathematical Olympiad can be chosen, we present the problems from one of our competitions.

Problems of Blitz Mathematical Olympiad on November 18, 2008

1. Two friends have not seen each other for many years. When they met, they talked about their family status.
"I am father of 3 children"- said one friend.

"How old are they?" – asked the other friend.
"The product of their ages equals 36; the sum of their ages equals the number of the bus that has just passed by!"
The other friend looked at the bus number and said: "there is not enough data to find out their ages!"
"My eldest son is blonde," – said the first friend.
"Then there is no problem,"- said the second one.
"They are ?, ?, ? years old."
What was his answer?
2. Let $\{a_1, a_2, a_3, \dots, a_{100}\}$ be an geometric sequence
(a) Is it possible that all elements except a_{100} are integers?
(b) Is it possible that all elements except a_{50} are integers?
3. Post Airplane

Every day at the same time a post airplane arrives at the airport. At the same time a car from the central post office arrives to the airport to get the mail. One time, the airplane came early and its luggage was put on a horse carriage, which was on its way to the central post office. Half an hour after leaving the airport, the carriage came across with the post car, which left the post office as usual, and was on its way to the airport. Somebody put the mail from the carriage on the car, which drove back to the post office and arrived there 20 minutes earlier than usual. How early did the airplane arrive at the airport (don't take into account the time of unpacking the mail and putting the luggage on the car)?

4. Find all the integer solutions of the following equation:

$$y^5 = x^2 + x$$

5. Compute the determinant: $\left|\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 0 \\ 4 & 5 & 6 & 0 & 0 \\ 5 & 6 & 0 & 0 & 0 \end{pmatrix}\right|$

6. A sequence is given by two first elements $a_0 = a_1 = 1$,

and $a_{n+1} = \dfrac{a_n a_{n-1}}{\sqrt{a_n^2 + a_{n-1}^2}}$ for $n > 1$.

Compute $\lim\limits_{n\to\infty} \dfrac{a_n}{a_{n+1}}$

7. Is it true that for any polynomial $p(x)$ with real coefficients there is a polynomial $q(x)$ with real coefficients such that $q(x^3)$ is divisible by $p(x)$?
8. A chessboard is given, the length of a side of each small square is 1.
Compute the sum of areas of all rectangles, whose 4 sides go along the lines of the chessboard.
Note that the majority of these problems are based on the problems written by Prof. Alexei Kannel-Belov and Lev Radzivilovsky, coaches of the Israeli student team on mathematics, while several other problems were taken from the known Russian mathematical journal "Kvant" and from the book "Puzzle-head" by Ben-Zion Erez.

Internet Mathematical Olympiads

Alexander Domoshnitsky, Dr.
Professor of Mathematics,
Dean of Faculty of Natural Science
Ariel University Center,
Ariel, Israel
adom@ariel.ac.il

Roman Yavich, Dr.
PhD on Education,
Ariel University Center,
Ariel, Israel
romany@ariel.ac.il

Abstract
Modern Internet technologies open new possibilities in a wide spectrum of traditional methods, used in mathematical education. One of the areas, where these technologies can be efficiently used, is an organization of mathematical competitions. Contestants can stay in their schools or universities in different cities and even different countries and try to solve as many mathematical problems as possible and then submit their solutions to organizers through the Internet. Simple Internet technologies supply audio and video connection between participants and organizers in a time of the competitions.

Introduction
One of the main problems in the organization of National and International mathematical Olympiads is their expensiveness for potential participants. Team's organizer has to find a corresponding found or a sponsor, which can support a team, in order to bring his students to different cities or even different countries and to organize their accommodation there. Note also that all money collected by team's organizers is actually passed to tourism companies rather than the persons who prepared teams to competitions for many months.
Innovative ideas based on the use of Internet technologies can essentially change all this situation. The proposed model of Internet Mathematical Olympiad is very cheap and convenient for participants and organizers.

Description of Realization of Internet Mathematical Olympiad
Problems are posted on a dedicated area in the website for 3-4 hours. Contestants can stay in their schools or universities in different cities and even different countries and try to solve as many mathematical problems as possible and then submit their solutions to organizers through the Internet.
Organizers arrange a broadcast to all participants through standard programs. Participants and specialists, who are interested in the competition, can also watch the opening and closing ceremonies of the Internet Mathematical Olympiads. All this strengthens the effect of presence in order to influence the motivation of students to study non-standard mathematical problems. Our main idea is to attract as many students as possible for this activity, and one of our aims is for students to enjoy it.

To Lessen Negative Psychological Effects of Losers
In every mathematical competition, corresponding psychological problems can appear. Although we invest a lot of time and efforts in choosing suitable competitors, we find that students, who lose in the competition, lose their confidence. As a result, these students do not want to participate in future competitions and are left out of our organizing efforts instead of getting additional motivation in studying mathematics, which can lead even to corresponding psychological problems. In order to lessen negative effects of losers, we propose two ideas.
The first one is that only the names of the best contestants in the top of the winner's list are published. The second one is that the list of problems should be long enough.

Our main principle in choosing problems is that they have to be interesting to solve or at least to try to be solved by as great a number of students as possible.

Grading System

The grading system is based on the principle that the points for each problem are graded according to its rate, which is inversely proportional to the number of students who solved the problem correctly. As a result, simple problems are able to bring only a small number of points. This avoids essential influence of simple problems on the final result. It is a possible situation, when the absolute winner solved less problems correctly than several other contestants, but among the problems he/she solved were problems with a higher rating. This factor is an element of competition game, what gives students of schools or first year, who know less, a chance to win. That way they can prove their creativity in solving nonstandard problem.

Our Experience

We started Internet Mathematical Olympiads for Israeli students in 2006. Since 2007 our Internet Olympiad has become international. Students from 14 countries participated in our Olympiad in December 2008. More than 35000 enters to the site of our Olympiad demonstrates a wide interest of students in our project. As an example, how the problems of the Internet Mathematical Olympiad can be chosen, we present the problems from one of our competitions.

Problems of the Internet Math Olympiad, March 19, 2008

Problem 1.

Calculate the following limits:

a) $\lim\limits_{n\to\infty}\sum\limits_{k=n}^{2n}\frac{1}{k!}$, b) $\lim\limits_{n\to\infty}\sum\limits_{k=n}^{2n}\frac{1}{k}$.

Problem 2.

Prove that for any 9 interior points of a cube whose sides equal to 1, at least two of them can be chosen such that the distance between them does not exceed $\frac{\sqrt{3}}{2}$.

Problem 3.

a) Calculate the sum of the infinite series $\sum\limits_{n=1}^{\infty}\frac{n}{(n+1)!}$,

b) Find the following limit: $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}\int\limits_{1}^{n}\ln\left(1+\frac{1}{\sqrt{x}}\right)dx$.

Problem 4.

Prove that

$$\int\limits_{0}^{\sqrt{2\pi}}\sin\left(x^2\right)dx > 0$$

Problem 5.

Prove that for any polynomial p(x) of the degree n and any point Q the number of tangents to its graph which pass through the point Q does not exceed n.

Problem 6.

Let A,B,C,D be four distinct spheres in a space. Suppose the spheres A and B intersect along a circle which belongs to some plane P, the spheres B and C intersect along a circle which belongs to some plane Q, the spheres C and D intersect along a circle which belongs to some plane S and the spheres D and A intersect along a circle which belongs to some plane T. Prove that the planes P,Q,S and T are either parallel to the same line or have a common point.

Problem 7.

For a square matrix A denote

$$\sin A = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \frac{A^9}{9!} - ...$$

a) Prove, that if the matrix A is symmetric $A = A^T$, then all elements of the matrix $\sin A$ belongs to the segment [-1,1].
b) Is the above assertion true for non-symmetric matrix A?

Problem 8.

All the position of a cellular tape are numerated by the numbers 0,1,2,3,… and in some of them one or more game pieces can be placed. Our moves are determined by the following rules:

1) If in all of the positions whose numbers are $n \geq 1$ there is no more than 1 game piece in each, we add 2 game pieces into position number 1.

2) Otherwise, the position $n \geq 1$ with the maximal number from all the positions which have at least two game pieces is chosen, and then 2 game pieces are moved from this position in two opposite directions: one of them is moved from the chosen position n to the position n-k and another game piece is moved from the chosen position n to the position n+k, where k is an integer number $(1 \leq k \leq n)$. This number k can be chosen arbitrary for each move.

What is the maximal number of moves that can be made so that no game piece will be in the positions with numbers greater than 2008?

Problem 9.

A matrix A 2008×2008 is given. All its elements equal 0 or 1. Assume that every two lines differ from each other in a half of the positions. Prove that every two columns in this case also differ in a half of the positions.

Problem 10.

Let $\frac{\alpha_n}{\beta_n}$ be an irreducible fraction of the form $\frac{\alpha_n}{\beta_n} = \sum_{k=1}^{n} \frac{1}{k}$

Let us call the prime number p a good number if it is a divisor of α_n for a some n. Prove that the set of all good numbers p is infinite.

Note that the majority of these problems are based on the problems written by Prof. Alexei Kannel-Belov and Lev Radzivilovsky, coaches of the Israeli student team on mathematics.

Essential Ingredients that form the basis for Mathematical Learning: What has 20 years of teaching mathematics to teenagers taught me?

Ruth J. Duffield, BEd (Bachelor of Education) University of Melbourne
Mathematics Teacher/Numeracy Coach, Deer Park Secondary College
Victoria, Australia duffield.ruth.j@edumail.vic.gov.au

Abstract
Educators strive to improve student learning outcomes and there are numerous theories suggesting how this is best achieved. However, application of these theories to the coal face of a classroom is often fraught with obstacles resulting in poor outcomes. Constraints imposed by educational policy, school systems, structures and the individual students themselves, realistically require adaptation of theoretical techniques if genuine learning is to be imparted to students. This paper discusses some of the issues surrounding the practical implementation of new methodologies into the classroom and identifies important factors that affect teenagers in their learning of mathematics.
Working within the constraints, constantly confronted with obstacles, can be frustrating and demoralising. This paper reflects on twenty years of classroom teaching of mathematics to students with relatively poor socio-economic backgrounds and the lessons learnt from them that may assist teachers to remain enthusiastic and creative with the energy to truly improve mathematics education. Key issues explored in the paper include: 'Realities of a teacher's working day', 'The learning of mathematics within a government secondary system', and 'What can be done to ensure mathematical learning takes place?'

Keywords Mathematics, learning, teaching methodology, teenagers

Disclaimer The opinion and comments presented in this paper represent those of the author and do not necessarily reflect the view of the Department of Education and Early Childhood Development in Victoria.

Introduction

Those passionate and dedicated to the development of mathematics education and the teaching/learning of mathematics, are constantly seeking for better ways to engage and teach the students entrusted to them. Our world is not static, new technologies, new things are constantly being discovered and educators must continue to be learners and embrace the changes occurring around them.

The temptation, however, to lose sight of some fundamental realities faced by many teachers in the classroom and to focus on the methodology and learning tools only, is very real. Educators should be wary of not letting the methodology or new technology cloud their perspective on the reality of each individual situation.

There are many obstacles and constraints a teacher faces in attempting to successfully engage and improve student learning. These detract from the optimum teaching/learning outcomes.

By reflecting on personal experience and observation over a period of twenty years of teaching mathematics to teenagers with relatively poor socio-economic backgrounds, this paper discusses some of the practical issues surrounding the implementation of methodologies into the classroom and some of the findings made.

Key factors contributing to the successful teaching/learning of mathematics are identified and strategies for managing and coping with the constraints and obstacles encountered are suggested.

Context

It is important to clarify from the outset that the students and school system referred to in this paper relate largely to what is considered to be an under privileged area in the Western suburbs of the City of Melbourne, Australia. Non professional occupations and relatively high unemployment characterise the socio-economic status of many of the parents. Thus, the situations encountered and discussed do not necessarily reflect the experience of all Victorian Government schools.

Over the last twenty years, many new ideas, educational methods, reporting systems, assessment procedures and overall strategies for teaching mathematics have come and gone. Methodologies and approaches to the teaching and learning of mathematics have constantly changed and many of these changes have been imposed on teachers within the government system.

Teachers have been encouraged to modify their practices and incorporate new teaching tools such as mind maps, graphic organisers and the like into their daily teaching. There have been suggestions to cease using numerical marking schemes and to use rubrics in order to more accurately measure deep learning. Project work involving lengthy reports that summarise and analyse certain situations and problem solving tasks have all been emphasised at different periods of time. Group work and the discovery of theorems/formulas from first principles along with a shift away from simply 'drilling' a skill have all been considered innovative and conducive to learning. In recent times, funding has been made available to those schools who demonstrate the use of integrated projects. The list could go on, and by naming some of the innovations no value judgement is being placed on them, but rather attention is being drawn to the fact that governments and educators over the years are fully aware that there is improvement to be made in the way students are educated and they have therefore been proactive in promoting new alternatives.

Realities of a teacher's working day

There is diversity in the ways schools function. However, there are many common experiences that are shared, e.g. the demands placed on teachers' use of time. Numerous meetings rob valuable time to prepare, correct and provide detailed feedback on student work. Many activities require specific setting up of rooms. These rooms are often being used by others for a different purpose and are therefore not available to be set up in time. Timetabling often places geographic constraints on teachers resulting in them rushing from one end of a school to the other, unable to meet the expected schedule. Carrying equipment (concrete materials are seen as very important especially in catering for the slower learner) to different venues can be hazardous. The preparation of materials and resources in themselves can be incredibly time consuming and expensive.

The use of IT has been assigned high priority in recent years, and students actively engaged in valuable learning activities using such technology is a situation that is highly rated. Unfortunately, the reliability of the technology functioning well is not guaranteed. How many times have teachers encountered the problem of them not working, just when they are needed or students have forgotten their passwords or run out of internet credits? Added to this is the continual frustration of limited access to computers. Often, software that was working beautifully the day before suddenly decides to play up and a teacher is frantically trying to rectify the problem whilst twenty five boisterous students are interjecting with 'helpful' advice or maybe just creating havoc.

So far, some of the physical components that can be the obstacles that hinder the successful implementation of a plan have been mentioned but what about the students themselves? What a collection of different tales enter the room every time a class of students appear.

"Miss, I've got a headache", "Sir, I've lost my book", "Miss, I don't have a calculator", "Miss, I've been away and don't know what to do", "Sir, I have to leave in fifteen minutes...."

So often, activities are made up of different parts that depend on each other so that when one component is missed it creates a missing link for subsequent sections. This becomes an issue for a student who has been absent. Activities often assume an interested, engaged and motivated student entering a classroom expecting to work, concentrate and learn (provided the material is stimulating and creative of course) but so often the reality is different and instead a teacher is confronted with non educational issues that are of much greater importance to the student than the learning of mathematics. Teachers are believers in education and place it high in their order of priorities, however, for many of the students, problems with friendships/relationships, chaos or instability at home, loom as much larger priorities and their emotional and overall mental state as they enter the classrooms impose far greater hurdles and obstacles for learning than teachers can often imagine. It should also be remembered that students, especially teenagers are not necessarily consistent in their receptiveness from one day to the next or even from one hour to the next. Therefore, when new methodologies and ideas are enthusiastically embraced, the many practical constraints that may influence the successful implementation thereof must be carefully taken into consideration.

The learning of Mathematics within the secondary school system

Regardless of all the constraints and obstacles encountered, teachers somehow manage to work around them and adjust as best they can to the situations they find themselves in. However, one particularly disturbing observation has caused a questioning of the emphasis placed on teaching methodology and teaching tools, to emerge.

With all of the improved awareness and consciousness of best practice when it comes to teaching/learning, one would expect significantly improved outcomes in student learning over the years. Sadly, this has not been evident. Instead the school has been confronted with the reality that those who have entered the secondary system with weak mathematics skills rarely show significant improvement in their six years of post primary schooling and leave with equally poor skills. (Note: In Victoria, children begin their formal education at primary school where they complete seven years before entering the secondary system). Students maybe demonstrate some mathematical skills at various times throughout those six years, but these skills are quickly forgotten and real mathematical learning/progress has not been achieved. This experience is supported by recent research that indicates that 80% of students who enter the secondary system with weak mathematics skills (in the poorer regions of Melbourne) continue to struggle with their numeracy skills and don't improve as they move through secondary school.[1]

Conversely, those entering the system with good mathematical skills, continue to develop and make good progress irrespective of the methodology or learning tools encountered.

This is disturbing in that it suggests that mathematical learning, when it concerns teenagers, is not as strongly linked to methodology as many would have liked to think and that there are actually other important factors that contribute to successful learning. It also gives rise to the question of why some students continue to progress while others continue to fall behind even though exposed to the same material.

If important concepts such as ratio/proportion, multiplicative thinking, place value, decimals and fractions along with fundamental number facts (e.g. times tables, grouping of numbers in 10s, i.e. $2 + 8 = 10$) are not in place or haven't been fully understood when students enter the secondary system, their future mathematical learning and development is severely hindered. The lack of these concepts means that there are no sound foundation blocks upon which to build and the whole mathematical learning structure remains shaky. To complicate matters, these students have often already developed a strong sense of failure with enormous 'blockages' towards mathematics learning due to seven years of previous unsuccessful exposure to mathematics. It is like a foreign language shrouded in mystical terms that have no meaning and continue to have no meaning. Avoidance and strong disguising mechanisms that protect these students from embarrassing themselves in front of their peers and also their parents and teachers make it difficult to address these deficiencies openly and so these students blunder their way through the system using different strategies like working hard at obtaining results from others, copying answers, simply modelling off examples without understanding the reasons why certain processes are being applied or simply convincing themselves that all mathematics is stupid and irrelevant and only for 'nerds'. With this mindset firmly ingrained, any activity no matter how exciting or innovative will struggle to positively engage and shift the learning momentum. No wonder, when these students are tested, using internal or external material, and left to their own resources, the results are poor. The temptation is to lay blame with the teachers and to grasp hold of the easy scapegoat 'Methodology' and focus on doing things differently. I.e. there must be something wrong with the way the material is being presented. It is interesting to note, that many of society's revered athletes, reach their level of success and performance through hard work, repetition and practice. E.g. a swimmer will endure hours of endless (seemingly boring) training swimming back and forwards in a swimming pool or a marathon runner will run endless kilometres enduring severe pain. It appears the overriding factor is not in the training model itself but rather with the person committed to a goal. In other words, a person will persevere and push themselves to improve if they see some intrinsic value therein or alternatively have a goal in mind that they are determined to achieve. For mathematics learning to take place, the student must see value in what it is he/she is involved in and also believe in their own ability to be able to succeed. Without these basic ingredients progress will always be hampered.

Improving mathematical learning

Before embarking on the question of what can be done to ensure mathematical learning will take place, the context within which this question is being addressed must be made clear. Mathematical learning can take on many different meanings and one needs to be clear what the aims actually are. Improvement of mathematics education worldwide embraces an enormous range of ability levels and cliental. For example, in aiming to improve mathematics education, the teaching of calculus to every student would not be deemed appropriate, so when working within a general statement like 'Improving mathematics education or mathematical learning' the cliental base needs to be specified since the aims and expectations will not be the same for everyone.

This discussion focuses on the mathematically low skilled teenager with relatively poor socio-economic background that attends a government secondary school in an under privileged area. The above discourse has briefly touched on:

(i) The fact that governments and those involved in education have been and continue to be proactive in improving mathematics education

(ii) Some of the practical constraints faced by teachers in their daily quest to successfully educate

(iii) The observation that the secondary system in its current form is not meeting the needs of the mathematically low skilled teenager as he/she passes through the system

It is at this point that one could be feeling largely deflated and somewhat despondent if working in a similar environment as described above. So how does one rise above the obstacles and remain enthusiastic, energetic and proactive for positive change?

[1] Farrah Tomazin and Carol Nader, 'Poor children less likely to improve through school', *The Age*, 18 April 2009, p.1.

The secret lies in the recognition of the key ingredients that form the basis of mathematical learning. It enables one to see past the actual teaching hazards and obstacles and puts them in a different perspective. Knowing that students can learn well in spite of their relatively poor resources and underprivileged status gives hope in what may often seem a hopeless situation.

So what are these essential ingredients? In the case of mathematics, a student needs:

(i) The fundamental concepts, as mentioned above, e.g. ratio/proportion to be in place. From this basis, further mathematical knowledge can be built. If these concepts have not been understood, sitting in mathematics classes for a further six years will not yield positive results, irrespective of the methodology. It is critical therefore that a student is brought to this realisation and by developing trust and confidence in the teacher has the courage to confront his/her low skills and go right back to the start. Ideally those responsible for the development of curriculum should devise a suitable course that addresses these fundamentals in an appropriate way that is suitable for teenagers as opposed to young children.

(ii) To see intrinsic value and purpose in learning mathematics. If he/she can make important links between the different areas and see how they connect with his/her current experience of the world, then motivation comes naturally.

(iii) To believe in his/her own ability to succeed in mathematics. This presents a huge challenge in that it involves a change of an already established mindset. Years of negative influences need to be undone in order to create a positive frame of mind, open to and keen to understand the language of mathematics.

Once these key ingredients are present, the basis for learning and progress is there and growth will occur with relative ease. Unfortunately these key ingredients require a huge investment in human time. I.e. they cannot be easily rectified by written programs or external mechanisms but rather require intensive human intervention. As stated previously, teachers need time to spend with these students. Teachers need time to thoroughly prepare and provide feedback on student work on a regular frequent basis and to attend to the individual needs of each student. Students feel valued and respond accordingly when they sense there is a genuine interest in their welfare. Governments and educators when addressing the short comings of low achieving students in the secondary system need to resist the temptation to blame the work force or methodology and rather be proactive in channelling significant human resources to those in the classroom.

So what can be done now within the constraints of the systems teachers work in?

Remember at all times that each student is an individual person with their own tale to tell and take the time to develop a positive relationship built on trust. Human beings learn and function best within positive relationships. Never lose sight of the fact that teenagers are potentially fragile beings who can’t be expected to think and function like mature adults, but have the potential to be both delightful and vibrant. Be willing to put everything into context and continue to let the students know both why there is value in learning whatever is being presented and why it is important in today’s society. Always prepare an activity with the question “What is being learnt here?” or “What learning value is there in this activity?” in mind. Consider the cliental before blindly applying someone else’s idea and question the basic assumptions underlying the ‘idea’. Be conscious of the fact that with every student there is the potential to create a spark and love for mathematics but be realistic in acknowledging that not everyone is going to love mathematics.

In an ideal world, there would be endless resources, endless time and opportunity to work with individuals and produce mathematically competent adults across all regions of the world, however, that is not reality. It remains important that teachers strive to improve their teaching strategies, constantly being prepared for change and innovations but at the same time keeping a true and realistic perspective on their own situation and how it can best be improved.

Conclusion

The tools used in the teaching of mathematics and the methodologies practiced by teachers do not stay the same. In twenty years of being in the classroom, many changes have been trialled. Teenagers in the secondary school system will make progress in their mathematical learning, if the concepts of ratio/proportion, multiplicative thinking, place value, decimals and fractions plus fundamental number facts such as times tables and grouping numbers in 10s are in place, and if their mindset is positive and they see purpose and value in the learning of mathematics. Without these key factors, methodologies or new innovative ideas are not likely to have a significant impact on the learning.

Being aware of the conditions under which a student learns, helps identify the appropriate strategies that will best aid the student in the quest for improved learning outcomes and gives renewed energy to a teacher despite the many constraints encountered. There is always a way forward, no matter how small the steps may be, once the direction is known. It is critical to understand and be realistic about each individual circumstance and to always be mindful of the human factor that plays a large role in the education process.

Exploring mathematical identity as a tool for self-reflection amongst pre-service primary school teachers: "I think you have to be able to explain something in about 100 different ways"

Patricia Eaton, Stranmillis University College, Stranmillis Road, Belfast, BT9 5DY, Northern Ireland p.eaton@stran.ac.uk

Maurice OReilly, CASTeL (Centre for the Advancement of Science Teaching and Learning), St Patrick's College, Drumcondra, Dublin 9, Ireland maurice.oreilly@spd.dcu.ie

Abstract

A study of students' mathematical identity was carried out in February 2009 involving participants from two colleges of education, one in Dublin (Republic of Ireland) and one in Belfast (Northern Ireland). All participants were pre-service primary school teachers in the third year of their B.Ed. programme, having chosen to specialize in mathematics. Data was gathered using a questionnaire (with, mainly, open-ended questions) followed by focus groups, involving the same participants, on each campus. This paper considers how students' exploration of their mathematical identity led them to deepen their insight into learning and teaching mathematics. Recommendations are made for how the methods used in this research might be beneficial on a larger scale, in different environments.

Introduction

Much has been written for over a decade on the links between the beliefs about, attitudes towards and self-efficacy in mathematics, on the one hand, and classroom practice, on the other. This discourse has flourished as understandings of the shift from a didactic to a constructivist approach to teaching mathematics have deepened. In this context, how is it possible to value the diversity of student teachers' own experience in learning mathematics as they prepare for their teaching careers?

It has been suggested that student teachers are more likely to teach mathematics in ways in which they were taught (Meredith 1993). To gain insight of this in the case of a particular cohort of students would in itself be sufficient reason to explore their previous experiences of learning mathematics. In the study about to be described, a broader view is taken. Instead of considering only their previous experiences in learning mathematics, student teachers' mathematical identity was examined in some detail. Moreover, participants were encouraged to reflect on how their mathematical identity evolved over time and on how they might draw on it throughout their teaching careers. By 'mathematical identity' is understood the relationship an individual has with mathematics, including knowledge and experiences as well as perceptions of oneself and others (Wenger 1998).

The study and the focus of this paper

The study was carried out in February 2009 involving participants from two colleges of education, one in Dublin (Republic of Ireland) and one in Belfast (Northern Ireland). All participants were pre-service primary school teachers in the third year of their B.Ed. programme, having chosen to specialize in mathematics. Data was gathered using a questionnaire (with, mainly, open-ended questions) followed by focus groups, involving the same participants on each campus, five in Dublin (but only four of whom participated in the focus group) and four in Belfast. The participants' mathematical sophistication was significantly higher than is typical amongst pre-service primary school teachers in Ireland. This afforded the opportunity to explore two mathematically motivated sub-populations in some detail, although, in this paper, no attempt will be made to distinguish between the characteristics of the two groups. Here the focus will be on how student teachers' mathematical identity can be harnessed as a tool for self-reflection and, in particular, how participants re-evaluated the teaching they themselves experienced.

On the questionnaire (which took about thirty minutes to complete), participants were prompted into revealing their mathematical identity by being asked: 'Think about your total experience of mathematics. Tell us about the dominant features that come to mind.' And later: 'Describe some further features of your relationship with mathematics over time.' The discussion in the focus groups (each lasting almost an hour) explored aspects of mathematical identity drawing on the responses to the questionnaires.

How participants re-evaluated the teaching they themselves experienced

In examining their responses, we note that participants are drawn into the interacting narratives of their mathematical identities. Of particular relevance to the current discussion are their own experiences as learners of mathematics and their emerging practice as teachers either in the classroom or as tutors to individuals. We can perceive moments of re-evaluation of these experiences as well as speculation on how participants might teach in future. In this section, participants' contributions (which are indented) are grouped around key ideas, highlighted in **bold**.

Participants engaged enthusiastically in the discussion exploring mathematical identity, **expressing surprise** at being stopped in their tracks to think about it; such exploration was new to them.

> I've never traced back before why I was ever interested in maths and where it actually came from, but I felt when I was doing out the questionnaire that it came from my family and it came from my teachers up through the years, their interest and their like of maths brought me to like maths.
>
> Yeah, and also the questionnaire kind of had, make you stop and think to say that, 'Oh, certain things did really influence me and other things didn't, I don't really pay much attention to.' But it does, like the questionnaire did kind of, made me think.
>
> But it's when you look back, you realize like, 'Well, that's why we did that certain thing' or 'That's when we …' It's kind of I had never really thought about it before.

They readily acknowledged that **satisfaction in mathematics does not arise without effort**. Speaking of her experience of giving private tuition to a student preparing for examinations at the end of secondary school, one participant recalled:

> And she, I've heard her saying, I don't know whether she knows herself, but I've heard her saying, 'Yeah, it took me ages, and then I finally got it.' And you can hear the enjoyment in a person's voice when they say that.

Her colleague emphasized the importance of **perseverance**:

> Just encouraging them to try it out, basically. Like instead of focusing on whether it's right and wrong all the time, get them to try it at least. And just keep trying it.

High expectations of a teacher in the mathematical ability of his/her student are very likely to contribute significantly to the student's appreciation of the subject, as one participant wrote:

> I think that my most influential teacher in maths was my secondary school teacher. Admittedly he wasn't the best maths teachers in the world, but his love of maths was clear and obvious to the class. Perhaps it was his love that rubbed off on me, I'm not sure, but one thing I know is that he always had high expectations for me. He always pushed me to do my best and I felt compelled to live up to these expectations.

Another explained that she came to appreciate later how one of her teachers **supported individual learning**:

> I had two different Maths teachers at second level and one in particular I think really helped my enjoyment & understanding of Maths. I didn't think so at the time as he was a hard taskmaster who gave great importance to his subject and for whom you could never do enough. Looking back on it, I admire how he placed great emphasis on individual learning and individual responsibility. Our Maths lessons were more 'workshop style' than other lessons. We worked independently until a common problem was encountered which he would then explain. If you required personal attention he would sometimes get another pupil to explain to you or do it himself. This meant that you could work at your own pace and level though he gave you a push when required!

The effectiveness of **working cooperatively** at mathematics in their university studies was discussed in both focus groups. Drawing on her experience, one participant is eager to incorporate it in her own teaching:

> And you can encourage them to work together more, now that you know that it works, as opposed to individual working on your own.

Amongst those in the Dublin focus group a discussion arose on the importance of relating mathematics to **practical applications**. This was absent from their experience in secondary school.

> Like if you're doing the practical thing in primary school, it should continue to secondary school because that's the kind of thing that they're used to.
>
> People think that bringing in a practical aspect into secondary school would dumb it down, or people say that all the time. But I don't think it would. … I think that if the practical aspect was in it, that they would be able to bring maths to a higher standard, rather than a lower standard.

This last participant criticized the apparent **disconnected nature of topics** in school mathematics in the light of her third level studies:

> And I found in secondary school that everything was not linked. Nothing was linked in maths, you know. The algebra and the geometry weren't linked, and where there's such a close, a big link between them or the algebra and the complex numbers were two totally different things that you'd never put together like ... But in college, we do. You do put them together then, yeah ... It's so linked, it should be linked.

The Belfast focus group also articulated the **absence of context** in their secondary school mathematics, in contrast to their experience in primary school.

> I remember when we sat there in the class, every day, we'd be like, 'What do you need this for?' And our teacher sort of wouldn't be able to turn round and actually tell us what we needed it for, you know. ... It was part of the syllabus. Had to learn it.
>
> In the primary school there was always a, like a relevant context. … Then you get to secondary school, it's more forget the context, just do the question.

Out of this awareness emerges a conviction that teaching can improve. When **understanding mathematics** is fostered amongst pre-service teachers at third level, it can be imparted to the children they will teach in future.

> I think in third level when we're learning why there's certain proofs and why certain things is the way it is, make us much better teachers because we can actually show the kids why there's a certain formula instead of just, 'Here, learn it off.' So I think it's our understanding of maths has really improved. And I really think it'll benefit the kids we teach.
>
> If they can put across the understanding, that's when they become a good teacher, I think.
>
> I think you have to be able to explain something in about 100 different ways. I think you have to give enough information for the kids to start a question themselves so that you don't give too much information to make it easy, but you give enough information that they can make a start. And then after that, deal with individual problems as they arise because you can't treat everyone the same because they all learn and will grasp ideas at different paces.

Discussion and recommendations

So, what is noteworthy from these narratives? In the study conducted in a period of less than three hours (across the two campuses), we have witnessed pre-service teachers' thoughtful reflections on aspects of their own learning. Their reflections gave rise to recommendations in areas such as the importance of individual learning, emphasis on the practical applications of mathematics, linking topics in the mathematics curriculum that are traditionally kept separate, and the enrichment of the context of mathematics lessons. Participants indicated how their experiences might orientate their own approach to teaching by valuing perseverance, encouraging cooperative study and fostering understanding.

Situating these reflections in a broader setting, we note that Papert (2009, p. 134) makes a strong case for doing and creating to happen in tandem with thinking and understanding, while Egan (2009, p. 150) argues that children make sense of the concrete better when it is tied to underlying abstractions. The general disposition of participants in our study is consonant with the thinking of these two authors. In considering Irish teachers' perspectives on mathematics, a high emphasis on memorising of formulae and procedures as opposed to understanding how mathematics is used in the real world has been reported (Lyons et al., 2003, chap. 9). Our participants appear to challenge this prevailing orthodoxy.

There are many factors that have a potential bearing on the future mathematics teaching of pre-service teachers. The authors consider the importance of the influence of others, such as teachers and family members, elsewhere (Eaton and OReilly, 2009a). Here we are making a case for exploring mathematical identity through narrative for re-evaluating pre-service teachers' own experience of being taught mathematics. We believe the approach adopted in this study can be gainfully applied to other groups, for example, to larger groups of pre-service primary teachers who do not necessarily specialize in mathematics, to pre-service post-primary (secondary) school teachers and to practicing teachers, at both levels, as part of continuing professional development (CPD). It would be fruitful to spend sixty to ninety minutes, in two sessions, the first prompting participants to focus on their mathematical identity by completing a questionnaire, the second exploring, through discussion, issues which arise from the questionnaire including re-evaluating the teaching they experienced. Depending on the group of teachers or students involved, the questionnaire might be informed by appropriate existing research such as Smith (2006) or Kaasila (2007) for pre-service teachers, and Back et al. (2008) for CPD.

Conclusion

So the process of using narrative to draw students' attention to their own mathematical identity is a very valuable one. From the experience of this study, we maintain it is an efficient method for drawing attention to the bigger picture of teaching mathematics. Such reflection is not the norm amongst pre-service teachers in (either part of) Ireland. To remember how they were taught, to discuss the memories and in doing so to tease out and distil important issues in the complex process of how they learned mathematics, will bring greater awareness to the professional practice which lies ahead of them. This exercise is also of value to lecturers in getting to know more intimately the formative context of students, and so lead students more effectively to a deeper understanding of mathematics, its learning and its teaching.

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Recognising Torres Strait Islander Women's Knowledges in their Children's Mathematics Education

Bronwyn Ewing
Faculty of Education, Queensland University of Technology, Brisbane, Australia
bf.ewing@qut.edu.au

Abstract
This paper discusses women's involvement in their children's mathematics education. It does, where possible, focus Torres Strait Islander women who share the aspirations of Aborginal communities around Australia. That is, they are keen for their children to receive an education that provides them with opportunities for their present and future lives. They are also keen to have their cultures' child learning practices recognised and respected within mainstream education. This recognition has some way to go with the language of instruction in schools written to English conventions, decontextualised and disconnected to the students' culture, Community and home language.

Introduction

This discussion paper is the first attempt by the author to put into words her early learnings and understandings of Torres Strait Islander women's involvement in their children's mathematics education. She is not at the same state of awareness nor understanding of Aboriginal and Torres Strait Island Peoples in Australia and therefore does not consider herself an "expert" on their ways of "Being-Knowing-Doing" as described by Veronica Arbon (2008, p. 29). To do this would be offensive and a substantial breach of trust and respect to purport to be an expert about such matters that she has not experienced. What she is attempting to do is to take small steps to learn about Torres Strait Islander women's involvement in their children mathematics education in the context of the Torres Strait Islands so as to work with this community in environments for mathematics learning. She is non-Indigenous, of Scottish/Irish Catholic heritage, a university educator of Early Childhood mathematics and researcher working with Aboriginal and Torres Strait Island Communities on ways to enhance the educational opportunities of their young people.

As a beginning point, the paper provides a general overview of some important features related to how women conceptualise their role in their children's mathematics education. It then discusses Torres Strait Island home languages and the learning of mathematics using formal mathematics language where possible in the context of Torres Strait Islanders' community and culture. However, an important caveat is needed here before progressing further with this discussion. The author recognises the term *Indigenous* as problematic because it collectivizes distinct populations of people whose experiences have been vastly different under imperialism (Smith, 1999). There is no disrespect intended where this term has been used.

Sharing Aspirations for Their Children

Torres Strait Islander parents share the aspirations of Aboriginal Communities around Australia, that is, they are keen for their children to receive a good education, one that includes literacy and numeracy (Schnukal, 2002, 2003; Mette Morrison, personal communication). Whilst there is literature that focuses on education in the Torres Strait Islands (see Schnukal, 2003 for comprehensive bibliography of Torres Strait Education) and women in the Torres Strait Islands (see for example Gaffney, 1989; Osborne, 1997) literature that focuses explicitly on the involvement of women in their children's mathematics education in the Torres Strait Islands is limited. Because of this limitation, the paper will explore beyond this region to develop understandings of how women conceptualise their role in their children's mathematics education and its associated language. It will also seek explanations of "both ways" environments as describe by Kathryn Priest et al., (2009) and Veronica Arbon (2008). Briefly, both ways is "where there is a blend of mainstream and Indigenous cultural knowledge being taught" (Priest et al., 2009, p. 118). This understanding will be addressed more fully later in this paper. But first important questions need to be posed.

How does a holistic definition of mainstream education accord with Indigenous Australian contexts? How does mainstream education accord with Australian Aboriginal learning systems as described by Karen Martin (2007, p. 18) given that Indigenous cultures are not homogeneous (Priest, 2005)? How does mainstream education acknowledge the influence of parents, extended family, Elders and community? These important questions are also raised by Canadian Indigenous people (Assembly of First Nations, 2005) who are calling for learning systems that are holistic, that is, culturally relevant regulations and curriculum.

> cultural values, beliefs, traditions and language must be interwoven in all early learning and child care programming. Culture has been acknowledged to play a key role in developing physically and emotionally healthy children with high self esteem that it must become an integral component of the everyday operation of these programs. First Nations clearly stated that Elders need to be involved as advisors and teachers in the development and implementation of First Nations early learning and child care programs. (p. 10)

McTurk, Nutton, Lea, Robinson and Carapetis (2008) highlight in their report of *The School Readiness of Australian Indigenous Children* the heterogeneity of Indigenous cultures. Indigenous people live across different geographical locations and live different lifestyles in communities, “awareness and understanding of the complex and delicate nature of the social and cultural issues at play within and between these communities is critical” (Clancy & Simpson, 2002, p. 54-55) if both ways education is going to work. What is similar however, is that they share similar aspirations for their children (Yunupingu, 1997; Mellor & Corrigan, 2004). Lester (2004, cited by Priest, 2005) emphasises these aspirations stating that indigenous families want their children to access quality education so that they can gain the knowledge, skills and capacity to succeed in education, employment and in their present and future lives. However, this does not mean that they give up their cultural identity. They do not. Indigenous parents see as paramount that their children’s cultural identity as an Indigenous person is sustained and maintained (Lester, 2004 cited by Priest, 2005).

Mainstream Education and the Recognition of Cultural Identity

Veronica Arbon (2008) questions the assumptions underpinning Western mainstream education as beneficial for Aboriginal and Torres Strait Islander people which assumes that it enables them to better participate in Australian society. She asks “how de we best achieve outcomes for and with Indigenous people conducive to our cultural, physical and economic sustainability as defined by us from Indigenous knowledge positions?” (p. 118). How does a mainstream education written to English conventions provide children with the knowledge and skills to participate in daily social life, if it does not recognise the cultural identity of Indigenous children as it should (Priest, 2005)? How can the over reliance on applying narrowly defined Euro American westernised ways of thinking about children’s learning be challenged? Priest (2005 cf. Fasoli, 2004) and Arbon (2008) state that this view is now brought into question with calls for both ways education where mainstream knowledge and practices is blended with Indigenous cultural knowledge of learning. Taylor (2003) explains this further by stating that both ways education must work within an “intercultural space” (p. 45). That is,

> . . . the meeting of two distinct cultures’ through processes and interactions which retain the integrity and difference of both cultures and which may involve a blending of elements of both cultures but never the domination of one over another. (p. 45)

It is crucial therefore that cultural knowledges and experiences of Indigenous people to be valued and respected and given the currency in the same way that non Indigenous knowledge is (Taylor, 2003) for both ways education to work.

The document *Preparing the Ground for Partnership* (Priest, 2005), *The Indigenous Education Strategic Directions 2008–2011* (Department of Education, Training and the Arts, 2007) and the *National Goals for Indigenous Education* (Department of Education, Employment and Work Relations, 2008) provide explicit ways to blend Indigenous cultural knowledge and mainstream knowledge so that Indigenous children receive the best possible literacy and numeracy education to enhance their opportunities for further education, training and employment.

A key theme from the above documents is the need to provide children with the best start to education and, the importance of contextualising literacy and numeracy to their community and culture (see Priest, 2005 for a detailed review). Here, community describes “a culture that is oriented primarily towards the needs of the group. This cultural orientation perceives that the whole community must be strong in order to adequately meet the needs of the individual” (Priest, 2005, p. 12). Karen Martin (2005) describes culture as about

> being related . . . it is being related to people, to the sky, the salt water, the animals, the plants, the land . . . that is how we hold who we are . . it is that we related to everything else . . . what is happening to our people now is we are not experiencing that relatedness . . . it is important that we pay attention to our responsibilities and keep our relatedness strong . . . we need that relatedness back . . . we need to re-present the stories of our relatedness (cited by Priest, 2005, p. 12)

Put another way, Martin Nakata (2007b) states that contextualising to culture is about that which already exists, that is, Torres Strait Islander community, cultural context and home languages (including the sky, the sea, the land and spiritual values) and “Indigenous knowledge systems” (Nakata, 2007a, p. 2). Continuing, Ezeife (2002) cites the work of Hollins (1996) who states that Indigenous people belong to “high-context culture groups” (p. 185). That is,

> High-context cultures are characterized by a holistic (top-down) approach to information processing in which meaning is “extracted” from the environment and the situation. Low-context cultures use a linear, sequential building block (bottom-up) approach to information processing in which meaning is constructed. (p.185)

What this means is that children who use holistic thought processing are more likely to be disadvantaged in mainstream mathematics classrooms. This is because westernised mathematics is largely presented as hierarchical and broken into parts with minimal connections made between concepts and with the children's culture and community. It potentially conflicts with how they learn. If this is to change the curriculum needs to be made more culture-sensitive and environmentally and community orientated so that parents can be involved in their children's learning.

Recognising Women's Cultural Learning Practices

Kathryn Priest (2005) states that for many years Indigenous women around Australia have struggled with gaining recognition for their cultures' child learning practices. A contributing factor to this issue is the typical characterisations, or the Euro American westernised view of Indigenous women's involvement in their children's education. Such involvement has reflected a deficit view of parental involvement in education. Indeed, the portrayal of parents as problems to be overcome and as uninvolved in their children's learning, upholds a particular view of parent participation in education (Jackson & Remillard, 2005). Now, according to Priest (2005, p. 19) Indigenous women in Australia are speaking out about what they want for their children, calling for recognition of their cultural knowledges and to be treated "on an equitable basis with Euro-American 'western' culture". Priest (2005) cites the work of the Warrki Jarrinjaku ACRS Project Team (2002) to explain

> the growing recognition of the need to have a 'both ways' approach to service design and delivery (Warrki Jarrinjaku ACRS Project Team 2002). An ideal 'both ways' environment places equal value and respect on quality of practices from both Kardiya (non-Aboriginal) and Anangu and Yapa (Aboriginal) cultures. (p. 123)

Whilst there is a growing recognition of Anangu and Yapa cultural knowledges, more work in the mainstream is needed to acknowledge, respect and learn about these knowledges (Priest, 2005). This issue for Indigenous women is not isolated to Australia.

In a study of African American mothers' involvement in their children's mathematics education Jackson and Remillard (2005) found that such characterisations have a strong tendency to privilege traditional westernised visible and invisible practices of education and schooling. As a consequence and because of stereotypical views of parental involvement in their children's education, they were confronted with challenges in relation to their children's education. This did not mean that the parents were not involved in their children's learning. They were. The parents took it upon themselves to create opportunities to support their children outside of school. By thinking proactively and strategically, the parents were strong advocates about their children's futures and the opportunities they wanted them to experience in their adults lives. That is, they used their daily lives and family activities as spontaneous opportunities to engage in discussions about mathematics and its associated language, informal and formal.

The Language of Mathematics – One way? No, both ways!

The previous discussion talked about both ways learning environments and the importance of recognising and valuing Aboriginal and Torres Strait Islander Peoples' cultural knowledges, community and home languages. Such recognition by non Indigenous people is crucial if they are to work with Indigenous Peoples in their communities to enhance the mathematics education of children and young people.

What is crucial here is the recognition that Aboriginal and Torres Strait Islander children be provided with quality education that recognises in explicit and implicit ways their culture, community and home language and that they are used as sustained entry points into all areas of the children's learning. Such an education needs to be both ways as described by Priest (2005) and Arbon (2008) earlier in this paper, and with the recognition that Anna Schnukal (2002, 2003) emphasises, "culture is still predominately oral with all important knowledge transmitted orally and in context" (p. 52).

The significance of recognising oral language is highlighted by Paul Herbert in his presentation at WIPCE in 2008

> Language is the conveyor of culture, through culture we add meaning to things based on symbols. When language disappears our symbols go with it leaving a group of people searching for symbolic meaning. These symbols are what we identify ourselves with. Without these symbols we are to an extent lost.

A strong point made in Shirley Brice Heath's (1983) work emphasises that from when we are infants language determines how we come to know and to be in the world. It is what binds communities, parents and children together "the adults which the children will one day become repeat the processes with the next generation of children" (Zeegers, Muir & Lin, 2003, p. 55). What happens then at the point of departure from home language when children are required to speak Standard Australian English in classrooms?

The Paradox: The Official Language of Instruction

As the official language of instruction, English is learned by Torres Strait Islander children as a second, third or fourth language (Shnukal, 2002). It dominates the Torres Strait Curriculum which is written to English conventions (Shnukal, 2002) even though it is being perceived by students as a "foreign language expressing alien and uncomfortable modes of thought" (p. 12). This point raises the question: How are children to find meaning in the symbols of Standard Australian English when their first languages are more likely to be Kala Lagaw Ya, Meriam Mir or Yumplatok? (Yumplatok is the current term used in the Torres Strait Islands for Torres Strait Creole, personal communication Mr Dana Ober, 2009).

Standard Australian English dominates the Torres Strait Curriculum as Schnukal (2002) has argued elsewhere, even though it is being perceived by students as a foreign language. Children's mathematics learning is further confounded by curriculum material that is decontextualised and lacking any practical purpose and connections to the children's culture and environment thus further reinforcing this perception.

Anthony Ezeife (2002) states that the differences between these two issues, decontextualised material and children's culture and environment, or put another way, mainstream and Indigenous cultural knowledge, would surface and influence the children's learning. He explains,

> If the instructional method favours the learning styles of students from Western cultures (as seems to be the case in contemporary formal school settings), then these students would perform quite well, while the performance of the disadvantaged students from indigenous cultures would not be as good. However, if indigenous students are given the opportunity to learn through an instructional medium that favours their learning or cognitive styles, then the likelihood is that learning would be facilitated and enhanced. (Ezeife, 2002, p. 180)

A more culturally sensitive way to enhance Indigenous children's learning would be to educate using culturally and environmentally based education that is contextualised to their culture. The effect of this process would be that children's have the incentive to learn for understanding because they can find meaning and links to their own cultures, their home languages and in the symbols used.

For children the mathematical concept may not be the difficulty, rather, it may be the language that is used to express it. For example, two categories of common nouns in English cause difficulty for Yumplatok speakers.

> The first is the count and mass (unbounded or non-count) distinction, so called because count nouns are thought of as units which can be pluralised, whereas mass nouns (e.g. "sugar", "wood", "flour", "cattle", "information", destruction", etc.) are thought of as substance and cannot be pluralised, except with specialised meaning. Thus, "two sugars" does not mean "two grains of sugar", but "two lumps/spoonfuls of sugar". Mass nouns take the quantifiers ("how/too) much/little"), whereas count nouns take "(how/too) many/few". There is not such distinction in Torres Strait Creole. All common nouns in Torres Strait Creole can be pluralised by using a number, the plural marker *dem* or a quantifier:
> *wan bred:* one loaf of bread*; tri bulmakau:* three head of cattle;
> *dem ud:* pieces of wood*; amass plawa:* how many tins of flour (Schnukal, 2003, p. 55).

Children who are speakers of Yumplatok are more than likely unaware of the circumstances with which English nouns can and cannot be pluralised and are uncertain of which quantifier to use (Schnukal, 2003). This uncertainty is likely to be influential to how they come to learn formal mathematics that is written and spoken to English conventions.

To further illustrate, puffing up shoulders and stating "he's big this kind way" means tall, while stating "I go . . . I go, go . . . I go, go, go", means "I went a very long way" (Nakata, 2002). Again, the problem may not be a mathematics or cultural issue but a language issue. Therefore, it is about having a specialized understanding of how children express their world as they see themselves in it—with verbalization the key to understanding concepts rather than simply having them manipulate objects that are not context related (Nakata, 2002; Shnukal, 2002). Torres Strait islander children's require explicit teaching via interactions with their teacher and other children and adults so that they become aware of the different grammatical structures from Standard Australian English.

Concluding comments

This paper has discussed the aspirations that Torres Strait Islander women have for their children. It has also emphasised the significance of recognising the cultural identity, home language and community of Australian Aboriginal and Torres Strait Islander women. This recognition by mainstream education is crucial if both ways education to going to succeed. Further, if Indigenous children are to have opportunities in their current and future lives, such recognition is deemed in this paper to be of importance. The paper takes the position that Torres Strait Island children's learning of mathematics can be enhance if there is a deliberate and explicit blending of Torres Strait Island cultural knowledge and mainstream western knowledge taught. Too many documents cite that the mathematics that Indigenous children are learning in school is isolated, disconnected and of little or no relevance to their daily life, their culture and home language. Whilst some effort is being made, more is needed to enhance the lives of young Torres Strait Islander children and recognition of the child learning practices of their parents.

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Proportional Reasoning Models in Developing Mathematics Education Curricula for Prospective Elementary School Teachers

Beverly J. Ferrucci, Keene State College bferrucc@keene.edu
Jack Carter, California State University East Bay jack.carter@csueastbay.edu
Ngan Hoe Lee, National Institute of Education Nganhoe.lee@nie.edi.sg

Abstract

A study of pre-service primary school teachers in Singapore and the United States revealed superior performance by the Singaporeans on proportional reasoning problems. Analysis of solutions showed the Singapore future teachers were more likely to use unitary and benchmark approaches than were their American counterparts. Conclusions include suggestions for programs intended to improve the performance of prospective elementary school teachers on proportional reasoning problems.

Introduction

Proportional reasoning is a benchmark in students' mathematical development (De Bock, Van Dooren, Janssens, & Verschaffel, 2002) and classroom data continue to demonstrate that students often perform less well on proportional reasoning problems than on other performance measures (Kaput & West, 1994, Van Dooren, De Bock, Hessels, Janssens, Verschaffel, 2005). Since proportional reasoning is a focus of the school mathematics curriculum in the elementary school grades, the capabilities of prospective elementary school teachers in solving proportional reasoning problems are critical for improvement efforts.

The importance of proportional reasoning in the mathematics curricula for prospective elementary school teachers is accentuated in national standards documents (National Council of Teachers of Mathematics, 2000; Ministry of Education, 2000). In the case of Singapore, one of the highest scoring countries on international mathematical comparisons, the teaching of proportional reasoning at the primary level was previously considered an integral part of 'Ratio and Proportion' in national standards documents. However, the latest national standards document, which reflects the revised 2007 mathematics syllabus, emphasizes the instructional importance of proportional reasoning by stipulating that it is now a fundamental aspect of the study of numbers, including whole numbers, fractions and decimals (Ministry of Education, October 2005, 2006d).

Despite many years of national attention in standards documents and other curriculum policy references, the performance of prospective elementary school teachers on proportional reasoning items remains problematic (Stacey, 1989; Swafford & Langrall, 2000). The current study compares the performance of prospective elementary school teachers on proportional reasoning problems in Singapore and in the United States. Since Singapore students have achieved an international reputation for high mathematics achievement and US students typically score less well on international comparisons, it was hypothesized that prospective elementary school teachers' performances and approaches on proportional reasoning problems would correspond to the differences reflected in international student performances. Such correspondences could then provide insights about, and models for, enhancing the performance future primary school teachers on proportional reasoning problems.

The present study investigated how prospective primary school teachers in both countries solved word problems which could be solved by using a proportion. These types of word problems are common in proportional reasoning or ratio and proportion sections of mathematics textbooks for future primary school teachers (Chan, 2007; Billstein, Libeskind, and Lott, 2007). Typically in these textbooks, a section on proportional reasoning is included as an application of students' work with whole numbers or with rational numbers.

Since whole and rational numbers, including mixed numbers, are major components in mathematics courses for prospective elementary school teachers, these two types of numerical representations were selected for inclusion in the proportional reasoning problems for this study. The Book Pages (BP) problem was one of the two problems investigated in this study. The problem statement for the BP problem contained only whole numbers and solutions to this problem could be carried out using only whole number operations.

The Tofu problem was the second problem investigated in this study. The problem statement for this problem contained only rational numbers and solutions to this problem could be carried out using only arithmetic operations with rational numbers.

Participants in this study were all enrolled in mathematics content or mathematics methods courses for prospective elementary school teachers. In the case of the Singapore participants, 24 were in their second year and 40 were in their third year of study in a four-year degree with certification program. Since the 24 Singapore second-year students had already completed a unit on teaching Ratio and Direct Proportion as part of their required mathematics methods course, all 64 Singapore participating prospective elementary school teachers had studied the concept of proportionate reasoning as well as methodology for teaching this form of reasoning at the primary levels in Singapore mathematics classrooms.

All participants worked on the problems for this study as an in-class assignment as part of their Singapore or US mathematics content / methods course. There was no time limit for students to complete their work on the problems and each student worked independently on the problems. In the instructions for completing the problems students were advised to show their work as they would to a class of elementary school students and to clearly indicate the solutions.

Results

For the BP problem, 64 Singapore and 167 US prospective teachers submitted solutions. An analysis of these solutions showed that 92% of the Singapore and 72% of the US future teachers submitted correct answers. The most commonly submitted approach by Singapore and US participants was to determine the amount read in one minute and to multiply the amount by 80 to get the amount read in 80 minutes. Notably, this approach, often called the unitary method, is a common approach taught to Singapore primary school pupils to handle problems involving proportional reasoning (Collars, Koay, Lee, Ong, & Tan, 2006, p.31, Method 2). Fifty-two percent of the Singapore participants used this approach and 97% of these solutions were correct, while 24% of US participants used this approach and 95% of these US participants solved the problem correctly with the unitary method. For Singapore participants the next most frequently used approach for solving the BP problem was to determine the amount read in ten minutes and multiply by 8 to get the amount read in 80 minutes. Twenty-seven percent of the Singapore participants used this approach and all solved the problem correctly. Again, this approach, frequently known as the benchmark method, is another method that is commonly taught in Singapore primary mathematics classrooms to deal with problems on proportional reasoning (Collars, Koay, Lee, Ong, & Tan, 2006, p.31, Method 1). Only one percent of US solutions used the benchmark method and, like the Singaporeans, all these solutions were correct. Koay and Lee (2006, p.38) provided insight into connections between the benchmark and unitary methods for solving proportions when they noted that when pupils have a "tendency [to commit] additive errors [in applying the benchmark method, teachers may want to] encourage them to use Method 2 [unitary method]".

Nineteen percent of the Singaporean solutions to the BP problem used a single ratio or a variation of one ratio in the solution and 67% of these solutions were correct. In contrast, none of the US solutions used a single ratio or a variation of one. Twenty-two percent of US participants determined the time needed to read one page and then multiplied by 80. This approach was successful 51% of the time. Three percent of Singapore participants used this approach and 100% of these approaches led to the correct solution. Another twenty-two percent of the US solution used a table, list, or chart, and 56% of these solutions were correct. However, none of the Singapore participants used a table, list, or chart. Twenty percent of the US solutions set up a proportion and 97% of these solutions were correct, while none of the Singapore participants set up a proportion to solve the BP problem.

A possible explanation for the absence of proportions in the solutions to the BP problem by Singapore pre-service teachers may be the perception that setting up a proportion is an algebraic solution. In the Singapore context, such an approach is not encouraged in teaching primary mathematics, as algebraic solutions of proportional reasoning problems are only introduced at the secondary levels (Ministry of Education, 2006a, 2006b, 2006c).

Among the US solutions to the BP problem there were 6% in which subtractive reasoning was used to obtain an incorrect solution, while no Singapore solutions used subtractive reasoning.

A final 6% of the US solutions contained four other approaches and each of these alternatives resulted in a correct solution.

For the Tofu problem, 63 Singapore and 25 US prospective teachers submitted solutions. An analysis of the solutions showed that 84% of the Singapore and 56% of the US future teachers submitted correct answers. The most commonly submitted approach by Singapore participants was the unitary method, i.e. to determine the amount needed for 1 kg and then multiply by the mixed number $4\frac{2}{3}$. In all 54% of the Singapore solutions used this approach and 91% of these solutions were correct. None of the US participants used this approach. However, 100% of the US solutions set up proportion with unknown component and solved for the unknown and 56% of these solutions were correct. In contrast, 2% of the Singaporeans used this approach correctly, a result possibly attributable to the perception held by many Singapore future teachers that this approach is algebraic and therefore not appropriate for primary school mathematics.

Seventy-eight percent of the Singapore solutions calculated $\frac{4\frac{2}{3}}{5\frac{1}{3}} \times 3\frac{1}{2}$ or variations of this calculation, and 87% of these solutions were correct. Another 13% of the Singapore solutions determined the amount needed for $\frac{1}{3}$ kg and then multiplied by 14 or variations, i.e., approaches that are basically variations of the benchmarking method. All these solutions were correct. The remaining 9% of the Singapore solutions included setting up a proportion with an unknown component, using $\frac{5\frac{1}{3}}{3\frac{1}{2}} \times 4\frac{2}{3}$ or submitting no work, a partial solution, or no solution. Of these solutions only the one that set up a proportion was correct.

Conclusions

The differences in the percent of correct solutions to the two proportional reasoning problems (92% vs. 72% and 84% vs. 56%) provides evidence that Singapore pre-service primary school teachers outperform their US counterparts on these type of proportional reasoning problems. These performance differences also provide evidence that proportional reasoning problems containing rational numbers may be more difficult than those containing whole numbers.

One of the more notable findings of this study concerned the greater use of the unitary and benchmark approaches by Singapore participants and the great success enjoyed by participants who used these methods. In the case of the BP problem, more than twice as many Singapore participants (52%) used the unitary approach as compared to 24% of the US participants. Yet, the percentage of correct solutions for those using this approach (97% for the Singaporeans and 95% for the Americans) was quite similarly very high. For those participants using the benchmark approach for the BP problem, the results were even more striking: 27% of Singapore future teachers used this approach as did only 1% of their US counterparts and 100% of those using this approach achieved correct solutions.

On the Tofu problem, 54% of the Singapore solutions used the unitary approach and 91% of these solutions were correct, while none of the US participants used this approach. Another 27% of Singaporeans used the benchmark approach or a variant of this approach and 100% of these solutions were correct, while once again, none of the US participants used either this approach or a variation on the Tofu problem.

Given the 20% and 28% differences in the percent of correct solutions by these prospective primary school teachers with Singapore future teachers consistently outperforming their US counterparts, there are grounds for incorporating the successful approaches used by the Singapore participants into mathematical content studied by US pre-service elementary school teachers. Particularly, the unitary and the benchmark approaches, commonly used successful approaches in the solutions by the Singapore prospective primary school teachers, were less frequently used by the participating US pre-service teachers. Consequently, efforts to improve the performance of prospective primary school teachers on proportional reasoning problems are apt to benefit from instructional programs that include the unitary and benchmark approaches to solving these problems.

The general decline in the percent of correct solutions by both countries prospective teachers on proportional reasoning problems that involved rational numbers as opposed to whole numbers is another notable finding from this study. This finding reinforces the results by Lim-Teo, Chua, Cheang, and Teo (2007) that even when future primary school teachers are seemingly well-versed in the unitary approach to solving proportional reasoning problems, they are often unable to come to terms with changes in problem difficulty as a result of rational numbers replacing whole numbers in solutions.

Strengthening a pre-service teacher's understanding of proportional reasoning problems is a more complex endeavor than merely ensuring that the unitary and benchmark approaches and rational number operations are emphasized in their programs of study. Nevertheless, these are apt to be key components of improvement efforts to ensure that teachers are prepared to teach proportional reasoning at the elementary level.

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Modelling in Mathematics and Informatics:
How Should the Elevators Travel so that Chaos Will Stop?

Andreas Filler
Professor of Mathematics education
Humboldt-Universität zu Berlin, Germany filler@math.hu-berlin.de

Didactic proposals on modelling in mathematics education mostly give priority to models which describe, explain as well as partially forecast and provide mathematical solutions to real situations. A view of the modelling concept of informatics, which also initiates rapidly generalised deliberations of models, can also make a contribution to the spectrum of models, which are treated in a meaningful sense in mathematics lessons so as to expand some interesting aspects. In this paper, this is illustrated by means of conceptual design models – and, here, especially of process models – using the example of elevator organisation in a multi-storey construction.

Modelling in Mathematical Education

During the past two decades, the use of mathematical models has been established for the processing of realistic situations and applications in mathematics lessons, in which step sequences matching the modelling cycle are performed as in Fig. 1 (see [3], p. 200). This cycle has been expanded around subjective aspects (see also [2]) during the past few years.

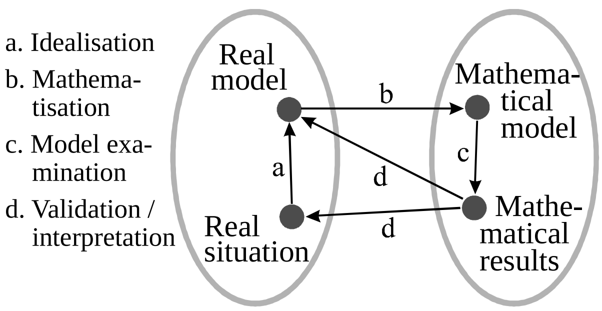

Figure 1: Modelling cycle

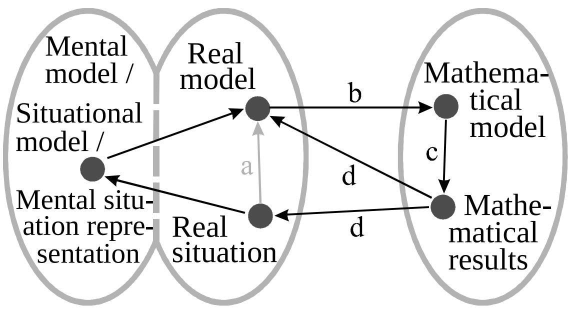

Figure 2: Expanded cycle

As seen in Fig. 2, it is apparent that deliberations, situation representations and mental models of subjects (e.g. students), which perform modelling, are only partially conceived in comprehensible diagrams. Therefore, solution options taken into consideration within the mathematical models pre-react to the model structure and have already affected real models and mental situation representations. An absolute distinction between reality and mathematics is often not feasible. Deliberations of the subject are mostly simultaneously characterised by real-world and mathematical aspects. Therefore, it can be established that:

Modelling cycles themselves are always models as well.

Modelling cycles related to mathematical lessons *reduce* modelling processes with the *purpose* of manageability in lesson conception and working out competences. Therefore, two of the *major attributes of the models* have been mentioned (in addition to the *mapping attribute)*: the *reduction* and the *pragmatic attributes* ([10], pp. 131ff).

Model Categories

The models relevant for discussion in mathematics lessons were subdivided by BLUM in descriptive and normative models ([4], p. 19). HENN refined this subdivision as follows ([5], p. 10):

- Descriptive models;
- Explanatory models;
- Predictive models;
- Normative models.

An exact separation of these categories is often not feasible. Predictions, in general, are based on descriptions of phenomena or processes. Descriptive models (describing, occasionally also explanatory or predictive) form the majority of models discussed in mathematical didactic publications.

Normative models ("prescriptive models") have a rarer occurrence, electoral systems and income tax rates are often mentioned as examples.

In this paper, based on an exercise of an elevator system control, models are introduced, which could be described as normative, although the customary terms used in informatics - "design model" or "process model" - appear to be more suitable in the characterisation. The successive "improvement" of a situation, therefore, almost inevitably requires a repeated run of the modelling cycle, which corresponds to a frequently elevated demand.

Modelling in Informatics

Informatics is often described as "the science of modelling", as expressed by SCHWILL: "*The deliverations .... show that informatics possesses much of a general model structure science*" ([8], p. 22). About the differences of modelling in mathematics and informatics, he wrote:

> "*Originals of mathematical modelling are mostly part of the natural world. … The associated situations possess a relatively low description complexity and are based on a few quantifiable, continuously variable data (at school) …*
>
> *Informatics primarily models situations which originate in an artificial world (e.g. office procedures, traffic, etc.). Therefore, it lacks natural simplicity. In fact, this original can be complicated, in which the complexity is essentially due to human arbitrariness and, therefore, barely underlies any reductionist rules. Likewise, the originals are, to a large extent,... discrete and their behaviour highly discontinuous."* ([8], p. 23)

The prospect (of an informatics educator), which is introduced by using this quotation as an expression of model structures in mathematics lessons, certainly corresponds to the models primarily applied in lessons, although it characterises no limits of mathematical modelling. In fact, mathematical methods are also efficient in describing not only artificial situations or processes, but also in designing or changing and optimising them.

Model Categories in Informatics

Models can fundamentally be concrete or theoretical *images* of entities available or *role models* for entities to be created. This classification corresponds with the difference between descriptive and normative model already mentioned. Informatics is concerned with a large variety of models. Applications (domains), work and technical processes, structures and construction of information systems or systems with IT elements[1] as well as human-computer interaction are modelled. In [11], THOMAS classified over one hundred(!) categories and subcategories of models related to informatics. The current paper elaborates more on conceptual models, especially designed for processes. It is also inevitable that investigation models (especially analytical models used in this context) will be of some significance.

A General Model Term

The previous sections have brought to light a variety of different approaches to the term "model", that may be confusing. In addition to this, for example, the issue arises as to what extent material models (e.g. cubes, cones, etc.) applied in mathematics lessons are related to the modelling concept initially drafted in mathematical education. Furthermore, there hardly appears any relationship between this concept and that of modelling in mathematical logic. A generalised view of the models will show such connections.

Modelling as a relation between the subject, purpose, prototype ("original") and the model

According to APOSTEL, a modelling process is identified by a four-digit relation.

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[1] It is observed that "systems with IT elements" are highly diversified and are not limited to computers or computer software in a narrow sense. For example, traffic control systems or even the elevator systems considered more closely in this paper are systems with IT elements

“*Let* R(S,P,M,T) *indicate the main variables of the modelling relationship. The subject* S *takes, in view of the purpose* P*, the entity* M *as a model for the prototype* T.”

([1], p. 4)

Here, the prototype (or original) T and the model M may be images, perceptions, designs, formalisms, calculations, languages or physical systems; they can belong to the same or different from these categories. In particular, the prototype and the model can exchange their roles.

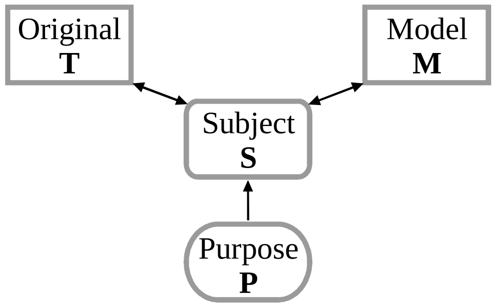

Figure 3: Model structure relation according to APOSTEL (SCHWILL: [8], p. 23)

The model design drafted here also integrates the model conception of mathematical education characterised by the cycle in Fig. 1, and the model term of the mathematical logic (especially of axiomatics). On one hand, axiomatic systems can be realistic models, which have been obtained from the latter by means of idealisation. This perception, for example, can assuredly apply in Euclidean geometric axiomatics. On the other hand, an axiomatic system can assume the role of a "prototype" and a model can be its implementation or interpretation in a "well-known structure". For example, the Poincaré and Klein models in non-Euclidean (Lobachevskian) geometry have come into existence like this. Detailed comments to the association between the model designs made by Apostel and models in mathematical logic are found in the work by WEBER ([12], pp. 55ff.).

Primary Attributes of Models

In his trend-setting book "Allgemeine Modelltheorie" (General Modelling Theory), STACHOWIAK constructed the following three primary attributes of models ([10], pp. 131ff.):

- The mapping attribute: *Models are always models* of something, *namely mappings, representations of natural or artificial originals, which themselves can be models in return.*
  The originals can pertain to the field of symbols, the world of perception and concepts or physical reality.
- The reduction attribute: *Models generally include* not all *attributes of their represented originals, but rather only those that ... appear relevant to model creators and/or users.*
- The pragmatic attribute: *Models are not only models of something, but also models* for someone*;* ... for a certain purpose.

These attributes also explain the fact that there is a high number of different modelling circuits for various purposes (compare e.g. Fig. 1 with [9], p. 29). The mapping attribute also emphasises that mappings are possible in different directions and, therefore, as already remarked, originals and models can "exchange their roles". Thus, an idealisation or abstraction process always forms the basis of the description of real spatial solid figures present with mathematical terms, such as "cube" or "pyramids" and the associated mathematical properties, are within the context of the circuit, as according to Fig. 1, Models of real objects. However, concepts or mathematical descriptions may, conversely, function as originals; associated models are, thus, real objects of the physical reality.

Elevators – an Exercise from the Netherlands Mathematics A-lympiad Competition

A complex modelling exercise and its processing steps by students are presented in the following. Both, descriptive mathematical modelling as well as concept modelling, especially process modelling of significance in informatics, appear in this case. The exercise has been set for four-member student teams of grades 10 to 13 within the scope of the Netherlands Mathematics A-lympiad Competition. Since the exercise actually contains complex modelling requirements, although it is very apparently formulated and processed using elementary mathematical means, their approach already appears possible and reasonable among younger students. Therefore, it has been set for grades 7 and 8 students with an interest in mathematics in a student circle. The experiences gained in this trial will be reported in the following.

Due to reasons of space, the exercise (which assumes the role of two-aspect original) is reproduced in the reduced form[2]

A multi-storey building with 1200 employees has a ground floor and 1-20 storeys, in which 60 employees work at a time. There are 6 elevators with a capacity of 20 people. When work commences, it leads to chaotic situations and long waiting periods. The management employs a supervisor who is assigned the task to let the manpower flow proceed smoothly. The following facts for the elevator speed are identified:

- Time requirement to travel from one stop to another for one storey located at an upper or lower level: 8 s
- From one stop to passing through the next upper or lower storey: 5 s
- Time between the transitions of two adjacent storeys: 3 s
- From passing through one storey to one stop in an adjacent storey: 6 s
- An elevator stops at one storey for an average of 10 s.

All employees arrive between 8.45 am and 9.00 am (consistent flow).

Exercise: How long can an elevator last in total in the worst case until it returns to the ground floor? Calculate the approximate length of time until all employees will have arrived at the correct storey.

In this exercise section, a descriptive model shall be constructed with the idealisation, that the elevators will stop at each storey.[3] An elevator trip must be described for mathematisation (Fig. 4), to which elementary calculations are connected.

The result (assuming the improbable worst case) is a travelling period of 7 min 15 sec per elevator, in which the total transport time lasts approximately 71 minutes.

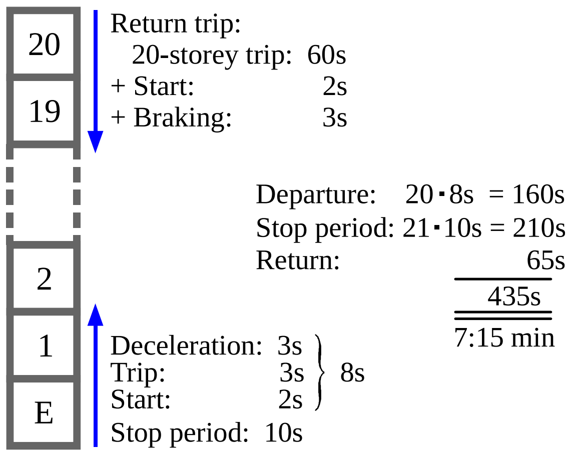

Figure 4

Although the sheer calculations are highly elementary, many errors (especially due to neglected stop and brake periods) appeared in the participating students, who could, however, be mutually corrected during discussion, in which a diagram similar to Fig. 4 was developed jointly. A second exercise section followed, in which three elevators were of service only for the first to the tenth storeys and three elevators for the eleventh to twentieth storeys and, as a result, already bring about a significant improvement of the situation.

The following exercises are kept more open:

Consider at least three travelling plans for handling the elevator traffic faster. For each model, bring forward arguments that agree with or contradict this.

Design a concept for the management, in which you present proposals, how human flow can be reconducted more rapidly. Support the concept by calculations.

Decide the extent to which it can accommodate the following circumstances:

- The employees do not wish to be much concerned and do not wish for complicated rules. But they just wish to arrive rapidly.

[2]The complete exercise can be accessed at http://www.fi.uu.nl/alympiade/en.

[3]This probability is extremely low: $\frac{20!}{20^{20}} \approx 2{,}3\cdot 10^{-8}$. However, the modelling assumption appears self-evident to the students (they were not familiar to probabilities and expected values yet). In my opinion, it is important that it concerns the worst case and the situation in general is less dramatic.

- The management is located on the 15th storey and would most be appreciative of the preferential treatment in your concept.

The following suggestions have been submitted by the participating students:

1. The three elevators, which first serve the first to the tenth storeys, are of assistance to the upper elevators when they are finished with the lower storeys.
2. Residents on the upper storeys are asked to change elevator in the tenth storey. Thus, the upper elevators require less time.
3. Each elevator serves only 3-4 storeys.
4. The three elevators in the lower storeys serve more storeys (e.g. 1-11) than the ones in the upper storeys.

Following the discussions, the students preferred suggestions 3 and 4. They calculated several examples, in which the realisation that a systematic approach is reasonable was achieved. A term for the travelling period of an elevator has been defined, which serves the *n* to *m* storeys (Fig. 5):

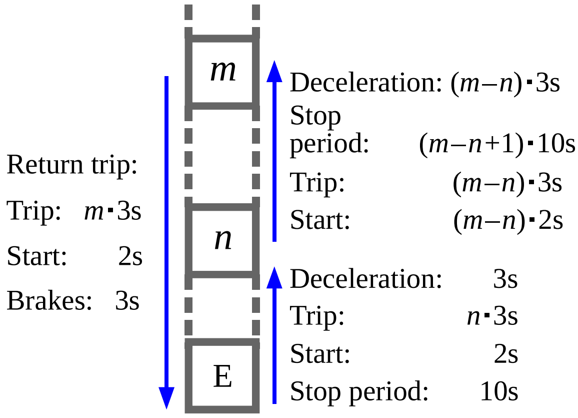

Figure 5

(1) $3m+5\ +\ 3n+15\ +\ 8(m-n)+10(m-n+1)\ =\ 21m-15n+30\,.$

Taking into consideration the number $(m-n+1)\cdot 60$ of employees working on the *n* to *m* storeys and the capacity of the elevators, the students could calculate the total period for transporting all employees in the *n* to *m* storeys in case only an elevator travels to these storeys:

(2) $3\cdot(m-n+1)\cdot(21m-15n+30)\,.$

Various errors also appeared in this case, which, however, could be clarified during the discussion.

By using the term defined, the students could yet compare and optimise many different variants by using a spreadsheet software (see Tables 1-3).

2 elevators per storey				
Elevators	from n	to m	Time (s)	in min.
1 and 2	1	7	1701	28
3 and 4	8	14	2142	35
5 and 6	15	20	2025	33

Table 1

1 elevator per storey				
Elevator	from n	to m	Time (s)	in min.
1	1	4	1188	19
2	5	8	1476	24
3	9	11	1134	18
4	12	14	1296	21
5	15	17	1458	24
6	18	20	1620	27

Table 2

1 elevator / storey, preferred management				
Elevator	from n	to m	Time (s)	in min.
1	1	5	1800	30
2	6	9	1548	26
3	10	13	1836	31
4	14	15	810	14
5	16	18	1512	25
6	18	20	1620	27

Table 3

The processing of the apparent modelling exercises outlined extended over two 90-minute lessons. Designing the most potentially favourable procedures represented an appealing challenge for the students.

Conclusions

The exercise described combines a series of aspects of mathematical modelling by using approaches which are typical for informatics. It has been demonstrated that many model structures in mathematics and informatics may appear in similar manners in different contexts. The deliberations delinerated for the elevator control are described as normative model structures, in which, however, the categorisation borrowed from informatics essentially appears to be better described as concept (especially process) modelling. From a mathematical pedagogical viewpoint, terms (1) and (2) are descriptive mathematical models which satisfy the predictions, whereas they are arranged in an information system model classification as (system) investigation models (specifically as deterministic analytic models) (cf. [11], p. 55).

When developing concept models, it is often emphasised that "the best model" does not exist, but rather benefits and disadvantages of different models are to be balanced against one another and priorities are set. A fairly high extent of openness in the exercise discussed here is the result of this.[4]

Concept models can also make a contribution to place more emphasis on the structuring as well as reassignment phases. In particular, the frequently postulated repeated run of the modelling cycle appears almost inevitable in the exercises to design processes, since optimal solutions in general are not found in a single step, but rather arise stepwise when investigating corresponding models and different models must be compared with one another.

In summary, based on the different facets of the exercises considered in this paper, the hypothesis is thus formulated that informatics modelling concepts can also enrich the modelling in mathematic lessons.

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MATRICES AND ROUTING

Dr. Ajda Fošner
Faculty of Management Koper, University of Primorska, Slovenia ajda.fosner@fm-kp.si

Abstract
The study of matrices have been of interest to mathematicians for some time. Recently the use of matrices has assumed greater importance also in the fields of management, social science, and natural science because they are very useful in the organization and presentation of data and in the solution of linear equations. The theory of matrices is yet another type of mathematical model which we can use to solve many problems that arise in these fields. The aim of this paper is to show how we can use matrices and their mathematical model to solve some problems in the process of routing. First we will introduce the term of routing and the new approach in the process of selecting paths. We will show some simple examples. We will also pint out how we can learn about matrices in the classroom. At the end we will discuss about advantages and potential disadvantages that may occur in the described technique.
Key words: selecting paths, routing, matrices
Area: applications of mathematics in the real world

1. INTRODUCTION

It is difficult to go through life without seeing matrices. For example, the company in Slovenia manufactures sofas and armchairs in three colors: white, black, and red. The company has regional warehouses in Ljubljana, Maribor, and Koper. In August shipment, the company sends 12 white sofas, 14 black sofas, and 4 red sofas, 20 white chairs, 24 black chairs, and 16 red chairs to each warehouse.

	white	black	red
sofas	**12**	**14**	**4**
chairs	**20**	**24**	**16**

Table 1

If we were to enter this data into the computer, we might enter it as a rectangular array without labels. Such an array is called a matrix A:

$$A = \begin{bmatrix} 12 & 14 & 4 \\ 20 & 24 & 16 \end{bmatrix}.$$

This matrix is a 2×3 matrix since it has two rows and three columns. In general, a matrix is said to have size $m \times n$ if it has m rows and n columns.

In everyday life we often confront with a problem of selecting routefor a multi-city or multi-country trip. Here matrices can help us. Namely, with matrices and their mathematical model we can easily answer some questions that arise in this problem.

Suppose that we have n cities and the diagram which shows the roads connecting these cities. Then we can represent this information with a $n \times n$ matrix A. In this matrix the entries give the number of roads connecting two cities without passing any other city:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

The first row of a matrix A gives the number of roads connecting city 1 with each of n cities without passing another city. For example, a_{12} represents the number of roads connecting city 1 and city 2 without passing city 3, or city 4, or …, or city n. Similarly, the second row represents the number of roads connecting city 2 with each of n cities without passing another city. For example, a_{24} gives the number of roads connecting city 2 and city 4 without passing city 1, or city 3, or …, or city n. And the last row represents the number of roads connecting the last city with each of n cities without passing another city. Of course, $a_{ij} = a_{ji}$ for all $1 \le i, j \le n$. In other words, the matrix A is a symmetric matrix. Note also that a matrix A is always a square matrix, which means that the number of its rows is equal to the number of its columns and this is equal to the number of cities.

2. EXAMPLES

Let us start with a simple example which we can use in the classroom.

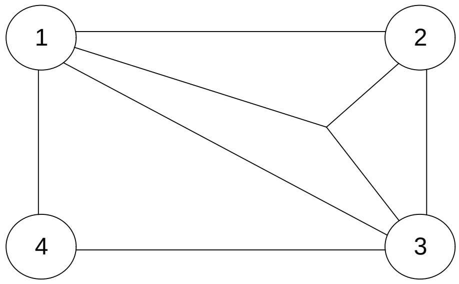

Figure 1

Suppose that the graph in Figure 1 shows the roads between four cities in Slovenia. As we already showed we can represent this information with a matrix A, which is written below. From the diagram we can easily see that there are exactly two ways to travel from city 1 to city 2 without passing through either city 3 or city 4. This information is entered in the first row and second column ($a_{12} = 2$) and also in the second row and first column ($a_{21} = 2$) of a matrix A. Similarly, we can go from city 1 to city 3 in exactly two ways without passing through either city 2 or city 4. This information is entered in row 1, column 3 ($a_{13} = 2$) and again in row 3, column 1 ($a_{31} = 2$) of a matrix A. Note also that there are no roads connecting each city to itself. Because of that we have zeros on the diagonal of a matrix A ($a_{ii} = 0$, $i = 1,2,3,4$):

$$A = \begin{bmatrix} 0 & 2 & 2 & 1 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

In the described problem of selecting route we often ask ourselves how many roads are there to go from one city to another by passing exactly one another city. For example, how many ways are there to travel from city 1 to city 2 by passing through exactly one other city? Of course, we must go through city 3 or through city 4. From the graph in Figure 1 we see that we can travel from city 1 to city 3 in two ways, $a_{13} = a_{31} = 2$, and from city 3 to city 2 also in two ways, $a_{32} = a_{23} = 2$. It follows that we can go from city 1 to city 2 through city 3 in exactly $2 \cdot 2 = 4$ ways. Because there is no direct road between city 4 and city 2, $a_{42} = a_{24} = 0$, we can not travel from city 1 to city 2 through city 4. It follows that we can travel in exactly 4 ways from city 1 to city 2 by passing one other city. It turns out that we can easily get this information also from a matrix A. If we multiply a matrix A by itself we get a matrix A^2:

$$A^2 = \begin{bmatrix} 9 & 4 & 5 & 2 \\ 4 & 8 & 4 & 4 \\ 5 & 4 & 9 & 2 \\ 2 & 4 & 2 & 2 \end{bmatrix}.$$

The matrix A^2 gives the number of ways to travel between any two cities by passing exactly one other city. The entry in the first row and second column of the above matrix is exactly 4 as we calculate from the Figure 1 (this entry represents the number of ways to travel between city 1 and city 2 by passing through city 3 or city 4). This number is found as follows ($A^2 = A \cdot A$):

$$4 = a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32} + a_{14}a_{42} = 0 \cdot 2 + 2 \cdot 0 + 2 \cdot 2 + 1 \cdot 0 .$$

The first product in the above identity is $a_{11} \cdot a_{12} = 0 \cdot 2$. This product tells us how many ways are there to go from city 1 to city 1 (0 ways) and then from city 1 to city 2 (2 ways). The 0 result indicates that such a trip does not involve a third city (city 3 or city 4). The only nonzero product ($a_{13} \cdot a_{32} = 2 \cdot 2$) represents two routes from city 1 to city 3 and two routes from city 3 to city 2 (there are 4 roads connecting city 1 and city 2 which go through city 3).

Further, if we multiply a matrix A with A^2, we get a matrix A^3:

$$A^3 = \begin{bmatrix} 20 & 28 & 28 & 14 \\ 28 & 16 & 28 & 8 \\ 28 & 28 & 20 & 14 \\ 14 & 8 & 14 & 4 \end{bmatrix}.$$

In a similar way as above we can show that the matrix A^3 represents the number of ways to travel between any two cities with exactly two intermediate cities. Moreover, the sum $A^2 + A$ gives the number of roads connecting two cities with at most one intermediate city:

$$A^2 + A = \begin{bmatrix} 9 & 6 & 7 & 3 \\ 6 & 8 & 6 & 4 \\ 7 & 6 & 9 & 3 \\ 3 & 4 & 3 & 2 \end{bmatrix}.$$

Furthermore, the sum $A^3 + A^2 + A$ represents the number of ways to travel between two cities with at most two intermediate cities. So, with a matrix A we can easily answer some combinatorial and also logistic questions in the problem of selecting route.

In our example the Figure 1 involves just four cities but in practice we can have many more cities, for example 100 or even more than 100 cities. In that cases the matrix A is of course much larger (the matrix A can be of dimension more than 100×100). Moreover, the Figure 1 can have many other interpretations. For example, the lines could represent aircraft lines or communication lines such as telephone lines and internet lines, the lines could represent lines of mutual influence between people or influence between nations, animals and similar.

Let us show one example more.
Suppose that a company, which manufactures sofas, has regional warehouses in London (city 1), Paris (city 2), and Barcelona (city 3). The company has its own small telephone system which connects all three cities. The Figure 2 shows this telephone system.

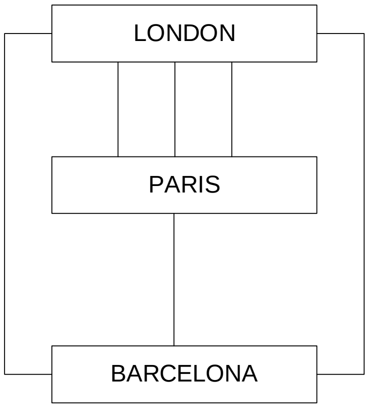

Figure 2

As in the first example we can represent this information with a matrix A:

$$A = \begin{bmatrix} 0 & 3 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

Of course, now the matrix A is of dimension 3×3 since we have just three cities. As before, this matrix can help us to easily answer some questions like how many lines are there which connect London and Barcelona and go through Paris or how many lines are there which connect Paris and Barcelona and similar. All these can be appropriate for student exercises, self-study questions, homework. We can also use these questions in the classroom before final tests.

3. CONCLUSION

Mathematics is about ideas and not just formulas, algorithms, theorems, and proofs. Knowing *why* is sometimes just as important as knowing *how*. Pupils in the classroom have to see not only why a given fact is true but also why it is important. For many, the mathematics has always been a body of facts to blindly accepted and used. And the notion that they personally decide mathematical truth or falsehood comes as a revelation. Promoting this level of understanding is the goal of this paper.

The theory of matrices is the branch of mathematics involving the properties of systems of simultaneous linear equations. Moreover, in the last hundred years the use of matrices has assumed greater importance also in other fields like management, economics, logistics, social science, and natural science because they are very useful in the organization and presentation of data and in the solution of linear equations. In this paper we have represented how we can use matrices and their mathematical model to solve some problems in the process of routing.

As we have shown with examples, matrices can help us to answer some combinatorial questions in the process of routing. For example, we can easily answer questions like: how many ways are there to travel between two cities by passing exactly one other city or how many telephone lines are there between two cities. So, the theory of matrices is now another type of mathematical model which we can use to answer some important questions and solve many problems that arise in the fields of management, social science, and natural science.

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Mathematics Professional Learning Communities: Opportunities and Challenges in an Elementary School Context

Douglas Franks
Faculty of Education, Nipissing University, North Bay, Ontario, Canada
dougf@nipissingu.ca

Abstract
School-based professional learning communities (PLCs) have become an important means of "building capacity" among teachers in a wide variety of areas, including those with a subject focus. Very often, these PLCs are mandated by administration, and operate under an established structure. This paper describes an attempt by a mathematics coordinator and school level "lead' teachers to establish relatively informal PLCs in mathematics in an effort to improve mathematics teaching, and thus student learning, in an environment that focused very much on literacy. The four PLCs created are discussed, as are the opportunities and the challenges that go with the relative freedom offered to the teachers. Sustainability is a central challenge to these groups.

Introduction

Professional learning communities, or PLCs, have gained considerable standing within the general education community in the last ten years or so. As DuFour, DuFour, and Eaker (2008) note, the concept of a PLC was once limited primarily to use by educational researchers, but has now become widely used to describe almost any gathering of educators. Most educational leaders insist, though, that professional learning communities must have student learning as their principal focus (e.g., DuFour, DuFour, & Eaker, 2008; Fullan, 2007; Schmoker, 2006). Interest in changing teaching practices must be for the purpose of improvement in student learning. In the province of Ontario, The Literacy and Numeracy Secretariat (LNS), established by the Ministry of Education in 2004 for the improvement of student achievement, has focused heavily in recent years on supporting the establishment of school-level PLCs for the purpose of "engag[ing] in processes of inquiry and learning focused on improving student achievement" (LNS, 2007, p. 2). Since school boards are expected to pursue this end quite vigorously, this often means they mandate that schools establish such PLCs to address particular school level issues related to student learning. A case in point is the school board in which the following study took place. What happens, though, when the initiative is much less formal?

This paper describes the development of four elementary school-level groupings of teachers formed to address issues pertaining to mathematics education. These were not mandated by the school board, but instead arose out of the interest of a mathematics coordinator and elementary mathematics "specialists" within the schools, and the feeling that "building capacity" (LNS, 2007) was not happening fast enough. Given the development parameters for these groups, which will be expanded upon further, key foci of the paper are purpose and sustainability.

Ontario and School Board Contexts

Provincial Context

In Canada, the formal school curriculum is set at the provincial level. The latest revision of Ontario's elementary (grades 1 to 8) mathematics curriculum occurred in 2005, with an increasingly explicit focus on "reform mathematics" and mathematical processes. The work of NCTM and reform mathematics educators such as Van de Walle (e.g., Van de Walle & Folk, 2005) have significantly influenced Ontario mathematics for the past several years.

The LNS was given the explicit goal to "help boost student achievement" in the province by "work[ing] directly with schools and school boards across the province to *build capacity* and implement strategies to improve reading, writing, and math skills" (LNS, n.d.). The LNS produces a large number of research-based resources for schools and teachers.

School Board Context: Literacy, Numeracy, and PLCs

The school board within which this study took place is geographically large, and predominantly rural, although a majority of its students are located in or near the one city in the region. Especially since the establishment of the LNS, the board has focused extensively on literacy. At the time of this study the school board had one elementary literacy coordinator and four literacy coaches working with teachers in schools. In addition, many of the schools had an in-school literacy resource or 'lead' teacher who had completed additional training in this area. In contrast, the school board had one mathematics coordinator to service all 36 elementary schools, although some schools did have an in-school mathematics resource teacher.

The school board has become a major advocate for school-based professional learning communities, mandating their development and implementation in a variety of contexts such as literacy and "turnaround," as reported by teachers. In these "official" PLCs, the focus is very much on student learning. For example, a poster created by the school board and displayed in at least one of the schools in this study reads, in part: "The Four Critical Questions of EVERY Professional Learning Community: What do we want students to learn? How will we know when every student has learned it? What will we do if a student has not learned it? What will we do if a student has learned it?" The flip-side of a focus on literacy PLCs, however, has been the absence of mandated, school-level, mathematics-based PLCs.

The Informal Mathematics "Professional Learning Communities'

PLCs: Their Establishment

To help increase school-level focus on numeracy, in January 2008 the elementary mathematics coordinator invited four teachers who had just completed additional "specialist elementary mathematics" training to become "lead mathematics teachers" in their respective schools. Shortly thereafter a second teacher joined to co-lead at one of the schools. The coordinator and the lead teachers themselves referred to these teacher groups as PLCs. Although informal, teacher release time was provided for all the teachers taking part (one half day per month), with more as planning release time for the lead teachers. The mathematics coordinator was eager to explore ways to "build capacity" in reform-oriented mathematics teaching and this appeared to be a way to start that process. Unlike many other school board based PLCs, the focus of each of these PLCs was determined by the schools themselves, usually by the lead teacher in consultation with the administration and with colleagues. The mathematics coordinator was to have a limited role in the ongoing process, meeting only occasionally with the lead teachers in sharing sessions.

It is important to highlight the informal, almost serendipitous nature of the establishment of these school-based mathematics groups. While numeracy levels among students was of interest to the board, no mathematics PLCs had previously been established. Only because he was aware that these five teachers had recently completed specialized elementary mathematics training was the mathematics coordinator able to take the initiative to invite them to assume lead roles at their schools, and arrange financial support for the groups. The question was, would these groups be able to sustain themselves, and "gel" as mathematics-based professional learning communities?

PLCs: Researching their Establishment

The mathematics PLCs were established and just beginning to function as the research began in late winter 2008. The intention was to tell each PLC's story, particularly from the perspective of the lead teachers, and to explore themes of (i) commonality and difference among the four schools, and (ii) sustainability and viability. Methodologically, the intention was to adopt a participant observer approach. As the researcher I was to observe as many of the mathematics community meetings as I could and was permitted to attend, remaining out of the experience except when explicitly called upon to offer a perspective on a mathematics education-related question. I was also to attend and observe meetings held by the mathematics coordinator and lead teachers. In both cases data were to be gathered through field notes. A final data source was audio-taped individual interviews with lead teachers.

The research reality was quite different. Teacher health issues, busy schedules, limited communication capacity, and teacher concern over outsider presence--all 'facts of life' in schools—meant considerable change. Through the end of the school year in June, four lead teachers, at three schools, were interviewed (two people twice), and one "PLC" meeting was observed. One active group of teachers did not want me present at their meetings, two schools had difficulty holding mathematics meetings for reasons such as teacher absence and readiness, and the coordinator and lead teachers were unable to find time to schedule meetings after the first two winter early organizing gatherings, which I was unable to attend. Nevertheless, the data gathered do help tell the story of these four mathematics "communities."

PLCs: The Participating Lead Teachers' Views

Opportunities to talk professionally with other adults are relatively uncommon in at teacher's typical environment: alone in a classroom with children. The power of a group such as a PLC is, as one teacher put it, “ a chance for teachers to get together to collaborate” (Teacher A). Teacher C described a PLC as “a group of teachers and colleagues that kind of get together and have the chance to talk about different learning...” Teacher B felt that the “biggest thing” about a PLC was the opportunity for teachers “to communicate” with each other. Teacher B particularly liked the non-mandated, voluntary nature of the mathematics PLC at school B--it was not “driven by someone else’s agenda.”

Emergent Themes

A number of themes began to emerge from the data collected through the spring of 2008, described in the following sections.

School-based Mathematics Community Building

While capacity building in mathematics at the school level was a major reason in all schools for initiating these local, teacher-based PLCs, the learning community meetings, the key to this capacity building, were highly dependent on the nature of the school and the teachers involved. Four sub-themes—mathematics lead teachers, continuity, focus, and participation--were identified.

Mathematics Lead Teachers: Credentials and Training

It was critical to the possibility of success that within each of the schools, qualified lead teachers in mathematics were available. The school board elementary mathematics coordinator was able to identify four, and eventually five, teachers with official “additional qualifications” in elementary mathematics. Thus, they had the credentials and training to be sanctioned by the board to serve as lead mathematics teachers, and be granted teaching release time in order to plan for learning communities.

Continuity: Ongoing versus Interrupted

Across the four schools the frequency and continuity of the PLC meetings varied extensively. Over the period February to June (the end of the school year) the number of meetings at each school ranged from one to four. Availability, and a willingness to commit and to remain committed--in addition to training--were also key to the success of the groups. Teacher absence, delays in completing requirements in preparation for a follow up meeting, ongoing demands on teachers’ time, and the degree of commitment to the process were among the major factors that contributed to this variance.

Focus: Connected versus Dispersed Sessions

I refer to a series of PLC meetings as “connected” when they focus on and develop a single mathematics theme, or are a closely linked set of topics. “Dispersed” meetings, on the other hand, are hose which have relatively few close links to each other.

The two most active groups provide examples of both. The volunteer group of teachers at school B chose problem solving as their focus, and successive meetings centred on developing and implementing a single model, with problems modified for grade appropriateness. The teachers at school A, whose attendance was requested by the principal, focused broadly on reform mathematics teaching. The specific topic for each meeting was chosen by the teachers at the previous meeting, usually with little continuity (e.g., fractions, division of whole numbers). The question is, does it make a difference to the nature and strength of the learning community?

Participation: School Mandated versus Voluntary

Although the overall professional learning community initiative was not mandated by the school board, in some schools (A and C) the initial topic was identified by the lead teacher in consultation with the principal, and certain teachers were asked by the principal to participate. On the other hand, at a third school (B), the lead teacher polled her school colleagues for a topic of most interest, and the response was a focus on problem solving. In addition, participation at this school was voluntary. At the fourth school (D), the teachers’ chose geometry as the focus. Does it make a difference to the development of the local mathematics learning community if the teacher group is requested to participate—potentially in a topic not of their choosing--or if it strictly voluntary?

Administrative Support

This initiative grew from the personal interest of a mathematics coordinator in seeing mathematics teaching strengthen within schools. Seizing upon the opportunity to invite recently credentialed “elementary mathematics specialists” to take on voluntarily the role of lead teacher, four mathematics professional learning communities were given the chance to establish themselves.

Passive Administrative Support

Teaching release time was critical to the formation of these communities--a monthly half-day for all participant teachers, with additional planning time for the lead teachers. This financial support (for "supply" or "substitute" teachers) was provided by the school board, on the recommendation of the mathematics coordinator, and was available to all the teachers who took part. His status as a professional development official working at the school board level was thus critical to getting these school-based communities started.

School-level administrators also provided passive administrative support for the development of the local mathematics communities by supporting the intention of the strategy, by specifically supporting lead teachers who agreed to accept the opportunities and challenges of becoming leaders in mathematics within the school, and by providing space for the teachers to meet for a half day.

Active Administrative Support—A Challenge

School administrators, especially at some of the participating schools, also took an active role in the process, working initially with the lead teacher to identify a mathematics topic on which to focus, and at times, identifying the teachers who should participate. On occasion, the principal also attended the meetings.

There were also challenges to the level of active support provided by administrators. At the board level, where student literacy levels were the dominant focus, the mathematics coordinator, working alone in mathematics, found himself generally unable to provide the ongoing support to lead teachers originally envisioned. Although it was never the intention for him to run the mathematics meetings at the schools, even arranging meeting times when he and the lead teachers could meet to share their experiences proved very difficult. Nevertheless, regular coordinator-lead meetings might have helped maintain focus and commitment.

The active involvement of school administrators, especially at the start, was also a double-edged sword. The authority of the principal ensured that teachers would take part, but sometimes was construed as "one more thing to have to do." In places where teacher ownership was able to take over once things got started, this concern appeared to fade.

What Has Happened Since?

Only two of the schools succeeded in holding meetings with any regularity between February and June, 2008. Some led teachers expressed a desire to see the projects continue in 2008-2009, but none did, at least in the form they were constituted in 2007-2008. A new mathematics coordinator took over in September 2008, and the lead teacher B, at one of the active schools, was transferred. While there had been administrative support, it was not enough to bridge the changes. The impetus to continue was unfortunately very much reduced in most schools. Commitment levels are clearly critical--commitment on the part of the board to ensure that the coordinator has time to work with lead lead teachers; commitment to ensure that the weight of the success of a group within a school does not fall entirely on the shoulders of a single person, a lead teacher, perhaps inexperienced at leading. A commitment by all to see that the groups are well focused, and have meaningful goals. PLCs have a strong focus on student learning; while all the groups were well-intentioned, they typically lacked that definition, potentially experiencing a critical tension in terms of sustained purpose. This study also suggests that a strong sense of purpose may be more important than being voluntary or mandated.

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Impact on Student Achievement of Teacher Participation in K-8 Mathematics Professional Development.

Todd Frauenholtz, PhD
Professor of Mathematics & Computer Science, Bemidji State University,
Bemidji, Minnesota, USA TFrauenholtz@BemidjiState.edu
Derek F. Webb, PhD
Professor of Statistics & Mathematics, Bemidji State University, Bemidji, Minnesota, USA

Abstract
The purpose of this study is to determine the impact on student achievement of elementary school teachers who participated in professional development in the content area of mathematics. Teachers participated in professional development courses and have accumulated a range from three to eighteen total credits from the summers of 1998 through 2007. The impact is measured by student achievement data collected on standardized tests.

Introduction

Beginning in the summer of 1998 regional teachers were invited to the campus of Bemidji State University (BSU), a small regional university in northern Minnesota, to participate in professional development in the content area of mathematics. The initial "math camp" was funded by federal money from the US Department of Education through the Minnesota Higher Education Services office. These funds have continued to support professional development of teachers in northern Minnesota through the summer of 2009. Teachers from many districts participated in the professional development; however, this study examines student achievement data from only one district.

Professional Development of US Mathematics Teachers in Grades K-5

The statement: "mathematics education in the United States needs some work" is putative! The mathematics education faculty at BSU sought to develop a professional development program in 1998 for elementary mathematics teachers in grades kindergarten through eight to address this national need on a regional level. Elementary school teachers generally are responsible for teaching several content areas; however, the focus of this program is exclusively mathematics. A professional development program at BSU was designed with these goals in mind: challenge teachers' traditional beliefs on teaching mathematics, be long term in nature, and fit the demographics of our region. One of the most influential groups in the U.S. calling for changes in mathematics teaching is the National Council of Teachers of Mathematics (NCTM) with their *Standards* documents (NCTM, 2000; NCTM, 1995; NCTM, 1991; NCTM, 1989). The professional development program at BSU was designed to follow the vision promoted in the NCTM standards documents and implement many of the lesson activities from the NCTM's *Navigations* series and National Science Foundation funded reform curricula (Hirsch, 2007).

Loucks-Horsley, *et. al.* (2003, p. 35) indentify the following features of professional development based on what researchers know of learning. They are:

- make useful connections between teachers' existing ideas and new ones;
- provide opportunity for active engagement, discussion, and reflection to challenge existing ideas and construct new ones;
- situate the learning in contexts teachers find familiar;
- challenge current thinking by producing and helping to resolve dissonance between new ideas and existing ones;
- support teachers to develop a range of strategies that address learning for all students.

In addition to challenging ideas about teaching, and how to teach mathematics, the program was designed to help teachers develop the specialized mathematical knowledge (Ball, Hill, & Bass, 2005) necessary to teach mathematics well. The professional development program addressed this need by using the following pedagogical model multiple times in each course: engage teachers in a mathematical activity, then follow it through to its conclusion, which involved multiple solution methods being described, explained, and examined, then analyze where the teachers struggled and where students would stumble in elementary and middle school classrooms.

To maintain engagement in one particular course, participants played games where keeping score looked surprisingly like addition. After several of these activities, participants no longer saw a set of rules or steps for the addition algorithm but rather a concrete understanding of place value and the concept of addition. Also, professional discussions took place where the university instructors discuss current research findings and policy issues relevant to mathematics education in the state and nation, including international comparisons.

A Long-term Professional Development Program in Mathematics

The program was designed to encourage K-8 teachers to pursue further study in mathematics. The overwhelming impression of the program designers was that having a series of professional development courses culminate in a master's degree would be necessary to encourage participants to persevere through the professional development series. The program was designed to have coursework on the following topics:

- algebra (patterns and functions)
- number sense
- assessment
- educational psychology.
- geometry
- probability and data
- discrete mathematics

The five process standards (problem solving, reasoning and proof, communication, connections, and representation (NCTM, 2000)) are addressed in each mathematics course by the manner in which the course is taught and instruction modeled. As an external program reviewer observed: "I was constantly struck by the parallels of the content of these courses with recommendations of the standards documents" (Martin, 2005, pg. 3). Each course was team taught by two instructors in a three week block on-campus with a face-to-face delivery method. Classes met five days each week for approximately three hours each day. One focus in the program was to have teachers actively engaged in doing mathematics and making sense of the solutions (Timmerman, 2003); hence, the three hours each day were filled with activities appropriate for the K-8 mathematics classrooms to which the teachers would be returning in the fall.

Loucks-Horsley *et. al.* (2003) make it clear that excellent professional development takes time; hence, the designed program would optimally occur over several years of the teachers' careers. Teachers begin with a wide variety of mathematical backgrounds and experiences, then study mathematical content and processes relevant to the K-8 mathematics classrooms. The program aligns well with both state and national content standards (Martin, 2005) while also addressing the process standards from the NCTM.

Professional Development in a Rural Setting

Any pragmatic professional development program must consider the geography of the participants. This program was designed for a small state university in rural northern Minnesota. The Minnesota Office of Higher Education (2008), using census data, identified the eighteen neediest school districts in the state of Minnesota and sixteen of them reside in the service region of BSU. In addition to high rates of poverty, the challenge of covering a large geographic region of the state also exists. Many of the teacher participants in the professional development program need to either drive long distances daily or reside in residence halls during the professional development coursework.

While on-campus teachers work collaboratively to develop a mathematical community with the goal of improving student learning (Timmerman, 2003). When they return to their classrooms in the fall, the teachers are often isolated from the professional community which the mathematics program attempts to promote. The professional development program may provide the only source of professional connections to our teachers and thus the cooperativeness of our program receives even more attention. Most teachers in BSU's service region are financially limited and thus the professional development program needs to be financially accessible to teachers. To address this financial concern, grant funding for the coursework was sought and obtained. The "math camps" were funded by federal money from the US Department of Education through the Minnesota Higher Education Services office. Courses were taught during the summer when teachers were able to be out of their classrooms and, if necessary, away from home. Timmerman (2003) noticed that elementary school teachers frequently lack confidence in their mathematical abilities, possess a procedural knowledge of the subject, and may have negative attitudes or even anxiety toward mathematics; hence, the courses were designed, and taught, in an intentionally welcoming and relaxed atmosphere to actively engage teachers in a long-term professional development program.

Purpose of this Study

The purpose of this paper is to describe the impact on student achievement of teacher participation in professional development in the content area of mathematics. The degree program was approved during the 2005-2006 academic year but participants began taking coursework in 1998. The courses evolved over the first several offerings but have now been sufficiently revised to represent a "final form" even though small improvements continue to be made with each offering. At this point, no teachers from the studied district have completed the requirements for the K-8 mathematics master's degree program, so this study focuses on the student achievement of teachers who have participated in some of the available coursework.

Research Methodology

This study utilizes Measures of Academic Progress (MAP) test data from the Northwest Evaluation Association. The MAP test data are norm referenced and this study analyzes data from the fall and spring testing sessions. The student achievement data that are available at this time are only grades K-5. Data from academic years 2000-2001 through 2006-2007 were obtained from one school district where teachers earned between zero and eighteen credits of the mathematics course offerings. The district averages *n*=73.3 elementary teachers and *n*=1686.6 elementary students each year (see Table 1).

Year	1st	2nd	3rd	4th	5th	Total	# of Teachers
'00-1	328.4	344.6	342.7	376.1	424.6	1816.4	97.50
'01-2	331.8	327.8	318.6	336.7	373.1	1688.0	76.00
'02-3	310.0	325.1	325.1	322.5	356.2	1638.9	64.00
'03-4	324.0	303.1	335.6	325.6	319.4	1607.7	64.34
'04-5	338.0	337.8	307.7	339.0	329.3	1651.8	68.27
'05-6	335.6	330.0	329.1	311.8	335.0	1641.5	70.00
'06-7	339.4	343.5	355.4	340.8	325.3	1704.4	71.00
'07-8	335.0	348.0	347.0	360.0	354.0	1744.0	75.00

Table 1: School District K-5 Attendance

The K-12 student population in the 2007-2008 academic year reported 19.6% minority students and 48% students of poverty and 14.8% of students qualifying for special education services. This study is looking for a relationship between teacher participation in the summer mathematics program and their students' achievement in mathematics.

Teachers were coded as 0 for having not participated in the mathematics professional development offerings, 1 for having participated in the past, and 2 if they participated in the future. For instance, a teacher who participated in 2004 would be coded 2 for the years 2000-2004 then coded 1 from 2004-2007 upon completion of their first credits from BSU. Teacher and student data are presented in Table 2.

Year	0 – No Math PD				1 – Past Math PD				2 – Future Math PD			
	#Teachers		#Students		#Teachers		#Students		#Teachers		#Students	
	Fall	Spr	Fall	Spr	Fall	Spr	Fall	Spr	Fall	Spr	Fall	Spr
2000-1	0	0	0	0	0	0	0	0	0	0	0	0
2001-2	0	0	1163	1178	0	0	0	0	0	0	0	0
2002-3	46	0	1240	1141	4	1	0	25	4	5	0	135
2003-4	31	44	1093	1085	5	4	97	100	3	4	105	106
2004-5	47	46	777	1107	5	5	125	127	2	3	73	79
2005-6	46	47	1121	1121	4	5	131	126	3	2	51	50
2006-7	51	47	1158	1173	7	5	109	131	0	2	75	52

Table 2 – Teacher and Student Participation in Mathematics Professional Development (PD)

Results

The computer program SPSS, version 16.0, was used to analyze the data. Student achievement data was a composite mathematics score which is an aggregate of number sense, algebra, geometry, measurement, and data sub scores. Initially the question "is there a difference between participation and no participation?" was examined. The group coded 0 (no participation in math program, mean = 199.50, N=14,803) was run against the group coded 1 (participants, mean = 211.97, N=1,149) using a two-sample unequal variances *t* test. The test was very significant (P-value = 0.000). These data clearly indicate that student mathematics achievement is different in the group whose teachers participated in the professional development when compared to the students whose teachers did not participate.

Next, the question "is there a difference between no participation and future participation?" was examined. The group coded 0 (no participation in math program, mean = 199.50, N=14,803) was run against the group coded 2 (future participation, mean = 208.31, N=726) using a two-sample unequal variances *t* test. The test was significant (P-value = 0.000). These data clearly indicate that student mathematics achievement is different in the group whose teachers did not participate in the professional development when compared to the students whose teachers would be future participants. Finally, the question "is there a difference between past participation and future participation?" was examined. The group coded 1 (participants, mean = 211.97, N=1,149) was run against the group coded 2 (future participants, mean = 208.31, N=726) using a two-sample unequal variances t test. The test was very significant (P-value = 0.000). These data clearly indicate that student mathematics achievement is different in the group whose teachers had participated in the professional development when compared to the students whose teachers would be future participants.

Conclusions

The data indicate that students whose teacher participated in the summer mathematics institutes achieved significantly higher when their teacher had participated in professional development than students whose teacher had not participated in professional development. This result does not explore the relationship between the number of credits of professional development taken by a teacher and achievement by students; however, there exists an opportunity for future research in this area.

Next, we compared the teachers who did not participate in any professional development (0) with the teachers before they did participate in professional development (2). Here again the data indicated differences in achievement between students of the two groups of teachers. We hypothesize that the teachers who participated in the professional development sessions were more highly motivated people, or had fewer personal distractions, enhanced general teaching skills from the beginning, or other desirable characteristics. These attributes would be independent and unrelated to the professional development.

The next hypothesis examined compared students of teachers before the teachers participated in professional development and after the teachers participated in professional development. The students will have matriculated to different grades, so the student-teacher association will change through time; however, the teacher's professional development is the variable of interest. Here again the data indicate student achievement increased with teacher participation in professional development. The teachers in these two groups should be, on average, equivalent on many confounding variables such as teacher motivation and other general teaching skills.

Limitations

The authors realize that the variability of number of credits taken ranges from zero to eighteen and is a large range. It is difficult to expect a small number of credits to have the same impact as a large number of credits on teacher performance and further study needs to be done in this area. This study did not have access to data indicating teacher experience. This variable may prove illuminating in future studies.

Additionally, it will be interesting to explore if the positive impact of the professional development fades as time passes. Perhaps it is a treatment that "wears off" over time and teachers need to revisit their professional development.

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One mathematical formula in the science textbook: looking into innovative potential of interdisciplinary mathematics teaching

Viktor Freiman, Université de Moncton, Canada, viktor.freiman@umoncton.ca and
Danis Michaud, Centre d'Apprentissage du Haut- Madawaska, Canada
in collaboration with *Sylvie Blain, Samuel Blanchard*, Université de Moncton, Canada, *Claudine Dionne, Brigitte Long, Roberto Gauvin*, Centre d'Apprentissage du Haut- Madawaska, Canada

Abstract

Our paper presents some preliminary observation from a collaborative exploratory study linking mathematics, science and reading within a technology enhanced problem-based learning scenario conducted at one French Canadian Elementary and Middle School. Presented in a form of dialogue between teacher and researcher, our findings give some meaningful insight in how an innovative mathematics teaching can be developed and implemented using a real-world problem solving. Instead of a traditional presentation of material about lighting up homes, participating mathematics, science and French teachers were working collaboratively with the ICT integration mentor and two university professors helping students investigate a problem from various perspectives using a variety of cognitive and metacognitive strategies, discussing and sharing the finding with peers and presenting them to a larger audience using media tools. Our preliminary results may prompt further investigation of how innovation in teaching and learning can help students become better critical thinkers and scientifically empowered citizens.

The project we are going to describe in this paper is a collaborative initiative of teachers, school leadership and researchers to explore innovative ways to teach students to become a real-life connected, problem-oriented and technology empowered learners of the 21st century. Following Lesh (2007), we ask what would be learning needed for successful citizen living in the world with the increasing use of mathematics, science and technology and how the traditional conceptions of reading, writing and arithmetic could be extended or reconceived to prepare students for such success.

Aldous (2007) is mentioning six key ideas that characterize engaging pedagogies in mathematics and science education : equity, service to humanity, literacy**,** knowledge dimensions and their changing emphases, affective as well as cognitive responses to mathematics and science, and connections to technology. The author mentions the need to identify and explicate the strategies of investigating and explicating the creative processes involved in solving novel real world problems and pointing to a way in which the content/ processes/ context and affective connections intrinsic to mathematics and science learning and teaching can be found. Our paper describes some preliminary observations on how innovative teaching, collaboration between teachers and researchers, and use of technology can change the way mathematics is taught and learnt.

While many researchers, practitioners, and government authorities call for the search of innovative approaches in teaching and learning, little is known of how innovation can be developed, implemented, and assessed in our classroom.

At the macro-level, its origins may be referred to the vision of today's society as 'knowledge society' one, based on innovation and ingenuity. Hargreaves (2003, p.2) argues that we can (and should) 'promote a high-investment, high-capacity educational system in which highly skilled teachers are able to generate creativity and ingenuity among their students by experiencing creativity and flexibility themselves in how they are treated and developed as knowledge-based professionals'. In this way, it is the changing society that forces governments to put pressure on educational systems to look for new and more effective methods preparing young generation to deal with the complexity of the modern world.

An analysis made of PISA data by Canadian New Brunswick's provincial government, showed in early 2000s that the educational system failed to adapt to new requirements, so the improvement is necessary (Freiman, Lirette-Pitre, 2007). Concrete steps have been taken since

2002 to improve mathematical, scientific and reading culture in our schools with the *Quality Learning Agenda* (GNB, 2003). The document introduced individual laptops in six middle schools (three French and three English) promoting an innovative use of technology to improve learning in mathematics, science and literacy.

At the micro-level, there are always devoted and enthusiastic teachers that are open-minded and critical to what they do trying to find better ways to reach every student in their classroom. In order to plan and implement new methods, they would turn to the research looking for ideas and innovative strategies. This is about how a sustainable collaboration between innovative teachers working in innovative schools and researchers who look into practical outcome of research may arise.

Freiman et al. (2007) examined an impact of individual laptop use on the middle school mathematics teaching and learning within the New Brunswick Individual Laptop Initiative. Research team of specialists in French, science and mathematics didactics has realized several interdisciplinary problem-based learning scenarios with Grade 7-8 (13-14 years old) middle school students that to worked on complex non-routine real life connected problems. Our major findings showed positive attitude towards the task, more autonomy, larger variety of strategies used, and good mastery of computer tools, constant discovery of new methods to use the Internet and different software, and the use of the rich and coherent mathematical vocabulary. At the same time, we found limited capacities to analyze problem context, lack of critical evaluation of computer produced results, lack of details in mathematical representations, and limited meta-cognitive links between different parts of the problem solving process that prompted us look for better teaching strategies together with teachers. The new initiative that arose from this project will be the subject of our recent paper bringing teacher's view on innovation in mathematics teaching, while linking it to the science and French curriculum so building richer interdisciplinary connections: *after having rich experience with an action research on 1-to-1 laptop use in 2004-2006 conducted with researchers form the Research and Development Center in Education from the Université de Moncton, we were willing to continue exploring the possibility to use project-based learning (PBL) in the classroom.*

A learning scenario was initiated by teachers willing to improve their students' learning in three major aspects of the New Brunswick reform school curriculum – use of better strategies of reading for better understanding of complex texts, developing better competences in scientific investigation and fostering a better understanding of mathematical relationships analyzing real life related problems. Teacher's comment: *In the context of project-based learning, a learning scenario presents to the students a real-life related situation. It puts emphasis on the practical use of mathematics and information communication technology (ICT). We had to find for students a research problem that would stimulate their scientific learning by putting them in a real context of investigation as well in a situation of reading and writing significant scientific texts.*

To construct a theoretical framework of our new study, we were looking at the combination of three new ideas that recently attract many educators going beyond problem based learning principles used in our previous study: collaborative research with teachers, interdisciplinary teaching approaches, and innovation in teaching (Novotná, J., et al., 2003, Korey, J., 2002, Lerman, 2004). As a starting point for our inquiry, we used a Grade 8 New Brunswick science curriculum that foresees a topic on light and optics. One of the notions that are being introduced is a source of the light and related costs of electrical energy. In a section of the OMNISCINCES 8 textbook (Clancy et al., 1999) the students are asked to reflect on the cost of the use of sources of light at home comparing incandescent and fluorescents bulbs. The textbook gives a definition of the watt and kilowatt as units of power shows on a concrete example how to calculate a cost of the use of 60 W bulbs during 10 hours if the power costs 8 cent per kilowatt-hour. The text concludes that fluorescent tubes would give a better energetic outcome than incandescent bulbs providing a corresponding mathematical explanation.

Being taught in a more traditional way, the text would probably be used by the science teacher as a narrative base for her classroom presentation with the example being solved together with students and some exercises helping to practice the procedure. Instead, we were looking for a different way to teach this in order to develop students' scientific curiosity, better understanding of mathematical concepts and their utility in science. Teacher's comment: *We decided to explore the incandescent and fluorescent bulbs. Students had to find advantages and disadvantages of both types of bulbs. The project developed by teachers together with researchers had natural links to the New Brunwick science, math and French school curriculum in terms of learning outcomes (general, specific and transdisciplinary) In this way, in April 2007, students from the Grade 8 of the and their Centre d'Apprentissage du Haut-Madawaska (CAHM) together with their teachers started this beautiful adventure. Some skeptics would argue why to do so? Our answer is to improve students' knowledge!*

From the point of research methodology, this exploratory collaborative study included data collection by means of classroom observations, reflective journals, Internet blogs, and interviews with students and samples of students' work have been used in our analysis that has been conducted together with teachers. The goal and the format of the conference and limited space led us to focus on descriptive aspects of mathematical part of the project. Future publications will report on its other aspects, namely, developing of reading skills and abilities of scientific investigations.

The text of the problem that students had to investigate has been published on the school's blog: http://cahm.elg.ca/prof/DanisMichaud/2007/04/les_ampoules_1.html
Here is the English translation: *The bulb in your bedroom is burned out. You are looking to replace it but there is no one at home. You decide to go to the local store to buy it but there are no bulbs of the same type available. Therefore, you need to make a choice among the types that are exposed on the shelves of the store.*

Teacher's comment: *Following the announcement of the problem and its investigation in several steps, the students had to experiment, solve and reflect on the problem. The results, conclusions and recommendations have been published in a local newspaper (Le Madawaska) and the blog of each student. The goal was to inform and the population and make it conscious on a better choice of bulbs to meet its need in lighting the house.* At the first stage of the investigation, students had to brainstorm the problem situation. Teacher's role is to nurture discussion about student's spontaneous ideas. Here is how the teacher describes it in her blog: *Students will have to compare different types of bulbs and make some hypotheses about the problem. My task is to question students putting them into cognitive conflict based on their intuitive ideas* (http://cahm.elg.ca/prof/DanisMichaud/2007/04/les_ampoules_1.html).

In order to foster student's reflective activity, the strategy KWLN (**k**now already - **w**ant to know – what is **l**earned – what do I **n**eed to know more) have been proposed by researchers. Students were also asked to analyze how they light up their own room. A schema on a scale was used to represent real measures proportionally. Teacher's blog suggests students to look for information about scales on the *wikipedia* using following link: http://fr.wikipedia.org/wiki/%C3%89chelle_(proportion). Looking for explanation why the distribution of the light is so unequal in different parts of the Earth, each team of students had to put a comment on the blog. Many students were using several terms expressing mathematical relationships. For example, one team wrote: *'In the North America, there are more big industries than in Africa because Africa is poorer than North America. There are much lighter parts in the continents that have more money and less on those who are less rich'.* In order to get more insight into relationship between the cost of the bulbs and their duration for both types, incandescent and fluorescent ones, students were asked to analyze a graph and conduct their own investigation (http://www.led-fr.net/images/fluocompacte_vs_incandescence_7000h.gif).

The role of the ICT integration mentor was very important to support for teachers' innovation. A note left in her blog by the ICT mentor clarifies this aspect: '*They found a formula* *to calculate the costs of enlightening in their science textbook but then realized that the information about unit costs is outdated. Further, students have learned form the recent bills that the costs have been increased and also (since the book comes form another province) that the costs vary form province to province in Canada. Using paper and pencil, as well as EXCEL software, they made necessary calculations. Working in this way, without being explicitly taught, students did calculations by themselves making tables in EXCEL. They surpassed our expectations. When we were thinking if the task can be accomplished by students, they did it better than we thought they would be able to do. Moreover, their tables were made with necessary information (titles, axes, etc.) with the use of nice fonts and colors making their work more attractive*'. Technology played yet another important role in the project providing a socially oriented medium for collaboration and sharing between all participants, students, teachers, mentors, school administration and researchers. Besides sharing material and comments on the blogosphere, a wiki-based collaborative tool has been used. Video documents with the results of students' investigations created by students have been shared through the youtube.com web community. Among comments left by participants in this common collaborative virtual space, one is particularly interesting because it presents another valuable aspect of this project - the support from the school principal: *What is fascinating to me is the time you devote to do this research. I know also that you work collaboratively as a team with other people who also try to contribute to the improvement of students' learning. The use of technology is still young and unknown field of teaching and your project would design new ways to advance in this in future. All this kind of project helps us to understand, to explain and to improve ourselves. Bravo and thank you for your interest. I appreciate and welcome your research and wish a success. http://cahm.elg.ca/classes/larechercheaction/2007/03/projet_de_reche.html* The parents were also involved in the project and were sometimes surprised by students' sudden interest in home lighting as it witness the following teacher's comment: *The final result has surpassed our expectations. The articles, videos, and comments in the blog have succeeded to inform the population and at the same time did a promotion of their new knowledge which lets us to conclude about the success of the project. There is evidence that students are able to make knowledge transfers and explain what they have learned to the members of their families. Some comments we collected from the parents can be resumed as following: 'My child has convinced me to change bulbs in our house … My child is interested now in electricity bills and is making a moral to us to save energy – what do to you do in your classroom?' – they look happily surprised.*

Students' work was very productive in terms going beyond mathematics curriculum making explicit links with language art and science. This observation is consistent with others made by researches. Already at the end of 90s, Schmitt (1997) noted about a close relationship between reading and mathematics: '*…I'm working with teachers who are purposefully emphasizing more realistic and relevant problem solving situations rather than the controlled one- or two-step word problems. As a result, math students are more engaged in reading, writing, speaking, and listening. When problem situations depend on gathering information from a variety of everyday sources, such as articles and advertisements in newspapers and magazines, prose literacy and mathematical literacy are hard to separate. Understanding the problem becomes much more complex than knowing a list of key words -- 'more' means add, 'less' means subtract, 'of' means times -- to solve formulaic word problems*'. An important role mathematics may play in scientific investigations have been underlined by Coulter (2002): '*By combining this (scientific, V.F.) background knowledge with thoughtful analysis of the data, they (students, V.F.) are able to achieve a deeper understanding of how tornado formation is influenced by specific weather patterns most commonly found in certain parts of the country at specific times of the year. Without this use of data, the textbook descriptions of how tornados form are much less likely to take root in students' growing conceptual understandings. As is true for students' development of science skills, these investigations provide opportunities to* *experience the power of mathematics in helping to understand the world—hardly an inconsequential lesson*'.

While a detailed analysis of our collaboration with teachers is still underway, the conclusive teacher's comment indicates some of its positive outcomes: *this research project allowed us to work as a team and to discuss with other people who want to improve students' learning. This kind of project allows students to develop the autonomy, communication abilities, transversal competences, as well as abilities to do transfers, to use their knowledge. As teachers, we can conclude that the project was a success.*

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THE ROLE OF THE MUSIC TO LEARN GEOMETRICAL TRANSFORMATIONS

Daniela Galante

G.R.I.M. (Gruppo di Ricerca sull'Insegnamento delle Matematiche) – Dipartimento di Matematica ed Applicazioni dell'Università di Palermo, via Archirafi, 34 – 90123 Palermo (Italy).

Abstract

This research studies the interaction among the following contexts: natural language, geometrical language and musical language and it can provide new instruments to accord didactical situations and for a deeper understanding of communication processes. It springs from the consideration that the geometrical transformations are usually used in the compositional processes and the "role of the music to learn geometrical transformations" is actually a new study. In the field of the theory of situations by G. Brousseau (1986) we can assume to be in front of a learning teaching-situation including non-teaching situation as the teacher of musical instruments, while transmitting the knowledge of musical language (theoretical-practical) didn't have the intention to transmit the geometrical transformation.

KEY WORDS: geometrical transformations, compositional process, mathematics, music, theory of situations.

INTRODUCTION

This article describes a part of the experimental work done for my doctoral thesis.[1] The relationship between mathematics and music has far fetched roots and geometrical transformations have played an important role and for certain aspects essential in the development of the language of western music (B. Scimemi, 1997). The aim of this research is to verify if the constant study of a musical instrument creates unconscious potentialities which are translated into strategies and methodologies for the solution of problems related to isometries. In fact, among the principal functions that the study of music is able to perform, besides the mere knowledge function, the linguistic-communicative function, the cultural, critical, aesthetical and affective function, a cognitive one is recorded because music exercises and develops the capabilities of thought: the productive-imaginative thought in the first place (in the activities of sound production) but also the analytical, logical and inferring thought (in the activities of reflection and interpretation). This experimental research is based on a comparison between secondary school students that study music at a conservatoire and secondary school students that don't study music at a conservatoire but which, anyway, have a basic knowledge theoretical-musical. This research has focused on spontaneous conceptions concerning geometrical transformations in general and their connection to music. The didactical experimentation was effected at Liceo Statale "Regina Margherita" of Palermo where two different samples of students were chosen:

Students of a music-oriented section of the same school which is connected to the state music conservatoire "Vincenzo Bellini" of Palermo. Number of students involved 70 between 14 and 16 years of age.

Students of a social- psychological-pedagogical section of the same school: number of students involved 70 between 14 and 16 years of age.

On the basis of these considerations the two following hypotheses of research were formulated:

H1 *In musicians students (music liceo-conservatoire) the constant study of a musical instrument creates unconscious potentialities which are translated into strategies and methodologies for the solution of problems concerning isometries differently from non musicians students (pedagogical liceo).*

H2 *Students possessing a knowledge of the musical rhythmic structures have a greater ability in recognizing the rhythm of geometrical forms for the construction of objects in comparison with those who do not have such knowledge.*

METHODOLOGY AND THEORETICAL REFERENCE

The study of situations/problem gets into the theory of didactical situations by G. Brousseau.
The experimental stages are:

- Formulation of the didactical problem;
- Formulation of the objective of the research;
- A priori analysis of the problem/situation which should take into consideration
    - The epistemological representation of both mathematical and musical concepts;
    - The historical-epistemological representation of the same concepts (variations which have interfered in the course of time);
    - The foreseeable behaviours of students towards the situation/problem.
- Research hypothesis
- Construction of the instruments for the falsification of hypotheses which consists in devising of an experimental apparatus through the preparation of:
    - Questionnaires;
    - Interviews to couples with the task of writing their common considerations written down after a common agreement (registration of interview protocols).
- Analysis of experimental data: correlation of experimental data in function of an a priori analysis.
    - Quantitative analysis about the problems of the questionnaires

    Application of:
        - Descriptive statistics;
        - Analysis of implicative statistics by R. Gras (1997,2000) with the help of software CHIC 2004;
        - Factorial analysis with the help of software SPSS 9.0 and others;
    - Qualitative analysis of the related protocols of the interviews of couples.
- Documentation and communication of the results of the research.

This research has focused on spontaneous conceptions concerning geometrical transformations in general and their connection to music. The didactical experimentation was effected at Liceo Statale "Regina Margherita" of Palermo where two different samples of students were chosen:

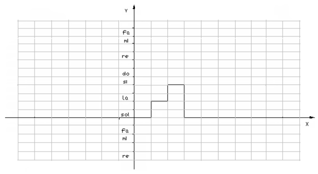
- Students of a music-oriented section of the same school which is connected to the state music conservatoire "Vincenzo Bellini" of Palermo. Number of students involved 70 between 14 and 16 years of age.
- Students of a social- psychological-pedagogical section of the same school: number of students involved 70 between 14 and 16 years of age.

THE TESTING

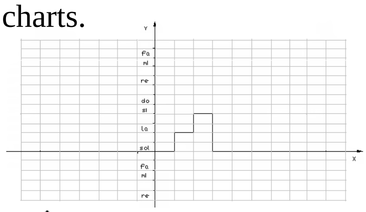
To verify the research hypotheses I proposed four sets of questions to both samples examined: the first two are about classical exercises on geometrical transformations present in any textbook for the first two years of upper secondary school; the third one is a problem regarding the reconstruction of a mosaic through the identification and iteration of geometrical figures and finally a last set of exercises regarding the application of the geometrical transformations in melodic tune bits.[3] I would like to concentrate my conclusions above all on the pupils' behaviour adopted towards the last set of questions. It is to be précised that both samples hadn't yet carried out in the classroom the study of geometrical transformations and this allowed me to pick their spontaneous conceptions on the subject. The set of questions proposed is the following one:

<<Let's consider a plane (x,y) and put the time on the x axe, which corresponds to a sequence of beats which have constant intervals (for example those ones produced by a metronome) and on the y axe the height of sound from the lowest to the highest. In this way any melody can be represented by a law f so that $y = f(x)$. After that let's choose as unit of measurement the second and match it to the musical crotchet figure for the x axe and the semitone[4] tempered for the y axe; in this way we can have the graphic representation through little squares which simultaneously indicate the duration of each sound that is how they flow through time (on the x axe) and the height they have according to a tempered scale (on the *y a*xe). Moreover, in musical writing notes written on the stave receive their names and indicate their height thanks to the use of the clefs[5]: for example to the treble clef corresponds the G note in the second line of the stave.
As consequence, if we consider as starting point of our system of reference, that is $y = 0$, the height of the correspondent sound to a G, the following melody: 

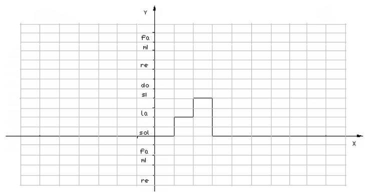
Is represented in a Cartesian plane in the following way:



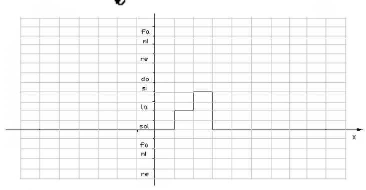
On the basis of these suggestions try to complete the following charts.



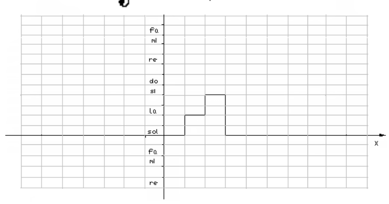
A) In the following Cartesian plane you see the original melody represented.
Draw, in the same Cartesian plane, this melodic tune bit:
Identify if there is a translation or a reflection of the original melody in relation to the x or y axe or to the origin of the axes. Give reasons for your answer.



B) In the following Cartesian plane you see the original melody represented.
Draw, in the same Cartesian plane, this melodic tune bit:
Identify if there is a translation or a reflection of the original melody in relation to the x or y axe or to the origin of the axes. Give reasons for your answer.

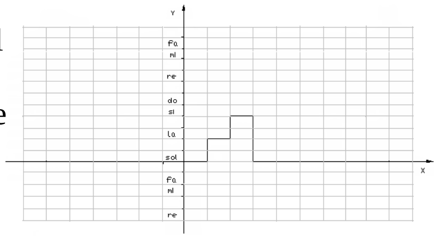


C) In the following Cartesian plane you see the original melody represented.
Draw, in the same Cartesian plane, this melodic tune bit:
Identify if there is a translation or a reflection of the original melody in relation to the x or y axe or to the origin of the axes. Give reasons for your answer.



D) In the following Cartesian plane you see the original melody represented.
Draw, in the same Cartesian plane, this melodic tune bit:
Identify if there is a translation or a reflection of the original melody in relation to the x or y axe or to the origin of the axes. Give reasons for your answer.

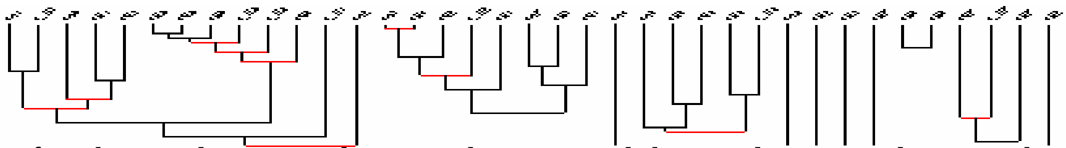
E) In the following Cartesian plane you see the original melody represented.
Draw, in the same Cartesian plane, this melodic tune bit:



Identify if there is a translation or a reflection of the original melody in relation to the x or y axe or to the origin of the axes. Give reasons for your answer.>>

ANALYSIS OF DATA

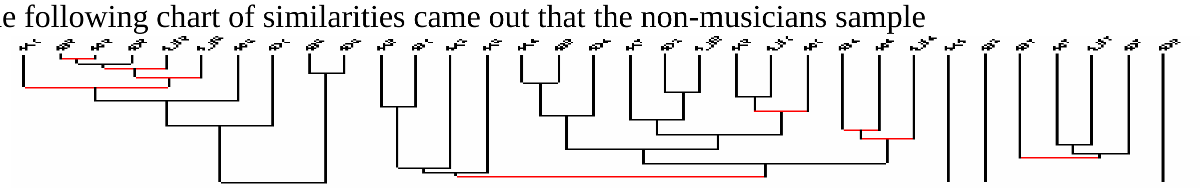
The set of questions was met with great interest and enthusiasm by both samples of pupils because they were made curious by the matching of geometrical transformation with music. The students who have elementary music knowledge[6] preferred to look for solutions in the field of music rather than in that of geometry, for example in the first exercise they said there was a translation because there is a pause. The sample of the musician students, in particular, used the term transposition to indicate the translation because in music translating a melody means moving it in time and height, therefore these students identified correctly the term transposition as a synonym of translation. In a double-entry pupils-strategies chart, for each student I have indicated with value 1 the strategies used and with value 0 the unapplied strategies. The collected data were analyzed in a quantitative way, using the implying analysis of the variables of Regis Gras by means of the Chic 2004 software. Observing the following Chart of Similarities regrouping the two samples examined and analyzing all the data collected from the two samples, four typologies of main strategies emerge:



- Identifies the translation in relation to the *x* axe and the translation in relation to the *y* axe (i.e. answers the A) and B) questions correctly) but confuses the concept of *translation* with *reflection* in C) and D) questions although answers the E) question correctly.
- Identifies the reflection in relation to the *x* axe and the reflection in relation to the *y* axe (i.e. answers the C) and D) questions correctly) but confuses the concept of translation with reflection in A) and B) questions .
- Draws the chart but does not say if there is a reflection or a translation;
- Does not draw the chart of the tune bit but affirms there is a reflection

From this first quantitative analysis I have stressed that in general a concept mistake is present between the terms translation and reflection both for musicians and non musicians.

To trace possible different behaviours I analyzed both samples separately and from the analysis of the following chart of similarities came out that the non-musicians sample

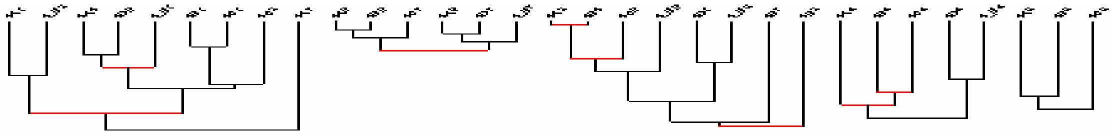


chose three typologies of main strategies:

- Identifies the translation in relation to the *x* axe (4A1) and the reflection in relation to the *y* axe (4D1), but considers the other charts as *identities*, that is neither *translation* nor *reflection*;
- Identifies the translation in relation to the *y* axe (4B1) and the reflection in relation to the *x* axe (4D1) and the reflection in relation to the origin (4E1) and is anyhow able to draw the chart but confuses the concept of *translation* with *reflection*;

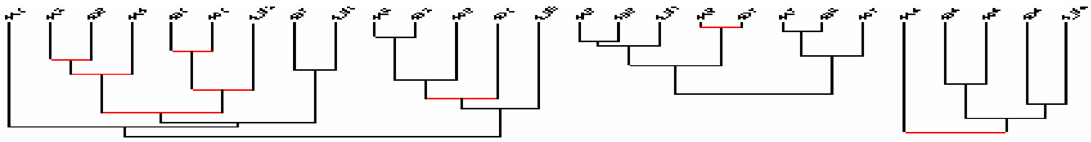
- Does not draw the chart of the tune bit but affirms there is a translation.

From the analysis of the chart of similarities of the musicians sample five typologies of main strategies emerged:



- Identifies the translation in relation to the *x* axe (4A1), the translation in relation to the *y* axe (4B1), the reflection in relation to the *x* axe (4C1) and the reflection in relation to the origin and is anyhow able to draw the chart but confuses the concept of *translation* with *reflection*;
- Always confuses the concept of *translation* with *reflection*;
- Confuses the concept of *translation* with *reflection* but is able to identify the reflection in relation to the y axe;
- Draws the chart but does not say if there is a reflection or a translation;
- Does not draw the chart of the tune bit but affirms there is a reflection.

Since the musicians sample is formed both by instrumentalists (winds and strings) and pianists I analyzed the sub-sample formed by pianists only and from the analysis of the following chart of similarities we can see that they chose three typologies of main strategies:



- Identifies the translation in relation to the *x* axe (4A1), the translation in relation to the y axe (4B1), the reflection in relation to the *x* axe (4C1), the reflection in relation to the y axe (4D1) and the reflection in relation to the origin (4E1) but confuses the concept of *translation* with *reflection*;
- not draw the chart of the tune bit but affirms there is a reflection or a translation but confuses the concept of *translation* with *reflection*;
- Draws the chart but does not say if there is a reflection or a translation.

CONCLUSION

From the analysis of the answers given to the set of questions proposed I have been able to find out a different behaviour, between the two samples taken into consideration, in facing the solution of problems concerning the geometrical transformations. In general, both for musician students and non-musician ones, a concept mistake between the terms translation and reflection is present (which we can hypothesize is a “misconcept”) and this is found also in the first two sets of strictly geometrical questions. From a quality and quantity analysis of the sub-group of pianists has come out that the “misconcept” concerning the translation-reflection decoding is less present and this is due to the characteristics of the piano. In fact, it is an instrument which naturally obliges the performer to use both hands symmetrically effecting both translations and reflections which are learnt in an unconscious way through the neuro-tendinous receptors of the upper limbs. Besides the reading of the score takes place in a polyphonic way and this helps the global vision and perception of the musical language, differently from other musical instruments (winds and strings) which, being monodic, develop in the pupil a vision and perception of the musical language of a punctual type, i.e. as a linked sequence of sounds, which excludes the understanding of the organizational criteria of the sound material. *In the field of the theory of situations by G. Brousseau we* can assume to be in front of a learning teaching-situation including *non-teaching* situation. In fact, we have the learning teaching-situation in the teacher’s wish (mathematics or music) to transmit the *specific* *knowledge of the discipline* (geometrical transformation or musical instrument's technique and repertoire); at the same time the student passes from a knowledge to another; simultaneously, we have a *non-teaching* situation, because the teacher, transmitting the geometrical situation knowledge didn't have the intention to transmit the knowledge relative to the musical composition technique, and, vice versa, the teacher of musical instruments, while transmitting the knowledge of musical language (theoretical-practical) didn't have the intention to transmit the geometrical transformation. Indeed, along students' educational teaching process, the two disciplines knowledge are always transmitted separately, clashing with the strong link they have. Theoretical-experimental research in this field might in the future allow a curriculum organization aware of mathematics of music in music high schools and in music conservatoires. Finally, we can affirm that with the music's help it's possible, not only to see the possible applications of the geometrical transformations, but we can also hear the effect these could have on a melody; this, in my opinion, makes the study of geometry definitely more fascinating. Vice versa the knowledge of geometrical transformations allows to musicians to understand the deeper aspects of compositional process used for ages by composers in the different musical field.

NOTES

1. Doctoral Thesis, University of Bratislava (Slovak Republic- 2006), Advisor: Prof: Filippo Spagnolo.
2. The epistemological and historical – epistemological reflections are part of the thesis work, they will be taken into account according to the discourse context of this treatment.
3. The test build with tunes bit is completely invented and elaborate by Daniela Galante.
4. It's the distance between any sound of the tempered scale and its immediate subsequent, either in ascending sense or descending one. It is the shortest interval of our musical system and it corresponds to the half of a tone.
5. They are graphic symbols that fix the position of all the sounds in a stave related to a sound fixed before them.
6. In the Italian school system music theory is studied in junior middle schools and in the high school pedagogical section, while professional study of music is entrusted to state music conservatoires.

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A Cross-Cultural Comparison of Algebra 1 Students' Achievement

Sofokli Garo, Ph.D.; Professor of Education
"A. Moisiu" University, Durres, Albania sofokligaro@gmail.com

Abstract
The purpose of this research was to compare American and Albanian students' achievement in Algebra 1. The study compared algebraic solving abilities of 219 students in a city of Albania and 242 ninth-grade American students, residents of an American region. Albanian sample did not use calculators on the test. Of the American sample, 97 students used calculators on the test, whereas 145 did not use them. The three research questions addressed: (1) students' mastering of the overall algebraic achievement, (2) students' mastering of specific domains of algebraic understanding: knowing, applying, and reasoning, and (3) students' preference of algebraic strategies for solving word-problems. The study found that Albanian students outperformed American students on the overall achievement. However, American students who used calculators on the test significantly outperformed not only the American group who did not use calculators on the test, but also the entire Albanian sample. In addition, Albanian students scored significantly higher than their American peers both on 2 out of 3 cognitive domains and on using algebraic strategies.

Introduction Various studies have focused on cross-cultural comparisons in the field of school mathematics. This study was designed to make a contribution to this field by comparing Algebra achievement of ninth grade students in the U.S. and Albania. The topic of this study was Algebra 1 because this mathematics course is required in every high school curriculum of every culture or country that has education as a priority.

Achievement and its Assessment

The object of this study is conducive to mathematics achievement of 9th graders. According to Ruiz-Primo (1998), mathematics achievement may be conceived as students' abilities in two component domains: *understanding domain* and *strategic domain. Understanding domain* consists of acquisition of algebraic facts, procedures and concepts, whereas strategic domain has to do with the abilities how to present word-problem solutions.

Assessment of "Understanding Domain"

For assessing students' understanding, this study adopted the framework used by the Third International Mathematics and Science Study (TIMSS), in which experts divided the *understanding domain* into three specific domains: *knowing, applying, and reasoning* (Mullis, 2004). *Knowing* refers to recalling definitions and properties, recognizing/identifying algebraic relations and functions, and computing the values of algebraic expressions. *Applying* deals with formulating algebraic situations, modeling problems, selecting appropriate algorithms to solve routine problems, and interpreting given algebraic models. *Reasoning* deals with conjecting, analyzing, generalizing, justifying, and solving non-routine problems.

Assessment of "Strategy Domain"

The strategic domain can be assessed by examining mathematical models used by students, as they attempt to solve word problems. Students can communicate their explanations for a mathematical strategy or solution in a variety of models: numerically, verbally, diagrammatically, graphically, by tables of data or symbolically (with algebraic symbols or equations) (Shield & Galbraith, 1998).

Research Questions
The following research questions were based on the need for comparing students' overall achievement, achievement in specific domains of understanding and achievement in the strategic domain:

1. Is the difference between the mean scores in the overall algebra achievement of students in the U.S. and Albania significant?
2. Are the differences between the mean scores in each specific domain of algebra understanding significant?
3. Does the variable of "country" significantly predict the students' preference of algebraic strategies when addressing the algebra word problems?

Literature Review
In the absence of research involving a direct comparison between the U.S. and Albania, this study focused on other available studies, no matter whether they were domestic, international or multinational. More

specific information about the overall achievement of the U.S. students in algebra is obtained from the Trends in International Mathematics and Science Study (TIMSS), designed to measure students' literacy that is dependent on school curriculum. The first TIMSS was conducted in 1995. It shows that, out of 41 participating countries, the U.S. eighth graders were outperformed in mathematics by eighth graders of 27 countries (Beaton et al., 1996). The second TIMSS Study (TIMSS 1999) shows that U.S. eighth graders performed below the international average, even though they improved their mathematics results of the first TIMSS Study. TIMSS 2003, using stratified random samples representative of each country's population, assessed 8,912 eighth graders in 232 schools. In the content of algebra, the U.S. students performed above the international average. They outperformed their peers in 25 countries, on average, and were outperformed by students in 9 countries (NCES, 2005).

TIMSS also used a three-type cognitive skill categorization of items: knowing facts and procedures, using facts and concepts to solve routine problems, and mathematical reasoning. TIMSS 1999 found that students in the industrialized countries that were not grouped into the highest achieving cluster tend to show weaker scores in items that require reasoning skills. These industrialized countries include Canada, Australia, England, and the United States (Mullis et al., 2000). TIMSS 2003 study showed that in knowing domain, the U.S. performed above the international average, outperforming 30 countries. In this cognitive domain American students were outperformed by 14 countries, including 7 European countries. In applying domain, the U.S. outperformed 28 countries and were outperformed by 16 countries.

The only reliable information regarding Albania achievement in multinational studies is related to Albania participation in PISA 2000, where students of this country scored second worst in mathematics (OECD, 2001). The Institute of Pedagogical Studies in Albania recently conducted two studies to examine, among other things, students' work with algebra word problems given on the National Leaving Examinations. The findings showed that the vast majority of students preferred a numerical mode of representation, 37% of answers were in verbal and diagram mode, and only 11% were represented in an algebraic mode (Lulja, 2003).

Methodology and Instruments

The sample of American students was chosen from Grand Forks county, state of North Dakota. Four schools were selected in consultation with local education authorities to represent the full range of the county's high schools. The total number of students included in the Grand Forks sample was 242. The American sample consisted of two groups. The first group included 7 classrooms, where students did not use calculators during the test and 5 classrooms, where students were allowed to use calculators.

The sample of Albanian students was chosen from Durres region. This sample included one outstanding school in the city, two average schools in rural areas, and one school in the countryside. Of the four chosen schools, three were comprehensive and one was vocational. The sample consisted of all Algebra 1 students present on the first and second hour period on the day each of the four schools were visited and included 226 students.

A Texas publicly-released standardized test was administered to all Algebra 1 students. A combination of 9 multiple-choice items and 5 free-response items was used to assess students' overall achievement, achievement in each understanding domain (knowing, applying, and reasoning) and the achievement in the strategic domain of Algebra 1. Multiple-items required students to circle a letter to indicate one choice among five alternatives, each of which might be a number, a word, or a phrase. Free-response items required students to construct their own responses. Four items (1, 2, 5, and 7) of the test were classified to match the knowing domain (see table 2). Six items (8, 9, 13, and 14) were qualified to assess the applying domain. Four items (6, 10, 11, and 12) were classified. Initially, the 14 items of the test were scored dichotomously. The "correct" or "wrong" results were used to measure both students' overall achievement and achievement in each understanding domain. Then 3 of the 14 items (items 10, 11, and 12), consisting of word-problems, were used to measure students' strategic domain. The answers on these items were considered for the second time, whether they were algebraic or not algebraic, regardless of being correct or wrong.

Results, Findings and Conclusions

Research Question 1

The research question is addressed by taking into account that some of the U.S. students used calculators on the test and some did not. The descriptive statistics of American calculator users, American calculator nonusers, and Albanian students (who did not use calculators on the test) are given in Table 1.

Table 1. Mean Scores on the Overall Test by the U.S. and Albanian Students.

	US students			AL students		
	n	M	SD	n	M	SD
Calc. users	97	7.70	2.450	-	-	-
Calc. Nonusers	145	5.63	2.674	220	6.97	2.964
Entire sample	242	6.46	2.775	220	6.97	2.964

The results of t-test procedures show that the difference between the mean scores of: (1) American calculator users and nonusers is significant, (2) American calculator users and Albanian students is significant, (3) American calculator nonusers and Albanian students is significant, (4) American students and Albanian students is not significant. Table 2 shows the percentages of students in each country, who were able to answer each item of the Achievement Test successfully.

Table 2. Percentages of U.S. and Albanian Students who Were Successful on the Overall Test by Item.

	US	AL
Item Number	n = 242	n = 219
1. Computing the value of an algebraic expression	66.1%	80.1%
2. Identifying a quadratic function	64.9%	87.2%
3. Interpret the solutions of a quadratic equation	39.3%	46.5%
4. Solving a linear equation	59.5%	64.6%
5. Recognizing the graph of a linear function	63.6%	76.5%
6. Finding the equation that represents the rate of reading	45.0%	38.1%
7. Recalling the properties of a parabola	55.0%	74.3%
8. Using the concept of slope	48.8%	53.1%
9. Finding the algebraic expression of a given situation	43.0%	46.6%
10. Finding the lengths of three wire pieces	63.2%	46.6%
11 .Finding the number of boys and girls in a classroom	22.7%	19%
12. Finding the number of saving-months to buy a fridge	58.3%	38.1%
13. Finding the graphical interpretation of an inequality	14.9%	16.4%
14. Solving a linear inequality with absolute value	1.7%	7.1%

Research Question 2

Table 3 contains the descriptive statistics that characterize the specific domains of understanding. A two-way MANOVA was conducted to determine the effect of country and calculator on the three dependent variables of knowing, applying, and reasoning. MANOVA results indicate that (1) "knowing" significantly differs for "country", (2) "applying" significantly differs for "country", and (3) "reasoning" significantly differs for "country" [F (1, 465) = .5.599, p=.018] and for "calculator use". Another analysis, conducted separately for the U.S. sample, was intended to examine the effect of calculator use. The results of t-tests showed that the differences between the mean scores of the U.S. calculator users and nonusers were significant at the .001 level for the the three variables.

Table 3. Mean Scores on the Specific Domains of the Achievement Test by the U.S. and Albanian Students.

	Knowing			Applying			Reasoning		
Calculator	n	M	SD	n	M	SD	n	M	SD
Albania									
Yes	-	-	-	-	-	-	-	-	-
No	225	3.18	1.01	221	2.37	1.48	220	1.42	1.278
US									
Yes	97	3.07	1.01	97	2.49	1.17	97	2.13	1.222
No	145	2.11	1.21	145	1.79	1.28	145	1.73	1.180
Entire Sample	242	2.50	1.23	242	2.07	1.28	242	1.89	1.211

Research Question 3

Table 4 represents the percents of American students (either calculator users or nonusers) and Albanian students who gave algebraic solutions to each of the three word problems of the test. The last row of the table indicates the percentage of students that managed to solve algebraically at least one out of the three problems.

Table 4. Percents of Students who Used Algebraic Methods.

Item	Albania	US	U.S. Calc. users	U.S. Nonusers
	n=219	n=242	n=97	n=145
10	31.9%	3.7%	7.2%	1.4%
11	29.2%	8.3%	13.4%	4.8%
12	19.5%	23.1%	33.0%	16.6%
At least one item	44.2%	26.0%	38.1%	17.9%

Results of the regression analysis show that the two countries significantly differ with respect to using algebra in items 10 and 11.

Conclusions

This result shows that students of both countries have difficulties with learning algebra. Below are presented the conclusive remarks about the main differences that correspond to the research questions of this study: (1) on average, Albanian students outperformed American students. However, Albanian students were outperformed by the American group that used calculators on the test; (2) compared with American students, Albanian students scored higher on 2 out of 3 cognitive domains, namely, on the cognitive domains of "knowing" and "applying" and scored lower on the cognitive domain of "reasoning". However, the group of American students who used calculators on the test scored higher than the Albanian group of students in the cognitive domains of "applying" and "reasoning", but not on the cognitive domain of "knowing"; (3) Albanian students were more inclined than their American peers to use algebra for solving relational algebra word problems. This conclusion holds for the three American groups, namely, the calculator users, the nonusers, and the mixed group.

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Can Early Algebra lead non-proficient students to a better arithmetical understanding?

Sandra Gerhard
Goethe-University of Frankfurt am Main, Germany gerhard@math.uni-frankfurt.de

Abstract

In mathematics curricula teachers often find the more or less implicit request to link the taught subjects to the previous knowledge of the students, for example using word problems from everyday life. But in today's multicultural and multisocial society teachers can no longer assume that the children they teach have a more or less equal background and thus everyday live can have a very different meaning for different children. Furthermore there is evidence that good previous knowledge in arithmetic can hinder the approach to other mathematical subjects, like algebra. In this paper I want to provide a brief overview on how previous knowledge in arithmetic can affect student's access to algebra and therefore present an early algebra teaching project which introduces elementary school children to algebraic notation by measurement in an action-oriented way. Thereby the chosen approach to algebra explicitly does not come back to the student's previous arithmetical knowledge but additionally may support non-proficient students in obtaining more insight in the structure of calculations and hence may help them to have more success in solving calculations and word problems.

Introduction

In the German national curricular standards ("Bildungsstandards"), the guideline for the curricula of the German federal states you can read the following:

"Der Mathematikunterricht der Grundschule greift die frühen mathematischen Alltagserfahrungen der Kinder auf, vertieft und erweitert sie und entwickelt aus ihnen grundlegende mathematische Kompetenzen. Auf diese Weise wird die Grundlage für das Mathematiklernen in den weiterführenden Schulen und für die lebenslange Auseinandersetzung mit mathematischen Anforderungen des täglichen Lebens geschaffen."(KMK, p. 6)

"The mathematical education in primary school takes up, deepens and extends early mathematical everyday life experiences and develops basic mathematical competencies from those experiences. Thus the foundation is laid for learning mathematics in higher classes and for lifelong examination of the mathematical requirements of everyday life."(translation by the author)

Everyday life in mathematical education

There are two contrasting ways to combine everyday life with mathematics: Looking at everyday life and trying to find mathematical content or learning mathematical concepts and applying those to ones everyday life.

The former, which seem to be more in line with the quotation above, you can easily find in primary school textbooks. The German textbook "*Das Zahlenbuch*" for 4th graders for example shows a map of Germany to motivate distances (p. 10), a handicraftsman to motivate calculating with money (p. 22) and a recreational lake to motivate calculating with decimal numbers (p. 71). There is also a double page about Christmas (pp. 122/123) and Easter (pp. 124/125) and a page about the benefits of mathematics (p.126) showing among others a doctor, a retiree and a consumer advisor, all talking about why they need mathematics. In the textbook you also can find a lot of word problems which are linked to the alleged everyday life of children, like car inspections (p.66) or buying lentils (p.73).

Looking at the textbook brings up some questions: Is this everyday life of all children in our multisocial and multicultural society? Can you really find everyday life that all children have in common? Is it necessary to base mathematical education upon everyday life at all?

There is no doubt that the mathematical background of children, the similarities and differences which arise by reason of children growing up in different quarters of a town to the point of totally different cultural backgrounds should definitely be part of mathematical education. But if you look at the background of children in a today classroom, one can easily see that there are a lot of differences and that it is hard to find a similarity for all of the children. The one everyday life which fits for all children in classroom does exist.

Instead there is to find a way to look at the everyday life of every child in the classroom. A way of doing this can be the latter mentioned above, teaching mathematical concepts and letting the child apply those to its everyday life. But the question "How can you use this in your everyday life?" is hardly to find in textbooks and classrooms.

Some reasons for putting everyday life on hold

Teaching mathematical concepts without coming back to the student's previous knowledge of everyday mathematics and applying those concepts to everyday life later can be a way to cope with the different social and cultural backgrounds of primary school children. This is all the more important because, at least in Germany, children with migrational background are disadvantaged in the educational system (see Auernheimer 2003).

But there are some other reasons, why it can be a good way not to build on children's previous knowledge while teaching mathematical concepts. McNeil (2004) observed, that "the activation of existing knowledge can interfere with the acquisition of new information". She explicitly refers to pre-school knowledge as well. El'konin (1975) differentiates between theoretical scientific and empirical knowledge. Empirical knowledge designates knowledge children extract from their everyday experiences, while theoretical, scientific knowledge is knowledge on a higher level.

"The adult – the teacher – is the key figure and helps the child to develop ways of operating with objects through which he can discover their essential properties – those which constitute genuine concepts." (El'konin 1975, p. 48)

Hasemann & Stern (2002) addressed their research to the question how to foster mathematical understanding of lower achievers. They worked with 2nd graders on different programmes and drew following conclusion:

"Die Auswertung der Tests ergab, dass bei schwächeren Kindern das alltagsnahe Programm eindeutig am wenigsten bewirkte, während bei den Kindern, die das abstrakte Programm durchlaufen haben, der größte Leistungszuwachs zu beobachten war." (Hasemann & Stern 2002, p. 222)

„The evaluation of the tests shows, that the program close to everyday life definitely had the lowest effect on low-achievers while children who worked on the abstract program showed the biggest learning progress."(translation by the author)

Thus we are looking for an abstract teaching program that gives children tools that can aid them with solving mathematical problems of everyday life and also with solving the word problems in their text books. Thereby it is important that abstract does not mean doing it without concrete materials. If young children shall cope with abstract knowledge this knowledge has to be taught in an action-oriented way.

An unconventional way of teaching Early Algebra

In the first years of school children usually spend a lot of time with calculating with natural numbers. They also adopt strategies that cannot be transferred to calculations with decimal numbers or fractions. With a teaching experiment in the 60s Davydov (1975) chose a different approach to mathematical education. His idea was teaching the properties of numbers while already using the common algebraic symbolizations and before introducing numbers at all. Therefore he chose an action-oriented way by using direct comparison of magnitudes like length, area, volume, mass, time and so on. The children used concrete material like water containers for comparing volume and balance scales for comparing mass and learned to write down their findings with inequations. The question, how big the difference between the compared magnitudes was, lead to equations. Aided by the concrete material, the children learned to manipulate and to interpret different linear equations. After the children have learned dealing with the equations properly numbers are implemented by introducing a unit. This way of implementing numbers not only works for natural numbers but for the whole real numbers.

Davydov's idea was taken on by the MeasureUp-Program (see Dougherty & Venenciano 2007), which showed that children can successful deal with abstract equations, achieve a deep understanding of properties of numbers and use them effectually for solving word problems.

Early Algebra as a guideline for word problems

Certainly starting mathematical education without using numbers would be a big change for the German school system and would hardly become accepted by teachers and parents. But the idea of teaching the abstract properties of numbers by the aid of concrete comparison of magnitudes while firstly excluding numbers deserves a closer look in terms of its usefulness for helping children deal with word problems and mathematical problems of everyday life. The main questions are: Will the MeasureUp-Program work for school children of different grades although the already have been introduced to numbers and arithmetical operations? Can they transfer the knowledge about abstract equations to mathematical problems of everyday life? And can this program lead non-proficient students to a better arithmetical understanding?

In a first project we modified the MeasureUp-Program for the use in a few weeks lasting teaching-experiment in grade three and fife. After the children have been introduced to the comparison of length, area and volume and the use of letters they learned how to set up and manipulate equations. To connect the abstract equations without numbers with word problems we gave the children word problems that contained letters instead of numbers. We the asked the children to make up word problems that are appropriate to given letter equations. Therewith we keep up our intention to firstly teach mathematical concepts and applying those to everyday life not till the children can handle the concept properly.

First results

Children of a 5th grade were given the equations L - R = U and N + M = B – J and asked to invent fitting word problems. Below we want to give some examples. The equation L – R = U resulted in the following word problems:

Lena geht in den Laden und will 10 Buntstifte von Pelikan kaufen. 10 Stifte = R. Doch es gibt noch so viele schöne andere, dass Lena noch mehr kauft. Sie kauft 23. Was für ein Wert hat U? Wie viele Stifte kauft Lena mehr? (Angelina)

Lena walks into a shop and wants to buy ten coloured crayons. 10 crayons = R. But there are so much other pretty ones, that is why Lena buys some more. She buys 23. What is the value of U? How much crayons more did Lena buy?(translation by the author)

Although the children learned to use letter equations only in the context of geometric magnitudes like length, area and volume Angelina chose the context of money for their word problems. We assume they chose money because it plays a major role in their everyday life and therewith a much bigger role than geometric magnitudes. The details on the brand of the crayons and the reason why she bought more are evidence that here we see an episode that really has happened or could happen in her life. The word problem fits to the equation which is revealed by Angelina as she is relating some of the letters to the values. Other children only used numbers or only used letters:

Kim hat 20 Blumen, sie verliert 5. Wie viele hat sie noch? (Axel)

Kim has 20 flowers. She loses 5. How many are left over? (translation by the author)

Horst hat L Boote geschnitzt. Ihm fallen R Boote ins Wasser. Wie viele hat er noch übrig?

Horst carved L boats. R boats are falling into the water. How many are left over? (translation by the author)

The above examples show that the children not only use the letters for magnitudes but also for numbers of objects.

The equation N + M = B – J resulted in the following word problem:

Lara geht zu Faberkastell und will einen Radiergummi von 2,00 € kaufen und einen Bleistift von 3,00 €. Sie hat aber nur 5,50 € mit. Reicht das Geld und wenn ja, wie viel bekommt sie zurück? (Lana)

Lara walks to Faber-Castell and wants to buy an eraser of 2.00 € and a pencil of 3.00 €. She only has 5.50 € with her. Is this enough money and if yes, how many money will she get back? (translation by the author)

The word problem fits to the equation. Lana invents values for N and M (2.00 € and 3.00 €) and B (5.50 €) and wants to know how big J is. She as well implicitly writes down, why it is important for her to know how big J is: she wants to know, if she has enough money for her buying.

Perspective

The next step is to explore if children will and can use their knowledge about abstract symbolic equations for solving word problems only containing numbers and no letters. First observations showed that low-achieving children who have not been able to solve a word problem directly came back to abstract symbolic equations. For example a low-achieving 3rd grader's first reaction after reading the word problem "A street has length 845 m. Hans has already walked 220m. How far does he still have to go?" was "I want to do that with letters."

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Problems to put students in a role close to a mathematical researcher

Nicolas Giroud, PhD Student,
Maths à modeler team, Fourier institute, University of Grenoble 1, Grenoble, France
nicolas.giroud@ujf-grenoble.fr

Abstract

In this workshop, we present a model of problem that we call Research Situation for the Classroom (RSC). The aim of a RSC is to put students in a role close to a mathematical researcher in order to make them work on mathematical thinking/skills. A RSC has some characteristics : the problem is close to a research one, the statement is an easy understandable question, school knowledge are elementary , there is no end, a solved question postponed to new questions... The most important characteristic of a RSC is that students can manage their research by fixing themselves some variable of the problem. So, a RSC is completely different from a problem that students usually do in France. For short : there is no final answer, students can try to resolve their own questions : a RSC is a large open field where many sub-problems exist; the goal for the students is not to apply a technique: the goal is, as for a researcher, to search.

These type of situations are particularly interesting to develop problem solving skills and mathematical thinking. They can also let students discover that mathematics are "alive" and "realistic".

This workshop will be split into two parts. First, we propose to put people in the situation of solving a RSC to make them discover practically what is it. After, we present the model of a RSC and some results of our experimentations.

Introduction

In 1969, at the first International Congress on Mathematical Education, Hans Freudenthal said in an address:

> *Mathematics is more than a technique. Learning mathematics is acquiring an attitude of mathematical behavior.*

In this workshop, we present a model of situation the goals of which are not to learn a specific mathematical notion but to learn mathematical behavior and especially mathematical thinking/skills related to problem solving. By mathematical thinking/skills related to problem solving, we mean : formulating conjectures, carring out experimentations, posing new problems, proving, defining new objects, modelling... So, here we propose a model of situations that can let students work on that.

As our goal is to let as many students as possible work on mathematical thinking, we also build our situations with the aim that mathematical notions are not an obstacle to understand the problem and to start the research.

The sequence of the workshop

First, we propose to the participant to try to solve a mathematical problem which satisfies the characteristics of our model. In the second step, we make a didactical and epistemological analysis of the problem which enable us to identify some characteristics of our model. Finally, the model will be given and we will present the results of our experimentations. At the end, if we have enough time, we will also present others situations.

An example of problem :

Consider a garden which is a "piece" of grid. There are flying bugs which land in our garden and it is riling us!

So, we decide to put traps (one trap takes one case) in our garden to stop them. But, unfortunately, a trap costs a lot of money. So, we want to use the shortest number of trap.

We give some examples where we use the garden of figure 1 and the bug of figure 2.

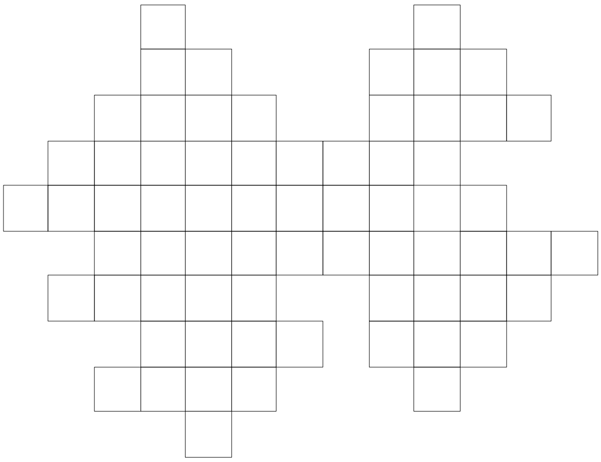

Fig. 1: An example of garden

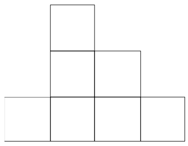

Fig. 2: An example of bug

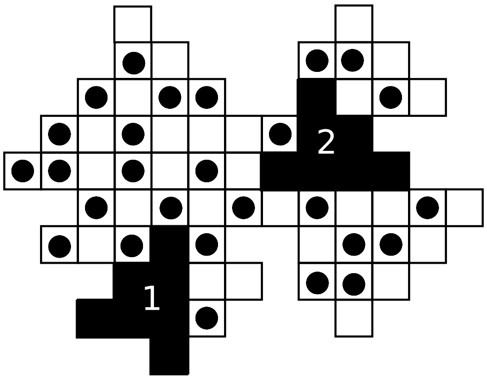

Fig. 3: A location of traps where a bug can land.

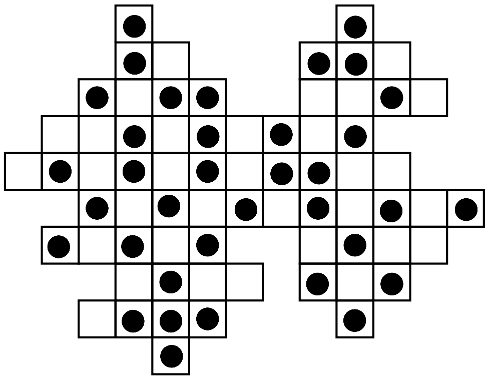

Fig. 4: A location where any of the bug can land

In Figure 4, the location of traps enables any of the bugs to land, but can we do better ?
This problem is linked to a research problem. Indeed, it was inspired by Pentomino Exclusion problem due to Gollomb (1994) and partially solved by Bosh (1998) and; Gravier and Payan (2001) and Moncel (2007).

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Toward Calculus via Real-time Measurements

Tine Golež, MSc
Physics teacher
St. Stanislav's Institution, Ljubljana, Slovenia
tine.golez@guest.arnes.si

Abstract

Several years of my experiences in the use of real-time experiments are now upgraded in order to enhance also the teaching of mathematics. The motion sensor device enables us to get real time *x(t)* and *v(t)* graphs of a moving object or person. We can productively use these graphs to introduce differentiation on visual level as well as to show the integration procedure. The students are fully involved in the teaching as they are invited to walk in front of the sensor. This approach motivates them by the realistic aspects of mathematical structures. The method could help to fulfill the credo of teaching: comprehension before computation. The steps of such an approach are explained and discussed in further detail below.

Introduction

Computers are more and more involved in the teaching. Physics teachers widely take advantage of them. It is important not to use them only for various kinds of physics simulations and applets but also as a part of measuring system. There is no doubt - the experiments should be the essential part of the teaching of physics. Virtual realty can not and may not replace them.

However, according to my experiences the use of computer has not reached its full potential in the teaching of mathematics. What could make mathematics more realistic, alive and accessible than real-time experiments? The answer to this difficult question is a major challenge for the mathematics teacher. Therefore I strongly suggest and prove by praxis that the physics teacher should support his colleagues – math teachers – in introducing some real-time measurement approach in order to enhance the teaching of mathematics. One of the most versatile equipment is motion sensor device.

Real-time measurements

An example *par excellence* of novel measurement techniques that can be used for this purpose is the so-called motion sensor, a device which uses ultrasonic pulse technology to measure the object's position. An ultrasonic transducer generates 40 kHz sound pulses and the device measures the time it takes for each pulse to travel out, bounce off a target and travel back to the sensor. The travel time of the ultrasound pulse is proportional to the distance. By connecting the sensor to a PC, it enables us to measure the position of a moving body; such as student walking back and forth in a straight line in front of the sensor, or a ball falling under the sensor. The computer plots real-time graphs of such linear motion.

The emitter can emit ultrasonic pulses up to 100 times per second, which is sufficient for a fast moving object. Some additional features like using two devices simultaneously to plot the path of a an object moving in a plane or following two objects during collision events increase the teaching value of this equipment. The necessary software was developed independently all over the world by firms and individuals. In the case of Slovenia [1, 2] its development was supported by the state and therefore it is available at no charge for all schools. The apparatus itself is in the price range of 120 EUR.

It is my opinion the "Homo Sedens" – a term which almost perfectly corresponds to today's student. Students spend the majority of the day in the classroom, sitting and listening to the lectures. Teaching is no longer a two-dimensional (blackboard) activity as it is still commonly practiced. For the best learning environment the teacher must find creative ways to engage the whole student. I have been successful using the motion sensor device during my lessons, which allows students to stand up and walk during learning. As a student is walking in front of the motion sensor he is active in creating changes and these changes (x(t)) are simultaneously displayed and seen on the screen. If he stops for several moments he/she can observe that the independent variable (time in this case) run actually independently – the position just doesn't change (Fig. 1).

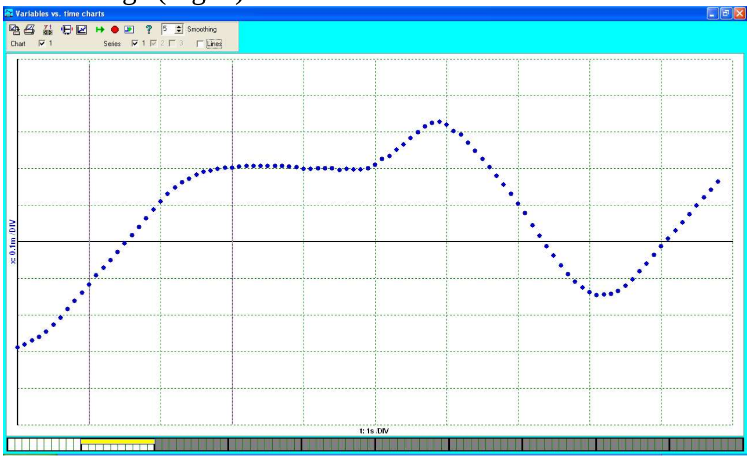

Fig. 1. Linear motion of a walking student; graph x(t) obtained by motion sensor connected to a PC.

Graph analyses-precursor of calculus (derivation and integration)

At the age of 15 or 16, which corresponds to the second year of high school in Slovenia, students are able to analyze a graph. As a result, they observe and examine the two graphs (*x* vs. time and *v* vs. time) for an object with variable velocity. They soon determine that the slope of the curve *x(t)* is exactly the velocity at that instant. Looking at the position graph, *x(t)*, they are able to sketch the velocity graph, *v(t)*. It should be noted that this is a qualitative task (Fig. 2).

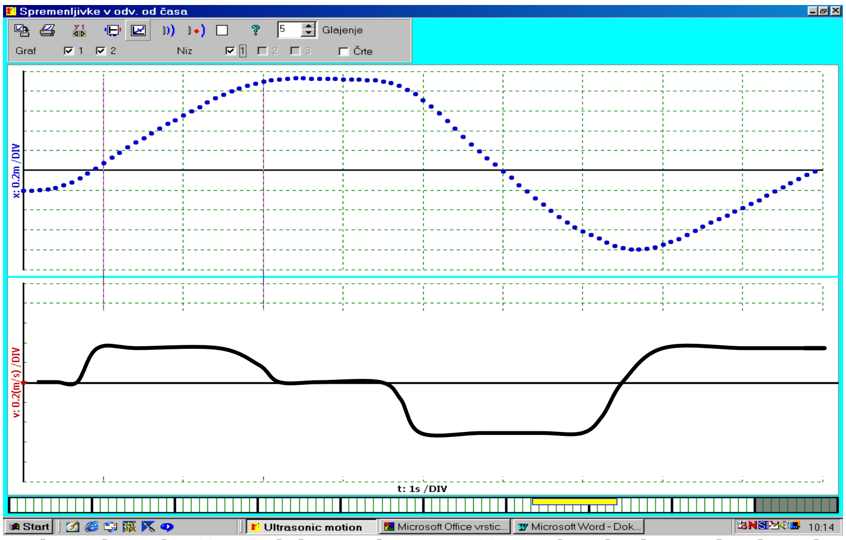

Fig. 2. A student has plotted *v(t)* graph by analyzing *x(t)* graph. The latter displayed *v(t)* graph computed by the measurement system confirmed the student's prediction of *v(t)* graph.

The quantitative task is to draw a tangent and calculate its slope in order to determine the instantaneous velocity [3]. The result is checked by the graph *v(t)*, plotted by computer. However, taking into account the geometric approach one can not expect that all the students will calculate exactly the same coefficient (slope). I am convinced these activities are the right moment to tell them that this process will be called differentiation and that it will be fully explained in mathematical terms two years later during their mathematical lessons (Fig. 3).

As the reverse process is also possible, we will use the opportunity to deal with it, too. Let us now focus on the velocity vs. time graph. We choose a time interval and observe the area under *v(t)* graph during the chosen time interval, Δt. Let us first consider the case of non-

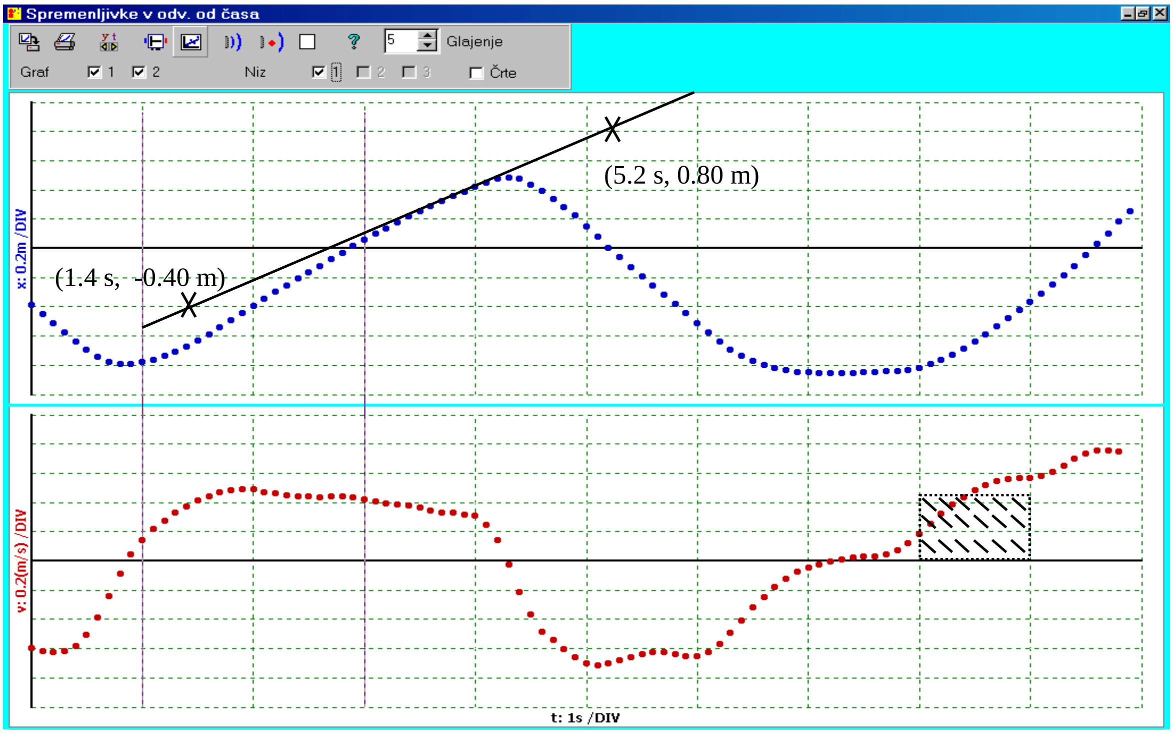

Figure 3: The slope of the curve *x(t)* is defined as the slope of the tangent at that point. The calculated slope at t = 4,0 s equals the instantaneous velocity at that instant ($\Delta x/\Delta t$ = 0.32 m/s). Graph *v(t)* confirms the result. The reverse process is shown on *v(t)* graph. The displacement during the time interval is determined by calculating the area (in our case Δx = 1,0 s·0,45 m/s = 0,45 m)

uniform motion, Fig. 3. The graph is not a straight line. We estimate the average velocity during the time interval and plot a horizontal line; therefore, we obtain a rectangle which has approximately the same area as the area under *v* vs. *t* graph during that time interval. The area of the rectangle is calculated and is equal to the displacement (during the same time interval). We can check our result using *x(t)* graph: $\Delta x = x_2 - x_1$. If the curve *v(t)* is under the time axis (negative velocity), the "area" is negative; it corresponds to a backwards displacement. Again, a physics teacher has an unique opportunity to explain to his students the importance of such a procedure, and one day they will call it integration.

But there is another quantity defined in kinematics. It is the acceleration. This excellent equipment enables us also to measure the most known acceleration, the acceleration of gravity. In addition, we can use very ordinary ball to measure it, no special physics equipment is needed.

The acceleration is defined as the rate of change in velocity. Hence we simply drop an ordinary volleyball or basketball from rest under the motion sensor, Fig. 4. It is right to assume the air resistance is negligible. The system displays both graphs (*x(t)* and *v(t)*). As the ball rebounds several times from the floor, a part of this motion is like an object thrown straight up. During one bounce it continues to move upwards for a certain time *t* and then drops back to the floor. Graph *x(t)* clearly shows the two time intervals, upwards and downwards, are the same. At the same height the instantaneous speed is the same. The velocities are opposite, as we can see from graph *v(t)*. And finally, it is possible to measure the acceleration – by calculating the slope of the graph *v(t)*. It is the same all the time, regardless of the ball moving upwards or downwards … with the exception of a short time interval when the ball hits the floor. What about graph *a(t)* for this experiment? The brightest students can draw it, so I expect the readers of this article will find this task as a minor challenge.

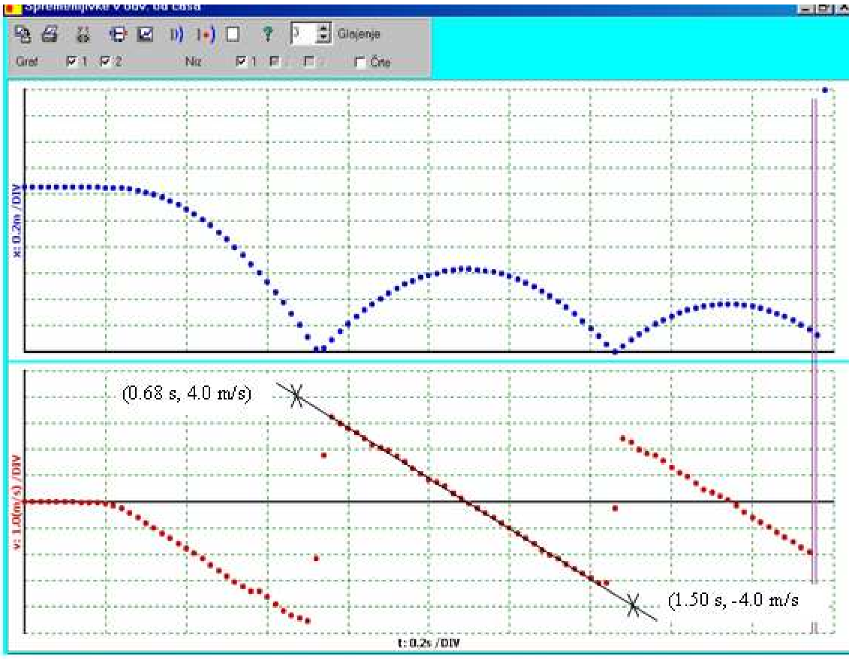

Figure 4. An ordinary ball was held under the motion sensor and dropped from rest. It bounced twice from the floor during the measurement. Graphs show that the instantaneous speed (= the magnitude of the instantaneous velocity) at points of equal elevation in the path is the same regardless of the ball moving upwards or downwards during the bounce (e. g., compare t = 0.80 s and t = 1.40 s). In addition, during one bounce the ball slows from the initial upward velocity to zero velocity. At the highest point it changes its direction of motion. Certainly, it experiences the same acceleration the way downwards. The acceleration, which is the rate of change of velocity, is constant. Therefore this part of the v(t) graph is linear. The slope of the line equals the acceleration. As calculated for this case:

$$a = \frac{\Delta v}{\Delta t} = \frac{-4.0\,ms^{-1} - 4.0\,ms^{-1}}{1.50\,s - 0.68\,s} = -9.76\,ms^{-2} \approx -9.8\,ms^{-2}$$

The software can also plot a tangent to the curves *x(t)* and *v(t)* and display its coefficient (slope). However, according to the expressed *credo* we use this option only after the students have mastered the same procedure themselves.

Once these basic experiments have been performed, it is possible to play with other options. The program allows us to select the place of each graph. In addition, it is possible to hide the legend. Therefore, students can check their understanding by “hidden graphs”. For example, we measured a pendulum motion. The three graphs were displayed without legends and not in the usual order (*x(t), v(t), a(t)*), Fig. 5. One can assume the first graph is *x(t)* and then examine its slope looking for the *v(t)* graph.

If one of the two graphs could be *v(t)*, the task is not finished yet. Now it is time to prove that the last graph corresponds to the slope of the second one. If not, one would simply assume the graph in the middle could be *x(t)* and repeat a similar procedure until the two consecutive graphs are “in slope relationship with the previous”.

Transfer from physics to mathematics

Mathematics is a compulsory subject for all students in the last year of gymnasium in Slovenia, while physics is already an elective subject. All students must pass a state prepared exam called Matura. Mathematics is one of three compulsory subjects, which gives students additional motivation to fully engage during the lessons.

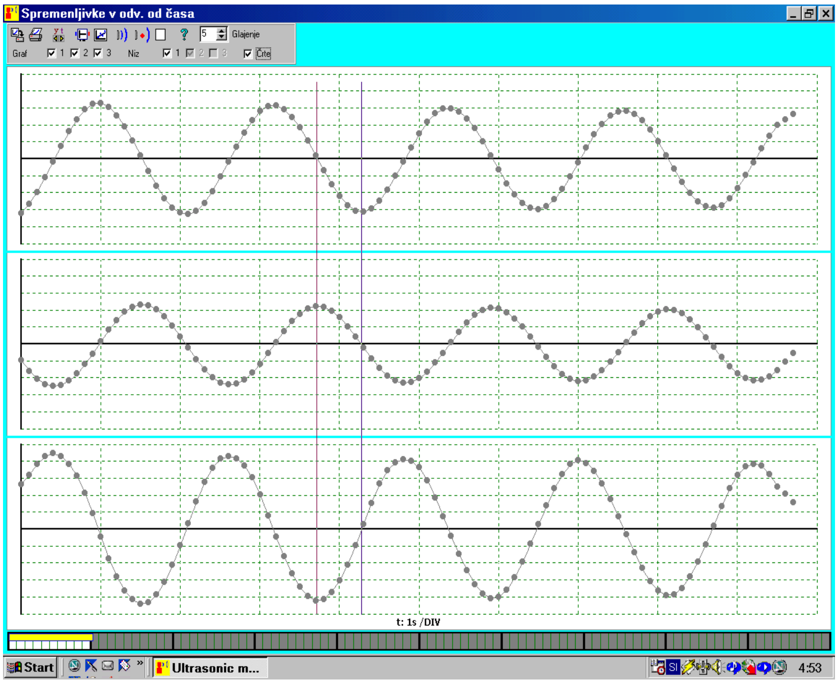

Figure 5: Motion sensor analyzed pendulum motion. The graphs *x(t)*, *v(t)* and *a(t)* are not displayed in this order. Students must find out the legend of each graph by investigating their slope and interrelationship between the graphs.

My colleagues, mathematics teachers at my school, are fully aware of the activities we are carrying out during the physics lessons. Consequently they are not at all reluctantly when I offer a combined lesson. I bring motion sensor at their lessons and students repeats some visual level calculus activities. Certainly some things can be done at higher level.

The bouncing ball experiment (Fig. 4) is now revisited using the advanced mathematical tool, calculus. First we find out the equation of parabola which corresponds to the first rebound. This equation is x(t). As the students are now capable of using derivative techniques they find the derivative of the parabola equation. Next derivative, dv/dt corresponds to the acceleration. They find out that the acceleration is constant and its value is exactly the acceleration of gravity [4].

Examining the same experiment as two years earlier but at higher level with a new mathematical tool is a superb longitudinal (regarding time) and transversal (regarding two subjects: mathematics and physics) approach. Such an approach will help to create less parceled knowledge. The abundances of connections put more logic and excitement in the learned topics and therefore helping to make these subjects more "alive", more "realistic" and more "accessible".

Conclusion The international ScienceMath Comenius project is the framework which strongly supports the cooperation between mathematics and other subjects. The project offers me both financial support as well as some directions to fulfill the ideas of combined lessons. I am convinced that such cooperation is a logical step. We must admit that during the developing of calculus (Newton, Leibnitz) there was much more connection between these two sciences. The curriculum of both subjects will soon bring more demand toward connecting both subjects via combined lessons.

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Problem Fields in Elementary Arithmetic

Günter Graumann (Germany)

Abstract Working with problems and making investigations is an activity one has to learn already very early. Therefore in primary school children should not only learn concepts and solve given tasks. They also should find out knowledge and reasons by themselves. Here you will find some problem fields in elementary arithmetic within children of primary school can make different investigations and find as well as give reasons for special statements. The topics concerned are partitions of numbers, sums of consecutive numbers, figured numbers, sequences and chains, table of hundred and number-walls.

Introduction

Working with problems and making investigations is an activity one has to learn already very early or better said one has to preserve it from early childhood. Therefore in primary school children should not only learn concepts and solve given tasks for example with using algorithms for computing with natural numbers. They also should find out knowledge and reasons by themselves and see that one can find different ways to work on a problem. The emphasis lies here on "find" and there are many things you can find out within mathematics already in primary school. Such discoveries may be not new for the teacher, it is only important that the children see it as discovery from their own. But sometimes even children can make discoveries nobody has done before or find a theorem for which nobody knows a proof until now. This all together can submit a more correct view of mathematics and support general aims like mathematical creativity or ability of structuring. Moreover from theory of learning we know that people keep things best by making discoveries and find connections by themselves.

In primary school mathematics there are many themes children can work on by themselves. Here there will be named some ***problem fields*** which start with a generating problem and by varying special aspects can lead up to different mathematical knowledge and conjectures.

1. Partitions of numbers in grade 1 or 2

One aim of arithmetic in grade 1 is to get good knowledge about the first natural numbers especially the connections between them. From empirical experiences we know that most children entering school know the numbers from 1 to 10. So after securing this knowledge already in the third or forth week in school it might be a good idea to let the children work on a little problem field concerning sums. For this we give every child a set of 5 (later on 6 and 7) concrete elements with the task to *find all possible partitions* of this set and to find on this way (from enactive via iconic to symbolic working) ***all sums the given number can be split into***. While discussing the results in the whole group with structuring them (by help of the teacher) we could get the following sums:

5 = 1+4 = 2+3 and 5 = 1+1+3 = 1+2+2 and 5 = 1+1+1+2 as well as 5 = 1+1+1+1+1
6 = 1+5 = 2+4 = 3+3 and 6 = 1+1+4 = 1+2+3 = 2+2+2 and 6 = 1+1+1+3 = 1+1+2+2 and
6 = 1+1+1+1+2 as well as 6 = 1+1+1+1+1+1
7 = 1+6 = 2+5 = 3+4 and 7 = 1+1+5 = 1+2+4 = 1+3+3 = 2+2+3 and 7 = 1+1+1+4 =1+1+2+3
and 7 = 1+1+1+1+3 = 1+1+1+2+2 and 7 = 1+1+1+1+1+2 as well as
7 = 1+1+1+1+1+1+1

We are completing this with the partitions of the numbers 2, 3 and 4 (here each child can be asked to find all sums by itself):

2 = 1+1
3 = 1+2 and 3 = 1+1+1
4 = 1+3 = 2+2 and 4 = 1+1+2 as well as 4 = 1+1+1+1.

By looking at the results the children can find out first that the number of sums is increasing if the basic number is rising up. This of course everybody would expect. But the amount of simple sums (i.e. partitions in only two parts) is the same for an even number and its following number and always half of the even number. A reason for this some of the children can find by themselves[1] and clear it with the others. The amounts of triple sums and so on we only can notice but not find reasons for them[2].

[1] The reason has to do with the rule of commutativity for the addition which some of the children already can explain on concrete examples I suppose.

[2] This is a problem for mathematicians within number theory. The above notations give a hint to find a solution e.g. for triple sums by looking out first all triple sums with 1 as beginning number, i.e. 1+1+(n-2), 1+2+(n-3), …

The children in this stage can produce a table and learn to work with tables via questions like "How many sums of four parts do have the number six?" They also can make conjectures (and give reasons for that) like: "For any basic number we can make a sum consisting only of ones and this is possible only in one way" or "For any basic number bigger than 2 we can make a sum consisting only of ones except one two and this is also possible only in one way" And they also can contradict the conjecture that in each row going from left to right the amount is increasing (see especially the row of 7).

Basic number	simple sums	sums with three parts	sums with four parts	sums with five parts	sums with six parts	sums with seven parts	all sums
2	1	-	-	-	-	-	1
3	1	1	-	-	-	-	2
4	2	1	1	-	-	-	4
5	2	2	1	1	-	-	6
6	3	3	2	1	1	-	10
7	3	4	2	2	1	1	13

2. Sums of consecutive numbers in grade 2 or 3

Another problem field with opposite handling could be the working with ***sums of two or three consecutive numbers***. We start with 1+2 = 3, 2+3 = 5, 3+4 = 7, 4+5 = 9 and can discover the following statement: "Only odd numbers do appear as results for all conceivable sums of two consecutive numbers". A reason for it can be found in different ways. For example we start with the above given four first sums. Then always the next sum gets two more - one for each part – so that we can reach only odd numbers. Or one can say that two consecutive numbers always consist of an even and an odd number and the sum of an even and an odd number always is an odd number. We also can represent the two parts of the sum as rows of dots which have nearly an equal length, only one has one dot more. So the sum is an even number plus one, i. e. it is an odd number. All these named reasons start with concrete examples but then lead to arguments which are independent from these examples and hold for all natural numbers.

We get a different conjecture if we take the sum of three consecutive numbers. Starting with 1+2+3 = 6 we always get three more. Therefore with sums of three consecutive numbers we get the multiples of 3. Moreover we can find out that the sum of three consecutive numbers always is equal to the threefold of the middle number. For this we can find a reason by shifting one unit from the third number to the first one (e.g. also in geometric representation).

After this we may look out for sums of four or five or more consecutive numbers. As variation the children also can look out for sums of only odd consecutive numbers or of consecutive numbers within the row of the multiple of a fixed number.

If we sort out the sums of two, three, four, five, … consecutive numbers starting always with 1 then we find the sequence of the so-called triangle-numbers which lead us to another problem field.

3. Figured numbers in grade 3 or 4

If we represent a number in a set with dots or little circles in geometrical shape then we speak of a figured number. Most well-known are the square-numbers which can be represented in a squared shape. Also known already with Pythagoras are the triangle-numbers which built an isosceles triangle or a right-angled equilateral triangle depending on the location of the dots.

Square-number • • •   Triangle-number • or •
• • •   • •   • •
• • •   • • •   • • •

For the symbolic representation of these figured numbers children easily find out that

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until 1+(n-1)/2+(n-1)/2 if n is odd and until 1+(n-2)/2+n/2 if n is even. Then all triple sums with 2 as beginning number like 2+2+(n-4), 2+3+(n-5), … and so on.

In elementary combinatorics one can find that for the number P(n/m) of partitions of a natural number n in m parts (with $n \geq 2$ and $n \geq m$) the following formula holds: *P(n/m) = P(n-1/m-1) + P(n-m/m)*. Together with the above found numbers we now can find the some next numbers like P(8/3) = 5, P(9/3) = 7, P(10/3) = 8, P(11/3) = 10, P(12/3) = 12 and so on. For all numbers n < 25 I could prove that for P(n/3) the explicit formula $[(n^2+3)/12]$ holds whereat [ ] is the floor function from C.F. Gauß.

- ***square-numbers*** are numbers like $1 \cdot 1$, $2 \cdot 2$, $3 \cdot 3$, $4 \cdot 4$, $5 \cdot 5$, … ,
- ***triangle-numbers*** can be described as 1, 1+2, 1+2+3, 1+2+3+4, … .

By putting the figures of *two triangle-numbers together* they also can find out that e.g. $2 \cdot (1+2) = 2 \cdot 3$, $2 \cdot (1+2+3) = 3 \cdot 4$, $2 \cdot (1+2+3+4) = 4 \cdot 5$, $2 \cdot (1+2+3+4+5) = 5 \cdot 6$, … or that the figures of *two consecutive triangle-numbers together* built a square-number like $1 + (1+2) = 2 \cdot 2$, $(1+2) + (1+2+3) = 3 \cdot 3$, $(1+2+3) + (1+2+3+4) = 4 \cdot 4$, … .

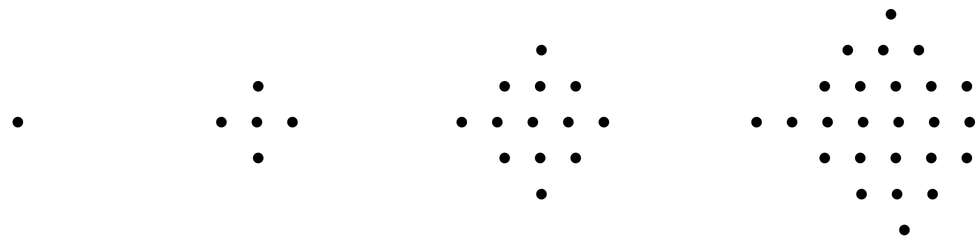
With looking at the square-numbers we e.g. can find out (and see at the figures) that the *difference of two consecutive square-numbers* built the sequence of odd numbers, at which any odd number is equal to the sum of the two basis numbers (or to the double of the smaller basic number plus one), i.e.

$2 \cdot 2 - 1 \cdot 1 = 3 = 2+1$, $3 \cdot 3 - 2 \cdot 2 = 5 = 3+2$, $4 \cdot 4 - 3 \cdot 3 = 7 = 4+3$, $5 \cdot 5 - 4 \cdot 4 = 9 = 5+4$, …
$= 2 \cdot 1 + 1$, $= 2 \cdot 2 + 1$, $= 2 \cdot 3 + 1$, $= 2 \cdot 4 + 1$,

From this we can find out that a square-number can be seen as sum of odd numbers like $1 \cdot 1 = 1$, $2 \cdot 2 = 1+3$, $3 \cdot 3 = 1+3+5$, $4 \cdot 4 = 1+3+5+7$, $5 \cdot 5 = 1+3+5+7+9$, … .

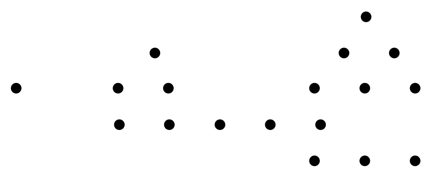
A *sum of two consecutive square-numbers* thus can be described also as sum of odd numbers which first increase and then decrease in the following way:

$1 \cdot 1 + 2 \cdot 2 = 1+3+1$, $2 \cdot 2 + 3 \cdot 3 = 1+3+5+3+1$, $3 \cdot 3 + 4 \cdot 4 = 1+3+5+7+5+3+1$, … . This symbolic description leads us to the following figured description:

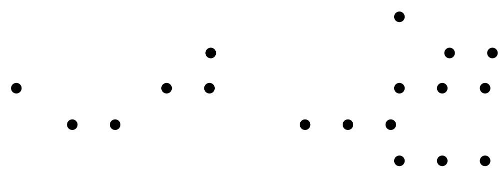


If we put *together a square-number and the triangle-number with side-length one less than that of the square-number* then we will get a ***five-angled-number*** (looking like a front of a house) or a ***trapezium-number*** (by using the right-angled form of the triangle-numbers). The amount of its dots can be computed in the following way:

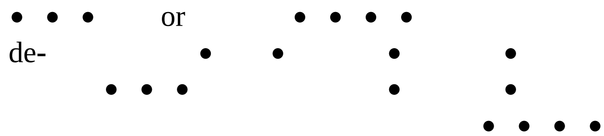
$1 \cdot 1 + 0 = 1$, $2 \cdot 2 + 1 = 5$, $3 \cdot 3 + (1+2) = 12$, $4 \cdot 4 + (1+2+3) = 22$, $5 \cdot 5 + (1+2+3+4) = 35$, and so on.



or



Building the *difference of a square-number and the square-number which lies two places before it* then we get a figured ***squared frame*** which in symbolic way with n as side-length can be described as $n^2 - (n-2)^2$.



Parting these frames into the sides (each without one vertex) we can find out that the sequence of these frame-numbers build the multiples of 4 with formula $4 \cdot (n-1)$. Parting the frames into two opposite sides (each including both vertices) and the two "rest-sides" (each without both vertices) then we get a third formula for the frames: $2 \cdot n + 2 \cdot (n-2)$.

Another task for the children[3] could be the working and looking out for different descriptions of *triangle-frames* and *five-angled-frames*. Or they could look out for three-dimensional figured numbers like *cube-number* or *pyramid-numbers*.

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[3] In upper grades you can get algebraic equations e.g. with T_x as triangle-number and S_x as square-number you can find $T_{2n} = 2 \cdot T_n + S_n$, $T_{3n} = 3 \cdot T_n + 3 \cdot S_n$ and $T_{4n} = 4 \cdot T_n + 6 \cdot S_n$, $T_{5n} = 5 \cdot T_n + 10 \cdot S_n$ which leads to the conjecture $T_{k \cdot n} = k \cdot S_n + T_{k-1} \cdot S_n$ and $T_{n^2} = n \cdot T_n + T_{n-1} \cdot S_n$ or $T_n - T_{n-m} = 2nm - T_m$.

The children also can make investigations about ***difference-sequences*** of a sequence of figured numbers (i.e. the difference of consecutive elements). Interesting in this context is that the difference-sequence of the difference-sequence is constantly 1 for the triangle-numbers, constantly 2 for the square-numbers and constantly 3 for the five-angled numbers.

4. Sequences and chains of numbers in grade 2 or 3

With the question "How does it go on" or "Call the next three numbers" children in grade 2 or 3 can be introduced into the ***concept of sequences***. Examples may be 1, 3, 5, 7,… or 3, 4, 6, 9,… or 2, 4, 8, 16, … or 1, 2, 1, 2, … or 5, 4, 6, 3, … or sequences of figured numbers. Afterwards they can produce such sequences of natural numbers by themselves and let find out by their neighbours the next five numbers or the rule it was produced. There are a lot of possibilities to build such sequences by using only basic operations in different ways and there are also often different ways to describe the rule for getting some next elements of a sequence[4]. In the example 5, 4, 6, 3, … e.g. you could say that the way from one element to next is –1, +2, –3, +4, … but you also could say that the sequence is existing of two sequences (5, 6, 7, … and 4, 3, 2 …) mixed up. Moreover this sequence will stop with its eleventh element because the following subtraction has no solution within N_0.
A famous ***chain of numbers*** is the sequence of Fibonacci. It starts with two ones and then always the next number is the sum of the two ones directly in front of it. A chain of numbers in general is defined by its first two elements and then always the next one is the sum the two ones directly in front of it.
Problems for discovery are e.g.: "Given two starting numbers find the next five." or "Find two starting numbers so that the fourth (or fifth or sixth) element is equal to 10 (or 50 or 100 or …)?" "How does a given chain change if you add to (subtract from) the first (the first two) numbers a 1 (or 2 or 3 …)?" "How does this chain change if you multiply the two starting numbers with 2 (or 3 ...)?"
For questions like these we first look at some examples whereat always the first two numbers[5] are given and the five next numbers are computed.

1st chain: 1, 1, 2, 3, 5, 8, 13 (Fibonacci sequence)
2nd chain: 1, 2, 3, 5, 8, 13, 21 (Fibonacci without first 1)
3rd chain: 1, 3, 4, 7, 11, 18, 29 (Second chain plus Fibonacci with 0 in front)
4th chain: 1, 4, 5, 9, 14, 23, 37 (Third chain plus Fibonacci with 0 in front)
5th chain: 2, 2, 4, 6, 10, 16, 26 (Double of Fibonacci)
6th chain: 2, 3, 5, 8, 13, 21, 34 (Fibonacci without 1, 1)
7th chain: 2, 4, 6, 10, 16, 26, 42 (Double of second chain)
8th chain: 3, 3, 6, 9, 15, 24, 39 (Triple of Fibonacci)
9th chain: 3, 4, 7, 11, 18, 29, 47 (Eighth chain plus Fibonacci with 0 in front)
10th chain: 4, 4, 8, 12, 20, 32, 52 (Fourfold of Fibonacci)

- Looking at the 1st, 2nd and 6th chain we get the following conjecture[6]: "If in a chain there are two consecutive numbers equal to two consecutive numbers of the Fibonacci sequence then all following elements are equal to the following elements of the Fibonacci sequence too." And it is not difficult to see the following a little bit more general conjecture "If two consecutive elements of one chain are equal to two consecutive elements of another chain then all following elements are also equal."
- Coming out of the rule of building the chain elements we also easily can find the following conjecture[7] "The difference sequence of a chain (difference between two consecutive elements) is equal to this chain without the first element".
- Discussing the 1st, 2nd, 3rd, 4th chain on one hand and the 5th, 6th, 7th on the other hand we can see that adding +1 to the second element influences +1 at the third element, +2 on the fourth element, +3 on the fifth element, +5 on the sixth element and +8 on the seventh element (i.e. plus Fibonacci with 0 in front).

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[4] And more: A sequence is not defined exactly with only a few elements of it. So principally there are also different rules (with different following numbers) possible. But that is normally not a problem for this age.
[5] Here we took the first two numbers a_1 and a_2 out of the set {1; 2; 3; 4} with $a_2 \geq a_1$.
[6] A proof or at least a plausible reason for this (and other conjectures) some of the children may find with words by their own. An algebraic way to handle with these chains then in grade 8 is a good way for deepening algebra. For example if we start with a, b then the next five numbers have the following algebraic description: a+b, a+2b, 2a+3b, 3a+5b, 5a+8b. It is interesting to find the numbers of the sequence of Fibonacci (twice) again here.
[7] In algebraic form the building role of such a chain is $a_{n+2} = a_n + a_{n+1}$ and thus we get $a_{n+2} - a_{n+1} = a_n$.

- Comparing the 2nd chain with the 5th one and the 3rd chain with the 6th one and the 4th chain with the 7th one as well as the 6th chain with the 8th one and the 7th chain with 9th one and finally the 9th chain with 10th one we can come to the conjecture that adding +1 to the first element influences for the following elements +1, +1, +2, +3, +5 (which is the Fibonnacci sequence again)[8].
- Looking at the 1st, 5th, 8th, 10th chain as well as comparing the 2nd chain with the 7th one we find: "Multiplying the two starting elements with one natural number yields the multiplying of all elements with this natural number." The 1st, 5th, 8th, 10th chain also can lead us to the conjecture that starting with two equal numbers we get the Fibonacci sequence multiplied with the starting number[9].

With these ten chains we already could make a lot of discoveries. The children later on can ***vary the rule of producing chains of numbers***. For example they can start with three numbers and then produce always the next number as sum of the three ones directly in front of it. Or they first multiply the last element with an integer n and the element before the last one with an integer m. The sum of these two products (instead of the simple sum) then is the next element. Or they use differences instead of sums. In grade 3 they also can change to multiplication[10] instead of addition or mix up different operations.

5. Discoveries within the table of the hundred in grade 2

Another problem field for grade 2 has to do with discoveries within the "table of hundred" (a table containing all numbers from 1 to 100 structured in rows of ten). With this the children can deepen the structure of the decimal system for example by walking one step (or more) down or respectively right or left with looking at the change of the numbers. They also could investigate all "horse-jumps" (jumps of a horse in chess – two steps in one direction plus one step perpendicular to it or one step in one direction plus two steps perpendicular to this). A look at a diagonal row or a parallel row to it also can lead to statements.
As variation of such discoveries we can use as basis a sheet of a calendar with seven columns or a table of natural numbers ordered in 5 or 6 or 8 columns. We also can use a table of only odd numbers instead of the normal "table of hundred".

6. Number-walls in grade 2 and 3

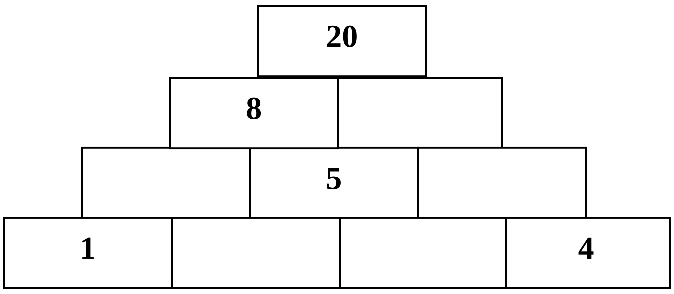
Number-walls represent a special format of tasks. You start with a first row with three or four (or more) numbers. The sum of always two neighbouring numbers then built the second row. In the same way you built the third row, and so on until you have only one number in the last row. The following example starts with the numbers 1, 2, 3, 4 given in the first row.

	20		
	8	12	
3	5	7	
1	**2**	**3**	**4**

If you start in this way with numbers in the first row this is only a special form for routine tasks. But if you start with numbers in different rows you open a wide field of different problems. In the following

[8] In algebraic form in a general case we find $a+n$, $b+m$, $a+b+(n+m)$, $a+2b+(n+2m)$, $2a+3b+(2n+3m)$ and so on.
[9] Children in primary school can give with examples and general arguments in words reasons for this statement. In algebraic form we can look at $a\cdot n$, $b\cdot n$, $(a+b)\cdot n$, $(a+2b)\cdot n$, $(2a+3b)\cdot n$, … [or with $a=b=1$ we get $n, n, 2n, 3n, 5n, \dots$] .
[10] In grade 4 or 5 we e.g. can find out that a chain working with multiplication instead of addition and with starting numbers 1, n (n any natural number) does produce a sequence with powers of n whereat the exponents are equal to the elements of the sequence of Fibonacci. The same (without the beginning 1) we get when we start with the two numbers n, n. If we take the starting pair 2, n then we will get the following sequence (noticed in an algebraic way): $2, n, 2n, 2n^2, 2^2n^3, 2^3n^5, 2^5n^8, \dots$ (again with the numbers of Fibonacci)!

example you only have to work back. You can make different considerations by starting with less of these given numbers.



- If you erase the 1 in the last row the number-wall also is determined with the same solution because in the first field of the second row you find that there must stand 3 and below of the three you can only have 1, 2 or 2, 1. If you take 2, 1 then the bottom row comes out with 2, 1, 4, 4 which leads to a 8 right of the 5 in the second row and a 13 right of the 8 in the third row which stands in contradiction to the 20 in the top row. Without using an argument of contradiction you also will come to the same solution if you work back and find first the 12 right of the 8 in the third row then a 3 and a 7 in the second which produces first a 3 in the third field of the bottom row and then a 2 in the second field of the bottom row and finally the erased 1 in the first field of the bottom row. (Similar consideration you can make if you erase the 4 instead of the 1 in the bottom row.)
- If you erase 1 *and* 4 in the bottom row then you have two different solutions. There are of course also different solutions if you erase two other numbers or even all numbers except the 20 in the top row. If you only replace the first 1 with another number then - as already discussed - you will get no solution.

You see that by varying the given task you get a problem field with different types of problems including also tasks which may have more than one solution or no solution. And you can differentiate for different abilities of the children by using different givings as well as different large number-walls. Good children also can find variations by themselves.

Moreover the children can make experiences in the direction of *functional connections*. For this we will discuss some examples:
- If you add in the given bottom row of a number-wall a fixed number to the first (or last) number so this fixed number also has to be added in the solution in the top row. But if you add this fixed number to the second (or third) number in the bottom row so you have to add the triple of the fixed number to the solution in the top row.
- If you double (or triple …) all numbers of the bottom row then all other numbers are doubled (or tripled …) too.
- If you take in the bottom row a fixed number at all positions then in the second row you will find always the double of this number, in the third row the fourth of the bottom number and in the top row the eighth of the given number in the bottom row (notice the powers of 2).

Another ***variation of these number-walls*** you get if you use multiplication instead of addition. We then will speak from “multiplication-walls”. This is of course possible only in grade 3 or later after being familiar with multiplication of bigger numbers because the numbers rise up very fast. In the case of given numbers not only in the bottom row you will get a lot more tasks which have no solution because many divisions have no solution within the set of natural numbers.
It is of course also possible to mix addition, subtraction and multiplication. But then it is better to change the wall to a tree (often used together with word problems) where we have beside the rectangular fields for numbers also round fields for the operation.

Disrupting linear models of mathematics teaching|learning

Barbara Graves, Associate Professor
Christine Suurtamm, Associate Professor
Faculty of Education, University of Ottawa, Canada
bgraves@uottawa.ca suurtamm@uottawa.ca

Abstract

In this workshop we present an innovative teaching, learning and research setting that engages beginning teachers in mathematical inquiry as they investigate, represent and connect mathematical ideas through mathematical conversation, reasoning and argument. This workshop connects to the themes of teacher preparation and teaching through problem solving. Drawing on new paradigms to think about teaching and learning, we orient our work within complexity theory (Davis & Sumara, 2006; Holland, 1998; Johnson, 2001; Maturana & Varela, 1987; Varela, Thompson & Rosch, 1991) to understand teaching|learning as a complex iterative process through which opportunities for learning arise out of dynamic interactions. Varela, Thompson and Rosch, (1991) use the term co-emergence to understand how the individual and the environment inform each other and are "bound together in reciprocal specification and selection" (p.174). In particular we are interested in the conditions that enable the co-emergence of teaching|learning collectives that support the generation of new mathematical and pedagogical ideas and understandings.

The setting is a one-week summer math program designed for prospective elementary teachers to deepen particular mathematical concepts taught in elementary school. The program is facilitated by recently graduated secondary mathematics teachers to provide them an opportunity to experience mathematics teaching|learning through rich problems. The data collected include questionnaires, interviews, and video recordings. Our analyses show that many a-ha moments of mathematical and pedagogical insight are experienced by both groups as they work together throughout the week.

In this workshop we will actively engage the audience in an exploration of the mathematics problems that we pose in this unique teaching|learning environment. We will present our data on the participants' mathematical and pedagogical responses and open a discussion of the implications of our work.

Introduction

A recent body of research on the teacher's role in the development of learners' mathematical understanding (Ball, 2003; Ball & Bass, 2002; Ball & Even, 2004; Boaler, 2002) has revealed that teaching through inquiry poses substantial challenges for teachers. For elementary teachers, the research suggests that they lack sufficient knowledge of mathematics to effectively implement inquiry-oriented mathematics programs (Ball, 1988, 1990; Ball, Lubienski & Mewborn, 2001). For secondary teachers, while they are more likely to have a background in mathematics, concerns exist about the procedural nature of their mathematics knowledge, and their inexperience in inquiry settings (Stigler & Hiebert, 1999). To complicate matters, few curricula are organized around meaningful problems. Among the research literature, some studies suggest that once teachers have experienced learning new mathematics through inquiry-oriented pedagogies, they start to incorporate the use of investigation and open-ended problem solving in their own teaching (Makar & Confrey, 2004; Shifter, 1998). Other researchers report that it is possible for teachers to deepen their understanding of mathematics and mathematics teaching through the activity of teaching itself (Hill & Ball, 2004; Ma, 1999; Segall, 2001). It has also been reported that when teachers engage in meaningful collaboration with other teaching colleagues, they demonstrate increased mathematical knowledge and more effective mathematics teaching (Crespo, 2006; Lachance & Confrey, 2003; Stein, Silver, & Smith, 1998; Wilcox, Schram, Lappan, & Lanier, 1991).

There are, however, a number of research studies which suggest that even when teachers have had some inquiry-oriented learning experiences and acknowledge this approach as supporting their mathematics learning, there remain visible tensions between inquiry-oriented and more traditional approaches to teaching mathematics (Graves, Suurtamm, & Benton, 2005; Hart, 2004; Jacobs, Hiebert, Givvin, Hollingsworth, Garnier, & Wearne, 2006). This is not surprising given that the changes that are being asked of teachers are neither easy nor straightforward. The movement from established practices of teaching which traditionally include a great deal of direct instruction to an inquiry approach that requires the interactions between teachers and learners be more dialogic, challenges teachers' beliefs and attitudes about learning, and what constitutes mathematics (Stein, Silver, & Smith, 1998; Suurtamm & Graves, 2007). With respect to teaching, these challenges are revealed when teachers struggle to decide when to explain, when to listen, and how much time to leave for ideas to emerge. When they worry about what question to ask next, we hear in their discourse familiar metaphors of linearity that underlie many established educational perspectives.

Theoretical framework

We orient our work within complexity theory (Davis & Simmt, 2003, 2006; Davis & Sumara, 2006; Holland, 1998; Johnson, 2001; Maturana & Varela, 1987; Varela, Thompson & Rosch, 1991) with respect to how we conceptualize and enable teaching|learning situations. Educational research on knowledge and learning that is oriented by complexity science focuses on the dynamic interactions emerging from diversity and variation, and suggests the value of viewing the interactions of teachers and learners as adaptive and co-emergent learning systems. From this perspective we resist viewing teaching and learning as separate entities and prefer to see teaching|learning as a mutually constitutive dynamic process in which the teacher is also the learner, and the learner is also the teacher. Varela, Thompson and Rosch, (1991) use the term co-emergence to understand how the individual and the environment inform each other and are "bound together in reciprocal specification and selection" (p.174). In particular we are interested in the conditions that enable the co-emergence of learning collectives that support the generation of new mathematical and pedagogical ideas and understandings. In addition, complexity theory provides models and metaphors of non-linearity that help us argue against the linear sequence of prescribing the next step which is so deeply entrenched in educational practice. In non-linear terms next steps are always contingent and co-emergent. This is the case not only in how we understand the teaching|learning dynamic but reflects our own research process. While we begin with specific research questions and a research design, we do not ignore emergent themes, participants or issues, but rather incorporate these into our study as they evolve with the research context.

The study

This research study began in August 2004 and is now in its sixth year. Each year of the study begins with a one-week summer mathematics program for prospective elementary teachers to deepen particular mathematical concepts taught in elementary school. The facilitators in this one-week program are recently graduated secondary mathematics teachers. Our objectives are twofold: to enable prospective elementary teachers to enhance their view of mathematics before they begin their teacher education program; and to provide beginning secondary math teachers an opportunity to teach mathematics through rich problems and inquiry. During five days, the prospective teachers have many opportunities to immerse themselves in mathematical discussion and argumentation as they work on interesting problems, and share solutions, strategies, and representations. They engage with a variety of materials including manipulatives and technological tools in order to explore mathematical ideas in numerous contexts, both within and outside the classroom. During the five days the facilitators actively participate in the inquiry process as they pose problems, provide encouragement, listen attentively and respond to mathematical ideas. In this way, the summer math program provides a teaching|learning experience in which prospective elementary teachers can expand their mathematical understanding and beginning secondary math teachers have the opportunity to experience engaging learners in mathematical inquiry.

Who are the prospective elementary teachers? In our recruitment invitation we state that "Many prospective elementary teachers confess to anxiety about doing and teaching math" and we invite them to join us in "a series of meaningful math activities in a convivial and supportive learning environment, [to] develop a better understanding of the mathematics that [they] will be teaching." Each year we accommodate 75 participants, the majority of whom either express that they are anxious about math or that they have forgotten, or never understood their previous school mathematics. There are also a small number who accept the invitation because they enjoy doing mathematics and welcome the opportunity to do more.

We collect data from the prospective teachers during the one-week program and throughout their teacher education program by means of questionnaires, and focus group interviews. Our focus is on how their mathematical understandings as well as their attitudes and beliefs about mathematics and mathematics learning become transformed in different settings.

Who are the beginning secondary math teachers? In the spring prior to the summer math program, we invite teacher candidates currently in secondary math to apply for the position of facilitator for the summer program. We receive approximately 20 – 25 applications each year, and we personally interview all applicants. We choose facilitators based on their expertise in mathematics, their openness and enthusiasm to think about mathematics and mathematics teaching and learning in different ways, and their conviction that mathematics is something that everyone can learn.

We begin our work with the facilitators during the week before the summer math program to collaboratively explore the mathematics in a number of rich problems and to prepare for the upcoming week. Facilitators come to the preparation week with a variety of expectation about what this teaching experience will be like. An important goal during this week is to begin a discussion and exploration of what it means to facilitate mathematics learning in a meaningful way and to think about posing problems, and listening to solutions rather than telling and explaining. The data collected during the preparation week include daily journals, and video-recordings of their work together. During the week of the summer program we meet with the facilitators for debriefing sessions at the end of each day. This provides all of us the opportunity to listen to everyone's experiences, engage further in mathematical investigations, and to understand how their views of mathematics and mathematics teaching|learning are evolving. The data collected during this week consist of daily journals and video-recordings of the de-briefing sessions.

Workshop format

In this workshop we will describe the summer math program and our research framework and will actively engage the audience in an exploration of the mathematics problems that we pose in this unique teaching|learning environment. In this context we hope to develop a substantial discussion around the nature of rich problems as well as the nature of problem posing. We will then share our data on the participants' mathematical and pedagogical responses to these problems and open a discussion of the implications of our work.

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Modelling tasks for learning, teaching, testing and researching

Gilbert Greefrath
Professor of Mathematics and Mathematics Education, Faculty of Mathematics and Natural Sciences, University of Cologne, Cologne, Germany g.greefrath@uni-koeln.de

Abstract

The article deals with a special kind of modelling tass. These problems are used for learning and researching as well. So the results of an empirical study on mathematical modelling of pupils in secondary schools are presented. Pupils of forms 8-10 were observed working on open, realistic problems. These observations were recorded and evaluated. The goal of the presented part of the study is a detailed look at the control processes of modelling problems. In this context changes between real life control and mathematical control during the control phases are studied and evaluated. We describe in detail the sub phases of controlling and explain their connection with modelling process. The problems used in this project can also be used in math lessons, so this kind of research can put teachers and researchers together. These tasks are suitable to support ongoing in-service development and teacher education.

Problems

Open reality-related problems can be used to analyse model-building and problem-solving processes. The following problems are examples of open and fuzzy problems with reality references used for teaching and researching.

How much will it cost to plaster this house?

The house-plastering problem

In order to characterize the house-plastering problem, I use the description of a problem as initial state, target state and transformation, borrowed from the psychology of problem solving (Bruder 2000, p. 70). The problem's initial state is unclear because the relevant information is missing. Also unclear is the transformation from initial state to target state, which students can employ. However, the final state is clearly defined, for instance, by asking for a price.


How many people are caught up in a traffic jam 180 km long?

The traffic jam problem

Learning and Research

Two pupils at a time were monitored while they worked on their problems (e.g. the house plastering problem). The students were asked to undertake the task in pairs – without any further help. The students' work was recorded using a video camera.

For evaluation, the entire video data were transcribed. Within the framework of open coding with three raters, the individual expressions of the pupils were allocated conceptual terms, which were discussed and modified during several runs through the data. These terms for individual text passages were then assigned to the following categories: planning, data capture, data processing and checking.

The process category 'planning' describes text passages in which the pupils discuss the path to complete the task or which – in the broadest sense – relate to the path of completing the task. The process category 'data acquisition' describes text passages in which pupils procure data for their further work on the problem. This can involve guessing, counting, estimating, measuring or recalling intermediate results that had been achieved earlier. The process category 'data processing' describes the calculation with concrete values. This can be done either with or without a calculator. For all problems the pupils were provided with a (conventional scientific) calculator. The process category 'checking' includes text passages in which the data processing, data acquisition or planning is questioned or controlled.

The choice of categories was done in such a way that the categories could be allocated in a consistent manner, independently from the problem. During this phase the preliminary categories were therefore combined and modified. Following this allocation, the entire transcripts were coded using the now finalized categories. Later we found that one rater was able to code all categories with adequate confidence on his/her own. The degree of agreement was checked by performing a sample correlation analysis (see Bortz et al. 1990, p. 460f), which showed statistically significant concordance at the 0.05 level. As a result, I then coded all of the remaining transcripts on the basis of the developed categories.

In the following, we present three observations, which were examined by analysing the central components of control processes.

Observation I: Global control

First two comprehensive school pupils of grade 8 are observed. They concerned themselves approx. 22 minutes with the task and obtained from a mathematical point of view a very exact result. The control procedures of the pupils contained the solution plan of the problem. Thereby also, larger sections of planning are taken into the view. The pupils did not control single computations and data capture. This was however in most cases not necessary.

Observation II: local control

Second two secondary modern school pupils of grade 8 are observed. They concerned themselves approx. 15 minutes with the task and obtained from a mathematical point of view an inexactly result. The pupils controlled their work only locally. For instance, they controlled only the last calculation. There are phases with data capture control, but they do not look at

the final result. They controlled the plausibility only in cases of partial results.
Observation III: multiple controls
Third two secondary modern school pupils of grade 9 are observed. They concerned themselves approx. 9 minutes with the task and obtained from a mathematical point of view an inexactly result. During the work with the problems, many control procedures take place. Many different kinds of controls are observed, e.g. controls of the data processing, data capture or planning. The control processes have local and global aspects.
To summarize the control behaviour of the pupils is extremely different. The idealized modelling cycle and the problem solving process are only partial suited to describe and differentiate between these control processes adequately.

Teaching and testing

Pupils can encounter difficulties while working on modelling problems in many places. Therefore, the modelling skills of pupils should be diagnosed exactly before a lecture series on modelling problems. By diagnosis, we understand systematic capture of individual conceptions, abilities, knowledge and learning ways. (s. Abel et al. 2006, p. 10, Büchter/Leuders 2005, p. 167). For this purpose, modelling tasks can be reduced to subtasks, which focus on special steps of the modelling cycle. By the following specifications, it can be determined whether pupils acquired these subsidiary skills.

Subsidiary skills	**specification**
simplify	Pupils separate important and unimportant information of a real situation.
mathematise	Pupils translate real situations into mathematical models (e.g. Term, equation, graph, diagram, function)
calculate	Pupils work with the mathematical model.
interpret	The pupils transfer the information gathered in the model to the real situation.
validate	The pupils verify the information gathered in the model at the real situation. They compare and evaluate different mathematical models for a real situation.
Evaluate	The pupils judge critically the used mathematical model.
Apply	The pupils assign a suitable real situation to a mathematical model and/or find to a mathematical model a suitable real situation.

Table: subsidiary skills and specification (s. Ministerium für Schule 2004, p. 28)

The specifications in the table have a double function. On the one hand, they allow the specification of the subsidiary skills of modelling and on the other hand, they are used to diagnose these subsidiary skills.

Development of diagnostic tasks

The production of tasks fitting to single subsidiary skills is not easy - in particular for subsidiary skills of modelling. By reducing of modelling tasks on subsidiary skills, the authenticity of the task can be lost. However, the authenticity is for modelling activities an indispensable condition.
By using diagnostic tasks, it is important to learn much about the ways of thinking of the pupils. Then we can learn something about the strengths and the difficulties of the pupils. The tasks should be open and the pupils should generate self-productions to prevent standard-answers. Another important aspect is the validity of the task relating to the relevant subsidiary skills (see Abel et al. 2006, P. 12, Büchter/Leuders 2005, P. 173).

A possibility for the development of diagnostic tasks to subsidiary skills of modelling is the limiting of existing modelling tasks by giving of further information. In the following, some examples of tasks are presented.

Example 1 (validate)

Katja and Toni want to compute, how many people are caught up in a traffic jam 180 km long. They assume 10 m space on the road for a vehicle and considered the following calculations

$3 \cdot 18000 \cdot 4 =$
$3 \cdot 18000 \cdot 2 =$

Compare the two calculations and evaluate them.

Example 2 (simplify)

Katja and Toni want to compute, how many people are caught up in a traffic jam 180 km long. The pupils were held to reflect for themselves, and consequently prepare a list of necessary facts. For which of these information's would you decide? Why?

- Vehicle length
- weather
- type of vehicle
- petrol consumption
- state
- distance to the next vehicle
- number of lanes
- day of week
- season
- age of driver
- number of passengers
- time of day
- road works
- holiday time

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Transcribing an Animation: The case of the Riemann Sums

May Hamdan, PhD
Associate Professor of Mathematics Lebanese American University, Beirut, Lebanon
mhamdan@lau.edu.lb

Abstract
In this paper I present a theoretical analysis (genetic decomposition) of the cognitive constructions for the concept of infinite Riemann sums following Piaget's model of epistemology. This genetic decomposition is primarily based on my own mathematical knowledge as well as on my continual observations of students in the process of learning. Based on this analysis I plan to suggest instructional procedures that motivate the mental activities described in the proposed genetic decomposition. In a later study, I plan to present empirical data in the form of informal interviews with students at different stages of learning. The analysis of those interviews may suggest a review of my initial genetic decomposition.

Introduction

I start by situating the study within my teaching experience. It is often the case that Calculus instructors do not emphasize the idea of Riemann sums, at least not symbolically as a limit of an infinite sum of rectangle areas because the justification of the "formula" involves many "unpopular" concepts. Although this section is covered just about as lightly as the delta-epsilon definition of a limit, yet the visual dynamic process that illustrates the area under the graph of a function as a limit of the area of a sequence of rectangles appeals to students' senses in general; they are quickly visually persuaded that the error decreases as the number of rectangles increases. An animation of the process can be supported by a simple graphical tool. Instructors face a difficulty in teaching the "transcribing" of this animation using algebraic symbols, to end up with a messy compact limit of an infinite sum. The lengthy ambiguous expression involves finding a limit at infinity, choosing an adequate parameter, lifting appropriate points on the interval, as well as other factors that will be detailed later. To circumvent this clutter of obscurity, many instructors are satisfied with a demo of the animation and putting forward the formula with little justification, if at all, applying it to a few simple cases, and hastily moving to the technicalities of the definite integrals. In this study, I discuss what it takes to mend those links between the visual and the symbolic representation of the process involved.

Framework for research

In my study I use as framework for reserach an interpretation of constructivism and Piaget's ideas on reflective abstraction (Dubinsky, 1991). According to Piaget mathematical knowledge consists of an interconnected collection of cognitive structures corresponding to individual mathematical concepts; and understanding a concept reduces to constructing one or more schema for it (Dubinsky, 1989). In order for learning to occur, Piaget considers that the student must become aware of some disequilibration and then try to re-equilibrate by reconstructing his or her own mathematical schemas. By equilibration he refers to a process by which a knower attempts to understand an item of information by situating it in his or her overall cognitive system. Such situating occurs as the knower constructs an understanding of the item through a process called reflective abstraction (Dubinsky & Lewin, 1986), which can be described as the general coordination between the different cognitive structures in order to construct more advanced structures. The cognitive tools involved are coordination, interiorization, generalization, encapsulation de-encapsulation, and reversal (Asiala et al., 1996). The purpose from the theoretical analysis is to propose a model of cognition for the concepts involved, referred to as a *genetic decomposition*, a description of how a student might make the constructions that would lead to an understanding of the concepts in question. Genetic decomposition is hypothesized from the researcher's own mathematical knowledge, but most importantly from observations and interviews with students in the process of learning. According to this theory (Asiala et al., 1996), mental constructions in the genetic decomposition are Action, Process, Object and Schema (hence APOS theory). An *action* conception of a mathematical idea is a description of an understanding that is limited to performing action on that concept. When an action is repeated and reflected upon, it may be interiorized into a *process*. A learner has a *process conception* of a concept when his or her depth of understanding is limited to thinking about the idea as a process without being able to execute an action on it. A process is said to be encapsulated into an *object* when the individual can perform action on it, and decompose it back to the matter from which it was initially formed. Finally a *schema* of a piece of mathematics is an individual's collection of action, processes, objects, and other schemas, linked in a coherent framework in the person's mind. (MacDonald et al., 2000). This paradigm has been applied to diverse topics including functions, mathematical induction, calculus, quantification, and abstract algebra, and equivalence classes and partitions (Hamdan, 2006) has lead to major curriculum changes.

Main Questions directing the research

During my observations I tried to focus on answering the following research questions:

1. Can students visualize the rectangles under the curve?
2. Are students persuaded that the error decreases as the number of rectangles increases? And consequently that there is no error when the number of rectangles grows indefinitely?
3. Can they choose the subintervals using appropriate indexing?
4. Also (later) are they aware, once the number of the rectangles grows indefinitely, of the irrelevance of the choice of those points on the subintervals that determine the height of those rectangles, as well as the uniformity of those subintervals?

Prerequisite schema

The objects or schemas that we consider a student needs to have developed prior to the introduction of the Riemann sums are:

1. A schema for functions of one variable.
2. A schema for real numbers where the objects are the numbers and the processes are the arithmetic and algebraic transformations on numbers (Trigueros et al, 2007).
3. A schema for the Cartesian planes including points as objects, and distances between points as processes and actions.
4. A schema for limits in general including limits at infinity.
5. Actions related to the sigma notation for the finite case. This schema should include basic formulas such as the sum of the first n integers and the sum of their squares.
6. A schema for basic geometry that includes areas of rectangles.
7. A schema for (finite) sequences including observing a pattern of objects and labeling them using appropriate indices. Observing similarities in a pattern and being able to recognize the *i*-th entry as a typical entry in the list.

(Conjectured) genetic decomposition

In what follows I list the stages of the proposed genetic decomposition:

I. First coordinate between the schemas for the real numbers with the schema for the Cartesian plane (including intervals and distances) through the action of subdividing the interval $[a,b]$ into n subintervals, for a fixed positive integer n. Then interiorize these actions into a process that gives all the equidistant n points x_i on that interval $[a,b]$, or just as well that gives the n subintervals $[x_i,x_{i+1}]$ of equal length, $d_n=(b-a)/n$. It will be agreed that a and b will be assigned the points x_0 and x_n, respectively.

II. Then coordinate between the schemas of functions of one variable and the schema of Cartesian plane through the action of evaluating the function f at those stops/spots x_i and then "lifting" those verticals of length $f(x_i)$ from $(x_i, 0)$ to $(x_i , f(x_i))$, on the graph of f. These actions then get interiorized into the process that results in an arrangement of adjacent evenly spaced vertical segments. This is followed by the action of connecting the tops of those verticals using horizontal segments from the point $(x_i, f(x_i))$, say, to the point $(x_{i-1},f(x_i))$, starting at x_1. This last action will be interiorized into the process that results in those adjacent rectangles of equal width but different lengths. These n rectangles are labeled $R\_1, R\_2,. .. R\_n$, using the previously mentioned indices.

Now the geometric set up is prepared, and the areas of the resulting rectangles can now be evaluated.

III. Next, coordinate between the schema for basic geometry and that of sequences together with the schema for the sigma notation through the action of evaluating the area ($f(x\_i)*d\_n$) of one "typical" rectangle (the i_th) from the finite sequence obtained in the previous step. This is followed by the coordination with the schema for sigma notation through the action of summing over all $i=1,\ldots,n$ to obtain the finite sum $\sum f(x\_i), dn$. This action is interiorized into a process that results into viewing that sum $\sum f(x\_i), d\_n$ as a function $S(n)$ of n.

Note that it is quite difficult for students at this stage to foresee that neither x nor i would figure in the last result.

IV. Next coordinate between the schema for limits and the schema for sigma notation through the action of evaluating the limit of $S(n)$ as n tends to infinity.

Comments on the genetic decomposition

Before officially conducting the planned interviews that will eventually produce empirical data, and following the setting of the genetic decomposition, I taught the course once more and was able to make a few observations, reflecting on the proposed cognitive model.

These findings were done informally by making close undocumented in-class observations.

I. Action level formula application

Some students seemed to work at an action level, memorizing some facts and trying to use them (Sfard, 1991). Once the formula is available, they know how to manipulate it by substituting values, but they are far from understanding it, or explaining it, let alone producing it themselves. In particular, they want the formula to produce an output in the form of a numerical result. They do not show evidence of having interiorized the mentioned mathematical objects in the genetic decomposition or having constructed relations between them and their intuitive knowledge of the process.

II. Is it a parameter, a variable or an unknown?

Dealing with parameters as indices in the sum was one of the insurmountable difficulties that were not predicted in the original genetic decomposition**.** The fact that it seems to be a common difficulty for most of the students points out that it needs further study. Also the large number of referents in the formula, n or $(b-a)/n$, i confuses students. It takes the development of a new layer of mathematical generality to believe that an object exists even though it is not numerically determined (Bardini, 2005). The problem boils down to accepting "indeterminacy". So, what is n? Is it an unknown, a variable (Drijvers 2001)? Is it the number of rectangles, or the width of the rectangle, or the position of that rectangle in question? Does it designate the position of the rectangle? Students feel they need to attribute a numerical value to n or i in order to progress in this activity. They want to associate the figure with a number. Some students think of it as a temporary fixed value, a placeholder which is changing, but like the variables, operations can be carried on it the same way as an unknown or a variable. There is a lot of research around the meaning of parameters and how students perceive of this genuine conceptual object that can only be referred to through signs (Bardini, 2005). This algebraic object encloses a unique paradoxical nature since it is fixed, yet it remains indeterminate in that it is not subjected to an inquisitorial procedure that would reveal its hidden numeric identity (as is the case with unknowns). Drijvers (Drijvers, 2001) refers to it as generalizer, an implicit that is conceptually more difficult than a variable, it is rather a *generic* organizer uses mostly in the description of a process.

III. Language used in transcribing

Some students' difficulties have their origin in Language. For example, one difficulty found involved the use of the word "lift" while determining the heights of those rectangles. It seems that this difficulty arises because the students assign a different meaning to the action invoked by the word "lift" from that intended. Building the necessary connections includes paying more attention to the use of language making the meaning of words explicit by means of performing the actions and reflecting and discussing the results.

IV. Separation Visual/Algebraic

Many students don't associate their visualization to the algebraic interpretation (Zazkis et al 1996) as if they separate them totally in their head. It became clearer to me that all students are convinced that the process undoubtedly generates the *exact* area, eventually. However, transcribing the animation into an accurate algebraic/symbolic compact expression is the major difficulty: in Piaget's terms, the description of a numeric schema allows students to obtain a formula by translating it into symbols. However, when translating the worded schema into an algebraic formula, students in general come up with answers that reflect some lack of precision in the meaning that they gave symbols. Their translation of the worded schema suggests that they do not interpret the letters "*n*" as standing for an unspecified figure (despite the fact that I tried to suggest it). Both, *i* and *x* are present in the same line, let alone *a* and *b*. As a broad closing, and in a way as to delay the agony of transcribing the process, it is always good to delay the formalization of the process, by exhibiting many diagrams and demos for the procedure that produces those rectangles and areas.

Conclusion To many, teaching implies a didactical transposition: a display of neatly presented material (Arcavi, 2003); which means the transformation of knowledge and adapting it from its scientific character to the knowledge as it should be taught. This explains how the students or receivers fail to see the way it was discovered which could be visual. All this is done at the expense of neatness of representation. This process (it is claimed) by its very nature, linearizes, compartmentalizes and possibly algorithmetizes knowledge, stripping it from its rich interconnections. As such, many teachers may feel that analytic representations which are sequential in nature seem to be more pedagogically appropriate and efficient (Arcavi, 2003).

In a later study I will conduct a formal quantified research where I plan to choose an initial group of students to participate in the first stage of the study. The students will be chosen from a group of undergraduate students at a private university who had taken the equivalent of a Calculus course in the previous semester. I plan to choose a good, an average, and a weak student to be interviewed. The results obtained will be independently analyzed. On the basis of the results obtained from these interviews, I plan to compare students' constructions of the Riemann sum to the constructions predicted (here) by the genetic decomposition. Based on those findings I will design new instruments for teaching that I will assess again. I plan to measure the level of their coordination predicted by the genetic decomposition that they seem to have constructed.

The good news is that it is unusual that in this case visualization is not a problem in this case and that difficulty didn't seem to inhibit the subsequent understanding of basic mathematical constructs.

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Each and Every Student: The Stamford, Connecticut Model for Change in Mathematics

Mona Hanna, MS, Director for Mathematics and Science
mhanna@ci.stamford.ct.us
Carrie Chiappetta, Ed.S, Secondary Mathematics Coordinator
cchiappetta@ci.stamford.ct.us Stamford Public Schools
Stamford, Connecticut, USA

Abstract The major aims of this paper are to: present the background of the mathematics education problem in the Stamford Public School (SPS) district which is common is most U.S cities; explain the need for change in mathematics education; describe the process to systemically transform both the curriculum and instruction of mathematics thereby ensuring that each and every students is prepared for the 21st century, for higher education, and for success in a global society; and provide ways to measure these changes. The K-12 mathematics education reform model presented can be replicated in other cities and for other academic areas.

Introduction A survey of recent studies and research show that mathematics education in the United States is in dire straits and when compared to their international counterparts and therefore, U.S students do not perform as well on international assessments. Much has been written about the need for reform in both curriculum and instruction as it relates to mathematics. But performance on international assessments alone is not the only reason why mathematics education in the U.S. should change; indeed teachers need to educate students so that they are successful participants of the 21st century, global society.

Revising the content in the mathematics curriculum which stresses fewer concepts each year to provide time for the depth of these concepts and providing professional development for teachers which focuses on instructional strategies is the change that needs to take place in the United States. Teaching for deeper understanding and teaching conceptually is not what most U.S. teachers are accustomed to doing; rather, many teach procedurally because teachers' preparation programs train educators this way. Therefore, a focused and precise plan is needed in order to train teachers on how to teach for the 21st century. Educators need to learn how to teach mathematics without solely relying on computational algorithms. They should be knowledgeable about why the algorithm works and about how to encourage students to critically think about strategies; teachers should support students to invent their own ways of thinking (Conyers & Wilson, 2006, p. 7). Once this is done, student achievement will improve.

Background

The state of Connecticut has standardized test for elementary, middle, and high school students. Elementary students, beginning in 3rd grade and middle school students in all three grades, take the Connecticut Mastery Tests (CMTs). At the high school level, students in 10th grade take the Connecticut Academic Performance Test (CAPT). Throughout the years, students in Stamford have not performed as well as expected on the state standardized tests in the elementary, middle, and high school grades. Although elementary students in the Stamford Public Schools have done relatively well on the CMTs in mathematics, the middle school students in the district seem to follow the national trend which shows a decline in students' math achievement during those three years. A decline in students' math achievement in SPS continues in high school.

The Goals for Change in Mathematics

There are many goals that are to be achieved with a change in mathematics curriculum and instruction in the district. The first is the creation of a common, district-wide mathematics curriculum that will be taught in each elementary and middle school grade and within each high school math course. This curriculum will not only provide educators with a foundation of what to teach and how to teach it but will also ensure that all students have the same mathematical experiences and skills necessary for the 21st century. The consistent curriculum will also guarantee that when students transition to schools within the city, they will not miss any math concepts.

Secondly, the creation of a consistent curriculum will mean that all teachers will have to acquire instructional strategies to reach all learners. Teachers will have to develop a student-centered classroom, will need to understand various types of learning styles, along with how to scaffold activities to make certain that all students are learning the content.

Third, the creation of a consistent curriculum will mean that all teachers will have to understand the content they are teaching. Prior to this, many teachers taught the concepts they were comfortable with or enjoyed leaving out some of the more "difficult" concepts or the ones they did not like. The professional development component will align with the curriculum and will ensure that all students will have teachers who understand the math content.

Process for Change

Teacher input is paramount when creating a district-wide, consistent, coherent, rigorous curriculum (Armstrong, 2003, p. 145). Therefore, tremendous measures were taken to ensure that all SPS mathematics teachers had an opportunity to participate in the creation of their curriculum. To begin the process of mathematics change in the city, committees of voluntary elementary, middle, and high school mathematics teachers and administrators from each of the schools were organized. The inclusion of administration on the committees was necessary since important decisions would be made about curricular programs, content, and monitoring of implementation.

There were two purposes of these committees. The first was to determine the key concepts that should be taught in each grade at the elementary and middle school levels and in each high school math course. This resulted in the creation of Grade Level Expectations (GLEs) for each K-8 grade and of key concepts for each high school math course. The second purpose was to choose a standards-based math program at the elementary and middle school levels. At the high school level, committee members did not look at programs but began creating syllabi, assessments, activities, and pacing guides.

Consultants from outside the district were used to facilitate the middle and high school committee meetings. Because the work includes a shifting of ideas and beliefs on the part of teachers and administrators, it was thought that it would be better to have people who have already successfully accomplished similar work to facilitate the meetings (Armstrong, 2003, p. 153). These two consultants from a neighboring district, although with different student population make-up, assisted with the meetings and shared their views and expertise with the committees

During the middle and high school committee meetings (which were held separately), international data were shared showing teachers that most states in the U.S. teach too many math concepts per year when compared internationally. This teaching of many concepts had led to less time to teach for deeper understanding. The data showed most teachers only superficially touched upon each math topic because they never seemed to have enough time for students to develop a conceptual understanding. The first assignment for the SPS committee members was to meet with the math teachers in their building and list the math concepts taught each year. Member reported out at the second meeting discovering that same concepts were taught in 6th grade, 7th grade, 8th grade, and Algebra I, and that all of these concepts were taught as if a student was seeing them for the first time. This had a major effect on student achievement.

Once the committees saw that there was redundancy in the mathematics being taught and that this led to less time to cover the materials for deeper understanding, they wanted to do something about it. This process of listing what was taught in each grade/course provided teachers with an appreciation of why there was a need for change in the district. Committee members, with input from the teachers in their buildings, determined which concepts were appropriate for their grade level/course based on state and national standards. By including all teachers in this process, this ensured that all teachers had the opportunity have their voices heard.

For the high school committee, members began working on the Algebra I curriculum. This would be the first course at the high school level to change. There would be two Algebra I courses; the standard Algebra I course and then an Algebra IA course. The intention for the Algebra IA course was for it to be a double period so that students have twice the amount of time to learn the concepts since students enrolled in this course were those who scored Below Basic on their 8th grade CMT scores. The Algebra I members of the high school committee, sequenced the key concepts, created pacing guides, syllabi, activities, resources, and quarterly assessments for the courses.

Professional Development

According to the National Math Panel Final Report, there is a direct relationship between teachers' mathematics knowledge and students' achievement (U.S. Department of Education, 2008, p. xxi). Therefore, providing professional development for math teachers that incorporate the content they have to teach and providing them with the conceptual understanding would begin to help change mathematics in the U.S and in the district. Prior to the implementation of a consistent, coherent and rigorous curriculum in Stamford, teachers at the elementary and middle school levels received math content training to make sure that they understood the mathematics they were expected to teach.

SPS teachers needed not only professional development on math content but also on instructional strategies to reach all learners since the expectation by the district is that all students will participate in at least grade/course level mathematics classes. No students will be learning content that is not grade appropriate nor will any teacher be using out of grade level materials. Therefore, teachers will need to learn how to address the various types of learners in their classrooms and also how to scaffold learning activities.

Each teacher received "just in time" training. This means that the training for teachers is given just before they have to deliver the instruction in their classrooms. The elementary and middle schools teachers followed similar models in that their professional development included not only content but also instructional strategies. The high school teachers follow a slightly different model. It was assumed that since these teachers are 7-12th grade certified there was not much need for content training; much of the training that the high school teachers received was on instructional strategies and pedagogy. The Algebra I teachers in particular received six hours of training on instructional strategies and how to differentiate instruction. Throughout the course of the school year, they received another fifteen hours of training in addition to having the consultant model lessons in their classrooms.

Leaders of Change

All of this change in mathematics could not be done without the development a "central guiding coalition" (Armstrong, 2003, p. 247). This group is able to "lead and manage" the change (p. 245). This group consists of teachers, math coaches, mathematics department chairs, central office personnel, and building administration. These people will lead the change within the schools which will then funnel through the district and into the community. The Stamford Public Schools is working towards creating leaders for mathematics change. At the elementary level, in spring 2009, twenty elementary teachers were training as *Everyday Math* Consultants so as to build capacity in the district. These leaders in elementary mathematics will support current teachers, train new teachers, and assist administration in explaining what to look for in an effective elementary mathematics classroom.

At the middle school level, mathematics coaches have been created. These coaches have received weekly training sessions with Central Office personnel and various consultants in how to effectively work with middle school math teachers on instruction, pedagogy, and math content. The mathematics department heads and the mathematics administrator at the high school levels have also been trained similarly to the middle school math coaches.

Certain schools within the district are schools who are have been listed "in need of improvement" for many years according to NCLB. For these schools, additional supports have been provided in that they have additional math and literacy coaches to support teachers in instruction and best practices.

How to Measure the Change in Mathematics

As the implementation of the common, coherent, consistent and rigorous mathematics curriculum proceeds and after the professional development for teachers, there is still a need to monitor the implementation and the collection data to see the effectiveness of the change. The data collected will include student scores on the district-wide common assessments and on state standardized tests, anecdotal evidence from teachers, students and parents, evidence produced by the state during their visit to the district, and evidence from the Central Office math team during their school visits.

The use of the data from the state standardized tests will show an increase in student achievement and therefore will show that the change in curriculum and instruction is working. Monitoring the implementation of the consistent, coherent, and rigorous curriculum will need to be done to see if there is fidelity of implementation. Building administrators and the Central Office math team will make classrooms visits and checks to see if teachers are following the pacing guides, curriculum, and using the instructional strategies learned in professional development sessions. Teacher surveys will continue to be conducted at least twice a year as these anonymous surveys provide teachers with multiple opportunities to give their opinion and feedback about the training they received, the curriculum they were using, and to provide suggestions for follow-up professional development sessions.

Next Steps

Creating a consistent, coherent, and rigorous curriculum is just the first step in changing mathematics within the Stamford Public schools. The next steps are to: create a common grading practice within the district; require the use of math notebooks at all levels; and provide time for teachers to discuss best practices, examine student work, discuss pacing and create assessments. This is a way to provide teachers with the chance to discuss their practice since education seems to be the one occupation where there is professional isolation.

There is a realization developing within the district that the curriculum is never static; it will constantly need to be revised and re-evaluated as students enter high school having participated in a standards-based elementary and middle school mathematics program, as they become better prepared for mathematics, and as the world changes. This is in accordance to the book *Curriculum Today* where it states that "curriculum development must be a continuous process that seeks to assure that instructional programs incorporate the best available contemporary information" (Armstrong, 2003, p. 18). The SPS committees of math teachers and administrators will need to continue in the upcoming years so they can re-evaluate and modify curriculum.

Reflection

Looking back on the process of creating a consistent, coherent and rigorous mathematics curriculum for the Stamford Public Schools, there are some successes and some issues that need to be revised. The first issue to be revised is providing consistent class time for mathematics. During the 2008-2009 school year, Algebra IA consisted of both 48 minute classes and 96 minute classes due to scheduling issues. Data were collected to show that those students in the double period classes did better than those students in single period classes. Therefore, for the 2009-2010 school year, all Algebra IA courses will be double periods (96 minutes).

A second issue that was revised was the implementation plan at the middle school level. Originally, the plan was to implement the standards-based math program in 6th grade during the 2008-2009 school year and then implement both 7th and 8th grade in the 2009-2010 school year. After looking at the implementation of the elementary program (kindergarten, 1st, and 2nd one year and then 3rd and 4th grade the following year), it was discovered that the students and teachers in the 4th grade had a more difficult time with the program than the other grades for a few reasons. For students, one reason there was difficulty was that they have gone through four years of school without a consistent math program. Teachers, parents and students felt a sense of frustration at varying levels. In order not to duplicate that sense of frustration in the middle school, each middle school grade will implement the standards-based program one year at a time.

A third issue was the monitoring of the implementation at all levels. Many of the administrators were not able to monitor the use of the pacing guides, the change in instructional strategies, change in content taught, etc. due to their numerous other responsibilities or their lack of knowledge about mathematics. The Central Office math team is focusing more on providing training and support for administrators in monitoring the curricular changes.

The district was successful in that it provided professional development to teachers which included instructional strategies and pedagogy simultaneously with the change in curriculum. Since the consistent, coherent, and rigorous mathematics curriculum newly created by the teachers and administrators of the Stamford Public Schools meant that teachers had to "teach differently," many were uncertain about what that meant. Therefore, professional development about the content, program (if one is used), instructional strategies, and pedagogy had to occur simultaneously in order for all the pieces to fit together effectively. The district was also successful in that each and every teacher was provided the opportunity to give input on the curriculum being created. Since not all the mathematics teachers in the district were able to be part of the committees, the committee members were able to share the work with the teachers in their building and were able to ask teachers for feedback and input. In addition to committee members asking for input and feedback, at all the professional development sessions teachers were reminded to provide input to the committee members, to the middle school math coaches, the high school math department heads, or to the Central Office math team.

Another success was the consistency in the mathematics content being taught. Parents, teachers, administrators, and guidance counselors all commented on the fact that students from various schools were working on the same math content. All who commented seemed to feel that this was the correct path for students in the district.

Conclusion The change process followed by the district included steps to gain teacher acceptance of the transformation of mathematics. In the beginning, teachers were asked to analyze their practices and the content taught. Once this was done, teachers saw that there was repetition in the concepts being taught and therefore there was a need for change.

Since the consistent, coherent, and rigorous curriculum means that for the first time all students will be learning at least grade level content, teachers need to learn a variety of strategies in order to meet the needs of all learners in their classrooms. Understanding various modalities of learning is important as well as understanding that the majority of students do not learn the same as students from years past. Continued professional development will focus on how to have a student-centered classroom and a change in instructional strategies focusing on pedagogy. In addition, there will be the continuation of course-alike meetings to look at student work and how it to use it to drives instruction.

To change mathematics in a city such as Stamford will undoubtedly take time; change of any kind takes years. There needs to be continued professional development for teachers and additional opportunities for teachers to talk with one another about the changes in mathematics and in their instruction. The continuation of the teacher leaders and the support for teachers by change agents at the building and district levels will be necessary to support and sustain this change. With the fidelity of implementation and the constant revisiting and re-evaluation of the curriculum, teachers will begin to see that the change in mathematics which is occurring is the best for students in that student achievement in mathematics will increase. This increase in achievement will mean that students from the Stamford Public Schools will be able to take higher level math classes and will be better prepared for the 21st century.

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Folding the circle in half is a text book of information.

Bradford Hansen-Smith
Wholemovement, Chicago IL, USA brad@wholemovement.com

Abstract

This paper addresses folding the circle in half and discussing some of over one hundred different mathematical terms and functions generated in that one fold. The simplicity of process in understanding fundamentals of mathematics by folding circles and observing what is generated is unknown because we only draw pictures of circles. Examples are given about observing and exploring relationships in the circle that are appropriate for first, second, third grade level and beyond. The traditional educational 'parts-to-whole' approach can only be fully realized through the comprehensive frame of Whole-to-parts by folding the circle. Wholemovement of the circle is not only direct; it is the only context inclusive to progressively understanding parts within unity of the Whole.

Introduction

The image of the circle is static; the circle disk in space is dynamic. This means the image does not generate anything and the circle generates information that can not be anticipated from the image. Through formulized and codified static images we have developed a particular kind of understanding about relationships, transformations, movement and change, using symbolic representations of parts in limited context. The generalized truths of mathematical functions are inherent in the movement of the circle and are revealed through a sequential process of folding, starting with the first fold.

The circle is the only shape that can through movement generate all other shapes and forms while remaining Whole. All circles inherently hold information for polygons, polyhedra, and mathematical functions. The circle is both Whole and part simultaneously. Nothing else functions in this way. This is much like a stem cell that is both an individual cell and yet holds all information for all other cell functions. We now know all cells contain this potential stem-information. The circle is a spherically compression and through folding the circle spherical information is decompressed in polyhedral form.

Logic Systems

Constructing a diameter through the circle image divides it in two semi-circles. Logic says joining two half circles makes one whole circle. When folding the circle in half, simply by touching any two points on the circumference together and creasing, we find six half circles. This goes against mathematical logic, a condition of how we generally think about things. The paper circle is a disk in space and by assessing the properties we find two circle planes and one circle ring around the middle. The three circle planes divided by that one fold makes six semi-circles. The folded crease is a diameter and given an infinite number of diameters in the circle, we can conjecture there is minimum two, if not six, times the number of half circles as there are diameters. This only when starting with the Whole. An infinite number of semi-circles will never complete the Wholeness of the circle. One form of the circle is not exclusive to the other. The circle and image co-exist as different forms; both show consistency within the limitations of the individual systems.

We did not totally accept Euclid's parallel postulate because it does not hold true for a spherical surface. It works only on a flat plane. Eventually we found a hyperbolic plane that happens between the movement from a straight line to the circle, from the flat plane to a spherical surface. None of these functions are exclusive from the others. The separation has been a convenience for lack of understanding the connections within a greater context. The circle and the image are simply neighbors in a larger community where conventions get formalized into rules and laws. All local functions in all communities are held in a global context on a cosmic map of mathematics development.

To understand the nature and the dynamics of the circle a shift from the image to the actuality of the circle is necessary. Using the circle image as a symbol for nothing, zero, is consistent to the nature of images, which have no inherent value. It is necessary to get beyond the concept image in math education if there is to be movement towards a more comprehensive and progressive understanding in a larger and more meaningful context.

We define the idea of the circle as image by the tool we use to draw it. We are stuck there. Thinking the circle has a center point reveals an inconsistency between the concentric nature of the circle and using a compass to construct the image. If the circle boundary goes infinitely out, and the circle boundary goes infinitely in, then there is no center or outer limits to the circle. The circle is the center, and the only limitation is origin and relative scale. Recursive set theory suggests infinity even bigger than infinity itself. That has been similar thinking about one universe, until recently having discovered other universes beyond universe. We get caught in the words, thus limiting our observations, holding us to past ideas, and static concepts represented by symbols. Words change to keep up with experience, but the underlying logic changes little and the inconsistencies go unnoticed. Only through active experience and observation can a system of logic be changed giving way to progressively uplifting our understanding.

Whole-to-parts

This observation of six halves would be of mild interest if four halves were the only difference between the circle and the image. When over one hundred mathematical functions describing a variety of relationships between just a few parts can be observe in this one fold in half, we need to rethink our understanding about the circle. The beauty is that all parts come from a single movement of equal division into the circle without separation. All parts function as context for all other parts, each multi-functional in relationships, yet individually identifiable. Nothing has been constructed; nothing added or taken away. These relationships are inherent to the circle and are given form by the movement of touching any two points and creasing one diameter; not difficult to do or understand.

The circle shows a comprehensive, self-organizing dynamic movement in space. When touching any two points accurately there is the appropriate alignment forming proportional interaction between parts that reveal many functional relationships to be discovered; even with parts not yet formed by the continuation of folding the circle.

It becomes problematic when each part is defined in isolation as a separate function. We are left with learning to construct relationships symbolically represented by images of concepts and abstracted ideas. We have trouble teaching the foundations of mathematics when we do not know the greater context, have forgotten origin, and being unclear about what is principle to the endless profusion of parts.

The diameter is multifunctional, as all parts are. It is a line of symmetry, a perpendicular bisector, a median, an altitude, axis to a spherical pattern of movement, the sixth edge of a tetrahedron pattern, codified it is the directional function of AB=BA, and it represents the number one when removed from the circle. the diameter reveals the ratio of one Whole to two parts (1:2).

For any one that wants to fold the circle in half and will take the time to observation and contemplate what is there that was not there before the circle was folded, there is much to be discovered. We do not usually look for relationships of movement; our education directs us to abstract marks on paper. We must learn to look closer for what we can not see, for patterns that have yet to be formed, for the intervals, into the space where movement occurs between the marks; the unseen rhythms of pattern shifting proportions in the regularity of measure. The connection between math and music is that the scores of each are not what they represent. What stimulates the mind and soul is the physical changes occurring in the space represented by the intervals between markings. It is within the intervals that music is made in both mind and heart. It is in the movement of unity found in the circle that reveals functional interactions that mathematics describes.

Classroom experience

What does this look like in practice in a first and second grade classroom? Young children will explore and talk about their observations without needing to be right or wrong in what they describe. The limitation of their language sometimes makes for a more direct expression of what they see, but often less clear. In the primary grades students have not yet fully acquired the language of cultural logic through which they interpret and express their observations. It matters little what first catches the child’s attention, in what order, or manner of description, since everything they observe is interrelated and consistent to the folded circle.

First have them discuss the difference between symbols and real objects; the image of the sun and the experience of the sun, the difference between a circle and a ball, an image of each? How is a sphere different than all other forms? Where do these things come from? Cutting a spherical object in half will show two parts, two circles. We can continue cutting a part getting more pieces, but the object has been destroyed and unity lost. If we compressed the sphere, squished it flat, like flattening a ball of clay to a flat circle disk, it will change the form without destroying spherical unity. Nothing is added or taken away; the unity of the whole is still there; simply reformed to three circles in the form of a disk.

Draw a circle on paper. Cut the image from the paper, or use pre-cut circles. By playing with different ways the circle moves it can be discovered that when two opposite sides of the edge are touched together it makes a hole, a thin odd looking donut that is squeezed in at one point. In rolling the touching points on the edge it changes from a cylinder pattern to a cone pattern. The solid forms are not there, but there is enough to see that the cylinder has one surface parallel to itself in a circle and the cone shows two open circle ends of different sizes where the surface is no longer parallel. Nothing of the unity of the circle has changed except it has been moved in relationship to itself and again becomes reformed. The two further-most points on the edge forming the cylinder forms an open circle plane. A relationship is seen between the diameter of one circle as it becomes the circumference to another. This can all be talked about in common language. Math terms can be introduced into the discussion for clarity, economy, and connecting new ideas.

Folding

The next thing we might notice is that no matter where the two points are touching on the edge, when folded and creased will always divide the circle in six halves. Count the front, the back, and side to confirm this. When touching the two most opposite points on the circle and creasing will form a square relationship; four equally spaced points on the circumference. The crease divides the distance, at a right angle, between the two points of the unseen diameter. The one question to students is; "What do you see that was not there before?" From this one crease there are many relationships and associations that students can observe and discover for themselves by talking about what they see.

How the question is asked will often direct students in what to look for. If triangles are the lesson for the day then the teacher might ask what kind of shapes they see, or it can be left open to see if shapes are the first things they do see. It all leads back to how they folded the circle. Putting two imaginary points together and creasing generates two points and a creased line. With a new circle mark two points anywhere on the edge, touch the marked points together and crease. The two imaginary points are now visible using two marks. Now ask the question about what shapes they see. If none, then ask them to draw lines connecting all four points. (If appropriate, we can introduce a side discussion about points and lines.) Have them count all the lines on the circle (six cords). Then ask the question about shapes again. Not only will they see the shapes, they will have the experience of where they came from and that nothing was added that was not there form the very first fold of two imaginary points. They were just unable to see what was there until the relationships were given form by marking the points and tracing the distance between them. This helps in looking for unformed relationships as well as those formed.

What kind of shapes, how many, what sizes? How many different ways can they be combined? We can observe and describe isosceles, scalene, right and left hand right triangles; they are all interrelated in the same context, parts one to the others. We can see altitudes to the triangles, perpendicular bisectors, medians, etc. It is all there when you start looking for the multiple relationships between those five points and six cords. When the circle is lifted from the flat plane there are now five cords and one unseen variable, the straight line relationship between the two touching points. We see the space get smaller as the circle is folded to a flat semi-circle position. This proves that all half circles of the same size are congruent, containing the full circle. What happened in the folding? The movement pattern of a semi-sphere was created without leaving a form. If we removed the circle from the table and turned the edge all the way around, allowing the diameter to function as an axis it would make a spherical pattern without leaving a trace of a spherical form. At the same time a tetrahedron pattern of movement is reveled by the four points in space and the six relationships between them. (The solid form of the tetrahedron comes eight creases later in reforming the circle.)

When asked what else they see in the spherical movement, a student will see an inside, suggesting to another an outside. When moving the circle in both directions we see the two insides and outsides change places; they are reciprocal to each other. One is the inverse to the other and can be talked about as a reflective movement of symmetry by folding the circle into halves. We can call one direction positive and the other negative. With two perpendicular bisectors to the kite we can then divide the radii into positive and negative directions out from the center point of crossing. When the circle is out flat there are two sets of right hand and left hand triangles. When the circle is folded shut the triangles are congruent, going in the same direction, no longer opposites of right and left, positive and negative. By saying one side is positive and the other side negative we can then see, one positive and one negative tetrahedron. There are many questions to be asked in guiding students to go deeper into their observations. It is not beyond five and six year old students to talk about what they see, and to introduce mathematical terms to clarify their observations. With older students this can serve as a good review of what they should know and be able to identify by connecting abstract book information to what they have folded. You find out exactly what students know by what observations they make, how they express what they see, and the connections they are finding.

Numbers

Looking at *one* circle, understanding there are *two* sides and the circle ring between the two shows *three* parts to the circle, (1,2,3). This is reflected in folding the circle where the Whole (O). when creased (1) forms two parts (2). One line and two parts can not be separated making three (3). Joining the total of three parts with the two previous individualized parts we have a total of five parts (5). This gives us the number of points, four points on the circumference and point of intersection when open flat. This counting process shows in a number progression, relationships from the undifferentiated Whole to points, to a folded line, and to areas. It is the points that define relationships of lines that show the areas. The number sequence looks like this (O,1,2,3,5) The next number is pretty simple; add three to the five, we get eight. Eight is the number of individual areas in this fold. You don't even have to know numbers to do it. This is not an adding one at a time sequence of numbers. We have added one to two getting three and the two to three getting five and three to five getting eight. That can be endlessly repeated in an orderly progression of one step backwards and two steps forward.

See how far a class can go with this sequencing. This is fundamental number progression and grows from the undifferentiated Whole of the circle/sphere. Then by adding one number at a time, or by two and threes, multiplying by sets and other kinds of groupings, many kinds of progressive sequencing can be discovered and played with.

This number sequence called the Fibonacci series, for the man who first discovered it, should not take away from students the opportunity to discover this for themselves. When many of the "higher" math concepts and functions are presented within the circle, where all parts are connected and interrelated, it is easier for students to understand from their own experiential observations, discussed in their own words.

Adding numbers that describe the properties of the tetrahedron pattern; four circumference points and six relationships, we get the number ten. Adding all the numbers of the properties of the tetrahedron four points, four planes, and the six edge relationships is number fourteen. By adding the one and four we get five, the primary number quality of the tetrahedron corresponding to what we have just counted. Now there are two tetrahedra in the movement of the circle. So two fives make ten, which is simple the diameter removed from the circle to form the symbol (1O), which represents the number count of four points and six edges of the tetrahedron pattern. Numbers are an easy way to see diverse connections between patterned relationships in different forms.

Conclusion

Folding circles along with drawing the image allows students a deeper and more comprehensive understanding of what it is to really observe what they are seeing, to discern the difference between pattern, form and design, to think deeply about something, to problem solve through observation in the largest possible context. It makes "higher level" math and geometry concepts easily accessible to young students giving them an experiential and progressive grounding. Nothing is necessary beyond a paper circle and the students own observations. Within the self organizing of the circle they now have a meaningful place to understand all the bits and pieces of fragmented curriculum information. They have experienced the multifunctional and interrelated aspect of parts where a change with one will always affect relationships to all other parts. There is a systematic organizational geometry inherent within the circle: nothing is arbitrary or random about what is generated. Students will have an understanding of the Wholeness of the circle that is principled to the patterns and arrangements of all other parts. Mathematics is a language used to formulize generalizations about combinations of relationships and associations of parts within the circle. This is only the beginning of folding circles. More complex mathematical functions are revealed in the continuation of folding, reforming, and combining in multiples, circle/sphere unity.

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CAS and calculation competence of students

Dr. Rainer Heinrich

Abstract

The use of new tools for mathematics at school wins increasingly Importance. It follows from this that they are consquences as well as on aims and contents of mathematics at school us like on methods in the lessons.

It is not unusual, that students and parents and also university professors are to be feared, that the calculation competence is decreasing with the use of CAS. In the lesson should be showed a possible way to developing such competences in the beginning phase of the learning process in Algebra. The examples refer to a level of school mathematics for students in the middle school age. The methods tell apart phases with and without CAS and shows a didactic principles of mathematics lessons in the case of use of CAS handheld technology.

Workshop-summery

The question of the use of technology isn't, as the history points, a question the "whether" but a question the "as". In the history this always called scepticism of some teachers and mathematicians.

In Saxony, one state of the Federal Republic of Germany, the use of graphic calculators is obligatory from the 8th form on in the gymnasium (high school). It is also necessary in the central school leaving examinations.

Introduce technology in math education has different methodical reasons.

- explorative learning – experimentation
- visualisation
- motivation
- calculator help
- change of assignment culture
- cross-curricular teaching and learning.

In the lesson I will look at the role of technology in the context of development of elementary calculation rules exclusively. I would like to ignore all the other aspects of use of technology.

Technology support the reform of teaching of mathematics.

But a lot of teachers and also some parents have doubts about the meaning of new technology in math education.

"Why do we need technology?" "Shall this what?" "The students forget the mental arithmetic. It is a big danger."

This isn't always simple.

Even if one has not aversion against the technology use in the mathematics lesson, the appreciation of sensible use of the technology is difficult from time to time as the following example shows.

The teacher provided the task:

The sum of the squares of three natural numbers, succeeding one another, is 590. Find the three numbers.

She expected the following solution way:

$$n^2 + (n+1)^2 + (n+2)^2 = 590$$

$$n^2 + n^2 + 2n + 1 + n^2 + 4n + 4 = 590$$

$$3n^2 + 6n + 5 = 590 \quad \mathrm{I} - 590$$

$$3n^2 + 6n - 585 = 0 \quad \mathrm{I} \div 3$$

$$n^2 + 2n - 195 = 0$$

$$x_{1;2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

$$= -1 \pm \sqrt{1 - (-195)}$$

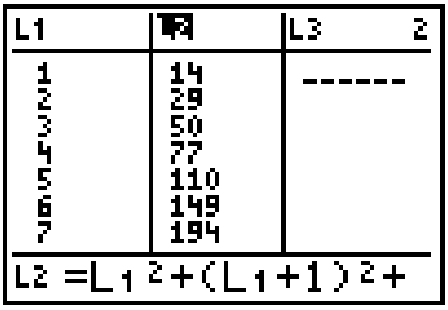
$$= -1 \pm \sqrt{196}$$

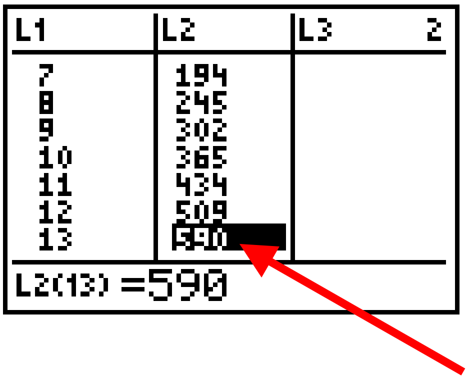
$$x_1 = 13 \text{ (trifft zu)}$$

$$x_2 = -15 \text{ (entfällt)}$$

Result: 13, 14 and 15. are the three numbers

A student worked on the task with the list menu of his graphic calculator and got the result very fast:





The reaction of the teacher was interesting now: **"Yes, this is correct. But now You hat to do this correctly again!"**

A basic concept of the mathematicians was also in the history, mathematics exempt of annoying algorithmic calculation.

Now it is a problem that every teacher, every school book author, every school administrator and also parents up to the acceptance must run through a cognition way. These phases of the cognition way can approximately be described as follows.

- developing fundamental interest
- discovering CAS as a hand tool for oneself
- use as a demonstration equipment in the hand of the teacher
- use as a calculation aid (most at well known problems)
- use as experimenting tool

→ at recognizing each the necessity of change of the mathematics lesson

In the school praxis often we can observe the classical way to introduce a new mathematical method.
In the first phase the teacher explain the new method with one or more examples.
In next step is the students practise it without technology on many examples.
If the students have solve enough examples, the teacher demonstrate the way with technology.
The problem is that students don´t experience the technology as a instrument in the realization process. They only know the power of technology as a calculator machine.
But in our point of view it is more effective to use the technology in the first phase of the realization process. In the next phase students should practise the mathematical methods without technology till a reasonable size. In the 3rd phase the mathematical methods should be applicable to practical problems or an other context. Technology is a useful tool in this phase.

Example (8th form, age of 14)
Aim: Set and resolve of brackets
Discovery phase with technology:
There is given the following number puzzle with the term: $(a\cdot 2+5)\cdot 50+m-365$
The variable a shall stand for your age, the variable m for your weight.
Multiple a by 2 and add up 5. Multiple the whole by 50 and adds up m. Subtract 365.
If the students do so and call the result, the teacher is able to calculate at once the age and the weight of the students. How is it possible?
If You give the term into a CAS, the result is 100a+m-115. If You add up now 115, you find the age in the first both numbers and the weight in the last two numbers of the result.
Such number puzzles are for students more motivating as calculating terms stubbornly. It is the Aim for the students to develop even such puzzles. But the students can not resolve any brackets at the beginning of the 8th class.
At availability of CAS the following task is provided:

Before a bracket can stand a plus, a minus or a multiply dot. You should find rules to resolve brackets by experimenting with the CAS orders "expand" or "factor". After the working phase in grozps You should give a presentation for the class.
An introduction to the CAS orders usually isn't necessary; Teenagers disclose themselves also the operation of a mobile telephone in shortest time.
The results which discover the students themselves finally have to be remembered more easily than predefined knowledge (activity level of the individual in the process of appropriation).
Depending on the situation the students will develop the need for grounds or confirmations of her assumptions. The teacher decides how far such grounds, plausibility considerations or proofs in the situation are necessary. Of course a a proof would always be necessary in subject science. In the mathematics lesson from time to time one can leave it with the confirmation of the teacher, too.
Exercise phase without technology:
Tasks are now practiced to obtaining the standard demanded in the teaching curriculum or of the teacher. Use of technology is not convenient in this phase of work.
It is possible to use CAS for result control, though. E.g. escapes the usual "announcements" of the homework results with that. The teacher only says: "Please ask your questions". The students could already check with the help of the CAS whether the calculations are right.
Application phase with technology:
It goes therefore now to set and resolve brackets in other mathematical and factural connections. The students develop corresponding terms and work on themselves. The students are appointing the technology as a calculation tool (Black box).
It is undisputed that classic calculation competence loses her place value here.
On the other side there are increasing "new" competences which are possible only in the CAS age.
These are such competences

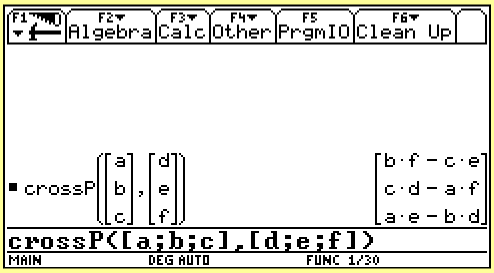
- Competence to find terms,
- Structure reconnaissance competence,
- Test competence,
- Visualization competence,
- Competence to use technology suitably,
- Competence to use computer work suitably for the formulation.

(Heugl, Herget, Kutzler, Lehmann)
The way of the change of phases with and without technology is used e.g. in the following part from a german school book at the introduction of the vector product (only the discovery phase with technology is shown here):

The multiplication of two elements of a number range is one element of this number range too in all number ranges known to us. If one multiplies two natural numbers, the product is a natural number. If one multiplies two real numbers, the product is a real number too. But the dot product of two vectors is a real number and not, like perhaps expect, a vector.

There is:
Sum: Vector +Vector = vector
Difference: Vector - Vector = vector
Multiplication by a real number: a * vector = vector
Dot product: Vector * vector = real number.

For mathematics it was necessary also to define a product which one a multiplication "vector*vector = vector " describes. You find it with the CAS order "crossP".



Working tasks:
1. Find a possible rule for a multiplication of vectors so that is: vector * vector = vector.

Examine your multiplication for qualities like commutativity and associativity. Interpret your result geometrically.
2. Find one vector which vertically stands on a given vector.
3. Find one vector which vertically stands on the two given vectors.
4. The picture in the introduction example shows the application of the order "crossP" on two vectors. The result is a vector again.
Examine what result the order "crossP" at application to two vectors would produce.
Check your assumption with the help of technology. Try to find a rule according to which one can calculate the coordinates of the product vector.
A negative example is the following table of contents contain in an other textbook.

1. Repeat
2. Limes of a function
3. Steadiness of a function
4. Difference quotient
5. Derivative of a function
6. Derivative of the power function
7. Derivative of a logarithm and exponential function
8. Derivative with CAS

Only the last passage contains parts with technology.

Summary:

The use of technology in math education don´t avert developing of calculation competences and literacy. On the contrary the use of technology is a basis for profounder comprehension of the mathematical methods so as calculation algorithm. In Saxony and Thuringia occurs in the last years surveys to compare calculation competences of students learning with or without technology. The results of the population learning with the use of technology were provable better.

But the kind of use had to follow an pedagogical concept. This kind of use can´t be a chance one. Technology should be integrated in the discovering phase of the learning process of calculation algorithm and should not be only a addition on the end of the learning process.

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Using Data Modeling at the Elementary Level to Make Sense of Doing Mathematics and Science

Marjorie Henningsen, Ed.D.
Head of School, Wellspring Learning Community, Beirut, Lebanon
marjh@wellspring.edu.lb
Nisreen Ibrahim
Classroom Teacher, Wellspring Learning Community, Beirut, Lebanon
nisreeni@wellspring.edu.lb

Absract

In this workshop, participants engaged with and reflected on authentic artifacts from data modeling projects related to the solar system and to deforestation that were completed by elementary students in grade 5 (average age 11). These authentic examples were used to ground a discussion of using a data modeling approach to help elementary students make sense of and meaningful integrated use of mathematics and science concepts and tools. School-based ways of helping teachers understand this approach in order to be able to use it in their classrooms were also discussed.

Introduction: What is data modeling and how can we use it with students?

For nearly twenty years now the field of mathematics education has placed a great emphasis on the importance of engaging children actively in doing mathematics (and using mathematics to make sense of science). One very promising approach that has arisen during the last few years is the *data modeling* approach (Lehrer & Schauble, 2000; 2002). In nutshell, such an approach involves conducting some pilot research in order to form a preliminary model that may be used to make predictions, designing systematic tests of the model that involve collecting, organizing and analyzing data, followed by a process of reflecting on whether the data support the model and revising or refining the model accordingly, thereby generating a new cycle of inquiry and experimentation.

Modeling approaches are grounded in the notion that the act of creating or inventing models, testing them, and refining them or adopting new models is akin to how real mathematicians and scientists create new knowledge in their disciplines (Gravemeijer, 1999; Izquierdo-Aymerich & Aduriz-Bravo, 2003). At the same time, many researchers also note that learners develop their understandings of concepts through continuous efforts to make sense using iterative cycles of testing and retesting mental models or ways of thinking to see how well the model represents realities we experience (Lesh & Harel, 2003). So this approach clearly resonates when thinking about how design mathematics and science experiences for students both in terms of affording conceptual development and also in terms of engaging students in authentic mathematical and scientific activity.

Our context and the examples we explored for this session

The school is located in the heart of Beirut, Lebanon. It is a new international school, currently beginning its third year and hosting classes for children from ages 3-13. The elementary and preschool classes follow the International Baccalaureate Primary Years Programme (IB-PYP, which places great emphasis on both integration of subject matter, concepts and skills, as well as the importance of student-driven inquiry. The school campus contains more green space than is typical of an urban school in Beirut and the campus has been designed to function as an extension of the traditional classroom. The school administration is highly supportive of teachers using a data modeling approach and is investing considerable resources into developing teachers' capacities to use it. The approach is viewed as completely

compatible with and supportive of student learning in the IB-PYP programme.
The two examples we looked at in the session were drawn from work completed by the grade 5 students (average age 11). The first example related to the Solar System. One group of students gathered data on the diameters of planets and distances from the sun. They selected a certain scale factor in order to create a scale model of the solar system. They reflected on how their model differs from what the scientific community tells us about the real solar system. Through that reflection they were able to pinpoint flaws in their model and also in their choice of the scale factor. They were then able to refine their model.
The second example is drawn from the students' work on a unit about the global problem of deforestation. The most concrete focus of the unit had to do with the cedar forest reserves in Lebanon. The students engaged in a variety of research activities, including a field visit to one of the cedar reserves, in order to gather data to help them understand the various physical, political, social and economic threats to the forest and to try to theorize sustainable models for solutions. The inquiry was guided primarily by students' own questions and they used a variety of mathematical ideas and tools to help them answer their questions.

Benefits and challenges of using modeling with children
It is often hard for teachers to imagine how they can use a modeling approach given all the traditional constraints and demands most are faced with such as the relative mismatch between the time they have and what they are expected to cover in the curriculum, lack of resources needed to engage in authentic hands-on investigations, and the demands of frequent or high stakes standardized testing. It is even more difficult to imagine using this approach at the elementary level because of the tendency to view younger children as unable to deal with complex reasoning and problem solving. In our view, however, it is not only possible to use a data modeling approach with children, it is essential and comes very naturally for children if they are truly engaged in trying to make sense of phenomena.

As a result of engaging in modeling processes, students can become critical assessors of their own questions and the questions of others, they can learn to be flexible when reasoning about data, they can reason about the effectiveness and necessity of various representations, they can make sense of mathematical ideas and tools in a meaningful context, and they can develop a strong sense of the nature of science. Put more simply, they can develop the tools they need to help them understand and explain "how the world works" and even invent ways to make it work better.

Helping teachers understand the approach
One of the challenges to using this approach in the classroom, aside from those mentioned earlier, is the extent to which teachers understand what the data modeling process could look like and to believe in its relevance to what they think students need to learn. They also need to believe that it's possible to do it in the context of their own school environment. In our school we have found some helpful tools and strategies for facing these challenges.
First, we have addressed the process of data modeling both implicitly and explicitly in our programme of professional development. A central tool we have used is the book, *Investigating real data in the classroom: Expanding children's understanding of math and science* (Lehrer & Schauble, 2002). The book describes the data modeling approach and illustrates it through a collection of classroom cases written by teachers for teachers. We have tried as much as possible to make use of these cases to help our teachers gain a better understanding of what an inquiry cycle might

look like or how it might play out in a real classroom, as well as to analyze the components of lessons and units in order to scaffold their ability to plan for inquiry. In both cases, the modeling approach in use was not the specific focus of professional development but was always present in the discussion because it was the backbone for each of the cases we used. Later we were able to revisit the same cases, but to specifically focus on understanding the nuts and bolts of using the modeling approach with students using the case contexts that were already deeply understood by the teachers.

Elsewhere, we tried whenever possible to use data modeling to solve real problems in the development of our curriculum in the school—and to make it explicit for teachers that we are doing so. For example, one of the long term aims of our school is to develop a strong bilingual program in English and Arabic. So early in the process of developing our language policy we asked teachers to keep logs of their instructional use of colloquial and formal Arabic outside of dedicated Arabic language instruction time. After 24 weeks of collecting these data, the data were compiled and summarized in order to be able to see patterns in the amount of time spent using Arabic outside of Arabic class along with the types of activity contexts in which it occurred. We used these patterns, along with our reading of the literature on bilingual instruction to construct a plan (or model) for how Arabic could be used across all grade levels and how that use should change from one grade level to the next. The data and the model were presented to teachers and teachers were given language use targets and strategies to pilot during the final instruction unit of the year. In those meetings the process was explained to teachers as an example of data modeling in real life. Data from the pilot work were then used to refine the model and pinpoint better targets and expectations for the upcoming school year.

Concluding discussion

In the concluding discussion, the hallmark features of the data modeling approach were highlighted and summarized with an emphasis on how student inquiry into mathematics and science was supported using this approach. Participants were also asked to share examples from their own experience as applicable. From our perspective, the most important outcome of the session was for all participants to appreciate the power and potential of using such an approach with young children in order to expand their understanding of what it means to do math and science in the world.

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The use of notebooks in mathematics instruction. What is manageable? What should be avoided? A field report after 10 years of CAS-application.

Mag. Peter Hofbauer

Commercial Highschool Horn, Austria , University of Education Lower Austria

ACDCA (Austrian Center for Didactics of Computer Algebra) *peter.hofbauer@schule.at*

Abstract

Computer Algebra Systems (CAS) have been changing the mathematics instruction requirements for many years. Since the tendency of using CAS in mathematics instruction has been rising for decades and reports have often been positive, the implementation of notebook classes seems to be the consequent next step of mathematics instruction supported by computers. Experiences that have been made with the use of CAS in PC-rooms can be transformed directly into the classroom. Hence the use of CAS is no longer limited to certain rooms. The permanent availability of the notebook with installed CAS offers the chance to realize these concepts that have already been approved with the use of CAS so far. The following speech shall show what these concepts could look like and that the use of notebooks is not only the further development of teaching in PC-classes. Examples from personal experience in teaching will especially show meanders and thought-provoking impulses in order to support teachers finding their way into teaching mathematics instruction in notebook classes successfully.

Please allow me to point out two things in the beginning: (1) Yes, I am a vehement supporter of the use of notebooks (and the use of CAS in particular) in mathematics instruction. (2) No, I do not believe that teachers who have chosen another path (or at least partly) are teaching badly.

0. Explanation

This English text is a short summary of a German paper that was published at the 100th MNU-Congress in Regensburg in April 2009. Main parts concerning the practical experience had to be abbreviated. The whole English version is available at http://www.acdca.ac.at (Website of the Austrian Center for Didactics of Computer Algebra)

1. Basic conditions

At my school, a Bundeshandelsakademie (BHAK in short, a commercial highschool) CAS have been inserted into mathematics instruction for more than two decades. In the beginning there were only a few classes held with the support of CAS due to the permanent overcrowding of PC-rooms.

After an adaption of the school now there are four PC-rooms available for approximately 350 students. Each of the rooms offer about 25 PCs and in addition to that there are also one media-and-network-room as well as one economical training enterprise where the real activity of a company is being re-enacted. Furthermore the school is wired with almost 1000 network connections which means that access to the school's intranet as well to the internet is possible from more or less every room. In particular every class is equipped with a teacher's PCc, a beamer and a printer. The decision to introduce notebook classes at my school has been made about ten years ago. Today every student of the BHAK possesses their own notebook that represents an indispensable part of the tuition. The notebooks are used in nearly all the subjects with different focuses, of course.

In this context it is important for me to emphasize that the notebooks are not the centre of tuition but only an additional device. Notebook-arrangements regulate the students' behaviour within the school's intranet (as well as the access to the internet). These arrangements are visible at http://www.hakhorn.ac.at within the part of "Verhaltensvereinbarungen".

2. Our expectations concerning the use of notebooks

The positive experiences we (that is the teachers of mathematics instruction at the BHAK Horn) have made with the use of Derive in mathematics instruction so far, gave reason to the hope that we would now get the opportunity to pursue the tuition that have been done in the PC-rooms continuously with computers and CAS. Up to that point we found the fact that tuition should be done without computers except most of the times for one or two hours per week very limiting. During the computer lessons we could observe a higher level of autonomy, willingness to do teamwork as well as more opportunities for discovering studying (not least via the internet) and finally a higher awareness of problems.

3. The first trials

A clear detachment between manual skills and the use of CAS had been approved in tuition so far. Especially the formulation of the basics of mathematics was done without the use of computers. Computer support was only accepted in succession when it came to application-oriented tasks. We wanted to stick to that procedure.

In tuition we set value on a good deal of application tasks that were answered mainly by using CAS.

4. The result – we got what we had deserved

Whereas the lead time for teachers has been rising considerably compared to the time before the notebook classes had been introduced (and there was no tendency of change recognizable), the expected changes within the tuition itself could not be determined.

Instead of a more open and student-oriented tuition the forms of tuition have almost not been changed at all.

Due to the additional time for the acquisition of the CAS handling we had to reduce time for the training of manual skills.

A partly distinctive decrease in the students' performance together with a rising discontent about the tuition was finally leading to a reorientation of mathematics instruction at our school.
In the following I want to present the theses on which our ideas were based as well as the deviated results that have been defining mathematics instruction at our school for many years (as we think: successfully).

5. What we changed – Theses for successful mathematics instruction with notebooks

Do you take all demands for mathematics instruction seriously likewise, the excessive demand of both students and teachers seems to be preprogrammed.

- What should the practical component aim at? (in a commercial school probably most likely at business and economic topics).
- What should the ability of studying aim at? (economic studies makes different demands on mathematics instruction than studies of mathematics or engineering).
- How far should application tasks for already acquired mathematical methods be searched for?
- How strongly should traditional topics of general education (catchword: definition – sentence – proof) shape the subject "applied mathematics"?

***Thesis 1:* You can not have everything!**

One of the first questions you should ask in time before the actual beginning of tuition planning is surely the question for the direction of mathematics instruction. It seems indispensable to carry out a clear assessment of the above mentioned aspects. The arguments for resp. against particular focuses in tuition are manifold, therefore it is not possible to carry out a general directive.

***Thesis 2*: stay simple!**

An intensive occupation with the question of which manual skills and abilities graduates have to improve in any case is indispensable before the use of notebooks.
The notebook changes the contents of mathematics instruction for sure, as well as it might have been the case with the introduction of the pocket calculator, the GTR (graphic pocket calculator) or the CAS. I can remember with amazement that when I was a student manual calculating of radicals was taught despite pocket calculators. The necessity of such contents can be discussed.
In this context I want to refer to a citation by Helmut Heugl, a pioneer in computer-assisted mathematics instruction, which is relevant not only for the common use of CAS (no matter if at the PC or with a handheld) but gains much more importance when thinking about notebook classes:

If It is not necessary to use computers, it is necessary not to use computers.

In order to realise this request especially when using the notebooks permanently one should add the following:

…. as far as one knows what they want.

The direction of mathematics instruction has a big influence of this point of using computers, since the matters of tuition defines when the computer should not be "necessary".
Even if the mathematical matter shows no need for the use of computers it can nevertheless be meaningful from a didactic point of view. The pro and contra of the computer-assistance should be considered with responsibility, whenever in doubt it should be tried if one could gain additional value with the computer (you can always go back).

***Thesis 3:* The fact that graduates have less manual skills due to the use of computers is irrelevant for their ability of studying.**

One of the loudest arguments against the use of notebooks in mathematics instruction (resp. against the CAS-application in general) concerns the fact that due to the rising implementation of technology arithmetic skills are pushed back. This argument is not new, but it does not become more correct only by repeating it again and again.
It is rather true that the moving of teaching matters away from the mere calculating is leading to the fact that certain manual skills are not being trained and hence are not being mastered in the usual way. Students from computer-assisted mathematics classes could make up for manual skills in a short time (as far as this is even necessary) they seem to have advantages in other areas (model-construction, interpretation of results,...)

***Thesis 4:* The subject mathematics must sell better!**

The meaning of natural sciences and mathematics has been pointed out in regular periods nevertheless mathematics importance in (the Austrian) public but also among the non-natural science subjects seems rather little.
"I was always pretty bad at maths!" is still considered as socially accepted statement, lacking knowledge about Goethe, Schiller or the ignorance of the updated orthography are admitted by contrast rather seldom.
The importance of mathematics for other subjects (apart from physics and chemistry) is also challenged at this point.
The right way – and here the cooperation with the colleagues teaching economic subjects is necessary – would have to be the treatment of tasks based (!) on economic consideration. Not the other way round! This would need the understanding of colleagues from other subjects that mathematics plays a certain part in their own subjects. To gain this insight can among other things be an aim of mathematics instruction with practical relevance.
In this context one should naturally not forget another argument against the practical relevance of mathematics instruction:

"Mathematics is more than just application!"
From the point of view of a vocational school the argument can be seen rather coolly since there are only 10 lessons a week (divided into four years of teaching) and focal points have to be set. Apart from this main mathematical concepts can be trained when dealing with the basics for these tasks anyway.

***Thesis 5:* Mathematics instruction currently still strongly follows the mathematic matters of the curriculum rather than the tasks of education and teaching!**

(The general and didactic tasks of education as well as the tasks of education and teaching for the subject "mathematics instruction and applied mathematics" can be seen in the document "Handelsakademie (Anlage 1) / Lehrplan 2004" (S. 3ff, 30ff) in the category Lehrpläne at http://www.hac.cc)
This statement applies with regards to content perfectly to

***Thesis 6:* The use of notebooks can support the movement to the direction of general (didactic) educational goals!**

We have already mentioned some possible reasons for the orientation at the matters of the curriculum. They may be based in traditional views of mathematics instruction (catchword: "Acquisition of the craft!")
If you are of the opinion that the matters of mathematics instruction should move into the direction of general educational goals, as they can be found in the curriculum (modelling, arguing, interpretation, critical challenge,…) you will surely find the use of notebooks an important tool. Once the transition has been made, mathematics instruction has clearly gained surplus as we think.

***Thesis 7:* Even if carried out skilfully the use of notebooks does not lead to additional knowledge but to a different kind of knowledge!**

As pointed out above the notebook can play a major role in moving the matters of mathematics instruction. It has to be mentioned that this movement can in big parts also be reached without the use of notebooks.
We want to warn you at this point from the notion that the introduction of notebook classes would automatically lead to an advancement of marks in the subject mathematics instruction. This hope has not come true – at least in our case. It is true that our students' approach to mathematics seems now more open and fearless and also other desirable characteristics have improved.
As comfort the fact remains that marks have not become worse either. Mathematics instruction has just become "different"- maybe not better but as we think in no case worse either!

6. If we could start anew – our advice for new comers resp. teachers switching

Contemplated retrospectively our ideas of mathematics instruction with notebooks were in the beginning misleadingly driven from the notion to have quasi the all-in-one device suitable for every purpose in front of us now. The thought of being able to make everything better that was so far and additionally integrate new ideas into mathematics instruction necessarily had to result in the above mentioned failure. This is a basic realisation from our beginnings:

Failure has to be allowed!

In the same way as students must be given the right to take wrong paths on their search for solutions and then change their minds teachers must be given the opportunity to learn from wrong concepts and subsequently make it better. Of course this needs a lot of understanding on the part of the school supervisory board, especially headmasters are in demand here when wrong concepts are to be accepted without detriminal consequences for teachers and tuition and if necessary stand behind the idea that new studying environments need new ways of studying and sustainable concepts have to be worked out. In some cases this is unfortunately only possible via trial and error.
The expenditure of time for preparation and revision in the introduction of notebooks should in no way be underestimated. It was true for the mere CAS-application that common forms and concepts of tuition could often not be kept the way they were, but this is true on a higher level for the tuition using notebooks. Thus the advice must be (not only for the beginning but in general):

Don't rock the boat!

Trying to adjust the whole mathematics instruction to the new tuition environment does not seem meaningful. This trial quickly leads to a massive congestion of the teachers as well as the students and does not lead to any positive effect.
It seems rather advantageous to slowly change from traditional tuition to the desired aims. In any case is it sufficient to take just a few new tasks and it is not necessary at all to create new tasks which are compatible to the new demands by oneself permanently. Mainly in the beginning it is more than enough to modify well known tasks in a way that they fit into the new concept. Students will very well recognise the changes but the tasks will somehow be familiar to them which is a quite comforting situation. A quick transformation swamps teachers as well as students and is only good for creating negative motivation from the very beginning.
For later on it is also a good advice to look for compatible tasks first before creating tasks by oneself. Of course it is satisfying and enjoyable for a teacher when a new task seems successful and makes its way also in tuition but one invests a lot of time into such work. Selective research offers enough material compatible with mathematics instruction due to the presented ideas, apart from the internet there are also certain (school)books that concern themselves with this form of tuition in particular (see bibliography).

Discard something old for every new matter

The use of notebooks does not increase the amount of time available for mathematics instruction. The decision for new focal points (e.g. arguing, modelling, …) is hence indispensably connected with the decision against other matters that have been taught so far. Considering which matters should be shortened or even discarded in order to give new ideas sufficient (temporal) room is probably one of the most important points for teachers preparing for notebook tuition. If this reorganisation fails you are on the best way to frustrating and discouraging tuition which may very well lead quite quickly to re-establishing well-tried traditional concepts and renounce the use of notebooks.

Concerning the organisation of your tuition it seems advisable to focus on some indications again and again:

No experiments in tuition if you are insecure (there are enough unexpected things happening anyway)

Be sure of the contents you want to convey and make sure to be familiar with the technology you use if you try out something new. Whenever teachers are insecure they are engaged with themselves and the introduction of their media so much that they are unable to commit themselves to the students and support them.

Never allow the notebook (or the CAS) to become the centre of tuition

The notebook itself will not enhance or revolutionise mathematics instruction neither has the pocket calculator. It just offers new opportunities that one should use extensively. Nevertheless the notebook is just an additional tool in order to present circumstances in a better way of to shorten steps in the calculation. In no case mathematics instruction is allowed to become a user training course for particular software products. It has been approved to use software just from the very beginning next to common tuition. Students learn how to handle it quite quickly in general. Most of the times they acquire the necessary skills themselves. The regular tuition should only pay attention to the handling problems if a great number of students is concerned. (note: this is not true for exams; here everything should be done to reduce handling problems – see the article “Tasks for tests and A-levels using CAS” by Heidi Metzger-Schuhäker in this conference transcript.)

Notebook excuses are forbidden!

There will always be students who have no working notebook at hand. Experience has shown us that you should not consider this circumstance, in the worst case students without notebooks should work together with other students for the time of the mending (or maybe there are arrangements with the dealers who offer replacements for that period of time resp. the school owns some replacements). Homework can be cone on other computers during that time it is rarely the case that no computer is available at all for students whose notebook has to be mended. Tuition should not pay too much attention to such situations.

Finally one major point for the introduction of notebook classes should be mentioned which may be decisive for the success of the tuition:

Above all, try to prevent frustration concerning marks in the first grades!

Certainly the information contained in this speech does not represent the silver bullet to the use of notebooks in mathematics instruction. But maybe one or the other thought-provoking impulse or one or the other information can be helpful for teachers to support their own way planning their mathematics instruction tuition in classes with notebooks or at least avoid some difficulties.

Altogether I want to point out in the end that we at the BHAK Horn meanwhile see ourselves encouraged to adapt and transform our tuition in the above mentioned form after difficulties in the beginning. (Even though I have to admit that the rapid technical development presents new changes for tuition again and again).

We are with Apostolos Doxiadis (Greek mathematician and author) of the opinion to have shown our students that actual mathematics is infinitely more interesting than solving quadratic equations or calculating the volume of bodies.

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Community Engagement: Home School Partnership

Marilyn Holmes
University of Otago College of Education marilyn.holmes@otago.ac.nz

Abstract
Five year old children starting their formal education in primary schools bring with them a range of informal mathematical understandings. Transitioning from an early childhood setting to the reception class at school can have a profound impact on their developing mathematical concepts. Traditionally their first teachers (parents, caregivers and whanau) gradually remove the support and encouragement and some of the familiar surroundings of their early childhood centres are no longer there. As children from 5 – 13 years of age spend approximately 85% of their time out of school it is important that their first teachers are encouraged to continue that support. This paper outlines a New Zealand project 'Home School Partnership: Numeracy' that gives one approach to enhancing children's mathematical learning through shared understandings between home and school.

Introduction

Home School Partnership in New Zealand refers to a partnership between schools and homes not to be confused with home schooling. Whereas home schooling involves parents/caregivers teaching their children in their homes, home school partnership is an attempt by schools and children's families to bring understanding and clarification about numeracy to both parties. With the changing of government new policies have been implemented and a timely one has been a call for plain language reporting to parents. Power and control is an unfair advantage when only one party understands the language.

With the numeracy project implemented in 2000 new terms have resurfaced or been born. Tidy numbers is a new one and belonging to that term are all numbers that end in zero. It seems an easy idea to understand and that many children will be able to give an example. What is not so easy for some children to understand is how useful tidy numbers can be. Making ten has been around for many years but connecting knowledge about 0 – 9 with larger numbers seems to be something some children miss out on and parents have not been privy to that knowledge.

Within the context of this paper the word 'parents' will refer to the collective body of parents, caregivers and whanau, who are the people schools communicate with in regard to the children.

Why Home School Partnership?

Overwhelmingly the evidence points to the fact that parental involvement helps children's learning. With the large amount of time that children are not in classrooms, schools need to develop strong supportive links with each family they have at school. Even with their high success rate in the Trends in International Mathematics and Science Study Foong (2004) maintains schools in Singapore are "beginning to see the advantage of engaging parents in the education of their children" (p. 49). Bull, Brooking and Campbell (2008) found research literature shows "parental involvement makes a significant difference to educational achievement" (p. 1). In their body of work Anthony and Walshaw also confirmed many educators and researchers believe if parents are involved in their children's education there will be positive outcomes (2007). The Ministry of Education (2008) in their booklet Home School Partnership: Numeracy also endorsed the importance of parental involvement "...when parents, children, and school staff work together, there are more opportunities for children's learning to improve" (p. 3). There does appear to be some debate about the degree of validity between involvement and achievement but whichever the way the discussion goes it is certain that parent involvement affects factors that impact on children's achievement (Peressini, 1998).

While asserting it is difficult to find the kind of involvement that makes the difference, Bull et al. (2008) did suggest common features that point to successful HSP:

- Relationships in successful home – school partnerships are collaborative and mutually respectful
- Successful home-school partnerships are multi-dimensional, and responsive to community needs
- Successful home-school partnerships are planned for: embedded within whole school development plans; well resourced and regularly reviewed
- Successful partnerships are goal oriented and focussed on learning
- Effective parental engagement happens largely at home
- There is a timely two-way communication between school and parents in successful partnerships (p. 1).

Pakeha and Asian children generally have higher achievement levels than Maori and Pasifika children but that does not mean only parents of the former group want their children to 'do well'. Schools are required to work with all parents to help fulfil their hopes of their child 'doing well'. Doing well in this instance means a child having the success they are capable of achieving. In his opening speech at a national numeracy conference Maharey (then Minister of Education) said, "...engaging families and communities in a meaningful way with the teaching and learning of their children, is an area where we need to do more work" (2006). A low decile rating in New Zealand is not an excuse for a child's lack of progress. It may mean schools may have to work harder and smarter to build children's confidence and motivation through other ways such as fostering close home school partnerships while taking into consideration the common features above.

Background to Home School Partnership in New Zealand

With increasing numbers of Pasifika people and other ethnic groups into the country over the last 20 years it became evident that cultural inclusion was an aspect schools needed to attend to. How could they communicate the ideas of how children learn to read, write and do mathematics in the New Zealand system with parents whose first language was not English?

A Pasifika initiative aimed at considering key literacy messages with parents was implemented in 2001; Home School Partnership: Literacy (HSPL). It aimed to use parents as leaders as a means of encouraging other parents to attend sessions, and to use the time to establish better relationships, closer communication ties and encourage ways in which parents could be involved in helping their children with reading and writing. Evaluations showed that schools that adapted the programme to suit their community needs had more successful outcomes than schools that stayed with the original model and 80% of the schools reported that "parental involvement had a positive impact on children's opportunity to learn" (Bull et al, 2008).

On the other side of the government's strategy Phase 1 of the Numeracy Development Project was being implemented so it wasn't until 2006 that a pilot for Home School Partnership: Numeracy (HSPN) was realised. Following on from the success of the literacy project, the positive remarks and growing interest made by the parents, HSPN closely followed the literacy model at first. Outside (of the school staff) numeracy facilitators throughout New Zealand began to attend every session their schools held in a supportive role. They were also used to offset any difficult questions that may be asked.

Forty schools were in the pilot, the same number the following year with increasing numbers in 2008 and 2009. Evaluations showed that schools did not stay with the original model but adapted to their school's needs and aspects of HSPN helped improve the way they connected with parents even if they discontinued the project (Fisher & Neill, 2008).

Home School Partnership: Numeracy based on Six Schools

Throughout New Zealand several schools are or have implemented the project but for the purpose of this paper the picture can be seen through the story of six schools in the southern reaches of the country and how they implemented HSPN by looking at commonalities they shared.

Rationale

Although the six Principals had differing ideas of why they wanted HSPN in their schools they were in agreement on one matter and that was there was little recognition of a partnership with families in numeracy. The same parents were often in their schools helping with reading groups, sports days, library work, parent teacher meetings and concerts but never in numeracy. While Posamentier said "mathematics is the only subject about which adults can cheerfully exclaim they know nothing and still be thought of as intelligent and even educated,' Principals were 'taken aback' at the number of parents who did 'cheerfully exclaim' they couldn't do maths and they expected that their children would be the same (Principals, personal communication, May 9, 2008). In a world when we are striving for excellence and teachers have high expectations it is imperative that everyone involved with children's education have the same expectations. Parents as well as teachers must believe children can do well with encouragement and support.

Partnership

In developing a partnership between home and school all schools discussed what sort of partnership they were trying to achieve. Schools wanted to encourage clear lines of communication between home and school. Newsletters, although they kept parents informed, were always going one way. Co-operation and collaboration were aspects the schools hoped would eventually build to a two way partnership. However, they realised as community sessions were held, that parents were interested in the school's numeracy programme and felt they had little to offer themselves. Schools reviewed their goal and decided they wanted parents to come to their sessions and enjoy themselves especially as some parents had expressed some negativity towards mathematics. Although the partnership was not equal schools felt it was something that would develop over time. One of the ways to help schools move to a more equitable partnership was to appoint a Lead Teacher in Numeracy (LT) for HSPN and two or three Lead Parents (LPs) depending on the size of the school.

Role of the Lead Teacher and Lead Parents

The role of the LT, who was a member of the teaching staff, was to support a team of LPs with planning, preparation and to co - facilitate community sessions. The LT also liaised with staff so that everyone understood what HSPN involved and what part they could play in promoting a numeracy partnership with the parents. An ability to relate well to people was an essential attribute for the LT and LPs. Lead Parents had a pivotal role to play in building the partnership and relationship with the community. In areas where large numbers of different ethnic groups lived LPs were sometimes chosen because they were bilingual. A key focus was to encourage those families to feel comfortable in the school settings by talking to them in their own language. Only one of the six schools had a significant number of Pasifika parents but they were first generation New Zealanders with competent English. However, they were encouraged to use both languages when counting or playing games.

Aspects that LTs and LPs agreed to foster in all schools were:

- to acknowledge parents as the children's first teachers
- to recognise parents' language and culture as important for their child's learning
- to learn from the parents as parents learn from the school
- to encourage ways in which parents could help their children in numeracy
- to create an environment where parents felt comfortable
- to encourage all teaching staff to participate in evening community sessions

Community Sessions

All schools followed the same format for their community sessions. Four planning days were set involving the outside facilitator, the Lead teacher in Numeracy and Lead Parents with four afternoons or evenings for delivery. Planning for a community session which typically lasted 1½ - 2 hours took a full day to plan. All schools covered the same topics:

Session 1	Session 2
sharing of mathematical experiences counting framework addition and subtraction	numeracy strategy stages part whole thinking basic facts
Session 3	Session 4
revision of strategy stages place value	multiplication facts multiplication and Division

Of the six schools three decided to hold a fifth session which was a celebratory night with parents, teachers and children participating in activities at different stations and competing for spot prizes.

Food played an important part especially if children were to be at all or part of the community sessions. It is culturally acceptable to offer food when visitors arrive at one's house and that tradition is practiced in schools.

During the sessions parents were given simple messages that would be useful when supporting their children. An example of one message was counting forwards and backwards. Many parents counted forwards with their small children but never counted backwards until they could see the relevance to subtraction. Another message was the difficulty young five and six year olds experienced learning the teen numbers. They could count by rote but not recognise 12 or 13.

In all schools LPs were nervous about facing other parents but to ease them into their first 'public role' they were given activities that they could introduce and play with the parents. "Incorporating school like activities, through providing parents with access to both additional pedagogical knowledge and information about finding and using local educational resources, can have dramatic and positive impacts on children's achievement" (Biddulph, Biddulph & Biddulph, 2003, p. v).

One game that LPs felt comfortable with was 'missing number bingo'. It was a game that helped children focus on number before and number after. Part of the success of the games was that parents were able to take a game home after the first session and then add to them after each consecutive session. A frequent comment from children at school was *"we've been playing those games at home* *and they're fun"*. As not all children's parent were attending the community sessions backpacks containing games, dice and counters were sent home on a rotation basis between the other children.

Teachers reported parents asked more questions, were knowledgeable and interested about the strategy stages and sought more ideas of how they could further support their children. Through 'postits' as they left or by comments about the sessions parents expressed their enjoyment and the growing confidence they felt when discussing numeracy with their children at home.

Conclusion

As part of their reflective review the schools compared their sessions to the common features (Bull et al, 2008). The schools felt they had achieved all with the exception of the two way communication between school and parents. All realised they still had some way to go before they were sure they had achieved a real partnership.

Good LTs, LPs and strong leadership from Principals, who turned up for every session, made a difference. Principals and staff attendance signalled to parents that they thought a home school partnership was valuable and likewise the number of parents at each session was an indicator to teachers that they were supported. Parents found different ways they could help their children with the new knowledge they had acquired and teachers found they were not alone in educating the children in their classes.

Building a partnership can take time as both parties move towards listening and learning about the numeracy they both have to offer. Ultimately HSPN is about the children and supporting them in their learning.

This paper has only given a glimpse of what HSPN is like in New Zealand. In pockets throughout the country facilitators, teachers and parents are working together to involve more of their community by engaging in their children's education. A connected partnership of home and school has a strong influence on children's achievement. Anyone who has shown an interest in children cannot help but see the impact a smile or a positive remark can make. Having one's parents involved in what is happening in school has the same effect.

> Building harmonious relationships between school, families and communities can have reciprocal benefits for all concerned. Parents develop more understanding of the school's programme and appreciate their children's numeracy knowledge while home and community environments offer a rich source of numeracy experiences on which to base and enhance that learning in school (Ministry of Education, 2008, p. 3).

Acknowledgement

Further information on HSPN booklet and games can be found on www.nzmaths.co.nz

The views expressed in this paper do not necessarily reflect that of University of Otago College of Education or the Ministry of Education.

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Modelling the Transition from Secondary to Tertiary Mathematics Education: Teacher and Lecturer Perspectives

Ye Yoon Hong
Suan Sunandha Rajabhat University
<yyipuni@naver.com>

Suzanne Kerr
AGGS, Auckland
<suzannek@slingshot.co.nz>

Sergiy Klymchuk
AUT University
<sergiy.klymchuk@aut.ac.nz>

Johanna McHardy
MAGS, Auckland
<jmchardy@mags.school.nz>

Priscilla Murphy
MIT, Auckland
<priscilla.murphy@manukau.ac.nz>

Sue Spencer
EGGS, Auckland
<suespencer1@gmail.com>

Mike Thomas
The University of Auckland
<m.thomas@math.auckland.ac.nz>

Peter Watson
AUT University
<peter.watson@aut.ac.nz>

Abstract The transition from school to tertiary study of mathematics is rightly coming under increasing scrutiny in research. This paper employs Tall's model of the three worlds of mathematical thinking to examine key variables in teaching and learning as they relate to this transition. One key variable in the transition is clearly the teacher/lecturer and we consider the perspectives of both teachers and lecturers on teaching related matters relevant to upper secondary and first year tertiary calculus students. While this paper deals with a small part of the data from the project, which aims to model the transition, the results provide evidence of similarities and differences in the thinking of teachers and lecturers about the transition process. They also show that each group lacks a clear understanding of the issues involved in the transition from the other's perspective, and there is a great need for improved communication between the two sectors.

Introduction

Concerns about decreasing numbers of students opting to study mathematics at university and beyond (e.g. the ICMI Pipeline Project) have encouraged research interest in the transition from school to university. A widespread decrease in levels of mathematical competence, including a lack of essential technical facility, a marked decline in analytical powers, and changed perceptions of what mathematics is, especially with regard to the place of precision and proof, have been noted in reports (LMS, 1995; Smith, 2004), with these difficulties even extend to 'high-attaining' students. Research on the transition period from school to university education in mathematics confirms the mathematical under-preparedness of students entering university (Hourigan & O'Donoghue, 2007; Kajander & Lovric, 2005; Luk, 2005), and the impact this has on students' success in university mathematics (Anthony, 2000; D'Souza & Wood, 2003). Moreover, the problem of a possible widening gap between school and university has been described by studies in a number of different countries around the world (e.g. Brandell, Hemmi & Thunberg, 2008; Engelbrecht & Harding, 2008).

In the research described here we have been using a developing theory by Tall (2004, 2008), which suggests that mathematical thinking exists in three *worlds*, the embodied, symbolic and formal, to examine the possibility of qualitatively different approaches to thinking about mathematics at school and tertiary levels. The embodied world is where we make use of physical attributes of concepts, combined with our sensual experiences to build mental conceptions. The symbolic world is where the symbolic representations of concepts are acted upon, or manipulated, where it is possible to switch from processes to *do* mathematics, to concepts to *think* about mathematics. The formal world is where properties of objects are formalised as axioms, and learning comprises the building and proving of theorems by logical deduction from these axioms. There is some evidence that one specific problem in mathematical thinking relates to an emphasis in school mathematics on symbolic world procedural understanding of algebraic material (Novotna & Hoch, 2008). Tertiary mathematics courses are usually trying to build formal world thinking based on a deductive, axiomatic approach; so if students are primarily symbolic thinkers, then tensions and difficulties will naturally arise. One outcome is that many students who are exposed to a formal deductive approach in mathematics for the first time on entry to university experience a significant amount of cognitive conflict in their first year (Tall, 1997).

Method

This study is part of a much larger research project entitled 'Analysing the Transition from Secondary to Tertiary Education in Mathematics' involving teachers, lecturers and students, that employs questionnaires, interviews and teaching observations. A questionnaire on the transition was sent to all 350 secondary schools and 31 tertiary Institutions (Polytechnics, Universities, Wanangas and Institutes of Technology) in New Zealand to be completed by all teachers who teach calculus in Years 12 or 13 (age 17-18 years) and by all the calculus lecturers.

The questionnaire was posted, complete with a stamped addressed return envelope and teachers and lecturers were given three weeks to answer. After this a follow-up copy was sent by email to remind the teachers and lecturers to reply. Using this approach we received a total of 178 teacher and 26 lecturer responses, and some of these were later interviewed. There are no figures available on the total number of calculus teachers and lecturers in the schools/institutions, which vary in size from fewer than 30 students (small country school, Polytechnics, Wanangas and Institutes of Technology) to 3000 (inner city schools and Universities), but we estimate the response rate at about 30% of the calculus teaching school population, and a little less of the tertiary one. In this paper we present and compare the teachers' and lecturers' responses to two questions from the questionnaire, along with some interview comments, in the light of Tall's (2004, 2008) model of thinking. The questions (22 and 23) asked "Do you think that there are any differences between Year 13 and first year tertiary calculus teaching in any of the following areas? If so please describe them." and "Do you think students have any problems on moving from school calculus to tertiary calculus." Of the 178 teachers and 26 lecturers who responded to the survey, only 154 teachers and 23 lecturers gave personal demographic details. Table 1 highlights some of the demographic differences.

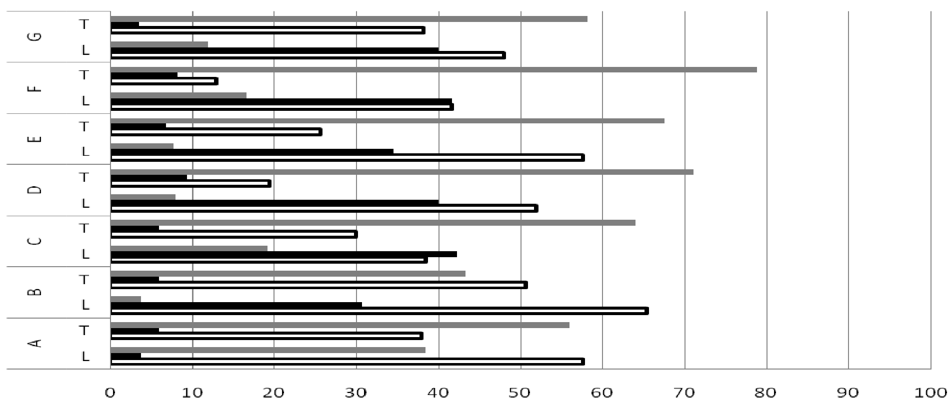
Table 1: A Comparison of the Personal Demographic Details

	Male	Female	Predominant age group(s)	English first language	Taught >11 years
Teachers (*N*=154)	82 (52%)	79 (48%)	41-50 (35%) 51-60 (29%)	90%	55%
Lecturers (*N*=23)	19 (83%)	4 (17%)	51-60 (44%) 41-50 and >61 (21.7%)	78.3%	17%

Results

In question 22 of the survey, teachers and lecturers were asked if they thought that there were differences in the following areas: A-assessment, B-teaching style, C-teaching resources, D-teaching emphasis, E-technology use, F-teacher preparedness, and G-students' experiences, and if so why. These areas were considered to be possible key variables in the transition and in order to be able to begin the process of constructing a model of transition we wanted to identify what variables are important, the relative level of their importance, and the relationships between them. Figure 1 (below) shows a comparison of percentages of the lecturers' and teachers' responses.

Figure 1: Percentages of lecturers' (L) and teachers' (T) responses.



	A		B		C		D		E		F		G	
	L	T	L	T	L	T	L	T	L	T	L	T	L	T
Don't know	38.5	56	3.8	43.3	19.2	64	8	71.1	7.7	67.6	16.6	78.9	12	58.2
No	3.8	6	30.8	6	42.3	6	40	9.4	34.6	6.8	41.7	8.2	40	3.5
Yes	57.7	38	65.4	50.7	38.5	30	52	19.5	57.7	25.7	41.7	12.9	48	38.3

While more than 30% of the teachers perceived differences in assessment, teaching style, teaching resources and student experiences, the most common response was to answer, “don’t know” whether there are any differences. This could be of concern when considering the transition from school to tertiary study since it implies a lack of knowledge of the tertiary situation. Some teachers alluded to this as a possible reason in their interviews, with one saying.

> I think that we don’t.. we haven’t got a lot of uniformity amongst schools in presenting to students what to expect at university, and I don’t think the universities do that brilliant a job in feeding back to schools what they want…I do believe that, where schools are trying to find out what’s required at university. (T018)

Comparatively speaking, a majority of the lecturers perceived differences in assessment, teaching style, technology and student experiences. However, for the assessment area 38.5% of the lecturers responded that they ‘did not know’, possibly implying a lack of knowledge of the school assessment system. This may be because many lecturers have not taught in schools, and even those who have may not have taught to the recently introduced National Certificate in Educational Achievement assessment system. Whatever the case, they seemed to have some knowledge of other areas such as technology use, teaching style and teaching emphasis, since the percentage responding “did not know” was relatively lower for these.

The following analysis will be an attempt to match Tall’s (2004, 2008) notion of mathematical thinking with these four variables: assessment, teaching style, teaching emphasis and teacher preparedness and support.

Assessment

For the assessment area the lecturers’ comments on differences presented only a vague perspective of school assessment in terms of types of assessment and how they are graded:

Assignments differ from NCEA internal assessments (L3)

NCEA does not require a student to get 'more than half' correct to pass (achieve) (L10)

The teachers who commented about differences in assessment between school and tertiary level made observations such as, “A lot more assessment” (4, 6.1%), although it is not clear whether they felt that school or tertiary had more. References were also made to the differences in assessment styles, such as “Standards-based versus norm-referenced” (4, 6.1%) and “Presumably universities are not using the type of marking used in NCEA [national] exams.” (2, 3.0%). In their interviews, teachers talked at length about the NCEA assessment and the attitudes of students and themselves in dealing with this summative assessment. A theme of tailoring work to assessment at a specific, often lower, level was prevalent.

I think that the internal assessments...because you know what you’re going to be assessing them and because of time constraints, you can teach the content that’s in the assessment. I’m afraid that that’s the sort of thing that has crept in. (T156)

Let me think of an example, let us go back to my expectations with the majority of the class, if I’m aiming at achieved or merit I might skip out the excellence part work at the end. (T134)

This suggests that teachers who tend to teach to the assessment may promote procedural, symbolic world mathematical thinking to achieve student passes.

Teaching Style

The prevalent perception of differences in teaching styles was agreement that the level of interaction between lecturers and students at tertiary level is not sufficient (41 teachers, 64.1% and 10 lecturers, 80%). The lecturers’ comments included “primarily lecture format less interactive than school” (L2) and “[our] teaching style [is] more formal less individualistic” (L3). Of course this is partly due to large class size, and this was evident in these kinds of comments “My class is 420 students! Determines style.” (L8) and “Lecture style is all one way for large 200+ classes” (L13). Some areas where the lecturers’ comments were consistent with the teachers’ comments included: “Tertiary students are taking more responsibility for their own learning. Teaching style is more teacher-centred” and “less personal interaction with students” (41, 64.1%).

Teaching Emphasis

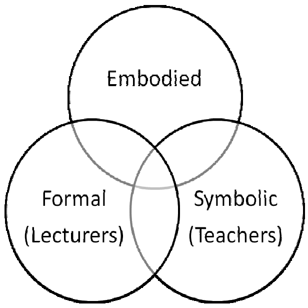
It is significant that 71.1% of the teachers answered that they did not know of any differences in teaching emphasis. Those who commented mostly felt there was greater depth to the understanding (2, 11.1%), an emphasis on the theory, and a more formal approach (2, 11.1%) at tertiary level than at Year 13. Some felt there were "Different approaches to certain sections, inclusion/exclusion of topics at school" (2, 11.1%), and "more on pure mathematics (and) less on applications." On the other hand, 52% of the lecturers also reflected the teachers' view that the lecturers "focus on understanding concepts rather than learning techniques" and have an emphasis on "applications in particular areas, example engineering, science". Question 10 of the survey considered the level of importance (1=Not important to 5=Very important) that lecturers attribute to various factors when teaching calculus. In particular, 92% of the lecturers valued applications in calculus teaching, but only 44% of them valued procedural learning. These results are consistent with Tall's (2004, 2008) model, which promotes the notion that procedural learning is more common and valued in schools, while formal thinking tends to be promoted, and valued, at tertiary level, whereas procedural work is less valued, and hence less common.

Teacher Preparedness and Support for Teachers

The results here showed that lecturers would be more prepared to teach in a formal way that engenders Tall's (2004, 2008) notion of mathematical thinking than the teachers who were over-burdened with administrative and classroom matters and possibly received lesser support. In the teacher interviews they discussed the predominant issue of being burdened with administrative work and classroom management. "If you're tired and you're wrapped off your feet because you're doing your reports and ninety thousand other things.., you don't prepare." (T156). The lecturers also echoed understanding of the teachers' frustration, "High school teachers are generally very under-prepared for their classes compared with tertiary teachers. They are often discouraged by the impossible situations which they face in the classroom." (L10). The teachers also stated that the lecturers get "possibly more support/preparedness at university and perhaps time." (2, 13.3%); the "University has more access to support for resource preparation." and there are "More colleagues and departmental discussion at university. "

By comparing the lecturers' and teachers' responses for the four variables, it reinforces the idea that lecturers tend to promote formal world of mathematical thinking while teachers may focus on developing symbolic world thinking in their teaching. Hence, based on the inter-relationships of variables, Figure 2 shows how they may embrace Tall's (2004, 2008) notions of mathematical thinking.

Figure 2: How lecturers and teachers fit in Tall's model.



Transition

Following question 22, the results from question 23 reinforce the cognitive conflict (Tall, 1997) faced by students as the teaching/learning paradigm shifts from symbolic to formal thinking during the school-tertiary transition. The results also show that most lecturers (60%) and some teachers (25.3%) tend to agree that students faced problems during their transition period. Based on Tall's (1997) notion of cognitive conflict, it would appear that the under-prepared first year students face problems in their learning, whereas the more prepared ones coped well during the transition. These statements show the teachers' perception of how the transition would be made easier or harder,"If calculus is well taught at school, the first year of university calculus can be 'too easy." and "Only if it were properly taught at school first year university mathematics is sometimes easier than L3 maths and there is little challenge for the top students in first year...".

Other reasons cited for under-prepared students were low achievement in school. The most common teachers' suggestion was that "students should aim higher to get merit or excellence as the tertiary education assumes they have a sufficient knowledge of Yr 13 calculus." It appears that these teachers observe students simply aiming to 'pass' rather than understand at a deeper level. Another possible problem faced by the students is the low lecturer-student interaction. Nearly 9% of the teachers believed that the amount and quality of interaction between lecturers and students was a problem, mentioning the importance of 'one-to-one contact and help'.

In summary, this paper uses Tall's (2004, 2008) notion of the three mathematical worlds of thinking in comparing teacher and lecturer perspectives on calculus teaching and learning. Both groups perceive differences between Year 13 and first year tertiary calculus teaching, including: there is a more formal approach to the teaching at tertiary level; secondary teachers interact more with their students; secondary teachers spend a large amount of time on administration at the expense of lesson preparation; and there is more procedural teaching, especially to the NCEA assessment, at school. There was a great deal of ignorance expressed about school and tertiary calculus teaching, notably by the teachers, and to a lesser extent, the lecturers. Clearly there are important roles for secondary teachers and tertiary lecturers to play in helping students with their transition. They can help to ease the cognitive conflict (Tall, 1997) faced by the students and be more aware of changes, including the shift in mathematical thinking, during the school-tertiary transition.

Acknowledgement

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Linking Teachers and Mathematicians: The AWM Teacher Partnership Program

Pao-sheng Hsu*, Columbia Falls, ME
Suzanne Lenhart, University of Tennessee
Erica Voolich, Solomon Schechter Day School, Newton, MA

Abstract
Within a professional organization for women in mathematics in the US, two mathematicians and a middle school teacher organize a program to link teachers of students at the pre-university level with professionals in the mathematical sciences in and outside of academia to promote collaborations among different communities in the mathematics education of students. This paper describes the program and its operations, some of its experiences, as well as some results from a formative evaluation conducted for the program. Some recommendations are given for potential organizers of similar programs in other countries.

Introduction
Founded in 1971 in the United States, The Association for Women in Mathematics (AWM) aims

- To encourage women and girls to study and to have active careers in the mathematical sciences, and
- To promote equal treatment of women and girls in the mathematical sciences.

It has more than 3000 members; membership includes men and women. One of the many programs (see http://www.awm-math.org/) that the association runs is a Mentor Network that connects students or career novices in the mathematical sciences with an experienced professionals for the purpose of mentoring. Such a program was an inspiration to connect mathematics teachers at the school level to practicing mathematical scientists. "The Mathematics Education of Teachers Project" team of the Conference Board of the Mathematics Sciences (http://www.cbmsweb.org) stated in their report that professional organizations of mathematicians have a critical role to play in the mathematical education of teachers by fostering discussion and encouraging greater involvement among its members. One of the recommendations in the report is that "There needs to be more collaboration between mathematics faculty and school mathematics teachers." In considering a program that links teachers and mathematicians in an informal setting, the organizers of our program decided that it should be a collaborative instead of a mentoring one. Hence the AWM Teacher Partnership Program (TPP).

The Organizers
Pao-sheng Hsu, Suzanne Lenhart, and Erica Voolich, members of the AWM Education Committee, are the organizers of the TPP. Hsu is a mathematician who has engaged in research in mathematics education, and has worked with middle school students in informal mathematics programs. Lenhart is a researcher in mathematical biology, has worked in outreach programs to schools at her university, and was a past president of AWM. Voolich is middle school teacher and is the president and founder of The Somerville Mathematics Fund that encourages achievement in mathematics in Somerville, MA, by giving scholarships to students and awarding teacher grants for mathematics projects and events. We also have help from the AWM web editor. All of us volunteer our time.

Program description
On its webpage, http://www.awm-math.org/teacherpartnership.html, the program announces:
The Association for Women in Mathematics (AWM) Teacher Partnership is intended to link teachers of mathematics in schools, museums, technical institutes, two-year colleges, and universities with other teachers working in an environment different from their own and with mathematicians working in business and industry. We invite individuals to join the partnership and will match members from different communities. Partnership activities will include:

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* Travel by the first author to the conference is partially supported by an AWM Travel Grant. The views expressed in the paper are those of the authors' and not necessarily those of AWM.

- electronic communications;
- teaching projects;
- classroom visits when feasible;
- informal educational activities.

The webpage also provides a request for a partner form for a prospective participant, together with a set of guidelines for participants and a disclaimer to release the program and organization of any liability. Participants in the program need not be AWM members.

How it is administered

The program was advertised in various newsletters and listservs of professional organizations of teachers and mathematicians** after the program was launched on the AWM website. When an applicant interested in joining the program fills out the "request for a partner" form, the organizers receive a copy. Periodically, the organizers meet on a telephone conference call to discuss possible matching of the applicants and other program issues. When a match is made, both sides are informed by an email that they should introduce themselves to each other.

A listserv was set up for the participants who have been matched to each other. In addition to being used to reach participants, it is also a forum for the participants to share their experiences and get information from each other.

History and our experiences

Almost immediately after the program was launched on the AWM website in August 2006, we received applications from interested people, including one from Azerbaijan and one from Turkmenistan. At the time of writing in April 2009, 125 people have requested a partner, and the program has made 68 matches involving 109 people. Requests came from Asia, Europe, Africa, and North America. In addition to the two from Central Asia mentioned above, other countries represented include China, India, Pakistan, Ghana, Cameroon, Uganda, Romania, Canada, and the United States.

In the original planning and proposal to AWM, we drafted a formative evaluation on the program to be administered after the program has run for a period of time. This we did in November 2008, using a tool, from SurveyMonkey.com, for which AWM has a subscription. Twenty-one participants of the program filled out a questionnaire that probed their experiences with the program.

What we have learned

Underlying the goals of the program—to connect teachers with mathematicians-- is the notion of building a community to enhance the education of children in mathematics. Teachers and practitioners in the mathematical sciences live in very different environments, each with a distinct culture and a language, and each has a sense of what is important. That "communication among communities" needs attention is indicated in the results of our formative evaluation: 8 of our 21 respondents reported that they had experienced some difficulty in communication. More significantly, "talking with someone outside of your milieu" is not part of these people's daily routines; we are adding a component to the professional lives of these teachers and mathematicians. Demands in one's usual professional and personal lives may take precedence of working with someone unfamiliar with your own situation, which may then become the first thing to be given up when other burdens are too pressing. In our various attempts to reach them, we realize that several of the partners have moved from their original addresses and that some participants have trouble reaching their partners. In the formative evaluation, 17 out of 21 respondents told us that they were not in touch of their partners at the time of the survey. To build a new collaboration needs our care. One respondent said, "…your email is what reminded me of what I had once done."

The fact that they are no longer in touch with their partner does not mean necessarily that they do not want to work towards the goals of the program, in theory. Two out of the 17 respondents not in touch with their partners wanted to continue with the program with the same partner, 7 wanted to quit the program, and 8 wanted a different partner. Earlier, we had realized that we needed to pay attention to this latter group who felt that their match was not workable. Our matching is limited to the pool of applicants that are available to us. One of our questions to applicants is whether they are interested in a partner working close to them geographically so that they could exchange visits. Some of our applicants indicated that they have such an interest, but we find ourselves able to satisfy this request only in a small minority of matches we make. Using a map as a guide, we had matched people who are within a couple of hundred miles from their schools. In one situation, the partners were able to meet at a professional meeting, but in most cases, partners reported that they were disappointed that visiting was practically not possible. Before we make a match that may be problematic from the point of view of the applicant, we have been asking them specific questions before a match is made. We also learned that sometimes an applicant may have expectations of the program other than what it offers. One of the respondents reported that she wanted to quit the program because her partner wanted a mentor and not a partner. One applicant left the program because he was expecting direct guidance from the organizers on what to do in the partnership.

One thing that seems to bind a person to the program is that the person has done a project for which the program has made a difference. An elementary teacher used our listserv to ask for ideas for doing a Science Fair in her school. Participants in the program responded. At the end of her event, she sent a message on the listserv, happily thanking people who had helped her to make the fair a success. She became convinced of the need for the program.

From the formative evaluation, we get a glimpse of how the program has been perceived by some of our participants. To the extent that the results we got from the 21 respondents represent those perceptions, we could say that, for at least some participants, the program has achieved many of its goals. That teachers and mathematicians would discuss together a mathematics topic (7 positive responses out of 20 who answered the question), an educational activity (12 positives out of 20), an issue related to teaching (11 positive responses out of 20), or issues related to life in our professions (7 positives out of 19 responses) was indeed one of the goals of the program. To a lesser degree, some of the participants also considered issues related to supporting students outside of the classroom (6 out of 20), gender issues in mathematics education (3 out of 20) and a joint project (3 out of 20)—activities suggested for the program.

What some of the participants have told us

To give some of the views of the participants, we quote some of the comments from our formative evaluation.

Eight out of the 17 respondents, who were not in touch with their partners at the time of the survey, wanted a different partner. Some who wrote said:

- At first we shared things of mutual interest. Then we got busy.
- The partner and I didn't have much in common. She was an administrator for elementary schools and I teach at the university level.

Seven respondents wanted to quit the program. Some of them said:

- My partner seemed to want a mentor rather than a partner.
- Seems like there is little time to do anything outside my teaching responsibilities. I honestly forgot about my pairing during the summer months and your e-mail is what reminded me of what I had once done.

Six of the 17 respondents wanted to continues with the same partner in the program. Some said:

- My position has changed in the last year making it difficult for me to take on new projects. My partner and I have become friends and now live closer, but I have even less time (I am not working much anymore and have limited daycare) making it harder for us to collaborate.
- All of the program participants have been helpful. I've received help from the listserv for my first Science Fair, which was a big success thanks for this group and their suggestions (even though I rarely spoke to my partner, mostly my fault). It does not matter if I have a partner, the whole group is helpful.

One successful partnership

We were fortunate in being able to match a high school teacher and an university faculty member working in close geographic vicinity that they could exchange visits. The mathematician visits the high school and talks with the students on mathematics and mathematical career choices. The high school has a team preparing students for the American Mathematics Competition (http://www.unl.edu/amc) and a contest in their state. The mathematician is now a member of the team and works with the students. The teacher has been invited to be a guest lecturer when the mathematician focused on a particular topic in his graduate course *Number and Number Theory* for students in the Mathematics Specialist Program (for K-8 teachers) in their state. (In the U.S., a teacher in a kindergarten through 5th grade class usually teaches all subjects; in some schools, mathematics in the 6th through 8th grades may be taught by a mathematics specialist.) Because of his prior experience, the mathematician introduced the teacher and students to a Science and Engineering Fair held regionally and nationally. Some students got very excited about some of the science and engineering projects he mentioned and started to work on them. Recently we got news from that partnership program that one of their student's project in Medicine & Health was the Grand Prize winner of the regional fair and is going to the 2009 Intel International Science and Engineering Fair (http://www.societyforscience.org/ISEF/) to be held in May. Another student is a runner-up or alternate to this national fair. Both students are female.

The students are the ultimate beneficiaries of the program. The story of this partnership is posted on the program website.

Recommendations

We feel that other countries would benefit from this kind of program for teachers and mathematicians in their local areas. Since there is definite benefit in partners exchanging visits, to be able to attract participants within a small area is an advantage.

Such a program needs a great deal of thought and nurturing. Keeping in touch with participants individually would be one way to help them to persist and the partnership to grow. Even though ultimately it is up to the people involved to work in their partnership in a direction of common interest to them, we wonder whether sometimes a little intervention may help in overcoming an impasse—something we have not tried.

Electronic communications is a big help: a listserv for the community to share its experiences and expertise. We also learned to use it with an awareness of its idiosyncrasies: the word "partnership" on the subject line of a message may trigger an anti-spamming tool to block the message; some people are just receiving too many incoming mails; sometimes we do not have a way of knowing when a message does not reach an intended person.

Collaboration requires a willingness to work with diverse perspectives and to negotiate an outcome that both sides are happy with. Even some of those respondents who wanted to continue working with the same partner reported that they had encountered some difficulty in communication. They seem to be telling us that the difficulty is not insurmountable. We should build on this wish that the benefits of a partnership will be worth the effort.

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Pao-sheng Hsu, mailto:hsupao@maine.edu Suzanne Lenhart, mailto:lenhart@math.utk.edu

Erica Voolich, mailto:voolich@aol.com

** "Mathematicians" in this paper include all professionals in the mathematical sciences such as statisticians, computer scientists and people in operation research and in mathematics education.

Project work *Is the Legacy of Ancient Greece and Rome really the Cradle of European Civilization?*

Darka Hvastija, Jasna Kos

Gimnazija Bezigrad, Ljubljana, Slovenia darka.hvastija@gimb.org, jasna.kos@gimb.org

Abstract

In this paper the project for 15-year-old students with the title *Ancient Greece and Rome* and the sub-title *Is the Legacy of Ancient Greece and Rome really the Cradle of European Civilization?* is introduced. It shows how to connect mathematics with art, history, physics, geography and philosophy by studying ancient Greek scientists and their achievements. Collaborative teaching is introduced. The major aim of the project was to show mathematics as a part of human civilization and to follow its development through history. Some topics from theory of numbers and geometry were studied. One part of the project was also a theatre performance, which should make the students aware of the difficulties of many dedicated mathematicians to find the answers to some problems from the ancient times.

Introduction

In this presentation we will describe a cross-curriculum project work, which was done with the firs-year students from the secondary school called "gimnazija". In Slovenia the secondary school lasts for four years and ends with an external examination. Afterwards most students continue their education at university. According to the syllabus students have four lessons of mathematics per week for four years. Last year we started with a new syllabus, which includes project work and team teaching as a new teaching method. For this project the topic was chosen which was interesting for different subjects, with mathematics as the central subject. The title was *Ancient Greece and Rome* and the sub-title *Is the Legacy of Ancient Greece and Rome really the Cradle of European Civilization?*

The project lasted for three days. During the first day the students were introduced into the science, especially mathematics, of the ancient times. The mathematics teacher taught in a team together with teachers of art, history, physics, geography and philosophy. During the second day students worked in different workshops. In one group they learned about the cuisine in the Roman time and they made "Roman" bread. Some groups studied and represented sport disciplines in the Olympic games from the ancient time and from the modern world. One group visited Greek and Roman remains in our town and the other compared the speech of Pericles with the speeches of different modern politicians. We offered three different mathematical workshops, but the most attractive one was a theatre performance which made the students aware of the difficulties of many dedicated mathematicians to find the answers to some problems from ancient times. On the last day the students introduced the project to their parents.

Mathematics as the central subject

"Arhimedes will be remembered when Aishilus is forgotten, because languages die and mathematical ideas do not. "Immortality" may be a silly word, but probably a mathematician has the best chance of whatever it may mean." (G. H. Hardy, A mathematician Apology)

Maybe that's why we chose mathematics as the central subject. Naturally, all the teachers of mathematics liked it, because we do not have enough time for teaching history of mathematics during our regular lessons. The major aim of the project was to show mathematics as a part of human civilization and to follow its development through the history. 15-year-old students have enough knowledge to understand the development of theory of numbers and geometry, at the appropriate level, of course. But in the project we also found time for additional topics, especially in the theory of numbers.

The project gave us the opportunity for collaborative teaching. This was very useful when learning about ancient scientists, because they were not only mathematicians but also philosophers, astronomers, etc. In the previous years our students heard about Pythagoras and his theorem, about Thales, when they discussed similarity and about Euclid, when Euclidean algorithm was treated. But nothing more. Consequently, one of our aims was also to present some other mathematicians, which the students had to arrange in the chronological order.

Mathematics and art

The art teacher opened our session. She compared the ancient painting and architecture with the modern art. The connection between mathematics and art was the golden ratio. The first clear definition of it was given 300 B.C. by Euclid, but it must have been known at least 200 years before in the Pythagorean time. The golden ratio is the best answer to our heading question *Is the Legacy of Ancient Greece and Rome really the Cradle of European Civilization.* The golden ratio presents

aesthetically pleasing qualities in art from the ancient to the modern times. The mathematics teacher then explained the golden ratio and the art teacher gave some examples like Parthenon (432 B.C.), of course, and the painting Modulator (1948) by Le Corbusier.
The art teacher finished the lesson with a very famous painting The school of Athens (1511) by the renaissance painter Raphael. This painting and the figures in it was a starting-point for our further work.

Mathematics and history

The history teacher was indispensable in studying history of mathematics. She described the government of Polycrates, which was so tyrannical that Pythagoras escaped from Samos. She presented the position of woman in the ancient time. By knowing their role in society we can better understand the importance of Pythagoreans, who allowed women to function on equal terms in their organization.
The history teacher reminded our students of the Punic war, because it was fatal for Archimedes, whose last words were "Do not disturb my circles" before he was killed by Romans. There was another murder of a mathematician some centuries later, which was described by the history teacher. Hypathia, the first notable woman in mathematics, was killed by Christian fanatics.
In the painting The school of Athens, we can also find Alexander the Great. He was known to the students from history lessons, but they did not know that Aristotle was his teacher. This is the example of team teaching. Teachers, who are together in the classroom, can supplement each other.

Mathematics and philosophy

Mathematics was closely connected with philosophy in the ancient time, so the philosophy teacher was very helpful in our project. However, profound philosophical questions can be very difficult for 15-year-old students. In this case the teacher had to substitute the theory with the anecdotes and other stories. But the teachers warned the students that these stories are based on poorly documented historical records. The mathematics and philosophy teachers explained together Plato’s philosophy and Platonic solids, because both topics are connected. It is the same with Zeno of Elea and his paradox “Achilles and the tortoise”. We can connect this paradox with modern mathematics because of the infinite process. In the 17th century it was shown that the infinite convergent series can have finite limit, which is the sum of the series.
Platonic solids also had influence on later mathematicians and physicists. In the 16th century Kepler used them in his solid model of the solar system. In the 18th century Leonhard Euler proved that the numbers of vertices minus numbers of edges plus number of faces of polyhedra (Platonic solids are included) is 2. In the 19th century the symmetry groups of Platonic solids were studied. The statement from the 20th century, that minerals and viruses have the shape of regular polygons, helps scientists to study their nature.

Mathematics and geography

During the project the geography teacher helped students to mark the birth places and other important places of the ancient scientists on the map. Then she gave a short illustration of ancient geography with stress on Ptolemy. Ptolemy for geographers is like Euclid for mathematicians. Together with the physics teacher they explained Eratosthenes' measurement of the Earth's circumference.

Mathematics and physics

The physics teacher added explanations of physics achievements. When talking about Archimedes, for example, we mentioned his excellent approximations for π and $\sqrt{3}$, his proof that the volume and surface area of the sphere is two thirds of the cylinder of the same height and diameter. The physics teacher added the information on many Archimedes’ inventions. We explained Heron’s Formula for the area of a triangle and the physics teacher explained his inventions and achievements.

Students’ activities

We have mentioned some students’ activities already. They formed the timeline and marked the important places on the map. They got handouts with the presentation and with the tasks that they did on their own:

- Archimedes: To estimate his approximations for π and $\sqrt{3}$
- Diophantus: To find one integer solution of the equation $12x + 42y = 6$
- Eratosthenes: To write the Sieve of Eratosthenes for numbers up to hundred
- Euclid: To find the first seven numbers of the form $N_n = 2^{n-1} \cdot \left(2^n - 1\right)$. Which of them are perfect numbers (the number which is equal to the sum of its proper divisors)?

- Hero of Alexandria: Water flows to the reservoir from four different pipes. The first pipe should fill up the reservoir in one day, the second in two days, the third in three and the fourth in four days. How long would it take to fill up the reservoir if all pipes are switched on?
- Plato: To check the Euler's formula on the example of Platonic solids
- Pythagoras: To prove, that 1184 and 1210 are amicable numbers (the sum of the proper divisors of one number is equal to the other)
- Thales: To prove the Thales' theorem (the diameter of a circle always subtends a right angle to any point on the circle).
- To arrange all mathematicians and other mentioned scientists in to the chronological order

More difficult tasks, like different proofs of Pythagoras' theorem, were solved with the help of maths teachers.

Influence of ancient Greek mathematics on modern mathematics

Not only Greek letters, in modern mathematics we also use many Greek words, like hypotenuse, ellipse, hyperbola and parabola. We know that Pythagoras found the name for the longest side of a right triangle. The conics got the name by Apollonius of Perga. We can find many mathematical terms, which have their origin in Greek language: axiom, arithmetic, asymptote, diameter, dodecahedron, graph, orthogonality, polynomial, sphere, trapezium, etc.

For every topic discovered in time of ancient mathematics we can find its impact in modern mathematics. For example in theory of numbers. It seems that perfect numbers and amicable numbers are unuseful, but for Pythagoras they were very important. In the 20th century it was discovered that these numbers are important for the assessment of capacity of computers. An other example can be found in geometry. Euclid gave five postulates. Mathematicians until 19th century tried to modify the fifth one, because it seemed less obvious than the others. So the Non-Euclidian geometries were discovered. The influence of ancient mathematics can also be found in music. In ancient Greek times it was recognized that consonant musical sounds relate to simple number ratios and the oldest system of the scale construction is the Pythagorean scale. It is a base for the equally-tempered scale we use in European music today.

"There is no doubt that anybody who grew up in a western or mideastern civilization is a pupil of the ancient Greeks, when it comes to mathematics, science, philosophy, art and literature. The phrase of the German poet Goethe-"of all peoples the Greeks have dreamt the dream the best"-is only a small tribute to the pioneering efforts of the Greeks in branches of knowledge that they invented and denominated."[1]

Workshop

One of the workshops was maths theater. Students performed a play with the title THE LAST FERMAT THEOREM. Of course, we did not prove it, but only showed its history and some other problems from Theory of numbers that interested the ancient Greeks and are still open now. Three groups of students were simultaneously on the stage discussing mathematics. The students from the first group presented Pythagoras and other mathematicians from 300 B.C. (dressed in white sheets), in the second group there were Fermat, Descartes and Pascal with white collars and wigs and in the third group there were mathematicians from Princeton University at the end of 20th century, admiring Andrew Wiles. The ancient mathematicians introduced some problems like Pythagorean triples, perfect and amicable numbers, twin primes. The "mathematicians" in other groups added the knowledge achieved during their time, Fermat was looking for the solutions of the equation $x^n + y^n = z^n$, Wiles was proud of his proof, the others complained that certain problems have not been solved yet. The students wrote the scenario by themselves, they also included some non-mathematical content. The aim of this performance was to develop the awareness of mathematics as a science, which is constantly developing, however some things that were once proved, will always be true. We think that some other topics like geometry, calculating the values of π or trigonometric functions, could also be presented in such a way.

Students did not learn a lot of mathematics but the aim was certainly achieved.

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A Model to Develop Mathematics Education: Modify the Public Traditional Perceptions of Mathematics-Case of UAE Schools' Principals

Hanan Innabi, Associate Professor of Mathematics Education
U.A.E. University, College of Education hinnabi@uaeu.ac.ae

Abstract

This paper addresses the idea that the successful of mathematics reform demands the support of the full educational community including school principals, parents, and students. One of the most important group that affect mathematics reform is school principals. A project related to modifying UAE principals' perceptions of mathematics is presented. This project consists three steps. In the first step, principals' perceptions of the nature of Mathematics and its learning and teaching were examined. Results showed that those principals possess many improper perceptions related to Mathematics. In the second step, a professional training program for promoting school principals' understanding of the new vision of teaching and learning mathematics has developed. This training program comprises two integrated phases: Clarification and conviction, and implementations for principal's role. It includes a package of paper documents, videotapes, discussion sessions, and group and individual activities. In the third step, the training program is applied on eight principals in UAE. An initial analysis of the qualitative data showed many positive improvements in principals' perceptions of mathematics education.

Introduction

This paper based on the idea that the educational community's awareness and understanding of the new vision of teaching and learning mathematics is a necessary condition for mathematics reform. Without this condition, all the efforts in changing mathematics curricula and training mathematics teachers will not be fruitful. We assume that the way *people* see and percept mathematics, affects the processes of teaching and learning mathematics. In this context, the word *people* does not mean just the teachers and supervisors of mathematics, but it also means all the educational community including parents, students, principals, administrators, and the other subjects teachers.

Apparently, there is a common public way in viewing the subject of mathematics. Unfortunately, most of the ideas of the public view contradict the real nature of mathematics. Such ideas as: mathematics is abstract, it is difficult, for boys more than for girls, just for clever people, it is a collection of fixed theories and rules, unchangeable facts, a tool for computations. Similar views about teaching and learning mathematics are held by the public; such as: facts should be taught first then the applications and problems, learning mathematics is a combination of a series of facts and mastery of procedural steps, always one specific right answer should be their. (Oster, Graudgenett, & McGlamery, 1999).

It is assumed that as long as those perceptions about mathematics are spread within the educational community, mathematics reform is facing considerable difficulties that might affected negatively. To clarify this point lets for example consider a situation in which the parents believe that their children are doing great in math because they can do all the computations quickly and precisely, or that their kids are poor in mathematics because they cannot do that. Let's also consider the situation where the principals expect from mathematics teacher to cover the textbook in order, step by step, and facts and knowledge should be provided first. In such situations we can conclude that the monitoring and assessment processes of teaching and learning mathematics will not be consistent with the new vision of Mathematics reform that sees Mathematics as a tool of thinking and communication and not as a accumulation of computations and procedures.

Principals Group

One of the most important groups that effect mathematics reform is the "school principals" group (NAESP, 2002). Assuming that perceptions affect people behaviors (Bandura, 1997), principals' perceptions regarding the new vision of Mathematics Education might significantly affect mathematics reform efforts (Lester & Grant, 2001; Spillane & Halverson,1998; Heck, Banilower, Weiss, & Rosenberg, 2008; Sassi, 2007; Hansen & Mathern, 2008; Williams, Tabernik, & Krivak, 2009).

Teaching Mathematics according to the new vision is facing many challenges. Teachers are challenged to provide opportunities and experiences that allow students to build their own mathematics. This requires a deeper understanding of the mathematical content than that developed by the traditional teaching methods. This also requires more time, conditions, and requirements (McDuffie & Graeber, 2003). Researchers found that, during the process of changing to reform-based approaches, teachers are likely to feel anxiety and frustrated (Campbell & White,1997). To meet these challenges, teachers defiantly need support. Without support, these feelings may cause teachers to return to more stable and comfortable, traditional, practices; but with continued support, successful implementation of reform movement goals is more likely (McDuffie & Graeber, 2003).

Talking about support, it seems that the school principal is an important supporter. Principals can profoundly influence teachers who are working to change the ways they teach mathematics. Brown and Smith, 1997, showed that teachers were concerned about their administrators' opinions and needed to know that the administrators understood reform-based approaches. Principals' actions also are needed to provide both pressure and support for change and time to plan and implement new strategies.

Research Project

United Arab Emirates has been active in mathematics reform over the last 8 years. A determined national effort to reform mathematics education has been underway since the publication of mathematics curriculum outlines in 2001 (MOEU, 2001). Within the frame of this reform and based on the previous section, a research project that contains three steps is presented in the following:

<u>First Step</u>

In order to understand how schools' principals in UAE percept the principles and standards that the new mathematics curricula is based on, a study was conducted by Innabi (2006). Responses from 244 school principals from all over the country were analyzed using an instrument prepared particularly for the study. The instrument was based on the standards that the Ministry of Education is adopting for teaching mathematics. Results showed that principals possess many wrong perceptions, such as:

- Mathematics is just computation that based on training and practicing.
- Mathematics is only for elite people.
- Math is a set of absolute truths that cannot be changed.
- It is a subject that always has right or wrong answers.
- It is not suitable to teach statistics and probability for the elementary stage.
- Problem solving is a verbal problem that comes as application at the end of the lesson.

These results indicated a need to help schools' principals in UAE to modify their view of mathematics to fit with the new vision.

<u>Second Step</u>

Based on the results of the previous study, a professional training program for promoting the elementary school principals' understanding of the new vision of teaching and learning mathematics has developed.

Two major principles underline the development of the program. The first is related to the conceptions of teaching and learning Mathematics that principals should possess. The other is the nature of the general strategy that has been adopted to help principals to modify their misunderstandings and misconceptions of the real nature of mathematics and its teaching and learning.

Regarding the first principle, the identification of conceptions of the new vision of teaching and learning mathematics targeted by the program were guided, in one hand, by UAE national standards for school mathematics (MOEU, 2001), on the other hand, by indicators gained from the previous study that mentioned before (Innabi, 2006) which examined the current conceptions of UAE principals of teaching and learning mathematics. The most wrong perceptions were found in nine aspects. These aspects were categorized in four categories; these are: nature of Mathematics, learning of Mathematics, teaching of Mathematics, and Mathematics curricula.

Table 1 shows these wrong perceptions within each category and brief descriptions of the desired conceptions as driven from the UAE national and international standards (MOEU, 2001; NCTM, 2000).

The construction of the program is guided by the constructivism theory- assimilation /reconstruction model. This method based on the following three movements:

1. Targeted person (principal in this case) will be asked to express her/his opinion or understanding or perception about a specific point. The aim in this step is to help the participant to see her/his own perception.
2. Several situations that propose alternative perceptions that reflect the real and right picture of mathematics will be presented to participants. The aim of this step is to give participant the chance to compare, reflect on their own perceptions and to modify it.

3. Discussions and reflections processes to promote perceptions.

The training activities was developed with the purpose of enhancing principals' understanding of (1) Nature of Mathematics (2) Learning of Mathematics (3) Teaching of Mathematics. The program combines10 focused groups' sessions; the duration of each section is two hours. Two sessions discuss the nature of mathematics, five sessions discuss the nature of teaching and learning mathematics, and three sessions discuss the role of school principals to support the new vision of mathematics. The program includes a package of paper documents, videotapes, discussion sessions, and group and individual activities. This program was built to be administrated in two integrated phases:

Phase 1: Clarification and Conviction

The nine points that were clarified in Table 1 were covered by using different activities such as: view and discuss videotapes of mathematics classes, explore student thinking through an examination of students work, read and discuss articles that provoke thinking about mathematics, engage in mathematics explorations and discussions. Figure 1 illustrates an example of the activities of this phase.

Phase 2: Implementations for principal's Role

During and after the activities and the discussions in phase one, the training program concludes implementations for principals role. In particular an answer will be provided at the end of the program to the following question: what principal can practically do for Mathematics reform regarding the following aspects: Supervising and evaluating teachers, professional development for teachers, communicating with parents & outside community, guiding and supporting school improvement efforts.

Table 1: Principals' perceptions of Mathematics

Wrong perceptions (found among UAE school principals)	**Category**	**Real perceptions (that the program aims to achieve)**
Mathematics is fixed unchangeable facts that should be accepted. To be good in Mathematics, one should have Mathematics talent and aptitude	Nature of Mathematics	Mathematics is a growing science that develops with industrial and technological advances; it is not a settled and unchangeable knowledge but any mathematical knowledge could be changed if contexts and axioms are changed. Mathematics is not a collection of some theories, procedures, and rules but it is a way of thinking that inquires, analyze and understand patterns and order. Mathematics is not a collection of vague symbols and rules that just elite and smart people can understand but it is a thinking and communication tool for all.
Learning mathematics is based on training and practicing, The most important indicator of students' learning in Mathematics is achieving the right answers	Learning of Mathematics	Learning mathematics should be based on understanding and not on repetition and training. Achieving the right answer is not necessary the evidence on students' learning. All students can and should learn mathematics in a meaningful way based on understanding.
Problem solving is a verbal problem that comes as application at the end of the lesson. Students sharing in solving problems could be negative as errors of some students could affect the other students.	Teaching of Mathematics	Teaching mathematics has to base on problem solving. Problem solving is not a verbal problem that comes at the end of lessons as application. It is unfamiliar situation that challenge the student and this makes learning happen. Communication is a very important standard that should be considered in teaching mathematics where students have to be encouraged to express their thinking verbally and written with each other and with teacher.

Mathematics curriculum is a collection of concepts, skills and algorithms, It is not suitable to teach statistics and probability for the elementary stage It is not suitable to contain all sorts of mathematical content in every single grade	Mathematics Curricula	Mathematics curriculum is more than a collection of activities. It must be coherent, focused on important mathematics, and well articulated across the grades. In order to learn mathematics with understanding all sort of content (number, algebra, geometry, measurement, data and probability) should be included in each grade.

Example 1:
The target of the activity: *Achieving the right answer is not necessary the evidence on students' learning.*
Context: Division lesson for grade 4
Question to be discussed with participants: How do you judge students' learning of this lesson?
(All answers will be written on board).
A video clip will be presented and discussed: the video contains two parts. The first part shows a forth grade student who solve- using the paper and pencil- a division problem that request dividing 20 by 5. Student here can solve the problem rightly and confidently. In the second part this student will be asked to explain what she just did using a set of cubes. It will be notice that this student fall to achieve this task.
Discussion: What this can tell you?
Example 2:
The target of the activity: *Mathematics is a growing science; it is not a settled and unchangeable. Mathematics is not a collection of some theories, procedures, and rules but it is a way of thinking that inquire, analyze and understand patterns and order.*
Context: Brain storming session
Question to be discussed with participants: what come to your mind when you hear the word "Mathematics".
All answers will be written on board, then categorized and summarized.
It is expected that participants will provide answers related to the nature of mathematics, its utility, their dispositions toward it,….
The discussion will be followed by providing examples (pictures, problems, facts from mathematics history) to show the real picture of Mathematics.

Fig. 1: Examples of phase 1 activities.

The target of the activity: *principals to be aware of the actions that they can do in their schools to be in line with the real view of mathematics.*
Context: this activity will be applied during conducting the program starting from the first session.
Principals will be asked to record all actions that they can do to support the real vision of mathematics.
A board will be assigned for this activity so principals can add their suggestions as follows.

What can I do regarding this?

1st session	2nd session	3rd session	………
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A discussion session will be conducted at the end of the program to categorize the suggestions in order to be distributed to participants.

Fig. 2: An example of phase 2 activities

Third Step

In April 2009, this program was applied on 8 elementary school principals in Alain city in UAE. The purpose of this application is to examine the effectiveness of the suggested training program on the principals understanding of the new vision of school mathematics. The following procedures were applied:

1. Pre monitoring: the perceptions of participated principals were determined by using two tools ; principals' perception scale, and semi–structured interview.
2. Implementing the program.
3. On going monitoring: by observations, analyzing documents.
4. Post monitoring: to determine the improvement in principals' perceptions. Same tools of pre monitoring stage were used.

The analysis of the qualitative data that has been collected from this training program is still under process. However, the general observations and impressions before, during and after applying this program are reflecting a very bright picture of changes happened in principals understanding of the nature of mathematics and the nature of its learning and teaching, and most importantly, the nature of their role to help mathematics reform.

It is hoped that this project will present a useful training program for schools' principals to change their misconceptions of the nature of Mathematics and about its teaching and learning. It is also hoped that this paper will get the attention of the issue related to the public perceptions of Mathematics that could affect mathematics reform efforts.

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A Four Phase Model for Predicting the Probabilistic Situation of Compound Events

Irma Jan and Miriam Amit
Ben-Gurion University, Israel janirma@gmail.com

Abstract

This paper presents an innovative construction of a probabilistic model for predicting chance situations. It describes the construction of a four phase model, derived from an intense qualitative analysis of the written responses of 94 mathematically talented middle school students to the probabilistic compound event problem: "How many doubles are expected when rolling two dice fifty times?" We found that the students' comprehension process of compound event situations can be broken down into a four phase model: beliefs, subjective estimations, chance estimations and probabilistic calculations. The paper focuses on the development of the model over the course of the experiment, identifying the process the students underwent as they attempted to answer the question. We explain each phase as it was reflected in the students' rationalizations. All phases, including their definitions and students' citations, will be presented in the paper. While not every student necessarily goes through all four phases, an awareness and understanding of them all allows for efficient, effective intervention during the learning process. We found that guidance and learning intervention helped shorten the preliminary phases, leading to more relative time spent on probabilistic calculations.

Theoretical Background

An important distinction on which this study rests is the separation between *simple events* such as one-dimensional experiments, and the more complex *compound events*, which include two-dimensional experiments and events such as $A \cup B$, $A \cap B$ (Polaki, 2005). The transition from simple events to compound events has been found to be difficult for students (Watson, 2005). For example, to understand compound events, such as the sums of two dice, one must be able to generate complete sets of outcomes for the events and use sample space symmetry and composition to make probability predictions (Polaki, 2005). Understanding sample space is complex in itself, since it requires the coordination of several cognitive skills: (a) recognizing different possible ways of obtaining an outcome, (b) being able to systematically generate those possibilities, and (c) being able to "map the sample space onto the distribution of outcomes (Horvath & Lehrer, 1998). Fischbein, Nello & Marino (1991) claimed that some failures in constructing the sample space of rolling two dice are due to the fact that there seems to be no natural intuition regarding the order of two dice. This paper advances research on probabilistic thinking by examining students' probabilistic thinking about compound events in depth. We present here a four phase model derived from our test subjects' reasoning process throughout the research, regarding the prediction of compound events that involve the throwing of two dice.

Methodology

Setting

Six groups of gifted and talented students from grades 6 to 7 (a total of 94), members of the "Kidumatica"[1] math club (Amit, Fried & Abu-Naja, 2007), participated in an extensive study that aimed to investigate how a teaching intervention, in a dynamic semi-structured learning environment, contributes to students' development and understanding of key concepts in probability. The learning intervention, which was based on the constructivist approach (Carpenter & Lehrer, 1999), was conducted during twelve sessions (75 minutes each) with a chain of consecutive probability assignments in the form of tasks, tailor made to prompt understanding and discussion. This study is an extension of our preceding studies (Amit & Jan,

[1] Ben Gurion University's "Kidumatica" math club was established in 1998 by one of the authors of this paper. Its goal is to create an after-school program in which students from 5-10 grades could develop their interest in mathematics and their mathematical thinking. Students participate in several "mini-courses," held weekly at the university campus. The mini-courses include: "Logical Problems," "Real-Life Mathematics," "Mathematical Games," "Number Theory," "Number Sequences," "Fractals," etc .

2006/2007; Jan & Amit, 2006; Jan & Amit, 2008) and aims, among other things, to create an infrastructure to build upon in a future, more formal approach to probability.

Source of data - The compound event question

The students were given a pre test questionnaire, to determine their initial perceptions concerning chance and probability. A post test was then given at the end of the study, to illuminate the changes in the students' perceptions. Both tests included a question concerning the theoretical probability of a compound event (see below). In particular, students were asked to predict the outcome of a compound event situation. (Note: The question in the pre-test involves 50 throws, while that in the post-test involves only 30.)

When playing a game with two dice, a player gets an extra turn when a 'double' is rolled, i.e. when he gets the same number on the two dice (for example, 2 on one die and 2 on the other). Explain and justify: how many doubles would you expect in fifty rolls of two dice? (Jones, Thornton, Langrall & Tarr, 1999)

The question presents the student with a compound event. The correct solution requires building a sample space for the simple event (rolling one die) and transferring it in order to generate the complete sample space of the compound event (rolling two dice).

In addition to its mathematical complexity, the question raises several affective/ psychological difficulties: (1) it is difficult to accept that random situations can be predicted mathematically ;(2) It is not obvious that outcomes of random situations can't be controlled, do not depend on subjective judgment, on individual's luck etc.

Findings

Students' written responses were analysed quantitatively and qualitatively. A qualitative analysis was executed according to four categories, which were identified as reflective of students' probabilistic reasoning: (a) types of strategies; (b) representation; (c) use of probabilistic language; (d) the nature of cognitive obstacle. In this paper we address the first category: *the development of students' strategies in their justifications.*

A Four Phase Model of Strategies

Intense analysis of the data provided a four phase model of strategies of justification: (a) beliefs as a source for justification; (b) subjective estimations of the compound event; (c) chance estimations; (d) theoretical calculations of the compound event.

Phase I-Belief Strategies

In the pre test 44.7% of the students used their everyday beliefs about chance and stated that it is impossible to predict the number of doubles. Examples are illustrated below:

> Tal: "*It is impossible to know the number of doubles since each time it will get a different number of doubles. We are betting, and in games of chance we do not know what would be".*
>
> Ben: *"Maybe the player had no luck and didn't get any doubles but may be he was lucky. It can be said that the chances for doubles are 50%-50%".*
>
> Nurit: "*Doubles aren't rare events but are difficult to get. Hence, it is impossible to know how many doubles to expect. It is possible to get 50 doubles and it is reasonable not to get any double. It can be assumed that in most of the throwing there would be no doubles".*

For these students it was impossible to predict a chance situation. Their explanations came from different perspectives: (1) Tal and Ben believed that chance situations relate to luck and luck isn't something that can be measured or controlled ;(2) From their experiences with board games, it is hard to get a double though it is still possible. This perspective is expressed in Nurit's justification: "*Doubles aren't rare events but are difficult to get*"; (3) Ben's justification suggests another strategy: chance situation either happen or not, and therefore they have a 50-50 chance. This strategy can be explained according to Konold's (1989) *outcome approach*: when children are required to make a prediction about a chance situation, they will respond that it is impossible to say "It's just a matter of chance". It can be expressed also as an incorrect use of language: the tendency to think in phrases like *50-50 chance* to describe unknown events (Amir & Williams, 1999). Though in the post test only a minor change occurred on the *belief strategy* (percentage of students decreased to 28.7%), still one significant change occurred. Students cease to use the word “*luck*” or phrases like: *“the chances for double are 50%-50%”* instead, they explain that since the chance of getting a double is low, it is impossible to predict the number of doubles in several trials of rolling dice. This approach indicates that a new comprehension was formed.

Phase II-Subjective Estimations

We assigned the term *subjective estimation* to all of the students' attempts to estimate the number of doubles using mathematical procedures without probabilistic consideration. The relative amount of subjective estimation used by the students remained constant (31.9% used this strategy in the pre test; 30.9% continued using it in the post test).

Examples are illustrated below:

> Omer: *“there are 50 dice rolls. If all are doubles then, there are 50* (doubles) *and if there are no doubles then there are 0* (doubles)*. I chose the number in the middle”.*
>
> Ron: *“since there are six possibilities for getting a double and* $6 \times 8 = 48$*, this is the closest number to 50, therefore in 50 rolls of two dice there will be 8 doubles”.*

These examples are evidence that students had no previous probabilistic knowledge connected to the question, and therefore searched for arithmetical and logical solutions. According to Omer, since the total number of throws is fifty, the number of doubles will be between 50 and 0 therefore the logical answer for the number of doubles was the number in the middle. Omer was trying to discern impossible totals from possible ones (cf. Fischbein et al., 1991), to accomplish that, he used a strategy that took into consideration the extremes of possible outcomes (Nilsson, 2007). Ron used the available numbers in the question (50 times rolling the dice and 6 sides to one die) and upon it built an equation. Taking available numbers and performing mathematical manipulations on them is a well known approach in the theoretical literature; having already been mentioned by Polya (1957).

> Liron (post test): “*there are 5 doubles to be expected because in a die there are 6 numbers and therefore by my estimation, every 6 rolls we will get one double”.*
>
> Ariel (post test): *“in my opinion by rolling two dice 30 times, 5 doubles are to be expected. I calculated it in this way: “I divided the 30 rolls (which I have to roll) by 6 possibilities in each die and got 5. So, I can get 5 doubles in the game. Sometimes maybe more and sometimes maybe less. This is more or less the number according to calculations. Eventually everything is a matter of luck”.*

The post test examples (see above, Liron’s and Ariel’s citations) show some change in students' thinking towards probabilistic thinking. Liron and Ariel are beginning to pay attention to all the possible ways to get double, but are still seeking for arithmetical solutions.

Phase III-Chance Estimations

11.7% of students in the pre test reasoned that to predict the number of doubles they had to connect the question to the probability of getting a double in one throw, i.e. to simplify the situation into a *simple event* and then transfer it to the *compound event*. They therefore used a strategy we referred to as *chance estimation*. These students built a method (though they used wrong estimations, the building process is important): (1) first they simplified the question and focused on one trial; (2) then they estimated subjectively the percentage of getting a double in one trial; (3) finally, to find the number of doubles in the compound situation (of several trials) they calculated the percentage out of the given trials. The percentage of students using the *chance estimations strategy* in the post test was very low (4.3%) since they progressed towards the formal theoretical probability predictions. Examples are illustrated below:

> Iaron: *In most of the trials we won’t get a double therefore, 20% that we get a double and 80% that we get no double. 20% of 50 are 10 doubles”.*
>
> Galit (post test): *“in each roll there is 10% of chance to get a double. 10% out of 30 are 3 therefore there will be at least 3 double in rolling two dice 30 times”.*

Phase IV- Probabilistic Calculations

In the pre test, only two (out of 94) students calculated the probability of “*receive a double in rolling two dice*” and connected it to the number of doubles to be expected in 50 trails. In the post test 19.1% of the students made progress towards finding the number of doubles to be expected (theoretically) according to the following strategy: (1) they generated a complete sample space of rolling two dice; (2) quantified the probability of getting a double in one trial; (3) multiplied it by the number of trails and got the number of doubles (theoretically) in several trails. This strategy is illustrated in Sharon's solution to the question (Figure 1): First Sharon generates a complete set of outcomes in rolling two dice, finds the sample space and writes a total of 36 possible outcomes. Then, he finds how many doubles are possible and writes 6 possible doubles. In the left side Sharon writes his justification in Hebrew. He builds a fraction [$6/36$] that represents the probability of getting a double as a ratio between the 6 possibilities to get double and the sample space (36). He reduces the fraction and receives [$1/6$].Sharon accomplishes his strategy in following manner: since 6 out of 36 is equal to 5 out of 30 he concludes that in 30 rolling of two dice there will be expected 5 doubles.

"If we reduce the possibilities to 30 then the number of doubles will be 5 since we have to keep the same ratio between the numerator and the denominator as in the first calculation".

Figure 1-Sharon's solution

Conclusions and Discussion

The four phase model that expresses students' typical strategies in understanding compound events is summarised in Table 1:

	The Four Phase Model for Predicting Compound Situations			
Definition:	A compound event is a two dimensional random experiment.			
Compound situation	When playing a game with two dice, a player gets an extra turn when a 'double' is rolled. How many doubles would you expect in fifty rolls of two dice?			
The four phases	**Belief strategy** ⇨	**Subjective estimations** ⇨	**Chance estimations** ⇨	**Probabilistic calculations**
Phase definition	Impossible to predict chance situations.	Estimating probability of compound situation using mathematical procedures without probabilistic consideration.	Simplifying compound situation and estimating the chance of its occurrence.	Using formal probability quantifications to calculate the probability of compound situations.
Students' citation	*"I don't know the number of doubles; it is a mater of luck and a gamble".*	*"There are 50 dice rolls. If all are doubles, then there are 50 (doubles) and if there are no doubles then there are 0. I chose the number in the middle".*	*"In each roll there is 10% chance for a double.10% out of 50 are 5. There will be at least 5 doubles in 50 rolls".*	*"In rolling two dice there are 36 possible outcomes, 6 of them are double, meaning, 6/36=1/6. 1/6 out of 30 are 5. this is the expected number of doubles".*

It is not necessary for students to cross all four phases, but we must be aware of their existence and address them in the learning intervention. We contend that extending the duration of the learning intervention will proportionally diminish the presence of *belief* and the *subjective estimation* strategies in favour of the *probabilistic calculation* strategy for compound situations, with the latter displacing the former more prominently as the time allotted for the intervention grows.

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From a textbook to an e-learning course (E-learning or e-book?)

Antonín Jančařík, Ph.D., Doc. Jarmila Novotná, CSc.

Charles University in Prague, Faculty of Education

Antonin.Jancarik@pedf.cuni.cz, Jarmila.Novotna@pedf.cuni.cz

Abstract

The main aim of this contribution is to introduce the potential that modern information technologies open to authors converting a teaching material from a printed to an electronic version. The authors come out of their own experience and propose options that are suitable especially for creation of study materials in mathematics education. Among others the contribution presents the use of flash animations, java scripts and Computer Algebra Systems.

Introduction

The second half of the twentieth and beginning of the twenty-first centuries are the time of massive development of information technologies as well as other branches of human activity. The amount of information that we come across every day does not allow anybody "to know everything". School, however excellent it may be, cannot hand on to its pupils all the information that they will need in their lives. That is why we must ask what one of the main functions that the school has in education process is. In our opinion it is to build a solid base on which the students will be able to build in the process of their life-long education. An integral part of this base is the ability to search for relevant information, to understand this information and the ability to further process it. In the time of an easy access to information sources on the internet this undoubtedly includes work with texts in an electronic form.

Many materials that can be found on the internet are mere publications of texts (e.g. [1] and [2]) that are of exactly the same form as the printed original. They do not make use of the unique possibilities that the interactive nature of electronic materials, their interconnectedness and the possibility to combine information from more sources offer. Undoubtedly even this type of materials is useful, if for no other reason, then for being accessible to a larger number of readers/users than in the case of the printed version. However, the goal of this contribution is to introduce materials and possibilities that opportunely use the advantages connected with modern information technologies.

If pupils and students are to be motivated to a meaningful use of computers and if their ability to look up and process information that they find in electronic form is to be developed, they must get familiar with these processes already in the time of their studies. We must prepare for them such environments and contexts in which the advantages of information technologies come to light and which offer to the users untraditional but very effective sources of new information. An active use of electronic educational texts is an efficient tool in this effort.

E-books

E-books represent the simplest form of using ICT. A book is converted – exported, or scanned into an electronic format and made accessible on the internet, or distributed on digital media, usually CDs and DVDs. Distribution of a book in an electronic form has the following principal advantages – price (the price of an electronic copy of a book is usually lower than price of the printed version), time (the book can be delivered almost immediately after the order) and environmental (no paper is needed, distribution of a book in its electronic form unless it is printed by the end-user contributes to sustainable growth). Another advantage of distribution of a book in an electronic form is that the end user may read the book not only on his/her personal computer, but also in his/her PDA and special readers. Many of these readers offer a function that is not available in case of a printed book – full text search.

The main problem connected with distribution of books in an electronic form is the problem of copyright and payment of authors' royalties. Attempts to find a way out of this problematic situation are being made (see [3]). For example the server Books.Google com makes it possible to state in advance what proportion of the book can be displayed on the internet, and this including the possibility to set different rights in different areas. This server also offers detailed tracking of how many times a book was displayed on the internet.

Antonin Jancarik Report for Apr 7, 2008 - Apr 16, 2009

Totals	574	565	4		7,776
Date ▾	Book Visits (BV) [?]	BV with Pages Viewed [?]	BV with Buy Clicks [?]	Buy Link CTR [?]	Pages Viewed [?]
April 2009	76	76	1	1.3%	410
March 2009	72	71	0	0.0%	606
February 2009	23	23	0	0.0%	376
January 2009	40	39	0	0.0%	1,007
December 2008	31	31	1	3.2%	625
November 2008	52	49	0	0.0%	1,041
October 2008	37	37	1	2.7%	712
September 2008	30	29	0	0.0%	399
August 2008	25	25	0	0.0%	606
July 2008	36	35	0	0.0%	690
June 2008	31	30	0	0.0%	584
May 2008	96	95	0	0.0%	366
April 2008	25	25	1	4.0%	354
Totals	**574**	**565**	**4**		**7,776**

Figure 1 – Report from books.google.com

Hypertext links

A great advantage of books in an electronic version is that besides full text search they enable the use of internal and external hypertext links. This enables creation of active tables of contents, indexes and links to other parts of the text within the book. Therefore for example if one wants to move to a definition of a term, it is possible by one click. The use of internal hypertext links significantly simplifies and speeds up work with the text.

E-books can also provide links to other sources, up-to-date data or search engines directly on the internet. Thanks to this the content of an e-book can by far exceed the content of the printed version.

Illustration, video sequences, audio samples

Another advantage of e-books is the possibility to use pictures in a much higher number without any increase in the cost price (and consequently the selling price) of the publication. Unlike printed publications, e-books may also contain video sequences and audio samples.

Video sequences may in some cases be much more illustrative than static pictures. In mathematics, video sequences may be used for example to document the change of a graph of a function in dependence on the parameter, to demonstrate the solving process of problem in 2D and 3D geometry, for modeling of phenomena such as flux of liquid or gas.

Flash animations, java applets

Further step on the way from a printed book to an interactive study material is the use of animations and java applets. These tools enable the user-reader to influence the content of the study material directly. He/she can change the parameters of functions, carry out simple calculations or even be tested.

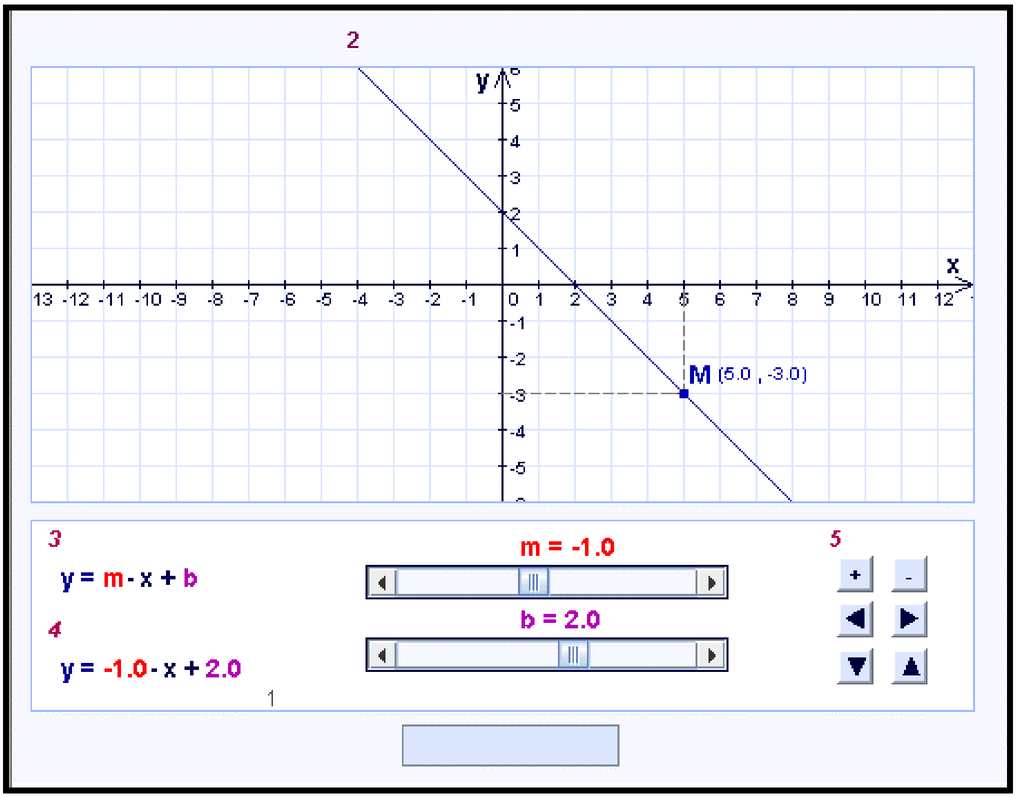

Figure 2 - Java Applet in the LeAM Calculus course

An example of a Java applet is the applet created within the project Active Math ([4]) which is a part of the study material LeAM_calculus (see figure 2). It demonstrates the changes of a linear function in dependence on the parameter.

The use of CAS

Yet more advanced step which however demands online access to the study material is the use of Computer Algebra Systems (CAS). Linking of CAS to a study text enables

- evaluation of test questions (see project ActiveMath – [4]), simple calculations:

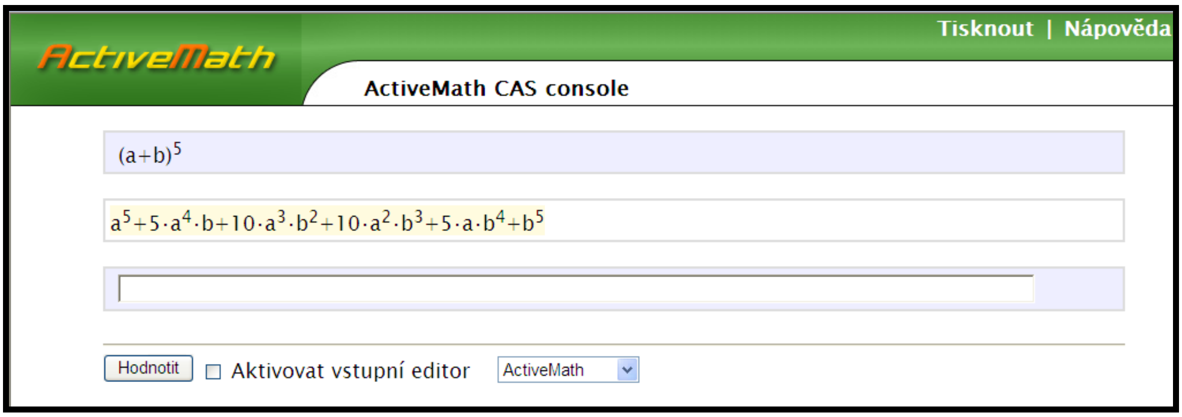

Figure 3 – ActiveMath CAS console

- input of one's own data into prepared calculations, functioning both on numerical and symbolic level:

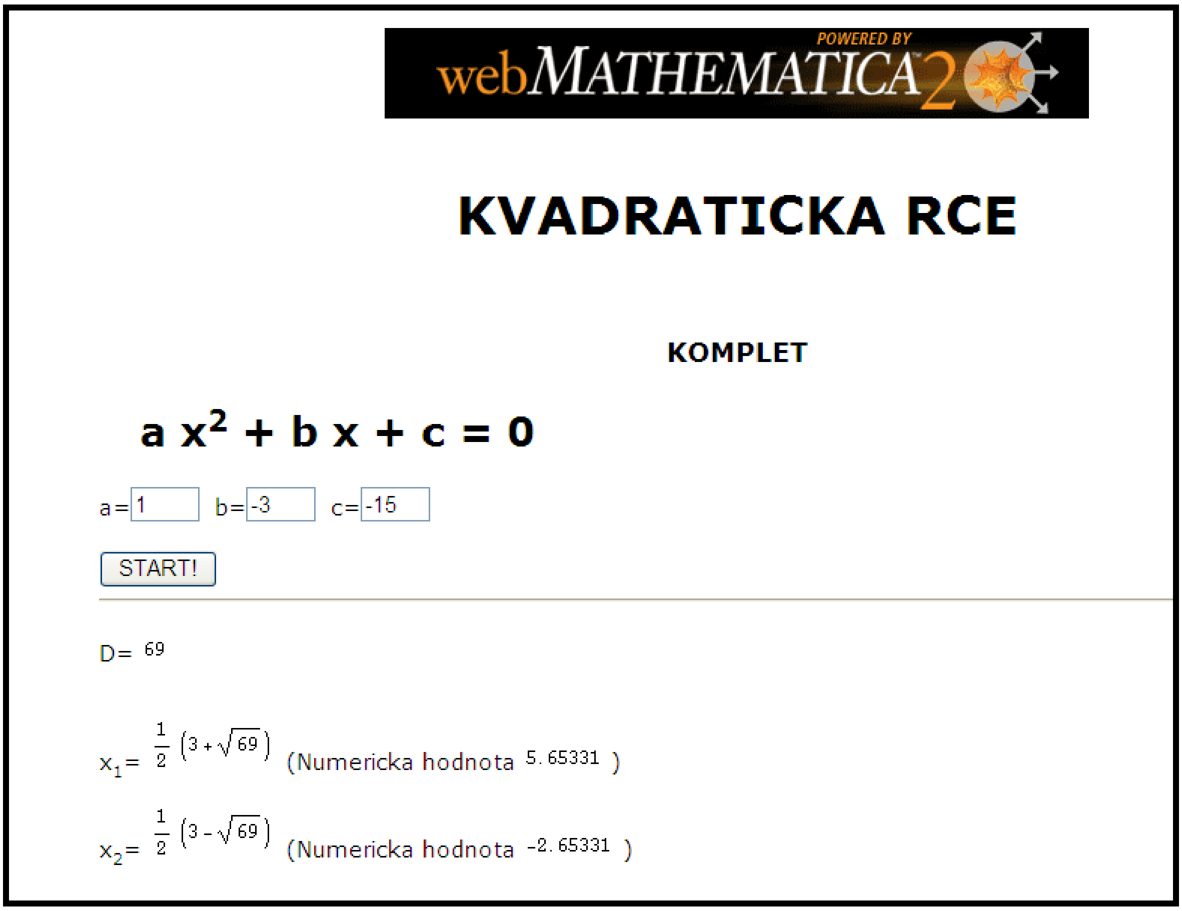

Figure 4 – The use of WebMathematica ([5])

- very complicated calculations including the step-by-step solution:

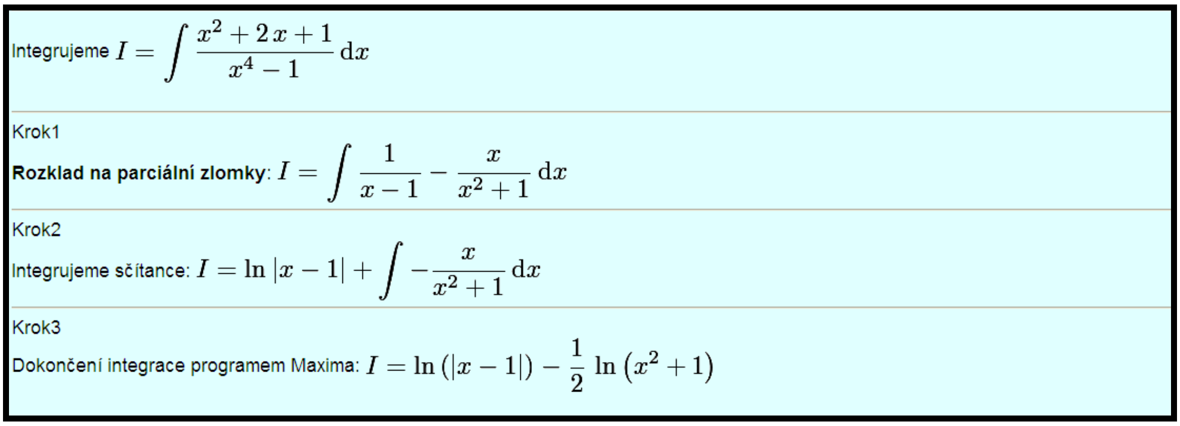

Figure 5 – An example from the project Mathematical Assistant on Web ([6] a [7])

Characteristics of a good study material

The first and very natural way of creation an e-textbook is the division of the text into various levels of difficulty and making hyperlinks between different parts of the text. It may for example be a return to a definition of a term needed for the solution of the problem, coming back to a particular computational process when solving the mathematical model of the problem assignment, link to visualization of the relevant object or of the solved problem etc. However, a good e-textbook should offer much more:

It should enable scaffolding. This concept and technique was used already by L. Vygotsky. In its original form, scaffolding may be defined as a process wherein the instructor, or a more advanced peer, operates as a supportive tool for learners as they construct knowledge. In case of an e-textbook, this role is partially taken over by the computer. For example a solver who manages to form an equation when solving a word problem but cannot solve it can "ask" the computer for help via an integrated link to CAS and get the correct result. However, it is crucial that the solver should not rely merely on the computer but should consequently fill in the gap in his/her knowledge. Another instance may be mathematical proof. A convenient structure of links may offer a graded help, links to processes necessary for solution of individual steps in the proof etc. In consequence the teacher ceases to be the only source of all help and advice. The computer, if the e-textbook is well structured – takes this role over. The use of an e-textbook enables every pupil to use such a type of scaffolding and in such an extent that is optimal for his/her level ([8]).

It should enable continuous feedback. Continuous feedback on the correctness, purposefulness etc. of the selected processes is one of the great advantages of a well created e-textbook.

It should offer graded help which is either automatic, or only on request, including automatic changes in the displayed content.

It should enable the teacher to monitor how pupils cope with the given topic, where they face difficulties, where they need help and on what level. This function, however, requires the teacher to have an immediate access to what the pupils are doing at a particular moment.

Conclusion

This article presents some of the possibilities that modern information technologies offer when creating electronic materials. In the light of the rapid development in the field of information technologies, the presented list cannot be under any circumstances regarded as exhaustive. The authors primarily try to reflect on their own experience with creation of electronic study materials which they gained when solving projects both on national and international level.

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Reflections on an Initiative to Improve Junior Secondary School Pupils' Understanding of Number

Noel Johnston
University of Otago College of Education noel.johnston@otago.ac.nz

Abstract
In 2005 the opportunity to apply the New Zealand 'Numeracy' approach to teaching Mathematics was extended into the secondary school sector. The goal was to alter teachers' pedagogy so that 'sense making' rather than 'instruction' was the core objective of their lessons. Ultimately it is hoped that along with a familiarity and comprehension of Number will come a relatively seamless acquisition of the fundamentals of Algebra. This paper will present details of this approach for teaching Number, the status of Number in the secondary school curriculum, the focus and ramifications of teaching for understanding, as opposed to assimilating and learning to apply algorithms, and will also consider evidence of the effectiveness of the initiative.

Introduction
The term 'Numeracy' as used above refers to a person's 'ability and inclination to use Mathematics effectively – at home, at work and in the community'. In New Zealand 'Number' refers to a particular strand in Mathematics in the New Zealand Curriculum. The focus of this strand is the structure of the number system; Number *Knowledge* and to consider the *Strategies* that can be applied to find answers to problems. The Strategies become the overt reasoning that pupils use when they perform arithmetic calculations. Out of this careful consideration of the Knowledge-Strategy interrelationship it is envisaged that the transition to Algebra will be less traumatic. The connection between in-depth number knowledge, flexible mental calculations and Algebra thinking was recognized by Irwin and Britt in their study (2005).

Historical Background
When the results from the Trends in International Mathematics and Science Study (TIMSS) released in 1996 identified a degree of underachievement by a significant number of our secondary school-aged pupils, compared to the performance of similar cohorts in the global Mathematics education community the government responded by establishing a Task Force closely followed by the Numeracy Development Project (NDP) implemented in 2001 after a successful pilot of Count Me In Too, in 2000. This NDP aims to raise pupils' achievement through teacher professional development.
Exploratory work carried out in secondary schools from 2001 through to 2003 found that pupils' achievement at year 9 level had been variable. The challenge of changing teachers' practice at the secondary level had also been noted by a number of teachers and educators. In the evaluation of the Numeracy Exploratory Study in 2001, Irwin and Niederer (2002) noted the "unexpected weakness, especially in the understanding of fractions, the ability to find a fraction of a whole number, and the meaning of large numbers" (p 97). One factor identified as a reason was the weak pedagogical content knowledge of teachers in the area of place value. In 2002 and 2003, pupils also performed below expectations in the National Certificate of Educational Achievement (NCEA). In response to the perceived ineffectiveness of teaching Mathematics in the junior area of secondary schools the Numeracy Development Project (NDP) was extended to secondary schools as the Secondary Numeracy Project (SNP) in 2005.

Theoretical Composition of the Numeracy Project
The NDP has its roots in constructivism, the Pirie/Keiren recursive model for teaching, along with the principles of cognitively guided instruction (Holmes & Tozer, 2004). A framework detailing the aspects of staged number knowledge that pupils will require was complied by a number of Mathematics education researchers (Frank, 1989; Fuson, 1998; Steffe, Cobb with von Glaserfeld, 1988; Young-Loveridge, 1999). Wright (1998) reworked this material and added aspects of number identification to the framework. The strategy part of the framework considering theories about pupil's numerical reasoning (Steffe et al, 1983) with refinements from Wright (1998) was further extended, giving teachers five graduated global stages of the structural thinking pupils use. In New Zealand the one framework was split into two (Knowledge and Strategies) and extended to Stage 8 with Stage 0 added. The Numeracy Project recognizes the

interdependence of Knowledge and Strategy and is delivered in the expectation that a pupil's mathematical understanding is nurtured and developed by carefully maintaining structured instruction in both areas. Teachers in the NDP are encouraged to move away from their traditional instructional style of teaching and teach for relational understanding.

Composition of SNP

The essence of the SNP is to promote quality teaching of Mathematics. Classroom lessons should produce deep conceptual rather than procedural understanding. The expectation is that pupils will exhibit flexible and imaginative explanations for the solutions to problems. Considerable attention is given to dimensions of quality teaching. Recent research has shown that quality teaching is fundamental to the improvement of pupils' outcomes (Alton-Lee, 2003). While readily acknowledging that classrooms are complex, demanding environments, it is possible to identify teacher dispositions and activities that will have a positive impact on pupils achievement: aspirational attitudes have been identified as affective and have been investigated by a number of researchers; the **inclusive classroom** (Cobb, McClain, & Whiteneck, 1995); **focused planning,** where lesson content and intent is compiled from a variety of assessment methods and is framed around pupil needs; **problem-centered activities**, especially those approaches where the expectation is that pupils will be responsible for new knowledge creation (Clarke & Hoon, 2005); **high expectations** built around a belief that a promotion of higher order thinking skills will motivate pupils to become independent critical learners; **connectivist teachers** with a propensity and capacity for linking mathematical ideas and vocabulary to actions on materials and an ability to parallel mathematical concepts to realistic contexts (Askew et al, 1997). Research on **quality teaching** (Shulman, 1987) has identified teachers' pedagogical and content knowledge as the principle components of quality learning.

Along with research on factors that underpin pupils' Mathematics education, considerable thought and planning have gone into designing a professional development project that will assist teachers to come to terms with the dimensions of the initiative. The model used to deliver the professional development is structured around an in-school facilitator working with teachers. Schools are invited to participate in the programme, and a contractual agreement is entered in to between the schools and the MOE primarily to ensure all staff participate fully. The contract also details the extent of professional development that teachers can expect.

The Professional Development

The teacher development model adopted for the introduction of SNP to schools in 2005 differed from that used in other numeracy projects in its use of in-school facilitators (ISF) supported regionally by an external regional facilitator (RF). The RF is charged with supporting the ISF to develop an appreciation of the appropriate pedagogy for complementing the project, especially around investigating ways to present Number Knowledge and Strategy. Further expectations are that the RF will model classroom delivery and will promote and assist with the development of games and activities. These aims are promoted by having regular in-school sessions as well as specific professional development opportunities. Evaluation of this model by Harvey, Higgins & Jackson, (2005) indicated that in general it is successful, particularly in the way that it has impacted on teachers and ISF knowledge, skill and practice. Another benefit was an increased occurrence in professional dialogue within mathematics departments about examining ways for optimizing their school's Mathematics programme.

Accompanying the professional development programme is a considerable amount of support material. To really facilitate change there needs to be material that teachers can use to assist their lessons and compliment their lesson planning (Fullan 1999). To this end the Ministry initiated the compilation of 8 texts that correspond to the stages in the framework. A web site and publications under the generic heading 'Figure It Out' have also been developed to complement teaching programmes. Accompanying these material supports for the project is an understanding that the teaching programmes are to be compiled to address deficiencies in pupils' Number Knowledge or their familiarity with using Strategies. In essence teachers are to continually assess their pupils, reflect constructively on their next teaching episode and then use the material provided to assist them to plan for the next lesson.

Material Support

The Number Framework is composed of two distinct but complementary sections: the *Knowledge* component ranks key number items that pupils can recall quickly with minimal effort, and the *Strategy* section considers the reasoning process by which children use their knowledge to find answers. Both sections are divided into 9 stages which correspond to expected development stages for pupils from the beginning of schooling through to the expectations for a pupil's Mathematics understanding after 8 years of education. Within the Knowledge framework the aspects identified as important are: number sequencing; grouping and place value; basic facts and written recording. For the Strategy section the 9 stages each demonstrate a progression in levels of sophistication of the rationale for solving problems. The strategies are applied over 3 operation domains: addition and subtraction; multiplication and division; proportion and ratios. The essence of the strategy section is a consideration of a pupil's progression from being "a counter" to being fluent and competent with multiplication and how they can be extended to thinking proportionally. Both sections contain considerable example detail about specifically which aspects constitute the progression at each stage. Classroom teachers are given detailed professional development about the content and design of the frameworks.

The Diagnostic Interview and the Knowledge Test

These are the assessment tools that teachers use to gain an understanding of where their pupils sit on the number framework and what teaching programme should be organized for them so that they progress through the levels. The diagnostic interview is an opportunity for the teacher to gain an insight into a pupil's reasoning capability and, more importantly, what aspects of number comprehension the pupil lacks. Up to 24 problems are presented in a non-standard pen and paper format and the pupil are invited to verbalize the process of how he or she has solved problems. Integral to this testing is the teacher's capacity to illicit responses from the pupil that put more emphasis on the mechanisms about how the answers are formed rather than the answer itself. Teachers detail their answers and consider carefully what the pupil's replies indicate about where he or she fits on the framework. The Knowledge test is presented as a 10 minute paper and pencil test examining a pupil's number understanding in the categories: number sequences and order, grouping and place value, basic facts. The results from the two tests enable teachers to set up learning trajectories that address specific gaps in understanding and knowledge. Both tests are repeated at the end of the year enabling progression to be identified. All the results are required to be entered in a secure national data base.

Strategy Teaching Model

At the heart of this model is the use of material to validate new concepts, explanations for working of the ideas, for example, demonstrating the addition of fractions or multiplying single place decimals using multi link cubes. Pupils are challenged to demonstrate the answers to a problem by exemplifying it with the cubes. This model requires that from the physical manipulation of the materials pupils will then be able to 'image' other solution paths to further problems. After practice the mechanisms they have used will then become part of their general repertoire of solution techniques. This is a fold back model based on the work of numeracy facilitators with Hughes (2002) who in turn has adapted it from P-K theory (Pirie & Kieren, 1994). Fundamental to the teaching model is an expectation that out of the use of material the pupil will construct understanding by moving from the tangible to the imaging after sufficient interaction and then to the generalized understanding. By cycling through this process a number of times and encouraging pupils to reflect on the material, the expectation is that the understanding will be more meaningful and personal. Success with this procedure is related to the extent of teachers' content knowledge and their ability to model content with materials. Much of the professional development is constructed around an expectation that materials will be used to assist with explanations and that their use will complement and help to verify the solutions to problems.

Outcomes

The SNP numeracy initiative was introduced in 2005 and was intended as a professional development project for secondary school teachers. The SNP initiative is constructed around an acknowledgement that the relationship between Number and Algebra is extensive particularly in the generalized use of strategies. By the end of 2008 approximately 120 secondary schools had been involved in the project.

Results indicated that teachers welcomed the opportunity to establish the extent and depth of pupils' Number Knowledge and Strategies and were enthusiastic about the teaching model, particularly the use of materials to complement lessons. There was also enthusiasm for putting more emphasis on extolling pupils to construct and articulate their own answers, rather than rely on explanations learnt directly from the teacher. Teachers appeared to be more prepared to put 'sense making' as the principle objective of their lessons and were questioning the reliance on teaching algorithm. Teachers welcomed the opportunity to negotiate solution strategies with their classes and have indicated a change in their approach to teaching, moving away from whole class instruction to adopting strategies that allow for differentiated learning programmes. The major area of concern often noted in schools is time. Many teachers feel pressured in their attempts to implement the curriculum and time management to address the intent of SNP has become a challenge.

The SNP continues to receive Ministry support. Measuring significant change is difficult as initially the primary aim of this project was to upgrade teacher pedagogical practices. Lately more emphasis and expectation has been placed on gauging the change in pupils' capacity to manipulate the Number and Algebra component of the curriculum. Some tension has emerged with a desire to see the expectation for rationalizing answers to problems reflected in the external exams that pupils sit in their last 3 years of schooling, particularly the Number sections

Conclusion

Inviting secondary teachers to examine and analyse their teaching practice and to then reflect on approaches that can have more positive outcomes for pupils is a challenge. The success of SNP is reliant on teachers recognizing that this approach will allow the Mathematics education of their pupils to be more meaningful and their transition to learning Algebra will be less cumbersome or problematic. This, along with an emphasis on teaching programmes that address individual pupil needs, has given the initiative considerable prominence in secondary schools throughout the country. The evolution of the SNP has involved the synthesis of ongoing research, data gathering and analysis, exchanging and sharing best practice, cultivating and examining schools that have successfully incorporated the initiative. Around half of the secondary schools in New Zealand have taken part in the initiative, giving a critical mass of practitioners who are able to critically reflect on the benefits of this approach to teaching Mathematics. The outlook is positive but there are a number of challenges that require further analysis and refinement, in particular, the implementation of the teaching model for the aspiration of catering for individual needs. The transition from Number to Algebra is not as seamless and smooth as has been purported, and there is still much work needed to assist pupils who have difficulty gaining traction with the approaches offered by this initiative. Preparing resources to complement lessons is seen as a demanding and difficult aspect of fulfilling the SNP aspirations. Finally, the success rates of Maori and Pasifika pupils in New Zealand schools is an ongoing concern.

Fundamental to this initiative was a desire to change the way that Mathematics was taught. Behind the notion that 'mimicking is not teaching' lies a big challenge, especially for secondary school Mathematics teachers who for the most part considered that their role in a pupil's Mathematics education was to instruct them in how to apply the most suitable algorithm to a particular problem. In stark contrast SNP is more about opening up the rich Mathematical world that lies behind much of what we teach. It is this Mathematics which runs parallel to what is taught but is often obscured that is one of the most tantalizing and satisfying aspects of the initiative. Having been involved for the past 5 years with this project my reflections upon its value, potential and overall benefits are considered from my experience of teaching Mathematics in secondary schools for 28 years. There are a number of approaches used in SNP for rationalising concepts that have interesting implications for the remainder of a pupil's Mathematics education at secondary school. Any initiative to alter pedagogical practice is a challenge but SNP brings a framework that complements the whole logical imperative that we associate with this discipline.

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Preventing 'Pushing for Privileged Passage': A study of a charter school working to step back from tracking.

Tina Louise Johnston, Ph.D.
Post Doctoral Researcher, Department of Science and Mathematics Education, Oregon State University
Corvallis, Oregon, United States of America Johnstot@onid.orst.edu

Abstract

One charter school's path to tracking and *pushing* for privileged passage is examined. The school as it increased in size began to track students first by grade level and then by ability. Realizing that moving mathematics out of the main school program compromised the teaching ideals of the school and potentially student learning. The school has embarked on a program to create a place-based, integrated curriculum developed around mathematics so that mathematics can be reintroduced to multi-age classrooms. Examining the data in terms of trust, size and the behaviors of administrators, teachers and parents at this school in this process are highlighted.

Introduction

There is much research chronicling the negative effects of ability grouping or tracking on both high achieving and low achieving mathematics students (Ballantyne, 2002; Boaler, 2002; Oakes, 1985; Slavin, 1995; Stevenson et al., 1994; Wheelock, 1992). In a variety of forms the practice has been found to limit the access that low achieving students have to rigorous mathematics content and place undue stress on high achieving students (Boaler, 2002; Callahan, 2005; Hahr, 2005; Lleras, 2008). For some years researchers have called for an end to the practice but to limited success (Carnegie Council on Adolescent Development, 1989; Oakes, et al., 2000). The reasons for this failure have been attributed to a myriad of sources including those related to social policy, administrative organization, teacher beliefs, and community factors (Oakes, et al., 2000; Spear, 1994).

Trust

School trust is closely linked to healthy and effective schools (Bryk & Schneider, 2002; Forsyth, et al., 2006; Goddard, et al., 2001). A lack of trust conversely has been linked to higher control mechanisms and highly controlled rules and regulations that isolate administrators, teachers and the community (Forsyth, et al., 2006). Lack of trust is also closely linked to the perpetuation of tracking (Johnston, 2006; 2008).

Trust in school can be defined as allowing vulnerability based on the belief that a trusted school party is honest, open, reliable and competent (Hoy and Tschannen-Moran, 1999). The kinds of and expected roles in these trusting relationships varies depending on whether the role group is a parent, teacher or administrator. The interactions occur across and within groups. The quality of the communication among these groups establishes feelings of trust and trusting relationships or not (Adams, et al., 2009). Failures to convey honesty, openness, reliability and/or competency create failed trust among one or more parties of the role groups (Johnston, 2006; 2008).

Lack of school trust has been linked to a focus shift by any role group member of increased advocating for specific students. Although this focus might be expected of parents it is not expected or desirable from teachers or administrators (Mann, 1848). The lack of trust manifests into increased scrutiny of programs. In mathematics that scrutiny turns to *pushing* behaviors that can form, exacerbate and perpetuate mathematical tracking (Johnston, 2006; 2008).

Size

School size can positively affect school trust. Although physical factors have only a minimal affect on school trust. Role groups can exhibit trusting relationships despite poor achievement levels or student heterogeneity, small schools may have an advantage in that small schools often draw from a more homogeneous community and the ability to clearly communicate common messages to fewer people is advantageous (Adams et al., 2009).

Size has an effect on student grouping practices. Small schools do not have the number of students or staff to be able to offer more than one mathematics level. The smallest schools exhibit even broader ability heterogeneity. At an extreme one room schools may have students of many ages learning math together. At the other extreme very large schools may have student divided into as many as six mathematics ability classrooms (Johnston, 2006). Growth in the number of students and staffing enables schools to create multiple classes, divide and assign students to mathematics classes by ability. It is common for schools to shift from heterogeneous grouping strategies to tracking as they increase in size (Johnston, 2006).

Pushing for privileged passage

Pushing can be defined as exerting oneself continuously, vigorously, or obtrusively to gain an end or engage in a crusade for a certain cause or person; in essence, becoming an advocate for a particular cause or person (Wordnet, 2006).This definition presents pushing as a positive action. In theory, educators are the pushers or advocates for all students (Mann, 1848). Parents are the pushers or advocates for their children (Crozier, 1997). So how do seemingly positive notions create conflict? The problem lies in who is deemed deserving of challenging material, all children or specific children? If all children do not receive access to advanced mathematics content, how are those children who should receive the attention and material selected? Although neither a plot nor scheme, pushers work to garner access into classes with students receiving advantaged instruction (Kohn, 1998; Oakes & Wells, 1998; Spear, 1994).

There are three levels of *pushing*. Some pushers may work at all of these levels over a period of time while others may only apply one or two in their quest to garner advantaged placement for their focus student. The levels have scope (foundational, combative, and strategic) and order (Investing, Pressuring and Lobbying). (Johnston, 2006; 2008).

School districts with varying trust levels among the differing role groups exhibit different levels of pushing behaviours. Role conflicts (teacher or administrators who are also parents) play a role in the kinds of pushing behaviours adopted. The greater the extremity of mistrust among the varying role groups the more pervasive the pushing behaviours can be (Johnston 2006; 2008)

The School

Oakview Community School (OCS) is a charter school that opened in 2007. Located in a rural school district in the northwest United States, the school serves 204 students in grades 1-8. The school has a mission of delivering an integrated curriculum to mixed-age classes using place-based, project-oriented instructional strategies.

The school is divided into four levels. Level I houses students in grades 1, 2 and 3. There are three teachers with approximately 20 students in each class who are evenly distributed from each of the age levels. Level II mirrors the level one configuration but works with students in grades 4, 5 and 6. There are two level III teachers. These teachers serve students in grades 7 and 8. One teacher specializes in math/science while the other does social studies and Language Arts. The level III math/science teacher is the only qualified mathematics teacher on staff.

Charter school staff is not held to the same staffing requirements as non-charter counterparts (Center for Education Reform, 2002; SRI, 2002). At OCS both administrators have little formal training in education (one is a journalist and the other previously worked in university admissions and has partially completed a teaching degree in mathematics and science). All of the teachers are certified to teach in the state except one who is certified in a Midwest United States. Teachers have from 3 to 25 years experience. Only one teacher on staff is certified to teach mathematics (the level III math/science teacher). The remainder have a much stronger knowledge of the social sciences and Language arts content areas.

OCS took a proactive role in fitting into the community in which it resides, proactively interacting with parents. The schools location, on the main street of the small university town, provides easy access for students' weekly 'out and about' experiences into the community to learn about food, ecology, conduct service projects and interact with willing businesses and university faculty. Administrators are very proactive in communicating with parents via twice weekly email messages. Quarterly whole school events are scheduled often at times when community events are also on the calendar so that students, their parents, and school staff can attend both. Grade-level teachers work together to organize various grade-level presentations for whole school events and less frequently do grade level or cross grade level events that culminate learning activities. Conference are held three times a year (one before school starts, one in the Autumn, and one in the Spring)

The mathematics program at OCS has been problematic over the two years since the school was started. Following year one instruction, teachers expressed anxiety about delivering standards-based mathematics instruction in an integrated manner. It was observed that mathematics instruction was neglected in many projects. Evidence was also seen in state standardized test results where students showed weak performance. 70.2% of the students in grades 3-8. This number was lower than both the district and state averages (75%, 77% respectively).

In response to these concerns, administrators with agreement of teachers changed math instruction. During the 2008/09 school year mathematics was taught by grade level using a purchased mathematics curriculum. Level 1 and 2 students left their main multiage classroom and travelled to one of the other teachers' rooms for mathematics instruction. Each of the three teachers at these two levels took on instruction of one grade. For grades 7 and 8 with only two teachers on staff, the separation of mathematics into grade level groups meant that multi-grade instruction at this level has been eliminated for all subjects.

Within 4 months of the change to grade level mathematics grouping teachers in the school began to shift specific students into new mathematics classes. These placements were changed when a teacher observed behaviors in a student that appeared advanced. One student was moved from a grade 1 mathematics to a grade 2 mathematics class. 3 students were moved into higher mathematics level classes among the Level II teachers and 7 students were moved into higher grade level classes in the level III.

Discussion

Even in the short time that this school has been in existence it has quickly shifted towards student tracking in mathematics. Although teachers and administrators did not plan to group students by ability in mathematics they did so. The initial grade-level division was a combined administrative and teacher driven decision. The movement of individual students based on judgments of their mathematical ability is a *pushing* behavior that was teacher prompted (Johnston, 2006; 2008).

The recent growth of the OCS has followed the course of many other growing schools (Johnston, 2006). As soon as it was big enough to start dividing students mathematically it did so. In this act it risked and begun tracking specific students into mathematics classes by perceived mathematics abilities. Despite the negative aspect of this growth the school size has remained small enough to maintain effective communication and a sense of community among the three role groups.

Examining trust issues of honesty, openness, reliability and competency among the three role groups suggests that both the nature of the charter school and the interaction between the role groups involved at this school have for the most part done a remarkable job of developing trusting relationships between the three focus groups. Issues of reliability and competency between the administrators and teachers as they relate to mathematics instruction have put trust at risk between these two role groups and among the teachers.

The nature of participation in a charter school assists in setting up a level of trust that is not present in public schools as all parties participate by choice (Belfield & Levin, 2005; Kleitz, et al., 2000). At this school the administrators participate because their initial vision and application were required to gain funding and district permission to start the school. Teachers work at the school because they feel some affiliation with the tenants of the charter. Some have done so for less pay and all have taken on the job outside the confines of union contract. Parents applied for lottery drawn slots for their child's acceptance the school.

The administrative staff and teachers actively work at bringing all role groups together through invited participation in school-wide and community linked events. The frequent and open communication that occurs between the school role groups fosters feelings of openness and honesty. Trust between administrators and teachers is maintained through weekly staff meetings where the administrators take on predominantly a facilitator role in decision-making but stepping in when decisions stall.

In one area there has been a break down in trust between administrators and teachers at the school. This is evident in the removal of mathematics from the adopted instructional program. The trust loss was valid. The teachers as a whole are not well prepared to teach mathematics in either an integrated or differentiated way to best meet the needs of students in their home classrooms.

Another potential problem identified in this study is in part a symptom of the new schools youth. The charter application lists educational goals and instructional methods that may not be clearly defined and may not be clearly understood by the role group members. Recent research has suggested that charter schools despite intentions to the contrary may not actually teach in ways any different than those offered at local public schools (Hanushek, 2007). How place-based, integrated and mixed-age classrooms looks at OCS is viewed by the three role groups has yet to be defined and yet to be carried out.

At the same time that teachers were beginning to track students by ability they were also revisiting both the appropriateness of the separated mathematics instruction and began working with advisors from the university and a regional place-based charter school to develop units centered on the state mathematics standards. Although the teachers are reticent to return the mathematics instruction to what they perceive as a failed integration. School administrators and the university advisor are actively working towards helping the teachers gain the mathematics content and pedagogical content knowledge needed by working with them to develop and teach place-based, integrated units with multi-age students. One unit is under development and will be implemented during the 2009/10 school year. The goal is to develop 3 units per year that incorporate strong mathematics content over a period of 3 years so that the school can return to the ideals in its charter for all subjects.

Conclusions

There have been many studies documenting schools often unsuccessful attempts to untrack schools (Hatton, 1985; Oakes, 1995; Wheelock, 1992). Alternatively there have been little to no studies of schools documenting the path into tracking practices and the work of these schools to resist that draw. This study provides a rare view of this process couched within a theoretical framework that suggests the importance of school trust in the process (Johnston, 2006; 2008).

It is important to note that this study was of a charter school. The fact that charter schools are designed to foster innovation cannot be ignored. It is probably the conflict between this schools charter and the practice that the parties involved are so willing to work on stepping back from tracking. It is notable, however that regular public schools are fully able to adopt similar teaching practices and in some cases have successfully done so (Boaler, 2002; Wheelock, 1992)

The very existence of choice in this process may have a large impact on the trust relationships between the administrators, teachers and parents involved in this study. There has been some research (and argument) on the effectiveness of charter schools but none about the relationship of charter schools and trust. More research needs to be conducted in this area.

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Creating and Utilizing Online Assignments in a Calculus Class

Veselin Jungic, PhD
Department of Mathematics
Simon Fraser University
Burnaby, BC, Canada
vjungic@sfu.ca

Deborah Kent, PhD
Department of Mathematics
Hillsdale College
Hillsdale, MI, USA
dkent@hilsdale.edu

Petra Menz, M.Sc., B.Ed.
Department of Mathematics
Simon Fraser University
Burnaby, BC, Canada
pmenz@sfu.ca

Abstract

The aims of this paper are to present some of the findings about the creation and utilization of online assignments and choice of support software for several calculus classes at Simon Fraser University (SFU) by considering the needs and perspectives of the instructors, students, and administrators. The term online assignment is used for a set of problems that are posted, submitted, graded, and recorded electronically through a course learning management system (LMS) of choice. The purpose of this note is to contribute to the discussion about a common question detected among research papers on the theme of online assignments; how can technology be used in teaching so that students benefit the most? Questions are provided to guide an instructor in choosing online assignment problems, and a list of necessary skills is supplied for an instructor to be able to deal effectively with this pedagogical tool.

Introduction

The Department of Mathematics at SFU decided to introduce online assignments for its science/engineering calculus course sequence in the summer semester 2003 and subsequently for the social science calculus course sequence in the fall semester 2004. The main reasons for this were: to provide consistent and on-going feedback on homework assignments for students in large first year calculus classes, to ensure that students practice concepts and skills sufficiently, and to reduce the high costs of handling paper assignments. Similar observations and moves to online assignments have been documented especially for physics courses [BDB03, D07] in the past decade and only lately for mathematics courses [SCX06, SS06, YHR08, LG09].

Managing Online Assignments

There are a number of educational software systems that support online assignments. Some of the software is commercial while some is open source. It has become standard for publishing companies to accompany their calculus and other math and science textbooks with software that supports online assignments. For example, the software called PHGradeAssist accompanies the book *Calculus* by C. Henry Edwards and David E. Penney developed by Prentice Hall. Another example is the software package called WebAssign, originally developed by North Caroline State University that accompanies the book *Calculus Early Transcendentals* by J. Stewart published by Brooks/Cole. There are also commercial companies that offer as part of their courseware packages to create questions or use existing question banks for online assignments such as Lyryx Learning based in Calgary, AB, and Maplesoft in Waterloo, ON. In addition, many commercial learning management systems like WebCT come with capabilities of creating online quizzes. Two examples of open source software are LON-CAPA developed by Michigan State University, and WebWorK, which originated at the University of Rochester Department of Mathematics.

Basic features of the existing software packages that enable online assignments include: a browser-based interface, question banks, various types of questions, parametrically generated questions, automatically graded assignments, an electronic grade book, and communication tools. The question banks may be provided or purchased as is the case with software that accompany textbooks, or shared as it is the case with all open source software. Almost all packages allow instructors to create their own questions. The questions might be multiple-choice, true/false, fill-in-the-blank with a formula, numerical value, string answer, or open-ended. Communication tools usually include chat rooms, a discussion board, and an internal mailing system.

It is our experience that an instructor preparing a set of problems for an online assignment faces the same challenges regardless of whether questions are created from scratch, or modified or chosen from an open source or commercial question bank.

- How can we choose online assignment questions that would best complement other elements of the course such as lectures, readings, paper assignments, and exams?

- Which types of online questions are best suited for the learning of various mathematical concepts and skills?
- Which types of online questions are the most appropriate for testing complex mathematical ideas?
- Can online questions be used to communicate mathematical ideas, i.e. be used to introduce new concepts that have not been seen in the lecture or the textbook?
- To what degree should online assignments be used as a drillmaster?
- Are the provided online questions suitable for all teaching and learning styles?
- How can we mimic online what we observed with paper assignments in our workshops, i.e. support discussion among students for the purpose of learning?
- How can we avoid cheating?

In our opinion, question banks are limited and cannot provide a resource adequately to address all of the above questions. However, these question banks do provide a platform to get started in the creation of online assignments. Publishing companies are paying more and more attention to this piece of the courseware and generally speaking their products are of a high quality. It is important to notice that commercial systems like PH Grade Assist or Maple T.A. allow users to create new or modify already existing problems. It may be the wish of many conscientious instructors to create problems themselves, but this is tied to investing much time and energy in the creation of questions, dealing with the time-consuming task of testing the questions and getting feedback from the students to improve their purpose, and needing a strong support by the administration for the creation and implementation of this new pedagogical tool.

During the five years of working first with a commercial question bank and then switching to an open source question bank, where we have been using and contributing to shared question banks, we have come to the conclusions that regardless of whether online questions are chosen or created the instructor must utelize this pedagogy with a whole new attitude and set of skills as listed below.

- Flexibility of Product: The questions are not tied to a particular textbook or a specific course. They are easy to change and to adjust to meet the needs of the instructor or the course. For example, a question created at SFU for the Introduction To Analysis course has been used in an algebra course at the Jerusalem College of Engineering (Jerusalem, Israel), a calculus course at Selwyn House School (Montreal, Canada), and a linear algebra course at the University of Applied Sciences (Braunschweig/Wolfenbuettel, Germany).
- Flexibility on part of the Instructor: The ambiguous use of terminology can create a stumbling block for students and even instructors. For example, not everybody agrees that a local extremum of a function might happen at the end point of an interval. Or, what one instructor calls “a critical number” another instructor might call “a critical point”. However, with an open mind these minor inconsistencies can be explained away or dealt with effectively.
- Looking ahead: When choosing a question from a shared question bank the instructor must be much more careful. Testing a few versions of the question is a must! A common problem with shared question banks is that a question might be created in an early release of the software and might not work properly in the newest version.
- Reaching out to students: By creating appropriate questions the instructor makes first year university students read the textbook in detail and regularly. For example, students might be asked to match parts of definitions and theorems from a given textbook section.
- Creativity: In creating questions for online assignments instructors can truly unleash their imagination. For example, a single problem might contain various types of questions: multiple choice, drop-down box, fill-in-blanks, formula and numerical responses. All this might be accompanied with dynamic plots and supported with hints that match most common mistakes.
- Community building: In our experience this happens at two levels, namely locally when a group of instructors from the same institution builds resources that the members of the group can use in their teaching, and globally, as in the example provided in “Flexibility of Product”.
- Self-Enrichment: Instructors can avoid having become routine and bored over the years by making changes in the way that we teach, by learning a new software, by creating new learning resources for students, and by reaching out to students by using a new medium for learning.

SFU Experience

Simon Fraser University was the first university in Canada to use PH Grade Assist, an online assessment tool developed by Prentice Hall, in teaching calculus classes. The tool was used from the summer semester 2003 until the summer semester 2004 for assignments in the science/engineering calculus course sequence. The initial idea was to simply put weekly assignments for the three courses online and let the system do everything else, collecting assignments, grading, and recording grades. All the instructors had to do was to create an online course and to assign questions straight from the textbook. Students, who beforehand had to register into the system and enroll in the course via a code provided with the textbook, would get one of the parametrically generated versions of the assigned question. Codes increased the price of the textbook packages. There was a possibility of purchasing codes separately in case a student had a used textbook.

This approach was extremely convenient for the instructors and meant no extra cost for the department. The onus of the work was on part of the students. However, soon it became clear that the teaching and learning were disconnected because the instructors' choice of questions was limited by the medium with no questions of the type "prove," "graph," or "give an example," and students' answers were constrained by the medium. In particular, they did not gain any experience with writing down their answers similar to their exams. At the end of this first try of the fall semester 2003 a survey was conducted, where students were asked to compare their experiences with online assignments and paper assignments for understanding, quality, and enjoyment. The 409 responses are summarized in Table 1. The survey results showed that with the complete switch to online assignments, the Department did not achieve its main goals.

	Online assignments better	Same	Paper assignments better
Understanding the material	16%	34%	50%
Quality of your work	13%	29%	58%
Enjoyment of the subject	22%	33%	45%

Table 1

The next step was to use a combination of paper and pencil as well as online assignments which, this time, allowed students multiple attempts to get a correct answer. Still there were three extremely sensitive issues that had to be resolved to the students', the instructors', and the department's satisfaction, namely the increased cost of the textbook package due to the codes required to register into the online assignment system, the lack of variety of mathematical questions, and students being forced to share their personal data with an entity that was not part of Simon Fraser University.

These challenges lead to the decision that, starting in the fall semester 2004, the Department of Mathematics would switch to LON-CAPA. At SFU, both the Department of Chemistry and the Department Physics had used LON-CAPA successfully since 1998. The use of LON-CAPA comes with no cost for SFU students and the access to LON-CAPA is automatically granted to all students enrolled in courses that use this system. LON-CAPA is an open source freeware web-based course management system featuring content sharing and content reusability, creation and grading of randomized homework, quizzes or exams, assessment analysis, porting content, one source/multiple targets, cross-institutional network, clicker device support, and much more. As of the spring semester 2008 there were 121,038 problems in the LON-CAPA shared content repository. The repository has been steadily growing since its inception in 1993 supporting various disciplines such as physics, chemistry, biology, mathematics, and more from over 70 participating institutions worldwide [LNC09].

In the fall semester of 2004, the Department of Mathematics offered its first LON-CAPA supported course, Calculus I for Social Sciences. By the fall semester 2008 seven calculus courses were supported by the system with five instructors involved in creating problems and managing courses.

One of the most beneficial aspects of LON-CAPA is its DISCUSSION TOOL. We have found that students use the electronic board extensively to discuss posted problems with each other. Since each student has a different variation of the problem, students need to analyze the problem and explain to each other what steps were needed to solve it. The instructors were quite elated about this outcome as the students were using more mathematical language than the instructors thought possible. Additionally, they were teaching each other the necessary concepts and skills. However, every now and then students were stuck with a particular problem or misguided help was given from other students and the instructors saw the need for assigning TA hours to the supervision and aid of the discussion board. This proved to be very effective and only costs about 40 TA hours per semester.

Paper and Pencil vs. Online Assignments

We propose that in math classes paper and pencil and online assignments should be used in conjunction. In our view of teaching math classes the main role of paper assignments is to give students experience of writing mathematics. As Kevin P. Lee states in [L0x], "You should not confuse writing mathematics with "showing your work". (…) Rather, you will be writing to demonstrate how well you understand mathematical ideas and concepts. So a page of computations without any writing or explanation contains no math." Secondly, paper assignments allow instructors to ask students to prove a statement, to demonstrate their problem solving skills, to graph a function, or to give an example or a counterexample for a particular mathematical phenomenon. An added benefit of paper assignments is that they give teaching assistants an opportunity to practice marking students' papers.

In our experience the most valuable feature of online assignments is that the medium allows the instructor to reach each student in the class at their will. For example, an online assignment that is due before the *second* lecture in the semester makes students do the coursework from day *one*. In our classes we use short multiple online assignments that are often closely related to class lectures and definitions, theorems, and examples from the textbook. Many of these problems were offered as multiple choice, true or false, and matching questions rather than in the fashion of back-of-the-chapter-exercises. Assignments are due before the next lecture and in this way students have to go through the course material after the lecture in which the material is covered. In a survey that was conducted in a Calculus I class in Fall 2006 semester 230 students were asked to respond to the statement, "To complete online assignments I had to read the textbook and lecture notes regularly." Only 3% of students strongly disagreed with the statement and 83% of the surveyed students agreed or strongly agreed with it.

We also use online assignments as a drillmaster. For example, as an assignment question we post a dozen limits divided in groups of three with one submission per group. We anticipate mistakes that students might make and we create hints accordingly. Multiple tries were allowed.

Research shows that online assignments help students manage their time better. See, for example, [D07]. Our experience is that online assignments also might cause substantial frustration for some students. In our surveys, students list as sources of frustration: entering answers in an acceptable form, internet malfunction, lack of hints, ambiguous wording, several multiple choice questions with a single submission, programming bugs in questions, and misuse of the discussion board. Regardless of these occasional complaints, it seems that the majority of students appreciate online assignments. In our Fall 2006 survey 82% of students agreed or strongly agreed with the statement "Online assignments helped me to learn the course material better."

As a conclusion we finish this note with a comment by an anonymous student from Calculus I class in Fall 2006, "I believe that the short online assignments definitely changed my homework experience. It forces the user to read the directions and follow them to the letter. The unequivocally ruthless marker (the computer) definitely enhanced my ability to be meticulous."

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Understanding Quadratic Functions Using Real World Problems and IT

Nakhshin A. Karim, Bsc, Msc
Department of Mathematics and Statistics
Zayed University
Abu Dhabi, United Arab Emirates
Nakhshin_karim@zu.ac.ae

Abstract

The concept of function is crucial to a great extent in modern mathematics and is considered a major barrier to many mathematics students. Students have difficulty interpreting information related to functions in general, and quadratic functions in particular. Quadratic Function is one of the topics which are covered in a course which is compulsory for a large number of students in the General Education Program of Zayed University. This program leads to different majors, including Mathematics Education, Business, Information Technology, and other majors.

The challenge in teaching Quadratic Function in a course like this is mostly based on the fact that many students think that Quadratic Function is a difficult topic to understand and learn, and some teachers would agree with them that it is difficult to teach.

In this paper, I demonstrate real world problems aimed to improve the students understanding of Quadratic Functions; life problems on this topic support developing student's knowledge, critical thinking, quantitative reasoning, and analytical skills. This paper also includes examples of the techniques used with graphing of quadratic function, the algebra, and inverses of the same function.

International move to improve mathematics curriculum have supported new goals for student's learning which highlights problem solving skills, reasoning, ability to work in groups and individually, and use of technology. Knowing that information technology plays considerable role in achieving the above goals, teaching students the concept of Quadratic Functions can be smoothly achieved by using Information Technology in solving real world problems.

Introduction

In this paper I will explore, present and discuss the syllabus and pedagogy of COL 111, Mathematical Modeling with Functions course, which is a course in the Colloquy for Integrated Learning at Zayed University. The emphasis in this course is on applications of Quantitative Reasoning in the context of real-world problems; the main objectives of the course are to provide students with material appreciate the role and importance of quantitative reasoning plays in the world; also to provide students with an appreciation of the nature and value of mathematics and to enhance critical thinking skills.

To achieve above aims and one of the objectives of this course is to introduce students to the concept of mathematical functions using real-world problems and there mathematical models.

This delivers students with real representation of the application of mathematical function to inspire and prompt their understanding and learning.

The syllabus includes Linear Functions, Exponential Functions, and Quadratic Functions.

I will take Quadratic Functions to show different methods of presentation and solution of functions. One way is by providing the students with real-world applications and from there move to theory, solutions, and interpretation of solution, teach students algebraic manipulation like factoring, changing from a general form to the vertex form of quadratic functions and vice-versa. The other way is by introducing the real-world problem, and use software to graph the function, from the graph explain and answer questions related to the problem, also use the same software to find solutions of the function.

COL 111 (Mathematical Modeling with Functions)

This course is designed to provide students with a broad general education in quantitative reasoning and critical thinking. It is also provide a foundation for the development of their ability to function competently and confidently in majors' programs. The focus of the course is on analytical reasoning and thinking to solve real world problems in business, finance, economics, computer science, education and the natural sciences.

The content of the course is delivered through classroom activities to introduce the students to the various topics. For some topics or case studies, data can be obtained from primary sources connected with other courses, such as Environmental Science, and Health Science. In each area, knowledge, analytical skills, critical thinking and understanding is developed using relevant examples for discussion, analysis and interpretation in class with follow up exercises or assignments of a similar nature to be done individually or in groups outside the classroom.

The content of the course is summarized in 5 Units:

Unit 1: In this unit we introduce mathematical modeling, we allow the students to think mathematically, and use technology to show some elementary mathematical modeling.
Unit 2: This unit is on Connecting data, graphs and functions, look for data relationships, using data to check rates of change, to look for trends, increasing, decreasing, both or neither, look for a patterns and relationships.
Unit 3:Linear Functions, Rate of Change of a Function, Function Notation for the Rate of Change, A General Formula for the Family of Linear Functions, Formulas for Linear Functions, Alternative Form for the equation of a Straight Line.
Unit 4: Exponential Functions, more details of Growth Factors and percentage Growth Rates, General Formula for the Family of Exponential Functions, Relation to Compound Interest function, Comparing Linear and Exponential Functions.
Unit 5: Quadratic Functions, more on Explorations in Quadratic Functions, The Equation of Quadratic Function in General Form, The Equation of Quadratic Function in Standard Form, Changing from Standard Form to General Form, Changing from General Form to Standard Form, and Zeros and y-intercept of Quadratic Functions.

Quadratic Functions

I would start this important topic of the course, likewise any other topic by introducing a real world problem which can showcase the Quadratic Functions. Through the problem I can explain, demonstrate and explore the basic properties of this function, also sketch the graph of the function which is great help for students to understand all aspects of Quadratic Functions and its properties.
A quadratic Function is a polynomial of degree two ($f(x) = ax^2+bx+c$) where a, b, and c are real numbers with $a \neq 0$; its graph in " U " shape curve that is called a parabola.
All parabolas have an axis of symmetry called axis and the point where the axis intersect with the parabola is called the vertex of the parabola or the function.
The Quadratic Function $f(x) = ax^2+bx+c$, has an extreme point (-b/2a, f(-b/2a)), when a > 0, the extreme point is a minimum point and the parabola opens upward. If a < 0, the extreme point is maximum and parabola open downward.
Also, an important point to be raised is the domain and rang of the function, the domain of the polynomial functions and quadratic function is one of them is R, while the range can be calculated as follows:

$Y = f(\frac{-b}{2a})$ which is either the highest point(Maximum point) or the lowest point (Minimum point)

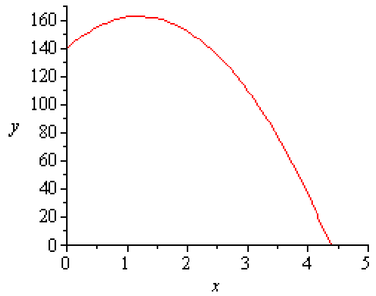
If a > 0 the range is [y min , infinity)
If a < 0 the range is (- infinity , y max).
Side by side with introducing the concept via world problem, using a soft ware to graph and solve the quadratic function is very important and make the understanding of this topic much easier. In this course I am using MAPLE as a tool.

<u>Example Of Application on Quadratic Functions</u>

A boy stand at a top of a 135 Foot high building, and throw a stone upword with an initial velocity of 38 feet per second , the stone travel up ward for a while then eventually be pulled by gravity down to the ground. The height of the stone above the ground is given by the function,

$F(t) = -16t^2 + 38t + 140$

Although the path of the stone is up and down, the graph of its height as a function of time is a concave down parabola, as per the graph below:



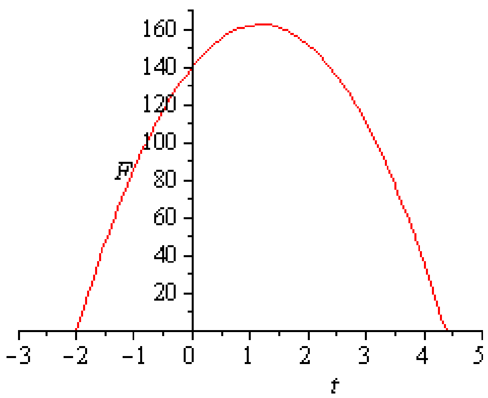
The leading coefficient **a** in the quadratic function indicates the orientation of the parabola. When **a** is negative, to be concave down, while when **a** is positive the parabola concave up.

With this problem and its graph, properties of quadratic function which was mentioned earlier can be explained, for that I use Maple to illustrate important points on the graph:

$$F := t \to -16t^2 + 38t + 140$$

$$t \to -16\,t^2 + 38t + 140$$

plot(F(t),t = -3..5, F = 0..170)



Looking at the graph would gives us a chance to explore the idea of y-intercept, x-intercept (zeros), axis of symmetry, and max or min approximately. Finding exact values can be done via answering few questions related to the problem.

How high is the boy from the ground when he throw the stone from the top of the building. Here, I explore the idea of independent and dependent variables, the time it takes the stone to reach the ground is independent, and the height of the stone dependent, as it depends on the time.

The height of the boy from the ground is as the height of the stone when it was in his hand before he throws it away, or at the time of zero second. In quadratic function its the Y-Intercept, on the graph it is the point where the curve cuts the y-axis. To find y-Interceot in Maple, use f(0):

F(0) 140

The height of the stone from the ground is 140 feet. To find how tall is the boy?

The boy is 140 – 135 = 5 feet tall.

Another question could be asked, when the stone hits the ground?

When the stone hits the ground, means the high of the stone is zero, or value of F(t) is 0, what will be the value of t which gives y zero?

X-Intercept or zeros of the same function in the problem is representing the point at which the stone hits the ground. On the graph zeros are points where the curve cuts the x-axis. `

Solve the equation for the value of t:

fsolve (F(t),{t})

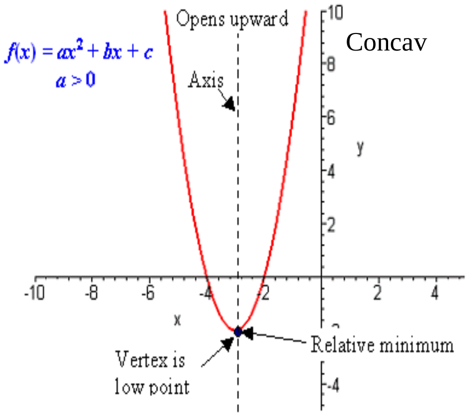
{t = -2.0000000000} , {t = 4.375000000}

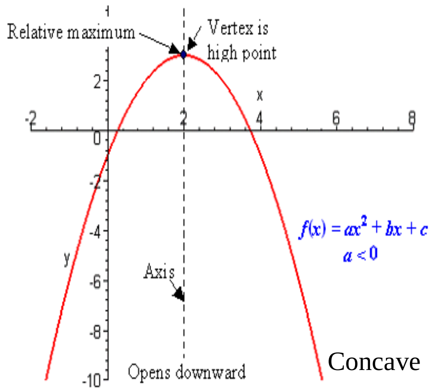
The two values are the roots of the quadratic function, in this problem we consider the positive value and ignore the negative value. So after 4.4 seconds the stone hit the ground.
Using the same command to find after how many seconds does the stone reach different heights. Example, when the stone will be at the height of 145 feet?

fsolve (F(t)= 145, {t})

{t = 0.1398090866}, {t= 2.235190913}

After 0.14 second on the way up, then on the way down at 2.23 seconds the stone will be at the height of 145 feet.
One of the important points on the graph of parabola is the extreme points(Maximum or Minimum).
When the stone will be at the highest point and how high?
This gives a good opportunity to explain different forms of quadratic function, in this problem the function is given in general form f(x) = ax^2+bx+c, change to standard form (vertex form) to find the max or min point, also a chance to illustrate how do we know if the graph of the function has a maximum point or a minimum point; again the value of **a** indicate (when **a** +ve) the parabola has a minimum point and (when the value of **a** -ve) the parabola has a maximum point.





Changing from general form to standard form, can be done algebraically, in this course I use Maple, as my aim is to have better chance for interpretation of those important points and their relation to our life.
with(student):
completesquare(F(t))

$$-16(t-\frac{19}{16})^2+\frac{2601}{16}$$

Compare between the approximate coordinate of the maximum point on the graph with the exact coordinate which you find from the standard form $(\frac{19}{16},\frac{2601}{16})$. Also what does the values in the order pair of maximum point means in respect to the problem we have, or in any other quadratic function.
When the quadratic function is expressed in the form y = f(x – h) + k we see that the graph is in the standard form ; has been shifted from the origin with the coordinates of the vertex being (0, 0) to a position with the vertex at (h, k).
In this form we can state the vertex immediately; it has coordinates (*h*, *k*).
The standard function is shifted *h* to the right and *k* upwards.

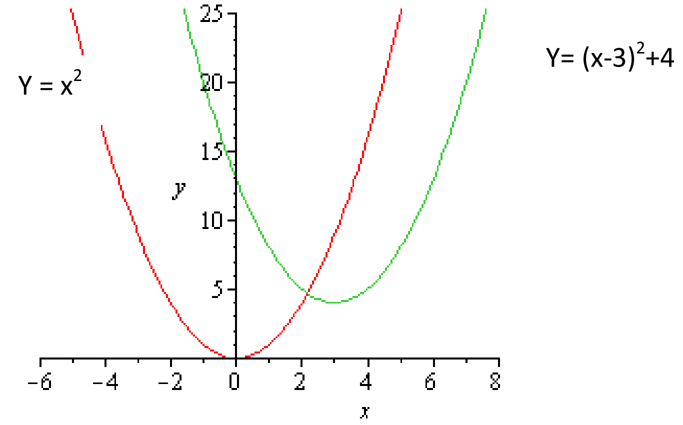
f:= $x \rightarrow x^2$

$$\boldsymbol{x \rightarrow x^2}$$

f1:= $x \rightarrow (x-3)^2 + 4$

$$\boldsymbol{x \rightarrow (x-3)^2 + 4}$$

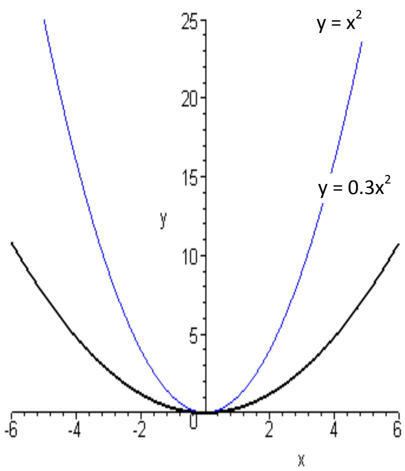
plot ([f(*x*), f1*(x)*] , *x* = - 6..8, y = 0..25)

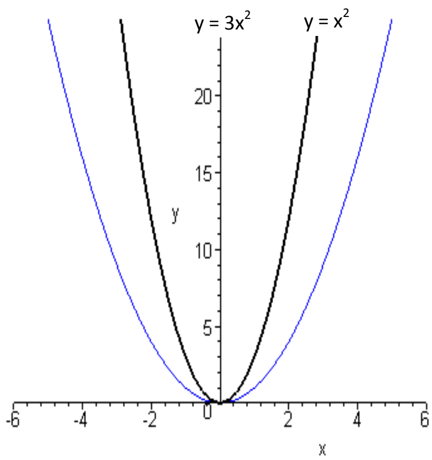


The stretch factor, *a*, affects the flatness of the graph of a function.

For example:

If $0 \le a \le 1$, the graph of a function is flattened
If $a \ge 1$ the graph of a function is squeezed upwards.





Effects/Conclusion

No doubt real life problem solving helps in making mathematical concepts more understandable; in order to successfully achieve this, through the above simple applications I was trying to show students how the quadratic functions can be used to find answers to questions that usually come across their minds and form barrier to their understanding, give them answers that are tangible, can be felt, and sense the real benefit of using such mathematical tool.

Thus, with the above example, along with the use of Maple as a tool to answer many questions in the application and others related to inverse of the same function , the approach to develop students critical thinking, quantitative reasoning, and knowledge has effect and cause the objectives to be achieved.

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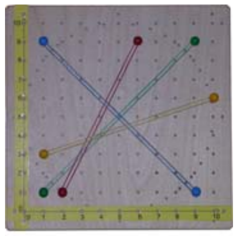
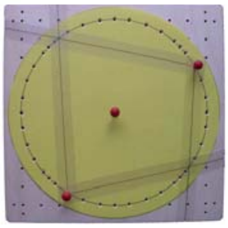
Individual Approaches in Rich Learning Situations
Material-based Learning with Pinboards

Michael Katzenbach
Institute for Educational Progress, Humboldt - University, Berlin
Michael.Katzenbach@IQB.hu-berlin.de, katzenbach@germanynet.de

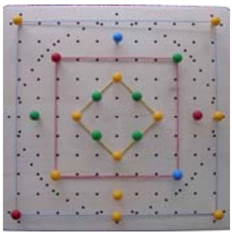
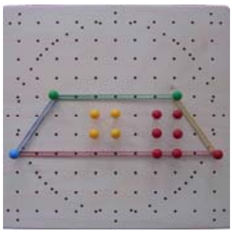
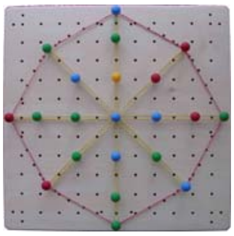
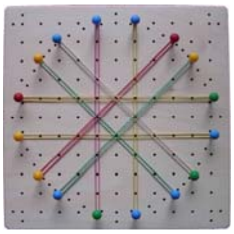
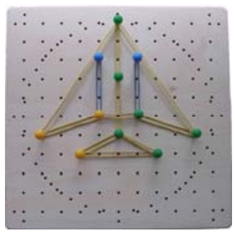
Abstract
Active approaches provide chances for individual, comprehension-oriented learning and can facilitate the acquirement of general mathematical competencies.
Using the example of pinboards, which were developed for different areas of the secondary level, workshop participants experience, discuss and further develop learning tasks, which can be used for free activities, for material based concept formation, for coping with heterogeneity, for intelligent exercises, as tool for the presentation of students' work and as basis for games. The material also allows some continuous movements and can thus prepare an insideful usage of dynamic geometry programs.
Central part of the workshop is a work-sharing group work with learning tasks for grades 5 to 8. The workshop will close with a discussion of general aspects of material-based learning.





Introduction

Imagine a classroom situation in 5th grade: Students work with a pinboard for the first time. Their task is nothing else but developing a figure they like, describing their process and explaining the meaning of their figure in the class using the pinboards for presentation. These are typical figures students have developed in such learning situations:

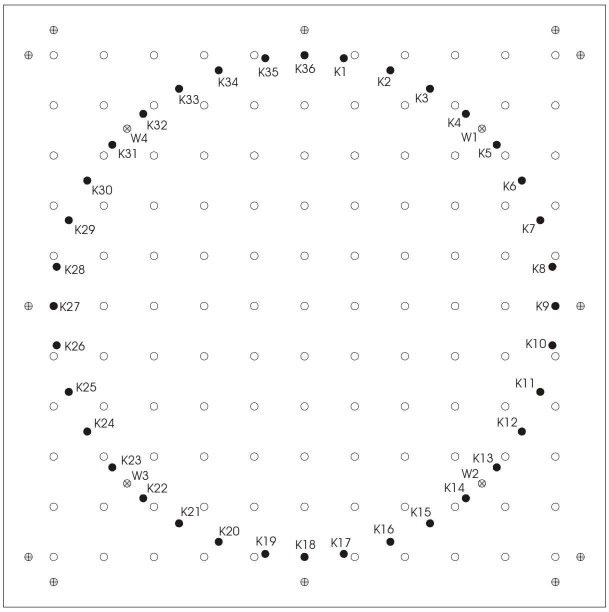
By working with the material and by presenting all figures to the class, the students discover structural elements of the pinboards: While some students mainly use grid points, other students also include some of the 36 points on a circle line in their figures. Students also use the pinboard to present something personal such as making model air planes. The presentation does not need any preparation like drawing the figure on an overhead transparency. The way students are presenting their figure and talking about figures of other students has a diagnostical value especially if the lesson is a beginning situation in a new learning group.
Students discover basic figures or symmetries. Based on the variety of the figures properties of special figures can be determined, patterns can be discovered and the different structural elements of the pinboard can be assembled..

The Material

The pinboards are made of birchwood.

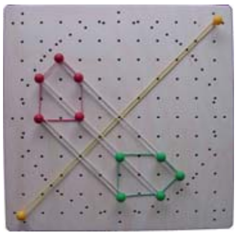
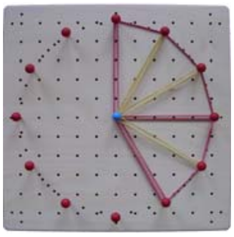
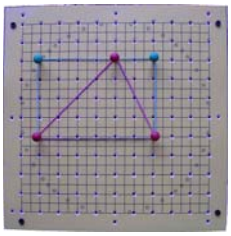
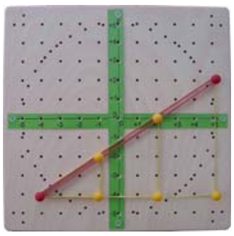
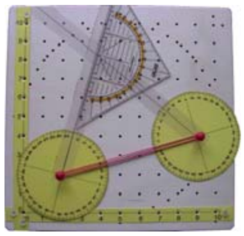
A pinboard contains

- 11 x 11 gridpoints (o) in the vertical and horizontal distance of 2 cm,
- 36 six points (●, K1 to K 36) on a circle line with the diameter 10 cm,
- 4 points (W1 to W4) on the circle line and on the bisecting line,
- 12 points outside the grid, which can be used to fix coordinate axes and other additional material.



A class set of pinboards (typically one board for two students), pins in five colors and rubberbands are stored in a wooden box, which contains the same pattern of holes on each side.

Additives like coordinate axes, lines, discs to measure angles can be produced by the students. The necessity to work accurately is self-evident. Thus the material can be used for application areas in geometry, functions, fractions and others.

Example 1: Material based introduction of a new concept:

Obtaining an agreement to a coordinate system

One wing of a blackboard is moved forward to make sure the two students on either side cannot see the pinboard of the other one. The class is watching. The initial question is:

Imagine you have found a figure on the pinboard you really like. Your friend in another city has the same kind of pinboard. You want to describe the figure for him by phone, so that he also can build the figure. Let us test how it could work with a figure of about 6 corners.

While several students are trying to communicate the position of the pins, the class is realizing successful and less successful strategies. The student, for example, who has one pin in the middle of the first row of the pinboard, gives the order “Put the Pin in the middle of the board.”, which makes the other student put it in the intersection of the diagonals. The audience realizes, how wrong positions are perpuated, as a student continues to describe the position of one pin related to the position of the previous pin. Reflecting on the observations, the students described reasons for problems in communications and criteria for successful strategies. Many students used the grid structure, for example “two steps to the left, three steps above”. It becomes evident, that a point of reference has to be agreed upon. So it is decided that the left grip point of the lowest row is to become the point of reference. Now it makes sense to introduce the notation for coordinates as the number of steps, which had to be moved from this point to the right and above. Paper Stripes are fixed on the pinboard and labelled to visualize the new concept. Hanni wrote a text showing that she did not only experience the situation on the cognitive level:

Pinboards
Next time we are playing this game we got to do it directly like Rouven and Pascal. Before you choose a point, at which you have to go back again and again. Because, if you always move from one point to another, you lose quickly and easily. I really like these new stripes for the pinboards. You can always say (4|3) or (9|10), this is is easier. I enjoy the funny game, for instance if something different happens as it should.

After the agreement on the coordinate system, the students draw the axes on paper stripes, label them and hammer holes in the stripes. Very often the first attempt does not bring the necessary result. The stripes cannot be fixed, because the holes in the stripes are in a wrong position. Thus the students have an evident self-control. The necessity to measure and draw precisely is here part of material based work.

Example 2: Intelligent Exercise
From a square to kites

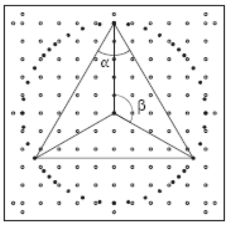
If you move one corner of the maximum square on the pinboard towards the opposite corner along the diagonal, symmetric quadrangles are generated. So a structured series of tasks evolves.
Which percentage of the area of the square has an emerging quadrangle?
Beginning with the first quadrangle, students find different approaches:
- They use the pinboard and a sheet with grid lines (100 small squares) and count the small squares.
- They dissect the square, determine the area of segments and subtract these from the area of the square.
- They calculate areas of triangles and quadrangles by formulas.

At least after determining the first two results, 90 % and 80 %, students presume that by moving the corner one more step the area decreases by 10 % of the area of the intial square. Now the question is, if this presumption is plausible or can be confirmed for other quadrangles in the series. If a result does not fit to the series of other results, a check of the procedure by the students makes sense.
Further problems help to leave the material base:
Imagine that in the middle between two holes in the diagonal is another hole. How does the series of results look like?
Imagine the pinboard would be continued at the top and at the right. Determine the percentage of the area of the quadrangle emerging, when you move the upper right corner one step on the diagonal to the right. Solving this question by an appropriate dissection of drawing is a challenge. Regarding the area of the quadrangles as a function of the length of the diagonal helps to get a general solution.

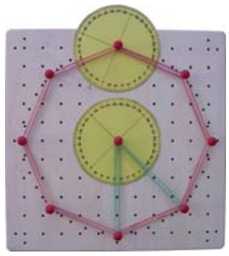
Characterisitics of this learning environment are:
- A low barrier at the beginning facilitates an access to the problem for every student.
- Different approaches, especially hands on activities for the students.
  Students individually decide if they use the material.
- By using the sheet with a hundred small squares, basis concept of percentages are enhanced.
- Self-control is possible because the results belong to a structured series.
- The limitation of the material and appropriate problems facilitate a transfer from the material based work to mental work.
- The structured series of problems contributes to building a concept of functions.

Example 3: Experimental approach – vertical integration:
The relation between the interior Angles and the central angles of regular polygons

"Investigate the relation between the interior angle α of a regular polygon and the central angle β of a partial triangle."
The pinboards allow one to measure the angles in the following regular polygons: triangle, square, hexagon, octagon, nonagon and dodecagon. Here is a typical result of the investigation:

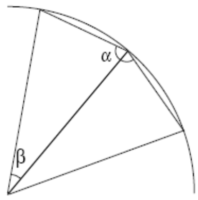


Number of Corners	3	4	5	6	7	8	9	10	11	12
Interior angle α	61	90		119		133	142			150
Central angle β	121	90		60		44	41			31



After the data collection from all student groups first presumptions emerge:
1) The larger the number of corners the larger the interior angle.
2) If the number of corners doubles, the interior angle also doubles, example 3 → 6, 61 → 119
3) The larger the number of corners the smaller the central angle.
4) The sum of the interior angle und the central angle seems to have almost the same value for different regular polygons.

A graph is a next approach to the subject showing for example that presumptions 2) is false.
When angles are measured in a certain context, a diagram of asymptotic curves occurs. There has to be a maximum of an interior angle and a minimum of the central angle because of the geometric properties. There are further presumptions concerning the missing values. The symmetry of the values can be discovered.
The determination of the missing data by the students drawing the polygons on paper is suggested, which brings the next challenge: How can one draw a regular pentagon? If students don't find an approach through calculating the interior angle, matches or sketches help to develop an imagination of the properties of a regular pentagon. By calculating the central angle it is possible to draw the missing regular polygons.



Yet the measurement is still imprecise. How is it possible to prove presumption 4) and determine the exact value. The pinboard may help to develop a figure for the proof.
As in the learning environment in example 2 all the students find an approach by hands-on activities and measuring angles. After some time some students will still measure angles whereas others will have proceeded to further tasks depending on the individual progress or on the progress of the work group. In contrary to dynamic geometry programms the imprecise measurement and the limitation given by the material is a chance to understand why a general argumentation is needed to prove a general proposition.

The Workshop

After a short introduction to present impressions of classroom situations, the workshop participants will have the chance to experience the material and the learning tasks working on 10 stations like the examples two or three in this article. An exchange of the experiences will prepare a general discussion on topics like the following:

- Does the learning situation provide an access to the subject for every student in the learning group?
- Can the students work on the learning task on different levels?
- Is it possible, to discover mathematical structures or patterns?
- How can students be supported in the transfer from material based learning to mental work?
- How can students document working- and learning processes?
- Which chances are given for individual products of students?
- How can material based learning activities be combined with the aquirement of general mathematical competencies?

Does the parameter represent a fundamental concept of linear algebra?

Stefan-Harald Kaufmann
Scientific Assistant, Seminar für Mathematik und ihre Didaktik, University of Cologne
skaufma1@uni-koeln.de

Abstract

In mathematics the parameter is used as a special kind of a variable. The classification of the terms "variable" and "parameter" is often done by intuition and changes due to different situations and needs. The history of mathematics shows that these two terms represent the same abstract object in mathematics. In today´s mathematics, compared to variables, the parameter is declared as an unknown constant measure. This interpretation of parameters can be used in set theory for describing sets with an infinite number of elements. Due to this perspective the structure of vector spaces can be developed as a special structured set theory. Further, the concept of parameters can be seen as a model for developing mathematics education in linear algebra.

Introduction

What is a parameter? In mathematics this question is often answered by: *A parameter is almost the same object as a variable.* Due to this perspective it is characterized as a special kind of a variable. Consequently, the property of this variable depends on the way of application. For mathematicians "parameter" and "variable" represent only different names given to the same object. There is no need for an exact definition of the object "parameter", because the use of the word "parameter" is acquired by individual mathematical experience, implying a mathematician classifies a placeholder as variable or parameter by intuition.

This problem was transferred to school mathematics by adapting mathematical structures and their concepts exactly into the curriculum of mathematics education. Especially, emphasizing the fact of the absence of an accurate definition neither of the word nor of the object parameter. Consequently students are unable to distinguish "parameter" and "variable". They use the parameter it in the same way like a variable, but they are convinced that there must be a difference between these two notions, which they can't explain[1].

Due to the different expressions, the question arises, if a model or a concept can be developed focusing on the idea of the parameter as an inferior kind of variable. This model must define or explain the use of parameters without distorting the general application. One way of realizing this complex task can be found in linear algebra.

The parameter and the description of sets

The fundamental concept of the following model is the classification of the parameter as an inferior variable by introducing the parameter as an auxiliary variable to describe all elements of a set. Due to the above, the parameter has found a somewhat exact definition. On the one hand, this declaration doesn't capture the entire meaning and use of this special term. But on the other hand students don't get misleading impressions which have to be modified later. This reduction represents only one possible perspective, how teachers can explain the complex matter. The point is that this perspective allows already a transmission to all mathematic applications employing parameters. Finally a comprehensible introduction of a parameter can't be given without some limitations.

The motivation is justified in enabling students to describe sets with an unlimited number of elements. For example: The linear equation

$$4x-2y=-6$$

features an infinite number of possible solutions. By the method of trial and error students realize and accept very fast that it isn't possible to note all solutions in an explicit way. A parameter, for example λ, can be used to capture all solutions:

$$L=\left\{(x,y)\in R^2 \mid (x,y)=(\lambda,2\lambda+3),\lambda\in R\right\}$$

The parameter leads to a description of all solutions. The special function of the parameter λ can be pointed out by the dependence of the components *x* and *y* which belong to any solution of the equation. For any figure chosen as λ a certain result can be found.

[1] Preliminary results of video-interviews recorded at the University of Cologne in March 2009.

Finally the description of infinite sets represents a very abstract mode to motivate the introduction of the expression “parameter” for specially used kind of variables. Once this part is understood by students, the most difficult part of the entire model and its application has been achieved.

The parameter as a fundamental concept of linear algebra

The parameter is introduced as auxiliary variable supporting the capture of infinite sets. In linear algebra, this view can support the construction of a vector-space-structure as a special set structure. In this case the parameter-introduction is realized with systems of linear equations. For example the system

$$\begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & -1 \\ 2 & 1 & -4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 8 \\ -3 \end{pmatrix}$$

can be transformed to

$$\begin{pmatrix} 2 & 1 & 3 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 14 \\ 0 \end{pmatrix}.$$

The above representation of the origin system shows that it has more than one solution. Now, you’re interested in capturing all solutions as easy as possible. The parameter in the function of an auxiliary variable offers a model to describe all components *(x,y,z)* of all solutions:

$$L = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3 \middle| \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \lambda \in R \right\}$$

In this way the parameter represents a tool capturing finitely an infinite set in a special way. The parametrization of the solution set distinguishes already the structure of vector-spaces. By generalizing the idea of parametrization, whole vector-spaces or sub-vector-spaces can be described.

The model of set parametrization simplifies comprehensively the structures of adjoining mathematics disciplines like analytic geometry. Due to the perspective of simplification by parametrization, the concept of planes and lines as point sets will be understood quicker than without set-parametrization. Especially the link between the intersection of planes and lines on the one hand and the solution set of systems of linear equations on the other hand can point out the meta-structure and the correlation of mathematic disciplines. Consequently the parameter-concept represents a vivid part in contributing to experience the elementary structure of vector-spaces as well as its relevance for other mathematics disciplines.

The generalization of the parameter-concept in mathematics education

The introduction of parameters in linear algebra, as shown above, reverts to the description of infinite sets by parametrization. Given that set theory represents the basement of mathematics, the model capturing sets by parametrization can be transferred to all mathematic disciplines in which explicit descriptions of sets are needed.

In calculus functions which are reliant on an additional parameter, for example $f_k(x) = k \cdot x^2$, represent a famous example in school mathematics. Here the transmission is done by considering a whole class or set of functions. Concerning the mentioned example of $f_k(x) = k \cdot x^2$ the graphs of f_k represent a set of parabolas which all proceed the point of origin. The realization of a whole set of functions and graphs respectively is done by the parametrization.

Due to the perspective of an elementary concept of linear algebra, the parameter represents a special kind of variable which enables the capture of infinite sets, in this case especially the capture of vector-spaces and sub-vector-spaces. Otherwise the parameter can be pointed out as fundamental object of mathematics, because of the importance of sets. Of course, this definition of parameters doesn’t represent exactly what many mathematicians dealing with parameters figure or imagine. The main intention of this model is to showvan easy way to explain the difference between the two expressions “parameter” and “variable” of placeholders in school mathematics.

Using the Media as a Means to Develop Students' Statistical Concepts

Marian Kemp
Head of the Student Learning Centre, Murdoch University
m.kemp@murdoch.edu.au

Abstract
In this era of increasingly fast communication people are being exposed to quantitative information, from national and international sources, through a range of media including newspapers, magazines, television, radio, pod-casts, *YouTube* and other areas of the Internet. Contexts include health statistics, environmental issues, traffic statistics, wars, gun laws and so on. It is becoming more and more important that citizens are able to critically read and interpret this information, and to do so requires an understanding of statistical concepts. Research has shown that students are motivated and engaged in learning through the use of authentic, real life tasks. The media provides current information, which can be used to help develop both students' awareness of how social issues are constructed as well as vital statistical concepts. This paper proposes that secondary school students' application of a model for statistical analysis to material taken from media sources, enhances their understanding of statistical concepts. This model, called the *Five Step Framework*, is described and exemplified for the particular context of opinion polling.

Introduction

Electronic and digital communication across the globe has brought with it more opportunities for transmission of information and the need for people to be statistically literate; to be able to make sense of the information presented to them in a variety of ways and formats. This paper will concentrate on the development of statistical concepts through the analysis of data from media sources. As Ben-Zvi and Garfield (2004) point out:

> Quantitative information is everywhere, and statistics are increasingly presented as a way to add credibility to advertisements, arguments or advice. Being able to properly evaluate evidence (data) and claims based on data is an important skill that all students should learn as part of their educational programs. The study of statistics provides tools that informed citizens need in order to react intelligently to quantitative information in the world around them. Yet many research studies indicate that adults in mainstream society cannot think statistically about important issues that affect their lives (p.3).

Given the indications that citizens are often unable to engage with the quantitative information around them, it is strongly suggested here that students be exposed to the interpretation and analysis of information from the real world, presented via the media while still at school. Interpreting material in the media provides a number of benefits to the students. Firstly, it provides authentic tasks for which students have to interpret the statistical language, deal with incomplete information, develop problem-solving strategies, do some research and make decisions for themselves. These authentic tasks, as advocated by Zevenbergen (1997), can engage and motivate students in a variety of contexts.

The literature indicates some agreement about the statistical concepts and skills that students should develop during their years at school. These include (i) developing appropriate research questions, (ii) collecting, organising and displaying data, (iii) selecting appropriate analysing tools, (iv) understanding probability in the context of uncertainty, (v) making inferences and predictions (e.g., Watson, 2006; Ben-Zvi & Garfield, 2004). There are pedagogically sound activities that can be undertaken in the classroom (e.g., Curriculum Council, 2009; Watson, 2004) to enable students to develop statistical concepts and skills. The author stresses here the importance of the integration of examples from the media in the process of students developing their ability to critically evaluate data in the media used in 'advertising, arguments or advice'. Given the space constraints, the paper will focus on the interpretation of data from opinion polls that are really prevalent in the media and usually reported in tabular form.

Description of the model and its adaptation to opinion polling

This paper proposes the use of a specific, tried and tested, model that appreciates people's need to interpret the statistical concepts and language that they meet in their everyday lives. It suggests that exposure to data from a range of media outlets can help students develop their statistical concepts and at the same time helps students to learn how to critically evaluate the information they are analysing, which is recognized as especially important (Steen, 1997).

This model is a *Five Step Framework*, designed to interpret tables and graphs (Kemp, 2005). The early development of the model, shown in Table 1 below, was reported in Kemp (2003). The framework was designed and refined to cater for the needs of university students in their study where they need to interpret data provided in academic journals, scientific reports, textbooks, lectures, media articles and other materials. The framework has been used successfully in workshops for science students in a first year statistics unit (Kemp & Bradley, 2006) and with pre-service primary teachers (Kemp, 2005). Pre- and post-tests before and after workshops for both sets of students showed that the strategies built through the *Five Step Framework* significantly increased students' ability to interpret tables of data. The framework has been successfully used with other students in Foundation Units and Alternative Entry Programs at Murdoch University, but the effectiveness has not been formally evaluated and reported.

Table 1. Five Step Framework (Kemp, 2005).

Step 1: Getting started Look at the title, axes, headings, legend, footnotes and source to find out the context and expected reliability of the data.
Step 2: WHAT do the numbers mean? Make sure you know what all the numbers (percentages, ‘000s etc) represent. Look for the largest and smallest values in one or more categories or years to get an idea of the range of the data.
Step 3: HOW do they change or differ? Look at the differences in the values of the data in a single data set, a row, column or part of a graph. Repeat this for other data sets. This may involve changes over time, or comparisons within categories, such as male and female, at any given time.
Step 4: WHERE are the differences? What are the relationships in the table or graph? Use your findings from Step 3 to help you make comparisons between columns or rows in a table or parts of a graph to look for similarities and differences.
Step 5: WHY do they change? Look for possible reasons for the relationships in the data you have found by considering societal, environmental and economic factors. Think about sudden or unexpected changes in terms of state, national and international policies or major events.

This paper focuses, however, on the education of secondary students and the integration of the use of media materials into their curriculum. It is really important that students appreciate that, especially in opinion polling, a critical element of reliability of the data concerns potential bias, for which there are at least three different possible sources.

The first way in which an opinion poll can be biased is directly related to the question(s) asked. These can be leading questions that can persuade the interviewee to give an answer in accord with the interviewer's point of view. For example, the question "Do you believe that our boys and girls should be withdrawn from Iraq?" poses a different question from "Do you think that Australia has a moral obligation to protect the oppressed in Iraq?" Even though both questions relate to whether people think that Australian troops should stay in Iraq, or be withdrawn, they are likely to elicit quite different responses.

Secondly, for the data from any poll to be valid, the sample needs to be a random representative sample. In this respect there is the need for samples to be taken at random, (where random sampling is a method of selecting a sample in which all possible subjects (or scores) in the population have an equal chance of being selected), and all possible samples have an equal chance of being selected. This can be done through using random number generators of some kind or putting names in a proverbial hat. By this mechanism of random sampling we hope that the sample will be unbiased and that the sample will include the same characteristics as the population. However, as Baldi and Moore (2009) point out "[r]andom samples eliminate bias from the act of choosing a sample, but they can still be wrong because of the variability that results when we choose at random (p. 220)". Thus, it can happen that when a population has a very wide variability random samples can give quite different results.

Thirdly, the size of the sample also contributes to the credibility of the results. Essentially, the sample size must be determined to give a sufficiently small standard error. Noting Baldi and Moore’s (2009) comments above, account should be taken of the variability of the different groups within the population to increase the chance that the range of opinions is captured and the sample is not biased by under- or over-representation of particular subgroups.

Therefore, it follows that in analysing opinion poll data it is essential that students explicitly address the three questions below:

1. What was the question asked?

(What did the researcher want to know? Was the question biased?)

2. What was the method of sampling and data collection?

(Was it a random sample? How was the sample collected and collated?)

3. What was the sample size?

(How many respondents are in the sample? How would this affect the variability of the results?)

These questions are appropriately located in Step 1 of the *Five Step Framework*, expanding the question about reliability referred to in Table 1. Step 1 in Table 3 below includes suggested sequences of questions to help students explore the data. Different levels of students' age and level of background should dictate the level of complexity of the data they are analysing. For simpler opinion polls and less sophisticated students it is probably more appropriate to concentrate on Steps 1 to 3. It is expected that for older students all five steps can be accomplished.

To illustrate the use of this model, consider an opinion poll conducted recently in Western Australia concerning daylight saving, This has been a topic of continued debate in the West Australian community over the last three years of a trial of daylight saving. The debate has included a wide range of perspectives including those of farmers, businessmen, surfers and families with young children. Table 2 represents a table of data published in *The West Australian* daily newspaper on 11 April 2009 in an article headed 'On a Knife Edge' (Phillips, 2009).

Table 2. Opinion poll results concerning daylight saving

Westpoll Do you support daylight saving?

	March '07	March '09	April '09
Support	34	42	47
Oppose	62	57	51
Don't know	4	1	2

Westpoll conducted April 6-8 through phone interviews with 400 voters across WA by Patterson Market Research

Table 3, which follows, indicates that Step 1 has been extended to take account of the questions described above. The questions in italics are those that have been routinely asked for any table of data.

As can be seen in Table 3 the five steps are amplified with typical questions that would be asked in the interpretation of any table of data. Once students have developed the strategies needed to interpret a table of data, after using the framework a few times, they can generate their own questions to explore the data. Emphasis has been given to Step 1 to indicate the importance of addressing the three questions described above. For some students the steps that are important to them in this context will be Step 1 to Step 3, while more sophisticated students would be encouraged to complete all five steps. It is vital to establish the links between the concepts and skills that students are learning in the classroom and data they explore in the media. Naturally, students can go to the website of Patterson Market Research (2009), or discuss with other students and their teacher to understand the significance of the potential bias in the sampling.

Applying the *Five Step Framework* to a more complex table

The context of an opinion poll on identity cards provides a much more complex table for analysis. Currently in Australia it is not a requirement for people to have an identity card and a survey was conducted in 2006 to investigate public opinion on this issue. The poll was conducted by *Newspoll* and the results published in the 27-28 January edition of *The Australian* daily newspaper. The table above is a reproduction of the data presented in the newspaper.

In this case the data is far more complex and involves more sophisticated comparisons to extract the meaning. The *Five Step Framework* gives students the structure to develop the strategies for interpreting this table of data. Naturally the interrogation of this table would be done by students who cannot only complete Steps 1 to 3 but also to Step 4 to examine the complex relationships in this table.

Table 3. Application of the Five Step Framework to Table 2 on Daylight Saving

Step 1: Getting started
Q: *From the title, what is the general topic being examined?*
Q: How are the variables being compared?
From the labels on the left column, how are the groups being compared?
From the labels on the top row, how are the groups being compared?
Q: What question was asked?
Do you think the question was biased?
Q: According to the table, over what timeframe was the poll undertaken?
Which data would have been collected at that time?
Are the collection dates evenly spread?
When were the data for March 2007 and March 2009 collected?
Q: *Is there any evidence to suggest that the data is reliable?*
How was the sample selected?
Who selected the sample?
Was it a random selection?
Q: How was the data collected?
How was the phone poll organised?
Were only landline phones included?
How many people in each household were interviewed and how were they chosen within the household?
How did the interviewer check that the interviewee was on the electoral roll?
Q: What was the size of the sample?
Is this is a reasonable number to gain a sampe of appropriate size?
Q: What else would you like to know about the sampling?
Spend some time researching this data collection Market Research Agency.
Step 2: WHAT do the numbers mean?
Q: What is the meaning of the 47 in the first column?
Q: Which year has the highest percentage of people in favour?
Q: Which category has the lowest percentage of people in favour?
Step 3: How do they change?
Q: How does the percentage of people in support change over the three collection times?
Q: How does the percentage of people who oppose change over the three collection times?
Q: How does the percentage of people who don't know change over the three collection times?
Step 4: WHERE are the differences?
Q: Compare the differences for 'in favour' and 'against' for each time frame, look at the relationship between the percentages.
Q: Compare the don't know responses with the support percentages.
Step 5: WHY do they change?
Q: Did any of the values in the table surprise you?
Q: Suggest possible reasons for the differences in the percentages for the responses considering that two of the dates are 3 years apart and the third is only one month later than the second.

Table 4. National identity card

Question: Are you in favour or against the introduction of a national identity card in Australia? If in favour - is that srongly in favour or somewhat in favour? If against-is that strongly against or somewhat against?

January 27-29, 2006

		Sex			**Age**		**Political**	**Support**
Percentage Support	Total	Male	Female	18-34	35-49	50+	Coalition	Labour
Strongly in favour	27	30	23	13	25	37	31	25
Partly in favour	26	24	29	28	26	26	29	27
Total in favour	**53**	**54**	**52**	**41**	**51**	**63**	**60**	**52**
Strongly against	12	11	14	16	12	10	14	11
Partly against	19	23	15	20	23	16	13	23
Total against	**31**	**34**	**29**	**36**	**35**	**26**	**27**	**34**
Uncommitted	16	12	19	23	14	11	13	14

This survey was conducted on January 27-29, 2006 on the telephone by trained interviewers in all states of Australia and in both city and country areas among 1200 people aged 18 years and over. Telephone numbers and the person within the household were selected at random. The data has been weighted to reflect the population distribution. The maximum margin of sampling error in the total sample is plus or minus 3 percentage points. Copyright at all times remains with Newspoll. More poll information is available at www.newspoll.com.au.

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There is a wide range of useful comparisons that can be made across the categories of age, gender and political persuasion and the levels of agreement or disagreement. In order to cope adequately, students need a strong grasp of proportions and the author found that special tutorials were needed for some students to develop those concepts and skills adequately to work with more complex tables. In relevant situations, particularly those involving social issues, the ultimate aim is to enable students to reach and successfully complete thoughtful answers to the kinds of questions posed in Step 5, which involve locating the data in terms of national and international issues.

Conclusion

Through the incorporation of material from the media in the classroom, students' understanding of statistical concepts can be enhanced. It has been observed from experience that without models of the kinds described in this paper, many students will not be able to interpret media reports. Students need support to develop the strategies necessary to do so. This paper has shown how a model can be adapted successfully for this purpose, both for unsophisticated and more complicated tables. It has focused on the particular needs concerned with analysing the data from opinion polls. Emphasis has been placed on consideration of the sample selection and collection. This seems to be best placed in Step 1 for all students. Tables with a simple structure can be tackled by younger students whereas more complex tables need an understanding of proportion and other means of comparison, more suited to more sophisticated students. Thus, not all students will reach Steps 4 and 5 but the aim over time is to enable students to become adept with thinking about relationships between the variables, and ultimately about why the data are as they are and how they relate to society in a range of ways.

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A program for reducing teacher's resistance to changes in curriculum in centralized education systems. An experience on changes of mathematics text books in Iran based on distinction results.

Zohreh Ketabdar, M.Sc Math Education Zohrehke_2002@yahoo.com

Abstract:

Curricula in concentrated educational systems are prepared from an upper-stream reference and hand over teachers' disposal . Curricula in Iran are compiled in the so-called math curriculum development office and then put at the disposal of teachers. The researches in this regard show that such plans provide some resistance against executing it which are named teacher - proof programs, even it changed to some extent for accept ion, its execution is suspeciable. This research first explains how math books of middle grade were changed as a result of investigations on TIMSS result in year 1995. These investigation show Iranian Books are weakness in problem solving methods as Polya had said. And so curricula developers in Iran tried to integrate Polya's method in math books and changed curricula based these frame work.

Then, it shows how teachers treated these changes. The finding of analyzing the data has been collected in this research through observations and interviews. It is intended that teachers resist against these changes. Finally according to this research, we suggest a model which we refer if it is used through concentrated educational system, we could expect teachers tolerance against the changes would be decreased and so compiled curricula further matched to executed curricula.

Key words: changes in curriculum – science – attitude – skill – problem solving – job training during service

Introduction:

Three types of education system could be recognized in different countries.

In centralized education systems curriculums as well as contents of books and methods of teaching is planned at a central governing organization such as a ministry or another government body.

In non-centralized education systems, teachers are free to plan their own method of teaching as long as it remains within a previously agreed upon framework. They even can choose the books and instruction material for teaching

In Iran education is just like other aspects of life centralized and text books are surrogates curriculum and they are prepared by ((The office of programming and compiling text books)) which itself is a subsidiary unit of ((organization for educational planning and research)).

The office of programming and compiling text books is responsible for preparing text as well as teaching plans for primary, secondary and high school levels.

Teachers are obliged to teach these books. Text book preparation is a lengthy process because only a handful of specialists are working on books and because they are responsible for preparation of books for all levels. Mathematics has a permanent presence in every curriculum and because of its importance in proper education these experts have been reviewed and modified math teaching programs. Changes have been made as to the volume, perspective, teaching plan etc. **Key words:** changes in curriculum – science – attitude – skill – problem solving – job training during service

Changes in teaching plans

Three types of teaching system could be recognized here.

A) Loyal execution scheme: In this scheme teacher should proceed exactly as the pre-planned teaching program dictates to him/her and he/she could not and should not exert any changes while he/she is implementing this program. This scheme is also called instructor-resistant scheme.

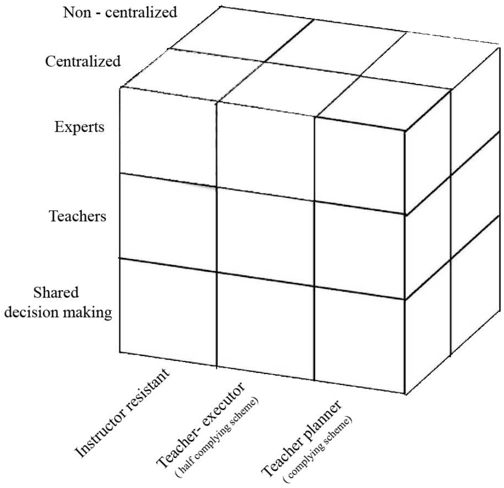
B) Half-complying scheme is a scheme in which the teachers can modify the teaching program in a way that it becomes more compatible to his/her pack of students. This scheme is also called Active Execution Scheme.

C) Complying scheme. Here the teacher not only can modify the plan but can also prepare a new teaching program and compile instruction material which he or she deems fit for his/her class. This is also called planner teacher scheme and here

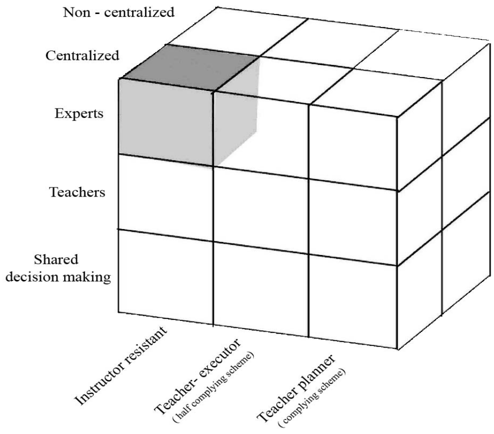
changes in teaching plan is made along with its execution. Teachers react to changes in teaching plans in 4 different ways: they resist, they comply, they co-operate and they lead.
A teaching plan consists of 4 components namely targets, contents, methods of instruction and evaluation. Reforming these four parts seems absolutely imperative and because in Iran text books are considered surrogates for curriculum reforming text books should be considered a very essential task.

Theoretical approaches to the process of educational planning

Educational planning consists of 3 main theoretical components. Experts in the field of educational planning have sought to offer [1]appropriate models for their approaches.



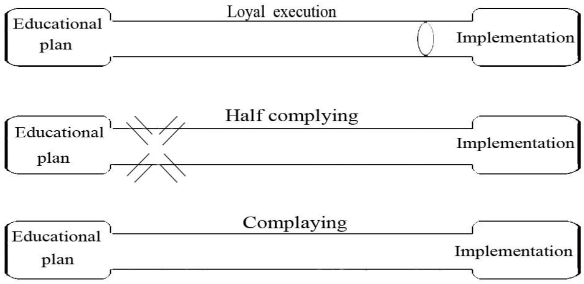
Among them Short has offered a relatively simple model based on his understanding of different theories of development of educational planning. His model is called Cubic Educational Model and in Iran it is usually presented in the following form:

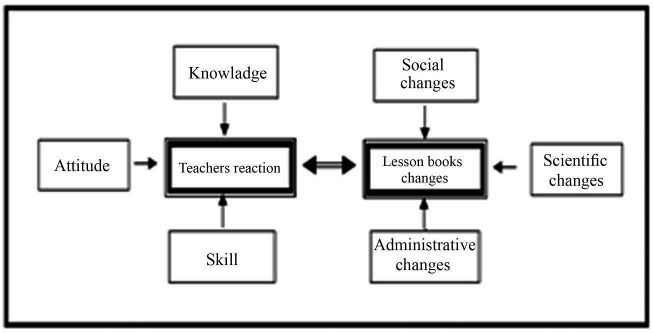


Because of the vertical gap between decision making bodies and execution units in Iran development of a plan which is not sensitive to teachers' attitudes could be easily expected. Recent changes in secondary school math text books are mainly related to implementation of 1) Polya method and 2) super cognitive learning.
New plans call for new requirements and those who execute these new plans should possess the required skills. These skills could be taught in relevant training courses.
A positive attitude toward these new plans is even more important than begin well versed in them. Willingness of those who execute plans will create an appropriate environment for them to improve their knowledge and skill.
Conservative nature of man always compels him to resist change and to oppose new plans and so it is very important to convince him that the new procedure is worth implementing. The following recommendations may be of use in this regard
1- Selecting those who agree with new plans as well as have adequate knowledge of them as implementers.
2- Educating those who are due to take part in implementation of new plans.

1-Mehr Mohammadi.M, Curriculum: Theories, Approaches, and Perspectives, Tarbiat Modarres University, Beh Nashr Publisher. Page 381

3- Involving them in evaluation process and detection of weaknesses and shortcomings.
4- Involving them in preparation of new plans for further changes
A combination of all or some of the above may be required to achieve the desired degree of co-operation. In any case, finding a way to persuade those who execute a plan that the plan is worth executing is imperative. In this regard the theoretical model for our study consists of a foundation and a body as depicted by the following diagram.





Research Theory Model

Process of change of educational plan

Most researchers divide the changing process into three main phases. These phases could be termed as: initiation, implementation and evaluation.
Words such as preparation, distribution, dispensing, dispensing and receiving are related to the initiation phase.
The most important concept related to the second phase (implementation) is execution. Execution could be done with total loyalty or may consist of compliance with constraints and alteration of procedures and routes.
Evaluation phase is when changes have been completely incorporated in the system and their effect could be readily observed.

Research method

Our method of research is qualitative[2] and the research is conducted through following steps
1-defining different aspects of the problem
2-defining methods of data collection
3-finding sources of information
4-collecting data
5-Description and analysis of data using inferential methods.
6-Conclusions and recommendations
We have come across changes in secondary school math text books ourselves and we were haunted by questions as to usefulness of these changes and to put the matter in a regulated frame work using previous theoretical studies we prepared questionnaires and conducted interviews with teachers and head-masters of a number of schools. Questions asked in these interviews were some pre-determined and some impromptu as dictated by previous answers given by a particular interviewee. Data analysis was performed based on 3 theoretical variables (knowledge, attitude and skill) and through inferential methods.

3- Education Of Research And Introduction Meredith D.Gall, Walter R.Borg, Joyce P.Gall, Sixth.Ed (2006)

Those who participated in the study

23 secondary school math teachers from an Iranian town (other than Tehran) were brought to our study. 3 of them were later put aside because of their insufficient co-operation. Two of the interviewees were head-master as well as teachers. Because we were mainly focused on changes made to text books in 2002 we have selected teachers with a teaching record spanning 7 years at least and up to 30 years. These teachers were teaching at public as well as private and distinguished schools. In one of sessions an experienced secondary school head master (my own mother) and some university students of mathematics as well as math faculty members and some members of the House of Math of the town of Saveh were present.

Method for data collection

Improvised questions were also asked by an interviewer during an interview. Each interview took between 45 and 120 minutes and interviews were done at schools as well as the math faculty office of the town's university and at the House of Math of Saveh. Two interviews were conducted at interviewees' homes. Interviews were conducted between December 2007 and March 2008 and were recorded on camera. Afterwards researchers watched and analyzed the interviews.

Tools for gathering data

Data was gathered through semi-structured interviews with teachers. Some teachers were interviewed in more than one session and main questions were always prepared in advance. During interviews and also during classes taught by interviewees notes were recorded by the researcher, also notes were handed over to the researcher by interviewees containing points which interviewees deemed useful as to the propose of the research.

Content of interviews

Some questions were concerned with interviewees' knowledge of the changes made to secondary school math text books. (The following questions for instance were meant to indicate the superficial knowledge level of interviewees regarding the changes)

* When were the last changes made to secondary school math text books?
* What parts were changed then?
* What method of problem solving was installed in the text book while modifications were done?

(The following questions were concerned with interviewees recognition and understanding of the logic behind these changes)

* Why these changes were made to secondary school math text books?
* What aim was sought by making these changes?
* What benefits were gained through these changes?
* What is the negative side of these changes?

Some questions were concerned with interviewees' attitudes towards changes. Examples are:

* Are you satisfied with changes made to secondary school math text books?
* Do you think changes were worth implementing?
* Do you see a need for further changes?

Some questions were concerned with the skills possessed by each interviewee to execute his or her teaching based on the new modified text book.

Examples are:

* How do you incorporate changes in your teaching style?
* Do you use the new problem solving method while teaching.

Method of analysis of data

To analyze the gathered data we separated and codified all points raised during filmed or tape-recorded interviews or offered in notes handed over by interviewees. On the basis of the theoretical model used in our research and discussed earlier we differentiated points pertaining to each of the three main subjects : a teacher's knowledge of the changes, attitude toward the changes and skill to implement the changes in his or her teaching. In a later review a fourth subject was recognized and incorporated as a variable. This variable was ((action taken by the subjects)). This was not due to any inadequacy of initial frame work and the decision was adopted only to make our study more rigorous and more complete.

The story telling analysis was used to enhance the validity of results. Based on the research it could be broadly stated that teacher's knowledge of changes made to secondary school math text books is inadequate and even those who possess such knowledge are not fully aware of the logic behind these changes. It was confirmed that those teachers with a more positive attitude toward these changes implement them in a more satisfactory way.

Results

Secondary school level is very important because it bridges primary and high school levels. Recent modifications in secondary school books have been based on results gathered from TIMSS[3].

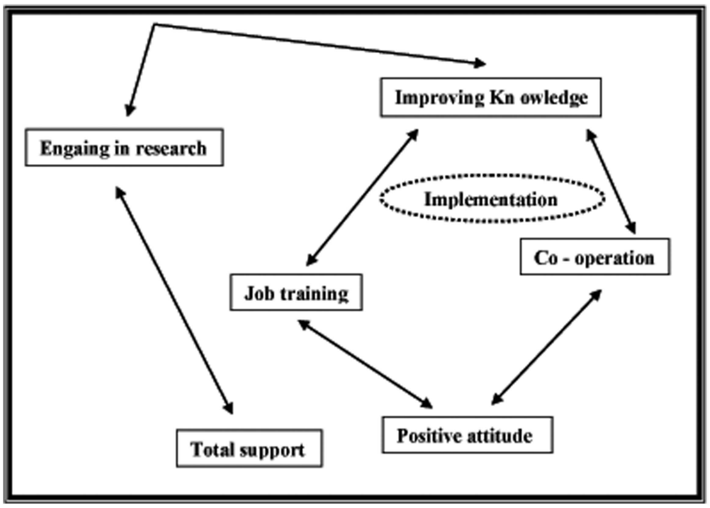
The main emphasis is on solving problems through Polya method.

According to (parsa-2005) teachers at present are not very familiar with Polya method or how to execute this method in their classes or how to use this method in solving math problems and they teach the new content in the old way. Some of these teachers resist the call for a change of ways and practices and it's partly because they are better tuned to the old method and have acquired necessary skills for its execution through their teaching experience. They feel more at home with the old model and feel somehow attached to it. Something however should be done in this regard.

One question is what would be the optimum model for reducing the resistance shown by some teachers to such changes and another question is how such a model could be implemented in Iran.

Of course it is not always easy to change some one's attitude toward something and again one way to achieve it is to educate teachers on the subject of changes in text books. Educating teachers could be done through on-the-job training workshop, distribution of booklets and pamphlets and the like. Another effective way of securing teachers' support of changes would be to engage them in the process of determining and planning changes. Most teachers themselves iterate that such involvement would probably improve the attitude they later may have toward these changes.

Considering the current situation we recommend that specialized training workshop with differentiated subjects such as organizational behavior, awareness of changes, resistance to change, using experience and opinions of older, more experienced teachers and other likewise titled workshops be held at relevant institutes to enhance both willingness and ability of teachers to comply with changes.



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3- Third International Mathematics And Science Study.

Professional Development for Mathematics Teachers Through Lesson Study

Azimeh Sadat Khakbaz
M.A in Mathematics Education, PhD. Student of Curriculum in Higher Education
Shahid Beheshty University (National University), Iran. Azimeh_khakbaz@yahoo.ie

Abstract: Lesson study is known as an effective Japanese professional development approach for teachers, since 1999.After that, this approach used dramatically as a way for improving teaching and learning in classroom through many countries and by many researchers. Review the literature of lesson study show that there is one common effort between the researchers to apply this approach. That is to make local model for using this Japanese approach as a localized one to release the unavoidable challenges of applying the method of another educational culture.
This paper first reviews the literature on lesson study as a way of teachers' professional development, which has been conducted since Meiji Period in Japan and is conducting for today's teaching in classroom. Then it clarifies how Iranian mathematics teachers encountered with this method. Finally, the finding of this research addresses a general translated model of lesson study that is preferred to apply lesson study for Iranian culture of education.
Keywords: Lesson Study- Mathematics Education- Professional Development

What is Lesson Study?

Many recent studies show the teachers' role in promoting students' learning. One of the most important studies about mathematics education is TIMSS. As two American researchers conducted a video study on some classroom which contributed in TIMSS, they noticed that the differences between students' results in different countries depend on education culture; and one of the most important elements of education culture is teacher professional development. Then after studying deeply about this important element, they noticed lesson study as a way of teacher professional development in Japan. So many researchers wanted to know if lesson study has good effect on their education culture. Lesson study has been a hot topic since 1999 in many countries. Many researchers used this Japanese approach to enrich classroom practices and provide opportunity of teacher professional development, so improve students' learning as a result (see: Stigler& Hiebert, 1999; Fernandez et al,2003; Lewis et al, 2004; Takahashi& Yoshida, 2004;Sarkar Arani,2006).

Lesson study is one of those professional development strategies that are deceptively simple on the surface and remarkable complex as you begin to probe beneath the surface (Richardson, 2004). Lesson study is a process consists of the study or examination of teaching practices. Teachers engage in a well-defined process that involves discussing lessons that they have first planned and observed together. These lessons are named study lessons or research lessons (Fernandez& Yoshida, 2004). These types of lessons are different from normal ones, they regarded not as an end in themselves but as a window on the larger vision of education shared by the group of teachers, one of whom agrees to teach the lesson while all the others make detailed records of the learning and teaching as it unfolds. These data are shared during a post-lesson colloquium, where they are used to reflect on the lesson and on learning and teaching more broadly (Lewis et al, 2006).

Lesson study, the major form of professional development in Japan, is a teacher-led instructional improvement cycle in which teachers work together to:

- Formulate goals for student learning and long-term development;
- Collaboratively plan study lessons designed to bring to life these goals;
- Conduct the lessons, with one team teaching and others gathering evidence on student learning and development;
- Discuss the evidence gathered during the lesson, using it to improve the lesson, the unit, and instruction more generally;
- If desired, teach, observe, and improve lesson again in one or more additional classrooms (Lewis & Tsuchida, 1998; Yoshida, 1999; Perry et al. 2002).

History of Lesson Study

Japanese education has changed dramatically in the decades since World War II. While rote, lecture-style 'teaching as telling' may have been common prior to World War II, recent observers are struck by the emphasis on 'teaching for understanding': on eliciting students' ideas sparking debate and discussion, and building comprehension through hands-on actives and reflections(Lewis& Tsuchida,1997). One of the major influences on this educational reform was creating lesson study as a traditional but unknown approach in Japanese educational culture. According to Isoda (2000), lesson study was a way to improve teaching process which applied in Meiji Period for first time.

Otherwise, although *konaikenshu* (School-based In-service Teacher Training Programs in Japan) is a newer practice, dating back only to the beginning of the 1960s, the strategy of combining *konaikenshu* and lesson study was already well stabilized by middle of the 1960s. A decade later the Japanese government, seeing the value of *konaikenshu*, began to encourage schools to engage in this practice, which at the time was solely a grassroots activity. It is estimated that today the vast majority of elementary schools and many of middle schools conduct *konaikenshu* especially *Jugyoukenkuu*(lesson study) as the most impotant feature of it (Fernandez& Yoshida, 2004).

Research Implementation

This research was done with cooperation of 12 mathematics teachers in middle schools of Kerman, Iran from October 2006 to March 2007. After the process of lesson study and aims of this project had been discussed generally for mathematics teachers in a meeting, these 12 teachers submitted engaging in project. Teachers divided to two groups except one, who wanted to contribute in two groups. So because of researcher contribution one group consisted of 5 partners, and the other, 10.

In each group partners chose a topic to concentrate on it: Linear Equation and Phythagoreth Theorem. These topics were selected because of students' misconceptions in understanding them.

Partners planned a lesson plan after introducing their methods of teaching these topics, and exchanging their experiences. A volunteer teacher taught this cooperative lesson plan in class and the other members of group were active observers, who had a role in class, like recording the process, examining students' group work, considering mutual communication between students and teacher, level of understanding each student and so on.

After this, partners came together in reflection meeting, which the group shared what they had seen in real classroom and provided their reactions and suggestions about lesson plan. They revised lesson plan to re-teach.

Above circle was rehashed three times about Phythagoreth Theorem and five times about Linear Equation.

Outside advisors, like members of the other group of this research, other teachers, and mathematics education students in Kerman universities came in some sessions and gave their suggestions. In some classes teachers wanted students to write their opinions about teaching-learning process and what they had learned. The entire reflection meeting was recorded and teachers' ethnographic notes were collected. Interviews were done with partners especially teacher who taught in real class about what they thought before engaging lesson study and what they think now and what has been changed through this process.

Research Findings

Analyzing data of this research showed three main trends which are as indexes of teacher professional development: teacher understanding, teacher practices, teacher assumptions. These indexes have their variables and each variable perceived through some indicators which would discuss at follow. Considering enhancing of all variables show that professional knowledge is create in school through applying lesson study in mathematics education.

1. *teacher understanding*: developing knowledge which cause increasing understanding of teacher about herself or himself, her or his students and her or his colleagues for knowing the professional environment with object of professional development is done, called teacher understanding. Subject content knowledge and teaching knowledge are also in this description. Teacher understanding have two main variables: teacher knowledge and evaluation knowledge.

Teacher knowledge is perceived through these indicators:

- Partners question about the content of lesson study process.
- Partners question about teaching methods in mathematics curriculum.
- Partners predict about students' thinking and problem solving in teaching process.
- Partners report from process of teaching in her or his class and her or his colleagues' classes.
- Partners study frequently in different references concerned lesson study and its delivering in the other cases and countries.

Evaluation knowledge is perceived through following indicators:

- Partners note in classes and analyze education process and cooperative lesson plan in meetings.
- Partners observe teaching process in real situation and evaluate the students' level of understanding.
- Partners analyze her or his implementation and others'.
- Partners evaluate effect of lesson study in her or his work.

2. *teacher practices*: practices which are done by teachers to expand professional development. These practices contain of collaborative ones to integrate theory and action in real teaching-learning situation. This index has two variables through this study: collaboration and proceeding.

Collaboration is perceived through theses indicators:

- Partners collaborate continuously in meetings, teaching session and other concerning programs.

- Partners make collaborative programs.
- Partners accept responsibility in collaboration implementations.
- Partners dialogue about implementation collaboratively.

Proceeding is perceived through following indicators:

- Partners theorize and act through theory to see the impact of if in real action.
- Partners apply lesson plan in different class and different situation.

3. *teacher assumptions*: attitudes and assumptions of teachers about professional development, professional improvement, cooperative learning, and abilities of criticize and accepting critiques which make her or him researchable person and independent thinker are in this index. Also teachers' assumptions through learners and nature of science are all variables of this index. So this study clarifies teacher assumptions as an index consists of five variables: acceptance her or his ability, acceptance learning from others, acceptance critiques, attitudes through learners, and attitudes through nature of science.

Acceptance her or his abilities are perceived because of following indicators:

- Partners making ideas and decisions and share them through others.
- Partners pioneer for implementations like teaching in class.
- Partners accept their colleagues in class with openness.

Acceptance learning from others could be appearing by:

- Partners accept that cooperative learning is more effectual.
- Partners come to colleagues' classes to learn, eagerly.
- Partners research others' experiences.
- Partners observe her or his teaching process, simultaneous others' and learn open-minded.

Acceptance critiques are also perceived through these indicators:

- Partners give opportunities to others to criticize.
- Partners hear others' views and criticize open-minded.
- Partners deal logical with critiques.

Attitudes through learners are perceived through following indicators:

- Partners try to develop learning environment to which learners could make decisions without stress and not fear from making incorrectness.
- All the meetings, dialogues, critiques, efforts and implementations are concentrating on students.
- Partners try to make connections between previous knowledge and new lesson.
- Partners let students make decision about general teaching process and even the admission of other teachers in class.

And attitudes through nature of science are perceived through:

- Assumptions about nature of mathematics.
- Assumptions about nature of education.

All described above embody in a coherence model to analyze data in this study, so authors claimed professional knowledge enhanced through applying lesson study in mathematics classrooms in Iran.

Review of literature show, engaging lesson study is not without challenges and naturally partners of the present study dealt with them. Authors characterized these challenges to three main indexes: time, assessment system, and support.

Time: this index has some parameters which caused time is a challenge in engaging lesson study in Iran:

- Some problems administratively like: limited time for teaching mathematics curriculum.
- Partners could not timetable for long-time, so they did it session by session.
- Partners could not adjust time because of circular work shifts.
- Accidental vacations caused time lag.

Assessment System: this index divides two main parallel parameters:

- Teacher assessment system.
- Student assessment system.

Support: main thing for continuing improvement in initiation and along a work, especially in education, is support of it. This support contains following main domains sponsor teacher to be independent thinker and researcher:

- Parents' support.
- Principals' support.

- Professors' support.
- Administrators' support.

Discussion and Conclusion

The results of this study provide evidence for the claim that lesson study can help teachers appreciate the importance of attending to students' thinking, as one of the partner acknowledged in interview:

> I think this was the first time that I know prediction of students' thinking is too important and I feel all our work was concentrated to this topic. I noticed students' thinking is the most important part of teaching-learning process.

Also partners acclaim lesson study because it made opportunity for them to meet each other, exchange experiences, observe themselves through different perspectives and learn in secure environment and with real situation. They accept abundant changes in their professional knowledge by applying this approach. A teacher said:

> It was like I could see myself in mirror. I could see my teaching problems and I could know I can learn in school and from teaching process. I can understand my students, my class and my teaching better. In addition I could know this topic of mathematics better and I can evaluate my knowledge resources and way of increase them from observing others and my work deeply.

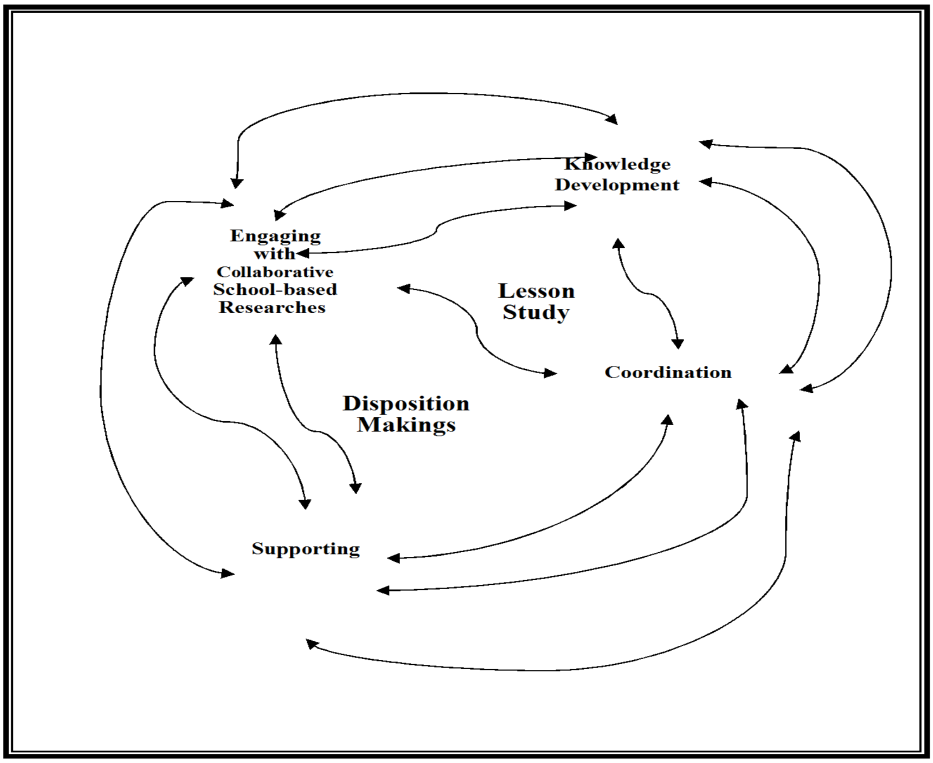
From this case teachers could communicate with their students and their colleagues more affably.
Evidences also show partners learn through lesson study as a collaborative research. Partners know their abilities and knowledge and improve them through collaboration tasks. As a teacher described this result:

> Lesson study rescued us from self-satisfaction in our drill in classroom. I did not like someone observe my class; first time I thought it is difficult for my students and me to open our class door on other teacher, even I concerned if it would drop students' concentration and confused me. Now everything has changed and I think my students and I could learn flexibly by cooperation with others.

She confessed:

> I thought I would drop it in small time incredibly I continue and I feel this was because of confederate spirit which is inseparable in lesson study as a collaborative research.

Figure1. Translated Model of Lesson Study to Iranian Culture of Eucation



Considerations of this research exhibit critical discussions with concentration on learning between partners which enhance self-assessment and cooperation assessment simultaneously that caused professional learning environment in school.
All this changes convert teachers from an individual person in work to shared professional person, from acquiescent about educational politics to active, acute and independent about professional life, from biased one to open-minded thinker and from a teacher only to a professional learner that spend professional life in a security professional learning organization, named school.
As said above, authors receive their first question of this research, and they claim lesson study made a sensational professional atmosphere in school through cooperative spirit as an inseparable part, which enhanced professional knowledge and caused enriching mathematics education in this research.

Although lesson study had its challenges and problems, and even more with Iranian educational culture because of absence of cooperative spirit, but partners respond positively to this innovation.

From this study, authors address a translated model of lesson study that is preferred as a model more conformable to Iranian culture of education, which is shown in figure 1.

This model suggests a spiral notion with no beginning and no end. The parameters of this model are not subsequent which could be described as stages so start point of description should not perceived as first stage.

For converting community to cultivated one that could be refined through cooperation, engaging in collaboration research in any cases is too important. As Stigler & Hiebert (1999) declared a part of teaching gap is between academic researches about school education and real situation that teachers are facing every day in schools in U.S, this gap is perceived dramatically in Iranian educational system too. As literature has shown researches specially lesson study those have cooperation as inseparable part, may be an effective approach to develop cooperative culture, so this model suggests engaging with lesson study. But it would be consider that Iranian education system have had many innovation those forgot after little time, so supporting collaborative researches like lesson study should be followed by administrators.

This model prefer lesson study for engaging collaborative research, because as the results of this study showed, it could ready its makings and reactivate to use them again as disposition makings. Indeed lesson study could make its resources and apply them to develop and create knowledge, like a knowledge management process.

The created knowledge in this model should be share to develop, so lesson study makes meetings for exchanging knowledge. Otherwise, open houses, outside advisors and information technology are three main apparatus in lesson study to develop knowledge by making connection between partners in all partners throughout world. When this created professional knowledge develop, it would revised and revitalize learning. Consider that this model is spiral not circular, because it would develop not limited.

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AN APT PERSPECTIVE OF ANALYSIS

Nanad Kishore M.Sc. D.Phil, Ramesh Chandra* M.Sc. PhD
Government College, Faizabad, 224001, UP, India
*Presenting author email: rameshchandra51@gmail.com

Abstract:
The discourse presented here is aimed at examining the justification of applications of current analysis to real world problems.

Technical Term: Application: By application we here mean to produce an apt model of situation of the real world.

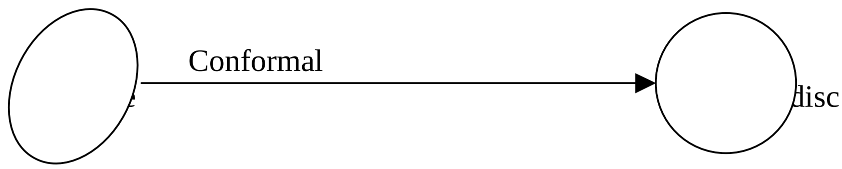
Introduction: The great Italian mathematician Galalio around 1600 AD had stated: “The world is written in the language of mathematics” I shall try to analyze as to how faithfully the current analysis is succeeds in reading the writings of the world.

Analysis in mathematics originated as a discipline of mathematics to study the geometry of physical objects through discovering the inter- relations between the constituents of these objects. In the process it led to conceptualize the techniques of limitations for infinite and infinitesimal cases. Thus in analysis emerged the processes of differentiations, integrations and infinite summations. In these processes, the concept of closeness is found to play the crucial role, and it was reckoned in terms of distance motion. The classical concept of distance, Known as Euclidean distance, was valuated in terms of non-negative numbers. Manrice, Rene Frechet in 1905 axiomatised this valuation, based on its essential characteristies. Thus, if X is a non empty set, than the valuation of distance between the elements of the set X is axiomatised as a function, say d, on XxX to the set of non-negative real numbers (R^x) satisfying the following conditions.

For x,y,zεX (i) d (x,y) = 0 iff x=y, (ii) d (x,y) = d (y,x)
(iii) d (x,z) $\leq$ d (x,y) + d (y, z)

It helped mathematicians to extend analysis to any set of objects, enabling them to develop important subjects like Functional Analysis. It also enabled mathematician to get deeper insight into several real world problems. As an example, I can cite two theorems of great importance.

1. Riemann mapping theorem Any simply connected complex domain whose boundary contains at least two point can be mapped in a conformal way onto the open unit circular disc



This Theorem has helped aeronautical engineers to tackle mathematically some intractable problems of their domain.

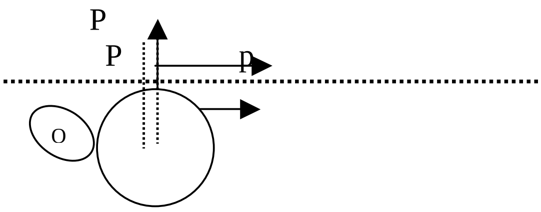
2. Poincare’s Conjecture : Every simply connected compact 3- dimensional manifold is homeomorphic to a 3- dimensional sphere.

In 2000, a Russian mathematician, proved that the conjecture holds true. This has very greatly simplified the problem of manifold geometry.

Perspective : I would examine here the position of current analysis from the point of view of applications First I would like to present some facts relating to the basis of the methodology of applications. For this I refer to Newton’s first law of motion. It states every object (bearing men) continues to remain in a state of rest or uniform motion in a straight line unless impelled by an external impressed force to change that state "This law prompted the eminent physicist and mathematician Poincare to proposed a general principal of physics 'physical phenomenon as noticed by a fixed observer must be the same as done by any other observer who possesses a uniform motion in a the straight line'

To provide broader setting to the principle, term inertial frame of reference has been introduced. An inertial frame of reference is a reference system (axes) in which the origin and the reference vectors (axes) move with the same constant velocity in a straight line. Thus we have a general premise of physics' accepted universally as "Every Physical phenomenon remains the same to observers attached to deferent inertial frames of reference".

Clearly the above premise holds true in Newtonian mechanics. However as we all know it is not found to hold true in Electrodynamics.

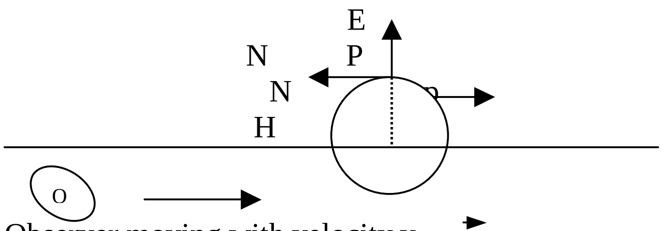


Infinite line of Uniformly distributed charges Q per unit length

Fixed observer

$$\vec{E} = \frac{Q}{2\pi \epsilon P} \widehat{OP}$$

For observer moving with constant velocity in the direction of line of charges the phenomenon appears as follows:



Observer moving with velocity $\vec{v}$

$$\vec{H} = \frac{I}{x\pi\mu p} \widehat{PN}$$

(Here current I is a Quantum of moving charge per unit time)

The above illustration shows that the fixed observer notices only the Fluid of force E, where the observer moving with constant velocity u in the direction of line charges notices an extra field of force w.

The solution to the above controversial problem is provided special theory of Relativity. According to this theory the valuation of distance between two fixed point is not a fixed non–negative number all observers attached to different inertial frames. The concept of distance valuation as postulated in the theory of relativity conforms well to all cases including those of Electrodynamics. Thus strangely enough Nature evaluates distance between two points, not in the simple Euclidean way, but in the relativistic way. Form the above presentation it is clear that base of analysis, as envisaged by Manrice Frichat's distance function does not conform to the nature's way. Hence the current analysis cannot provide an apt model of real physical situations, and it faces the same handicap of Newtonian mechanics did. As the theory of relativity has given rise to development of relativistic mechanics we will have to evolve Relativistic Analysis to conform to the application of the real-time situations. This will provide an apt perspective for the development of correct analysis.

Models for harnessing the Internet in mathematics education

Barry Kissane
School of Education, Murdoch University, Murdoch, WA, AUSTRALIA 6150
B.Kissane@Murdoch.edu.au

Abstract

In recent years, the Internet has increasingly been used to provide significant resources for student to learn mathematics and to learn about mathematics, as well as significant resources for teachers to support these. Effective access to and use of these has been hampered in practice by limited facilities in schools and the limited experience of many mathematics teachers with the Internet for mathematical purposes. This paper offers models for understanding the effective use of Internet resources, based on typologies of resources for learning and teaching mathematics. Six categories of Internet resources for mathematics student use are identified: (i) Interactive resources; (ii) Reading interesting materials; (iii) Reference information; (iv) Communication; (v) Problem solving; and (vi) Webquests. Similarly, five categories of Internet resources for mathematics teacher use are identified: (i) Lesson preparation; (ii) Official advice and support; (iii) Professional engagement; (iv) Commercial activity and support; and (v) Local school web sites. The paper recognises that web resources can be used in a range of ways, including supporting both teaching and learning. The prospects for sound use of the Internet are briefly described in terms of these models of use.

Introduction

Although the situation is different from one country to the next, the Internet has continued to rise in significance across the developed world, with the mounting prosperity of the last two decades and with the rapidly declining infrastructure costs to individuals and to schools of having reasonable access. In this paper, we consider some of the ways in which the Internet has been harnessed to date for use in mathematics education, in order to construct a model of a productive relationship.

The Internet has the potential to support the learning of students and the teaching of teachers. In short, the problem addressed by this paper concerns the productive use of the Internet for mathematics education: how can students and teachers best exploit the potential of the Internet in their respective roles in learning and teaching?

Description of the model

The Internet has provided previously undreamt of possibilities for connections within mathematics education, both for students and for their teachers, now part of a globalised world. It is appropriate to consider the twin activities of teaching and learning separately for the purposes of understanding the significance of the Internet for mathematics education. Accordingly, the model described in this paper begins by separately identifying the kinds of opportunities potentially afforded by the Internet for students and teachers. These are different because the roles of students and teachers are different. The paper then proceeds to consider some of the connections between these affordances as well as some of the constraints on their implementation in practice.

The Internet for students

A recent paper (Kissane, 2009b) described in some detail a typology of ways in which the Internet might be a useful learning resource for students. Six categories of Internet resources were described briefly and examples given from within most categories. The companion website (Kissane, 2009a) provides links to many examples, and is maintained to take advantage of new opportunities that become available. In this paper, it is appropriate to describe each of the categories quite briefly.

The first category involves the use of interactive resources, taking advantage of the possibility of designing web objects that students can manipulate directly, using software platforms such as Java and Flash. Virtual manipulatives are one example of such a resource. Within this category, there is an increasing number of well-designed materials intended for direct student use, across a wide range of student ages and levels of sophistication. Some of these, such as the National Library of Virtual Manipulatives (Utah State University, 2009) in the USA, comprise large collections that are well organised, with adequate online help to make them independently accessible by students.

The second category involves the provision of worthwhile reading materials for students. In many circumstances, students seem to have access to a remarkably thin range of contemporary materials regarding mathematics, in part because such materials are not routinely available in many homes and even many school libraries. (Kissane, 2009c). A range of recent and well-written materials in various forms has been produced over recent years (not always for an audience of students), some of which have become accessible to students through the Internet.

The third category of materials involves reference materials of various kinds, such as dictionaries, encyclopedias and databases of mathematically related materials. Once again, it is rare for either homes or schools to provide good reference information that is mathematically informative for students, in stark contrast to the wealth of such materials now available on the Internet.

A fourth category of materials is concerned with communication among students and others. The Internet provides mathematics students with an opportunity to communicate with other students and, unlike the use of email, does not require students to know each other beforehand. In this context, the emphasis is of course on communications related to mathematics, not personal communications, although communications between people necessarily are personal to an extent. Communications typically are concerned with understanding mathematical concepts or solving mathematical problems. Similarly, the web can be used to connect students with teachers (or virtual teachers), prepared to respond to their questions or provide advice and feedback concerning mathematics.

The use of the Internet for problem solving presents a fifth category of learning opportunity for students. Some web sites offer regular problems and puzzles for students of different levels of sophistication, as well as advice and hints on solving them (and guides for teachers). While few school environments or school curricula are bereft of problem-solving opportunities, the Internet can provide an environment for problem solving that has some distinctly different characteristics than those available through a book of problems, including the opportunity for solutions by other students to be presented and discussed in a virtual world.

Finally, the Internet can be used to provide students with structured explorations of situations of mathematical interest in the form of webquests. These generally are constructed (by teachers) to support a group of students tackling a task that has a mathematical flavour and which uses the web as a source of real data relevant to the students' context, as well as a stimulus to work together with a team to tackle a contemporary problem that is connected with mathematics.

Although these categories are presented here as if they are mutually exclusive, in practice some Internet opportunities for students involve more than one of these at once, such as a website that engages students in exploratory activity, using a virtual manipulative, motivating work on a suitable mathematical problem. However, it is argued that the categories each offer distinctively different opportunities to support and encourage student learning, as well as fostering interest in mathematics; these opportunities were not readily available prior to the development of the Internet.

In terms of addressing the central problem addressed by this paper, it is suggested that this typology captures the key opportunities for learning afforded by the Internet, and that there is an already large, and increasing, collection of good examples of most of these. While no classification of this kind will allow each potentially useful web site to be located in one category and not another, a model for Internet use ought sensibly recognise that quite different kinds of opportunities are available: that the Internet for learners is a multi-dimensional object.

The Internet for teachers

While many of the Internet sites that are relevant to students are also likely to be of interest to mathematics teachers, the Internet provides further and different opportunities for teachers. In this section, these are briefly identified and a first attempt at a typology presented for discussion. A website (Kissane, 2009d) supports the development of this model with online examples. The order of the various categories proposed is not intended to be interpreted as meaningful.

One category of mathematics teacher use concerns accessing direct lesson materials or ideas. Of course, many web sites might suggest ideas to teachers for lessons and classroom tasks: such is the nature of the craft of teaching. However, some websites offer detailed collections of lesson plans for mathematics teachers, usually written by teachers themselves, sharing their more successful lessons with others; the Australian *Maths 300* collection (Curriculum Corporation, 2009) is a good example of a subscription-based provision of this kind. While some lesson collections are too strongly related to local curricula and contexts to be very useful to teachers elsewhere, there are inevitable differences in teaching styles and there are various standards of quality assurance used before posting, such lesson collections can offer helpful and practical advice to teachers. In a similar vein, some sites have focussed recently on the provision and sharing of teaching materials (such as software files) for new technologies such as interactive whiteboards; two good examples include the site maintained by an Australian teacher (Boggs, 2009) and the excellent subscription site maintained by Keele University staff in the UK (Miller, 2009).

A second category of use for teachers concerns official communications regarding the curriculum or the governance of education within their environment. At least within Australia, but also elsewhere in the developed world, the Internet is increasingly being harnessed by authorities to provide both guidance and support for teachers undertaking their mandatory roles. This use of the Internet has a clear advantage over previous mechanisms for doing similar tasks: that materials can be readily updated and corrected, so that the most useful advice for teachers can be made available very quickly. A possibly unexpected advantage of this means of supporting teachers is that the information available in one jurisdiction (such as in one of the states in Australia or Germany, or one of the countries in Asia) can be efficiently and quickly accessed by those who reside elsewhere. Although this has not been the intention of the developments, it has certainly been a useful by-product, permitting good ideas for teachers or good ideas for curriculum developers to be rapidly shared. A disadvantage of this category of information dissemination is that the burden—and the costs—of printing are passed from authorities to schools and individuals, which is sometimes quite problematic. An alternative of course involves reading materials online, although many do not find this satisfactory, for a range of reasons.

A third category of Internet use for teachers concerns professional engagement. Over the past decade, voluntary professional associations of mathematics teachers have increasingly come to rely on their Internet sites to communicate effectively and helpfully among an organisation, and more widely to the entire community. Thus, national associations such as the Australian Association of Mathematics Teachers (AAMT), the National Council of Teachers of Mathematics in the USA, the Association of Teachers of Mathematics in the UK and similar bodies in other countries have developed strong web presences to support a variety of functions that were previously neglected or handled with great difficulty. An example of this is the use of the AAMT website (AAMT, 2009) to provide a mechanism for the development of policy responses to official documents such as, in the case of Australia recently, the developments towards a national curriculum for mathematics. Prior to the accessibility of the Internet, it was too difficult in practice for such organisations to democratically seek advice from a wide membership. While it is still difficult, as professional people are usually very busy, the constraints of access have been lifted so that a genuinely collaborative effort at policy development is possible. Professional websites serve a number of other purposes as well; in the case of the AAMT website, a lively email list regularly allows professionals to connect together to discuss issues of the day in a collegial manner with colleagues from around the country; such professional collaboration was simply not possible a generation ago. As well as (voluntary) professional associations, recent years have seen the development and maintenance of large web sites intended to support the work of teachers as professionals in a range of ways. Two exceptional examples of this are the Math Forum in the USA (Drexel University, 2009) and the government-sponsored National Council for Excellence in Teaching Mathematics (2009) in the UK, with an interesting *Mathemapedia* for teachers, among other forms of professional support and collaboration.

A fourth category of Internet use for teachers concerns websites that have a commercial element, yet still offer opportunities for teachers that were previously inaccessible. This category does not include sites whose *sole* purpose is to provide opportunities for teachers to buy goods or services (as these are strictly commercial sites, not educational sites), but rather includes a range of sites that offer significant support of various kinds to a professional client base. One example of this is the *HotMaths* (2009) site in Australia, which offers significant curriculum materials for students, providing a school purchases a subscription that gives them access. While subscription websites might be seen solely as commercial exercises (and, indeed, in some cases, correctly so), in other cases, the subscriptions raised are used essentially to fund development of suitable innovative learning materials, for which developmental costs are generally very high. An excellent example is the *Maths 300* website in Australia (Curriculum Corporation, 2009), which offers detailed advice for teachers regarding a large set of innovative mathematics lessons, together with other professional materials and professional developmental help, provided a school subscription has been purchased. Another example of commercially connected Internet use comes from companies marketing computer software, computer hardware (including interactive whiteboards) or calculators. Several sites offer significant support for teachers using, or planning to use, the technology involved, in the form of demonstration software, activities to use with students, access to technical advice and help, upgrading of equipment or software and information about communities of users and opportunities for local professional development. In a similar vein, many school textbook publishers now offer privileged access to materials developed and stored on the web to support their text materials. While some of this sort of activity is doubtless part of the promotional activity of the publishers, in order to capture or retain a market with the lure of extra materials, it also offers students opportunities to learn that were not previously available, targeted to a particular set of curriculum materials; without doubt, this also helps their teachers as well.

A fifth category of teacher use of the Internet concerns local use within a school or a school district. Increasingly in developed countries with high (and rapidly increasing) Internet access, the possibility of teachers using a school Internet site for teaching purposes has become real, and of significance. With some expertise and local support, teachers can use their own websites, or the school website, in order to post lesson materials for students and to engage students within the community of the classroom. In some cases, still fairly rare, more structured learning management systems (such as *WebCT* and *BlackBoard*) are used to focus instruction, although this is more common in universities and colleges than it is in schools, which, by their nature, have students in full-time attendance during childhood and adolescent years.

To return again to the central problem addressed by this paper, it is clear that there are many opportunities for supporting the work of teachers via the Internet. As for the case of students, the proposed categories in this model do not provide mutually exclusive sets of opportunities, as several websites might be fairly described as helping in more than one way. The model also helps clarify the complex craft of teaching, especially teaching in the twenty-first century. As well as engaging in what many outside education regard as the only role of teachers, that of designing and executing worthwhile activities for the pupils in their immediate classroom, teachers are simultaneously engaged in other activities as well. These other things include working within an official regulatory environment, sometimes with high public visibility. They also include working with others as an active professional, collaborating independently with like-minded professionals, as well as keeping up to date with recent publishing and technology changes and directions. Even teaching itself has changed a good deal in the age of the Internet, with teachers expected, routinely so in some cases, to develop new skills to handle construction and maintenance of school websites and operate within the constraints of a learning management system.

Connecting teaching and learning

The two models for students and teachers on the Internet are presented here as if they are distinct. In fact, of course, teachers are themselves interested in both models, since a key role of the teacher is to frame the experience of students, and it is now important that teachers have a sense of what the Internet offers to students, and which of those offerings are worthy of attention in a particular context. The relationship is not, of course, reciprocal: students are for the most part disinterested in what the Internet has to offer their teacher.

The connection is even stronger than that, however. The teacher in search of lessons is likely to make use of the same websites that are helpful for students learning. Indeed, it is usually through the teacher becoming aware of the websites, evaluating their potential benefits for a class and using them in some way, that students are offered access to them. This access might take a number of forms, ranging from advice to use a particular website for some purpose, using a website for a classroom activity of some kind, making links to relevant websites on a school web page, including a website in a webquest designed by the teacher or even assigning homework to students based around a particular website (in the circumstance where good home access to the Internet is assured).

The categories of Internet use that relate to students accessing the Internet for information or for reading interesting materials are also of relevance to the teacher. As noted earlier, many of these are in fact designed for a wide audience, which certainly includes professional mathematics teachers. Many mathematics teachers have reported informally and favourably to the author about the excellent materials of interest to them as mathematically educated people (not necessarily in their role of teacher) in these two categories. Indeed, some of the material is written for this audience.

Description of the model success

The models offered in this paper are theoretical models, designed to provide a sense of perspective on what the Internet might offer mathematics education. As such, they have not yet been subject to formal testing or validation, although they reflect a good deal of the author's experience in using the Internet from the perspectives of both teaching and learning. For some years now, the author has maintained a website (Kissane, 2009b) with annotated links to good examples of Internet sites for school students (and their teachers or prospective teachers). Work with teachers in contexts of professional development and with student teachers suggests that the categories suggested identify distinctly different and important kinds of use of the Internet by students. It is clear, too, that the detailed examples of websites in these categories are recognised by teachers, both experienced and less experienced, as plausibly related to effective Internet use.

Transfer of the model to different environments

By its nature, the Internet has a measure of portability across environments, although of course it is difficult for this to occur across linguistic divides. Successful use of the Internet by either students or teachers depends a good deal on local circumstances, including especially the ease with which Internet access is available to students (at school and at home) and to teachers (inside and outside their classrooms). Excellent overviews of the range of ways in which effective use can be handled are provided by Alejandre (2005) and Galindo (2005), both of whom recognise the inevitability of constraints on good practice and offer helpful ways of working around them.

It is clear that effective use of the Internet in mathematics education brings new professional demands for teachers. Although it is clear that mathematics education can be supported, informed, improved and inspired through appropriate use of the Internet, achieving these aspirations includes high expectations for support of teachers, significant resource needs in schools and a need for new paradigms for both teaching and learning to be developed. In the best of circumstances, the prospects for harnessing the Internet to improving mathematics education are already promising, although in practice the circumstances are still far short of optimal in many classrooms in developed and affluent countries.

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Cryptography and number theory in the classroom --
Contribution of cryptography to mathematics teaching

Katharina Klembalski
Institute of Mathematics, Humboldt-Universität zu Berlin, Berlin, Germany klembals@math.hu-berlin.de

Abstract
Cryptography fascinates people of all generations and is increasingly presented as an example for the relevance and application of the mathematical sciences. Indeed, many principles of modern cryptography can be described at a secondary school level. In this context, the mathematical background is often only sparingly shown. In the worst case, giving mathematics this character of a tool reduces the application of mathematical insights to the message "cryptography contains math". This paper examines the question as to what else cryptography can offer to mathematics education. Using the RSA cryptosystem and related content, specific mathematical competencies are highlighted that complement standard teaching, can be taught with cryptography as an example, and extend and deepen key mathematical concepts.

Introduction
Cryptography fascinates. Concepts such as secrecy, espionage, code cracking are often associated immediately (and not only among students). Hence, the motivation to work on this topic is high.
Modern cryptographic methods in particular are based in large parts on elementary number theory and are therefore accessible to secondary school students. Corresponding publications may be found both in mathematical as well as in computer science education literature (e.g., [1], [2]). An implementation into teaching practice in Germany is currently found predominantly in computing rather than in mathematics classes. This has effects on the nature and extent of the mathematical foundations presented. These often take a back seat in favor of the (partial) implementation of individual algorithms. Similar approaches for teaching specific mathematical content by means of cryptography that go beyond the pure cryptographic algorithms or protocols are rare and are often only implicit contained (typical [3], exception [4]). In this way, essential mathematical potential remains unused.
To show which contributions cryptography can make at school, the paper first introduces cryptography-related, general educational objectives which can be associated primarily with media competency and do not necessarily require mathematical expertise. The paper will then discuss the added value of considering cryptography within mathematics. The main focus is on the encounter with unsolved problems in mathematics, on the experience of mathematics as a living science as well as on cryptography as an application of mathematics. These considerations will be linked exemplarily to the RSA cryptosystem and its mathematical background. Finally, the paper outlines how this approach to cryptology deepens and extends known key mathematical concepts.

Why cryptography in school?
Due to the increase in electronic data traffic, cryptography is of practical relevance to everyone – be it through online banking, e-mails, electronic health cards, electronic passports or the protection of personal data. Within these applications, cryptography not only ensures the secrecy of data exchanged but also provides reliable means for the authentication of communication participants and for the verification of the integrity of data. Hence, as an element of media competency, students should acquire basic knowledge on the use of cryptographic applications.
General learning objectives in this regard are:

- raising awareness in relation to data security, especially the knowledge that data exchanged on the Internet can in principle be monitored and is thus insecure;
- derived from this, the insight into the necessity of encryption and the ability to perform and verify encryption;
- the knowledge that and how the identity of communication participants may be verified;

Integration of cryptography into the school curriculum differs among the German federal states as different standards exist in each state. Nevertheless, none of these contents are mandatory for all students, as cryptography in mathematics is only an elective subject. At junior secondary level, these elective subjects are usually found in years 9 or 10 and predominantly feature applications from classical cryptography (historical symmetrical ciphers like Caesar and Vigenère cipher). At senior secondary level, cryptographic content is almost exclusively taught in computer science. The guidelines here range from non-binding references to cryptography up to teaching units containing essential principles of modern cryptographic algorithms. Exceptions are the so-called "Seminarkurse", which can be taken as elective courses in the Abitur (mandatory in Bavaria but voluntary in other federal states). The schools set up such courses according to their capacity in subjects that are in demand. The content of these courses may be chosen rather freely compared to standard courses in the subject. The author of this paper taught such courses (2 semesters, 3 hours per week) in cryptography and number theory in 2007/2008.

Why cryptography in mathematics?

To give students an authentic image of the mathematics as a science, it is necessary to show current developments in scientific maths [5]. In this sense, cryptology[1] is a lucky coincidence [6, p. ix] as many of its modern techniques and algorithms can be fully explained and require little mathematical background (predominantly elementary number theory) in order to be understood. In the following, the example of the RSA cryptosystem and some closely related subjects are used to demonstrate how to extend the image of mathematics acquired in secondary school education.

a. Unsolved problems in mathematics

The contents of school mathematics do not extend beyond the scientific knowledge of the 18th century, with the exception of probability theory and some formalities. The contents of the classical branches of school (mathematics, arithmetic, algebra, calculus and geometry) may be found under the broader terms arithmetics and geometry as early as 1905 and have been intensely formed out and worked on since then [7]. Hence, the opportunity of having the students face unsolved scientific problems hardly exists within the scope of the standard mathematics curriculum. This fact has a very practical reason: Most of the open scientific questions within the aforesaid branches are difficult to describe at school level and thus hardly accessible to the student. Some open questions within other branches such as discrete mathematics, number theory or numerics are understandable, but the curriculum does not offer chances of a natural encounter with such issues. Therefore, it is advisable to take advantage of this when teaching cryptography.
Unsolved problems in number theory include, for example:

(1) Is there an infinite number of prime twins?
(2) Is there always a prime between n^2 und $(n+1)^2$?
(3) Is there an efficient way to find the prime factors of large numbers?

The first two questions are easy to grasp and are accessible by experimentation. They appear in connection with the distribution of prime numbers. [2]
The third question is particularly interesting from a cryptographic point of view. Consider the function E with $E(x) = x^e \pmod n$ where n and e are natural numbers, and n is large. As long as it cannot be answered in the affirmative, the function E may be seen as a one-way function, i.e. a function which is *practically* impossible to invert.[3] *Theoretically*, the inversion is solvable, because of the congruence $c = x^e \pmod n$ – for example by testing all x with $x = 1, 2, ..., (n-1)$. In practice, there is no efficient approach to this problem for large moduli n. The computation of x from e and n is equivalent to the knowledge of the prime factor decomposition of n ([8], p. 141). If n is chosen as a product of large, secret primes, the function cannot efficiently be inverted without this additional information. This fact is used in the construction of the RSA cryptosystem (see below).
The inability to solve (3) in this case represents no flaw but is essential to the security of the function E used in the RSA cryptosystem.[4] This gives the students the opportunity to deepen their understanding of integers in the context of cryptography and lets them experience that mathematical science is still incomplete. Additionally, the utilization of lack of knowledge supports a sparsely used approach in students' school experience.

b. Mathematics as a living science.

Cryptography has been used for several thousand years ([9], p. 105). A significant problem had always been the key exchange.[5] The solution to this problem was found approximately 30 years ago – namely by use of

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[1] This text does not distinguish between cryptography and cryptology. Formally, cryptology is often used as an umbrella term which encompasses cryptography (science of designing ciphers) und cryptanalysis (art of breaking cipher systems).
[2] Questions on the existence and quantity of large primes arise for example when choosing suitable moduli n in the calculation of the function E described below. An estimate can be determined in the classroom with the use of the prime counting function π, where $\pi(n)$ denotes the number of prime numbers up to n, and its approximation by $n/\ln(n)$. To get an additional impression of the distribution of primes, the theorem on arbitrarily long prime gaps is available.
[3] The one-way property of a function arises from *experience*. To date there is no one-way function for which a proof of this property is known, i.e. the proof of the non-existence of an analytical inversion.
[4] Similar considerations apply to other functions, i.e. there is no general approach for calculating the discrete logarithm for large moduli n, which is exploited for cryptographic applications (Diffie-Hellman key agreement, ElGamal) ([8] p. 153).
[5] Until 30 years ago, information was encrypted solely according to the following principle. A message M is encoded by an invertible function E and a secret key K into a cipher text $C = E(M)$. The recipient decodes/decrypts the message with the inverse function D so that $D(C) = M$. The parameter K for the construction of E and D must be transmitted via a secure channel (personally in advance, by a courier, etc.). Secure data exchange between strangers (e.g. online shop and customer) over an insecure communication channel (internet) is not possible in this way.

mathematical knowledge that was already several hundred years old. Why so late?
The rise of computers and with it the new applications of cryptography were crucial for this development. The predominant historic use of cryptography had been the exchange of military secrets between two parties. In contrast, the use of computers increased in particular the exchange of sensitive data over multi-party communication networks. New tasks that had to be solved were the authentication of communication participants and the verification of the integrity of transmitted data.
In 1976 Diffie and Hellman published the idea of a public key algorithm which overcame the old key exchange problem [10]. An algorithm that implements this idea was published by Rivest, Shamir and Adleman in 1978 [11] and is known as the RSA cryptosystem. It is mainly based on Euler's theorem[6], which allows the generation of the keys $K_E = e$ for encryption and $K_D = d$ for decryption. From the knowledge of one of the keys one cannot derive the other, therefore K_E can be transmitted via a public channel (hence the name: public key algorithm). The key exchange problem has thus been solved.
Crucial to the effective application of the RSA cryptosystem was its simple computational implementation. Encryption and decryption (exponentiation with exponents e and d modulo n) as well as the key construction (determination of e and d) can easily be performed by square & multiply or the extended Euclidean algorithm.
The computer is thus tool and at the same time the occasion for the application of classical number theory in cryptography. This makes cryptography and the RSA cryptosystem a suitable example for pointing out the development in scientific mathematics triggered by the use of computers. This very direct link between application and mathematics is also an important fact which can extend students' perception of mathematics.
Computational mathematics, in particular numerical methods and discrete mathematics, has increased in importance in recent decades. A major contribution of the computer consists in shifting problems from computability to the development and implementation of suitable (and in particular efficient) algorithms. On the other hand, computers introduced new problems, such as the need for data compression (information theory).
Such developments are typical in the history of mathematics. They are often triggered by questions posed by other sciences, technological advances or even social developments. Physics and its influence deserve a particular mention here, for example for its influence on the development of calculus (mechanics), functional analysis (quantum mechanics) or differential geometry (general theory of relativity). This interrelation appears in teaching practice especially in the context of calculus. Demonstrating that such processes also take place today does, however, require a move beyond the standard curricula. Cryptography offers an opportunity to do so and can broaden the students' view of mathematics as a living science that is still developing.

c. Cryptography is applied mathematics

More to the point, modern cryptographic algorithms, and especially the RSA algorithm, are based on very old mathematics: mostly basic number theory, a branch of mathematics long known for its beauty rather than practical use. Essential elements like modular arithmetic, the Euclidean algorithm (at least 4th century BC) or Euler's theorem (18^{th} century) were already valuable instruments within mathematics before their joint *practical* applicability for the RSA algorithm was recognized and became accessible through the use of computers. The application of mathematics in this case can be seen as an interdisciplinary transfer of mathematical knowledge to contexts other than those in which the insights were obtained.
The question arises as to which role the presentation of this transfer could or should have in the implementation of teaching. Number theory is, at least in Germany, not part of the curriculum. Its basics are generally only provided in connection with cryptography – usually simplified and isolated from the mathematical context. The depth of this presentation also depends on the temporal extent of the teaching unit and the weighting of cross-disciplinary relations, and primarily aims at providing an understanding of the cryptographic principles. This cannot be considered an application of existing knowledge. In the worst case, mathematics is reduced to an auxiliary science which permits the implementation of ingenious ideas seemingly by chance. What remains is cryptography and the insight "everything has maths inside".
That said, an extensive education in number theory and group theory at school is not feasible. However, the value of the mathematical content used should be made a topic beyond its cryptographic use. Otherwise,

[6] Euler's theorem was first proven by Euler in the context of modular arithmetic and later generalized to finite groups (G): Let $\gcd(a, n) = 1$, then $a^{\varphi(n)} \equiv 1 \pmod n$, where φ is the Euler phi-function. I.e. $\varphi(n)$ gives the number of nonnegative integers which are prime to n. In the case $n = pq$ the theorem can be simplified to $a^{(p-1)(q-1)} \equiv 1 \pmod n$. It follows $a^{k(p-1)(q-1)+1} \equiv a \pmod n$ for nonnegative integers k. Hence, for $ed = k(p-1)(q-1)+1$ the function E with $E(x) = x^e \pmod n$ presents a one-way function. This function can only be inverted (decrypted) with the knowledge of either d or p and q using D with $D(x) = x^d \pmod n$.

there is a risk that the problem solving appears arbitrary. Number theory should thus be connected with contexts beyond cryptography to illustrate its wider significance and use. Links to known curricular material but also to overarching contents are useful here. Examples:

- modular arithmetic: (i) Subsequent justification of the divisibility tests for 3 and 9 known from earlier years at school; (ii) Generalization of the proof technique known from differentiating between the cases "even" / "odd" for various moduli *n*; (iii) Outlook / consolidation: (**Z**/*n***Z**)* as one of several groups on which one-way functions can be defined; [7]
- Euclidean algorithm: (i) Comparison of the effectiveness of determining the gcd by comparing the prime factors (known in connection with the introduction of fractions) with that of the Euclidean algorithm; (ii) Outlook / consolidation: computation of multiplicative inverses in (**Z**/*n***Z**).
- Euler's theorem: Outlook / consolidation: order of group elements.

Conclusion

Cryptography is suited to exposing the students to unsolved questions in mathematical science in an authentic context (a). At the same time, it is an example that counters the common student perception that mathematical knowledge is a fixed structure and only has to be extended (b). Instead, mathematics appears as a science which vividly generates new branches and is engaged in a constant exchange of ideas with practical applications. These in turn arise from applications of knowledge already gathered and are interlinked with them in multiple ways (c). This is partially transferable to the classroom as cryptography is not independent of conventional classes but extends and deepens the pre-existing knowledge and perceptions of mathematics. This concerns in particular the key mathematical concepts[8] of *number* and *algorithm* and, depending on the implementation in the classroom, the concepts of *functional relationship* and *data analysis and probability*.

Key concept *number*: The knowledge of natural numbers is deepened. The occasion for this are questions of key construction for the RSA algorithm and its security. These lead to primes, their distribution, prime number tests and the problem of the factorization. In addition, familiar divisibility tests are proven with the help of modular arithmetic and are thus legitimized retrospectively. In particular, this presents an opportunity to highlight elementary unsolved problems in mathematics.

Key concept *algorithm*: The question of the key generation leads to the Euclidean algorithm, which the students can discover themselves. This and the square & multiply algorithm illustrate the advantages of algorithmic problem solving; questions about the efficiency of algorithms will arise almost by themselves. The importance of computer use for the application of these and other algorithms can be connected with the history of cryptography. This contrasts with the normal use of algorithms in secondary school, which is usually limited to solving systems of equations or, in the wider sense, to the processing of calculus problems (curve sketching).

Key concept *functional relationship*: The concept of the one-way function required in cryptography augments the invertible (if only by limiting their domain) function types already known. Furthermore, if the topic is covered in depth, students get to use the functions φ and π from number theory as two functions without closed expression.

Key concept *data analysis and probability*: the internal link to probability theory within mathematics originates, among others, from the need for suitably large primes for the key construction in RSA. The comparison of the probabilistic prime number test (Miller-Rabin test [8], p. 128) to the classic test for divisibility of all primes smaller than the square root of a candidate *p* is of interest in this context. Another opportunity to dig deeper concerns the construction of (pseudo-) random numbers used in conjunction with the security of the RSA algorithm.

The author investigates as part of her thesis how these ideas may be implemented in practical teaching. An example of the content and related key concepts for a practical implementation may be found in the appendix. The example refers to the first semester of a two-semester Seminarkurs in mathematics at senior secondary level.

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[7] A similar consolidation is recommended to experience, that the property of being a one-way function is not bound to (**Z**/*n***Z**) and exponentiation. For example, cryptographic techniques which base their one-way property on the difficulty of inverting the discrete logarithm within (**Z**/*n***Z**) may be realized on elliptic curves - e.g. Diffie-Hellman Key Exchange ([8] p. 153).

[8] The notion of the key concept (Leitidee) is used here in the same sense as in the framework curriculum for senior secondary level mathematics in Berlin ([12], Kap.2). It combines the mathematical competencies to be acquired in school into competency areas. There is a distinction between process-related competency areas (reasoning and proof, problem solving, modeling, representation, use of procedures and tools, communication and cooperation) and key mathematical concepts (functional relationship, approximation, geometry, data analysis and probability, measurement, algorithms). The key concept "number" is, in contrast to the corresponding curriculum for the first 10 years of schooling, no longer listed as no specific content on extensions of numbers or the concept of numbers is included.

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Appendix

Content	Key mathe-matical concept	Unsolved problems in mathematics	Mathematics as a living science. (Focus: algorithm)	Cryptography is applied mathematics
Classical cryptography				
- History and basic expressions				
- Key exchange problem			x	
Number theory				
Divisibility	N			
- Gcd, Euclidean algorithm (EA)	N/A		x	x
- Bézout's lemma **(**incl. proof) /extended EA	A		x	x
- Numeral system in different bases	N			
Primes				
- Proof: infinity prime numbers	N			
- Sieve of Eratosthenes	N/A			
- Distribution of primes, Prime number theorem (without proof)	N/F/A	x	x	
- Fundamental theorem of arithmetics	N			x
Modular arithmetics				
- Calculation rules	N			
- Exercise: proof of divisibility tests of 3 and 9	N			x
- Fermat's little theorem (incl. proof)	N			x
- Fast exponentiation: square & multiply algorithm	N/A		x	x
Modern cryptography				
Principles of public-key cryptography	F		x	
- RSA – principle (experimentation with CAS)	N/F	x	x	
- RSA – key generation (extended EA)LA	A			x
- RSA – proof of correctness	N			x
- RSA – for authentication	N/F			x
- *Primality tests*	N/A/DR			x
- *Chinese remainder theorem– fast signature*	N		x	x
- *Attacks on RSA*	N/DR			x
- *RSA – on a debit card: weaknesses in the implementation*	N			
- *Factorization: quadratic sieve.*	N	x		

Table: Core topics for the first semester.

Optional topics prepared and presented by the students are shown in italics. This allows both for the variety of possible in-depth topics as well as for personal interest of students and teachers. On the other hand, it serves as a preparation for writing an extensive assignment at the end of the second semester of the Seminarkurs.

Using History to Teach Mathematics

Jacqui Klowss B.Ed (Mathematics/Information Technology)
Year Coordinator (Clayfield College)
District Review Panel Chair (Queensland Studies Authority)
Clayfield College, Clayfield, Queensland, Australia
jklowss@clayfield.qld.edu.au

Abstract

Students today need to be taught not only the real life context of their mathematics lessons but also the historical context of the theory behind their mathematics lessons. Using history to teach mathematics, makes your lessons not only interesting but more meaningful to a large percentage of your students as they are interested in knowing the who, how and why about certain rules, theorems, formulas that they use everyday in class. Students are captivated by learning the history behind mathematicians, rules, etc. and therefore can link the lesson to something in history and a concept. Even learning the mathematics behind historical events motivates and interests them. They cannot get enough!

Introduction

How do we get our students interested in learning mathematics? There are countless times that I have dealt with students who have stated that they “didn’t like maths”, “couldn’t understand why we need to ‘do’ maths” etc. I have even taught very bright students who have a natural ability in mathematics who “didn’t like it”. How do we help them to understand that mathematics can be fun and, hopefully, encourage them to enjoy it? The two major questions I felt needed to be asked were: What do students need/want to know? and How do we, as educators, make it interesting?

I first thought about this problem and asked myself what **I** found interesting and enjoyable about Mathematics. I have always regretted not taking up history during my high school years, although the essay aspect would have killed me! I have had a keen interest in history and I thought that maybe we could link both mathematics and history, to begin with, especially when we have national curriculum that encourages the teaching across the curriculum.

What use could using history to teach mathematics be? I know that we have all taught about Roman Numerals and how the number system changed throughout the years, but what about bringing into our teaching of the mathematics the answer to WHY?

Why is it that we have a ‘number’ zero? Why do we use x’s and y’s to represent variables in algebra? Why did **that** mathematician come up with **that** theorem? Where is/has that mathematical concept been used?

One method of “Using History to teach Mathematics” involves telling the class small stories from maths history books or using class activities that can really engage the students at any stage within a lesson. For example, in a year 8 (12/13 yr old students) classroom, we were beginning a unit on algebraic equations, so I asked the class if anyone knew why we used x’s and y’s to represent our unknowns. I could tell that they were thinking “Why would we know something like that?” So I told them the story of Descartes’ first publication. I told them that back in those days, they had a printer who had to use tiles to print out newspapers, books etc and I explained how that happened. I then told them that the publication went to the printer and the printer then contacted

Descartes and asked him if he needed all the y's and z's in the text. Descartes asked why he wanted to know and the printer told him that he didn't have enough z's. Descartes then asked him what letter he had plenty of and the printer replied 'I have plenty of x's' and the rest is history as they say.

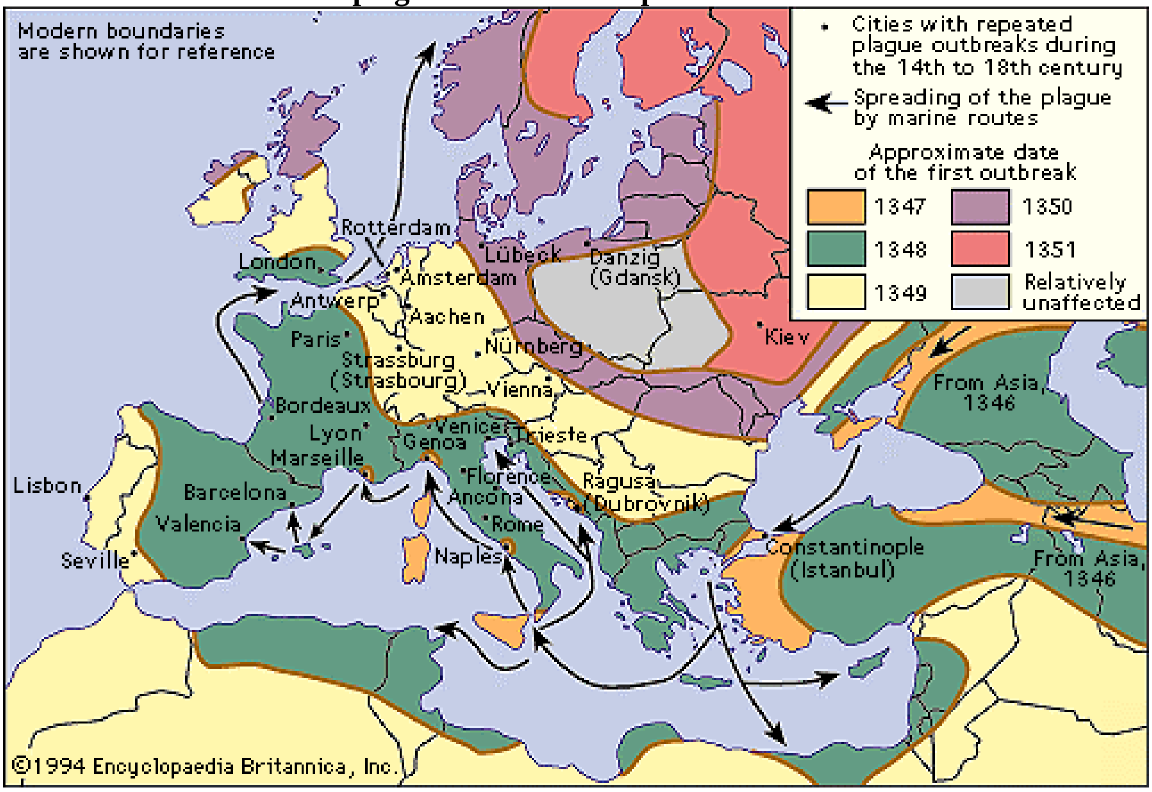
You don't have to be someone who has undertaken a history degree. I just started collecting maths history books and read them. Picking out bits of information that I thought I could use in the classroom. I got this small piece of information from a maths history text book. I thought it was just a little interesting and maybe it would work with the students. About a month or so later, I was speaking to a parent of a student in my class and they mentioned the story. The students had loved hearing it. It seems like a pretty small concept, but it also worked in helping me to make their mathematics learning fun!

The second method of "Using History to Teach Mathematics" is taking actual historical events and extracting the Mathematics from them. One such example is as follows:

WORKSHEET: The Bubonic Plague

The plague is a disease that was rampant in the 1300's and killed almost 20 million people throughout Europe, this was nearly one-third of Europe's population. It was spread to humans by fleas that lived on Black rats. The plague was around for some time but by the time 1665 arrived, it was rampant in London. The table shows the Number of people to die from the Plague from selected years leading up to the infamous 1665 Plague of London. http://www.britainexpress.com/History/plague.htm

The plague moves in Europe 1347 – 1351



1. Using the map above, calculate the area affected by the plague for each year, commenting on the exponential effect the plague had on Europe.
2. By what percentage did each year increase/decrease?

3. Which year had the biggest percentage increase?
4. What area was 'relatively unaffected'?
5. What is the percentage of area unaffected to area affected?
6. For the whole of Great Britain, England, Ireland, Scotland and Wales, what percentage was affected in 1348?
7. What was the increase in area affected in Great Britain for 1349 compared to area for 1348?

Number of Deaths from the Bubonic Plague, London 1592 - 1665

	Year				
Month	**1592**	**1603**	**1625**	**1630**	**1665**
January	123	103	45	0	0
February	94	53	32	0	0
March	87	11	23	0	0
April	140	26	85	0	2
May	200	83	224	0	43
June	1333	362	564	19	590
July	2986	2996	5887	235	4127
August	2404	8922	16455	242	19066
September	1820	12500	9969	310	26220
October	1236	3943	1514	233	14373
November	503	1390	256	133	1414
December	196	160	37	44	0

http://www.schoolshistory.org.uk/plaguedata.htm

8. Draw a line graph with all the yearly periods on the same graph. Use an appropriate scale.
9. How many people died in total for each year shown?
10. Which year had the greatest number of deaths?
11. Looking at your graphs, do you see a trend? Give a possible explanation for this trend
12. Looking in particular at the year 1665, where the plague reached epidemic numbers, draw a sector graph showing the number of deaths each month.
13. From your sector graph for the year 1665, what do you notice? Give a possible explanation for this occurrence.

The above worksheet has been written with the idea that a classroom teacher can change it is any way in order to make it easier or more difficult depending on the need of the class they are teaching. As you can see, there are many, mathematical opportunities within such a worksheet and therefore it can be used to help 'extend' the brighter or the more interested students within the classroom.

I also have an interest in coding and cryptography. This topic is one of my favourites and can be used in all areas of Mathematics , especially statistics. It is also a wonderful historical topic that a teacher can manipulate in order to engage young students or older, high-level mathematics students.

Overall, I have found BOTH methods of using History to Teach Mathematics interests students and engages them in your classroom. I do not believe you have to use these methods *every* lesson, but it certainly makes Maths FUN!

The use of technology to motivate, to present and to deepen the comprehension of math

Damjan Kobal
Department of Mathematics, Faculty of Mathematics and Physics,
University of Ljubljana, Slovenia damjan.kobal@fmf.uni-lj.si

Abstract
The aim of the workshop is to present and discuss several ideas which relate to technology as well as to creative teaching. Educational experience, common sense and educational research have all proven how important for comprehensive understanding different cognitive representations are. We will present and discuss several elementary mathematical ideas of which mechanical realisations mean ingenius technological inventions (for example: 'car differential' and 'digital sound technology'). Technological insights can provide deep intuitive understanding of otherwise abstract mathematical concepts and therefore yield also better comprehension of mathematics. Besides that we will use and present the technology in the form of *dynamic geometry programs* to show, provoke and motivate rethinking and deeper understanding of several elementary mathematical concepts.

Introduction
The discussion and explanations within the workshop will focus on several simple mathematical ideas which have an incredibly intuitive and useful meaning in modern technology. In each case we will start by posing real life (technical) questions which seem to have nothing in common with abstract mathematics, as we know it. While researching the problems, we will go ever deeper and closer to simple and abstract mathematical formulations. All the problems will be addressed by clear and elementary mathematics with an emphasis on the deepness of meaning that already a simple mathematical ideas can carry.
In the second part we will focus on the use of dynamic geometry programs and computers in general. The technology can be used to elegantly present intuitive mathematical ideas and concepts and probably most importantly, to deepen the comprehension by different representations. Furthermore, by its technical challenges to prepare smart dynamic representations it offers enhanced awareness of the mathematical meanings. We will show some smart applets with simple and elegant outputs but with a rich array of mathematical challenges in the background, which can be an ultimate challenge and motivation for a devoted teacher.
We will gradually develop the workshop discussion and exploration adjusted to the knowledge and interest of audience along these titles and questions:

Car differential

- How is a bicycle powered?
- How is the rotation from the pedals transmitted to the powering wheel?
- How can two wheels welded to the same axes rotate and move?
- Why such pair of wheels can only move straight (on a plane)?
- How would such a pair of wheels move on an arbitrary (not plane) surface?
- How is a three or four wheel vehicle powered?
- How do we turn?
- What happens with a power transmission on the left and right wheels while turning?
- Can we explain 'the danger' of snow driving?
- And several other more sophisticated but real life questions related to the movement of a car. The discussion will get us even to a very intuitive understanding of a mathematical concept as abstract as *geodesics*. And the beauty of the whole discussion will reach its climax in the realisation that all the multi faceted meanings of our

discussion is hidden in the understanding of a simple math formula of the arithmetic mean.

Bicycle gears

- How do we shift gears on a bicycle?
- How many gears do we have? How many do we use?
- How different are the gears?
- Ordering the gears.
- Explaining ‘smart gear shifting’ on a bike.
- Clog’s teeth ratio as gears and slopes.
- Very intuitive view of a line slope.

Car lights, satellite dish and the ancient concept of a parabola

- Classical geometrical properties of a parabola will be analysed by the use of computer simulations.
- Insights into preparation of the applets will reveal interesting elementary mathematical challenges which are no easy task even though they require only the basic math knowledge, but skill and precision, which is a true treasure of mathematical thinking.
- We will challenge ourselves to reach the same results analytically and geometrically and compare the two. Comparison will be an interesting provocation of our ‘neglect geometry – promote calculus’ approach.
- The abstract geometric properties of a parabola will be challenged and given intuitive meaning in a computer simulated ‘parabola shaped billiard table’.
- The shape of a satellite dish will be analysed (after knowing the parabola shape ax^2, why a is not bigger, why not smaller?).

Communication technology

- Intuitive understanding of an analog and digital signal.
- How can several conversations be transmitted over one line by digital signal?
- How sensitive are human senses?
- What defines the quality of a sound?
- The idea of a sound as an intuitive function
- The idea of a discrete function
- Where is the limit? How many conversations can be squeezed into a single wire?
- Computer animated simulation of a ‘miracle of transmission of several conversations through one wire’

Computer technology as a teacher of understanding

- Computer animations will be presented, where problem is understood visually.
- Simple questions like ‘What is going on’ will guide us through the fruitful jungle of understanding and comprehension.
- Answers to self posed questions will reveal several otherwise unnoticed mathematical meanings, which will deepen and enhance understanding.

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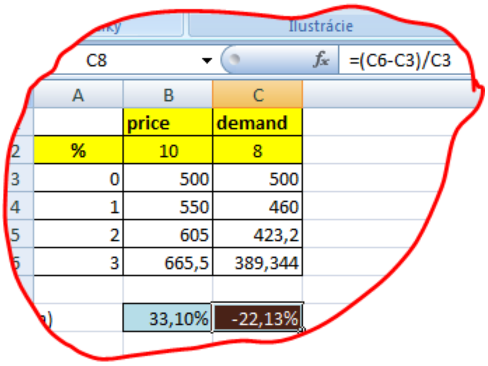
Experience with solving real-life math problems in DQME II project

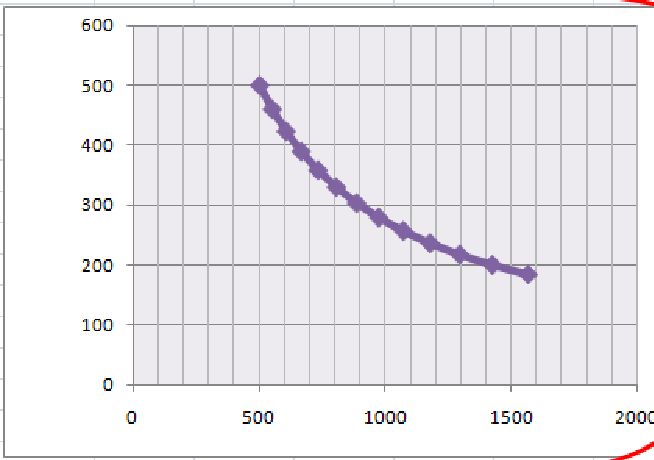
Koreňová L., Dillingerová M., Vankúš P., Židová D

The network "Developing Quality in Mathematics Education II" is a continuation of the associated project "Developing Quality in Mathematics Education" (http://www.dqime.uni-dortmund.de/). In this project participate universities, teacher education institutions and schools from 11 European countries. Cross-cultural cooperation and exchange of ideas, materials, teachers and pupils support developing quality in mathematics education, especially in the area of mathematical modelling.
The quality and application of the developed learning materials is also guaranteed by using, comparing and modifying them in eleven different countries. This comparison leads to an agreement about contents of mathematical learning and teaching in eleven European countries. Thus we want to establish a "European Curriculum for the teaching and learning of mathematics" in the 21st century.
A special feature of this project is the strong connection between theory and practice and between the research and development of mathematics education.
In this project our Faculty of Mathematics, Physics and Informatics of Comenius University Bratislava manage testing of translated teaching materials at the high school „Gymnazium Sturovo“.
We know that using ICT and didactical software in schools is almost present and wide spread. So we try to focus on several possibilities in solving real-life tasks using this technologies, regard to the fact technologies are hard upon the young generation of students.
Tasks (http://www.dqime.uni-dortmund.de/index-entry.php?language_chosen=0)
The price for tobacco and demand. Author: Heinz Böer
Medics want expensive cigarettes
The British journal for medicine "The Lancet" warns about a worldwide lung cancer epidemic and pleads for a drastic increase of cigarette prices. Lung cancer had become the most common kind of cancer, the oldest journal for medicine emphasizes. According to calculations of the world bank a price increase of 10 percent could decrease the demand for tobacco by 8 percent. Westfälische Nachrichten, 14.05.2005
Assume that the relation price/demand is true for a wide range of areas. At which increase/decrease will the parameters be after 3 changes? "12 times decreasing by 8 %, the remaining demand is 4% and nearly gone“- Is that true? Write down a term of a function for the price development and one for the development of the demand for tobacco. Also write down what x stands for. How many steps does one need to double the price (≈). What is the demand then? After how many steps has the demand decreased by 50%? What is the price then? "The demand has to be lower than 1% to make the risk of smoking negligible for the population’s health“ the opponents of smoking say. – The price? Think of other scenarios for prices and demands.
Solution: The author’s solution in a classic way is on the website shown above.
Using MS EXCEL – students can make their own experiments.



Students have a possibility to make experiments, design a sheet computing the demand and offer in each year. Their mathematical thinking can so be enlarged and this is important in solving problems in every day’s life. Here mathematical models of thinking and presenting are requested.

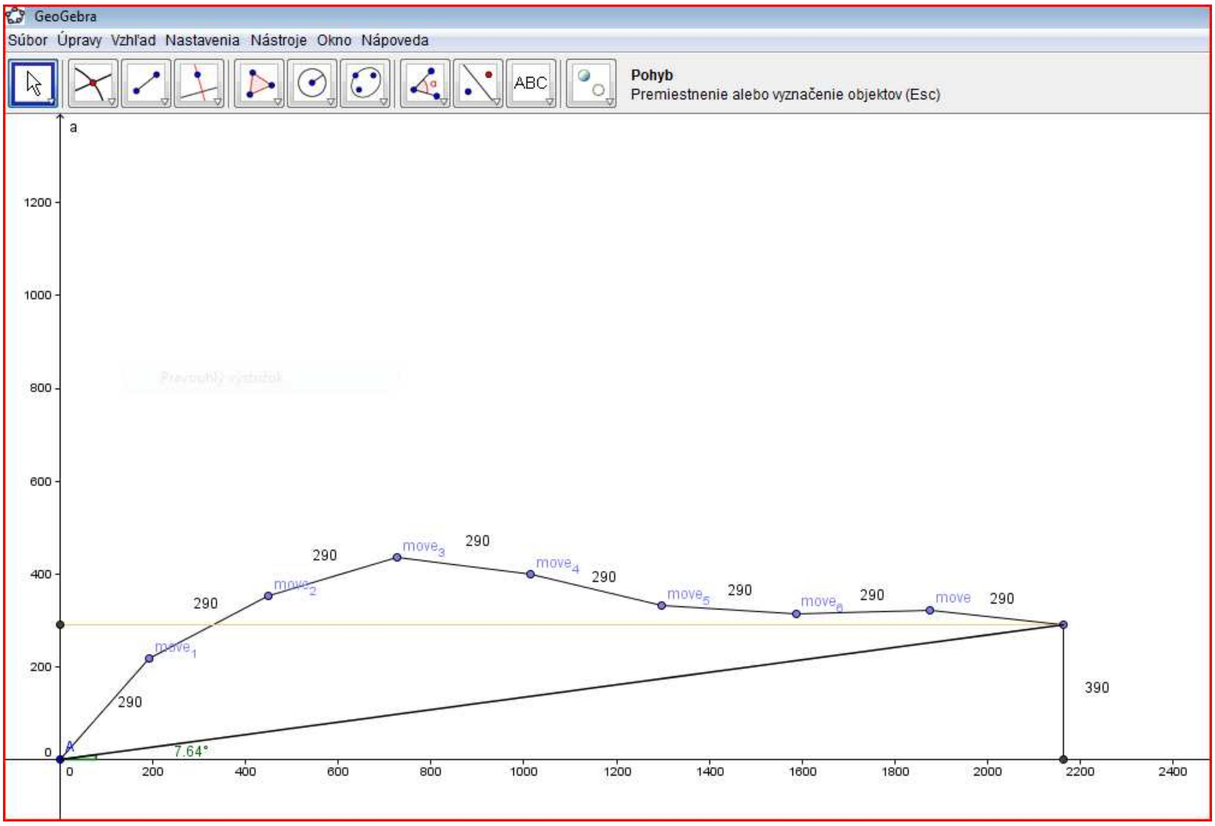
	price	demand
%	10	8
0	500	500
1	550	460
2	605	423,2
3	665,5	389,344
4	732,05	358,1965
5	805,255	329,5408
6	885,7805	303,1775
7	974,3586	278,9233
8	1071,794	256,6094
9	1178,974	236,0807
10	1296,871	217,1942
11	1426,558	199,8187
12	1569,214	183,8332



Tasks (http://www.dqime.uni-dortmund.de/material/Sledging_in_every_weather-10-sin-tan.doc)
Sledging in every weather
Author: Heinz Böer
After about five months of test phase the longest all weather sledging course was officially opened in Todtnau in the Black Forrest. The 2900 metres long course overcomes a height difference of 390 metres and will be opened all year long said owner Adolf Braun. 100 bobsleighs are at disposal. Every hour they can carry up to 350 passengers - children and adults.
Westfälische Nachrichten, 13.06.2005
1. What is the course's average gradient angle?
2. With which percentage can the slope be described (as it is the custom with streets)?

Solution:
The author's solution in a classic way is on the website shown above.
Teaching maths has to be done with an effort of gaining new knowledge by solving problems with various contexts. They should create hypothesizes, check their verity. Another goal is to be able to use several methods of representation of an mathematical content. The students should enlarge their abilities to orient themselves in plane and space. Maths has to develop algorithmical thinking, working with guidelines and create them
In this task we will use GeoGebra (http://www.geogebra.org/cms/) – students can make their own experiments again. They create with some help of their teacher the trajectory built-up of some segments of different length (in the picture are 10 segments each with 290 m length shown), so they can change the sledge course, but the full length will stay 2900 m and the height difference always remains 390 m. They investigate the average slope. In such a process of investigations the students pose a lot of questions and they gain not only some social and informatics competencies, but also knowledge about statistics and trigonometry.

In the DQME II project, there is a huge amount of tasks related to real life experiences. We could show many more, each solved by another type of ICT or software. But the aim of our paper was rather to present few of this tasks but let you much more see the complexity of such tasks and to let you know, ICT and maths software are useful helpers in obtaining experiences and develop competencies.



Workshop: Some interesting math problems for high school students solved by graphic calculators CASIO

Korenova L., Zidova D.

Abstract

The complete solution of real/life problems starts with the specification of the problem, its expression using mathematical concepts, solving it using a mathematical apparatus and interpreting its results using the terminology of the original problem area. In this four-stage process, graphic calculators can be efficiently used for speeding up its third ("purely mathematical") stage. The application of ICT will free the teachers' hands and allow them to concentrate on the pre-solution and post-solution relationships between the problem and its mathematical classification, representation, and meaning. During our 60-minute hands-on workshop, this principle will be demonstrated on examples from financial mathematics and other real-life problems using CASIO ClassPad. The workshop participants can play the role of learners. They are also invited to discuss and express their opinions on even more effective exploitation of this flexible tool. No previous experience with CASIO ClassPad is needed.

Good classroom practice – how a new journal supports this

Rüdiger Vernay

Teacher in Bremen, Germany and editor of the journal "Mathematik 5-10"

("Maths year 5 –10")

In December 2007 a new maths journal in Germany started with a really innovative concept: The journal articles are written by expert teachers with ample classroom experience. All articles show good classroom practice with innovative ideas and teaching methods.Teaching suggestions can be put into practice immediately. Materials needed are provided in a materials pack that comes along with the journal. Schools can order a teacher training service held by members of the "Mathematik 5-10" editorial board and based on particular issues of the journal. This concept will be presented and explained in detail supported by examples taken from the journal.

Accompanying “in-service teaching” internships of prospective mathematics teachers – a model for encouraging exchange between theory and practice using the triple coaching approach

Sebastian Kuntze, Anke Wagner, Claudia Wörn
Institute for Mathematics and Informatics, Ludwigsburg University of Education, Ludwigsburg, Germany kuntze@ph-ludwigsburg.de

Abstract: Developing professional expertise of prospective teachers not only in terms of theoretical knowledge but also in terms of competencies of designing challenging and cognitively activating learning opportunities in the mathematics classroom is certainly one of the key aims of internship phases in pre-service mathematics teacher training. As mathematics-related theoretical contents of teacher training and practice-related learning opportunities of school internships are not always linked in an optimal way, this paper aims at discussing a model of an intensive internship phase combined with a triple coaching approach partly integrated in a course accompanying the internship phase.

1. Introduction

School internships of prospective mathematics teachers do not always accomplish their goals: Short or one-day-per-week forms of internships tend not to give a realistic workplace experience to the prospective teachers. Moreover, in these forms of internships, it is hardly possible to establish strong links between realistic workplace practices in classrooms and mathematics education theories provided in university courses. There is the risk that students may see these contents as purely theoretical and develop patterns of instructional practice almost without taking advantage of academic contents that could support their professional growth.

At Ludwigsburg University of Education, for several semesters, a so-called semester with practical focus has been created, emphasising practice (four days a week for more than three months) and linking it with academic teaching (one day per week at the university). This form of intensified internship offers the possibility to accompany the professional growth of prospective teachers in a potentially crucial phase, and to support the development of classroom routines that conform with research results of mathematics education. Accordingly, from the point of view of mathematics education, the question of how to design learning opportunities at the university for accompanying the teachers’ professional growth is crucial for the quality of this form of “in-service teaching” internship phase.

Consequently, this paper presents a model for encouraging exchange between theory and instructional practice in the accompanying university course. The model uses the triple coaching approach, integrating reflections and feedback of researchers, expert teachers, and peers. The focus contents of tasks/materials, visualisation, and classroom interaction are addressed.

2. Theoretical background

The theoretical background for research on supporting professional growth of prospective teachers is associated with the notion of professional knowledge of mathematics teachers. Professional knowledge and epistemological as well as instruction-related beliefs of mathematics teachers encompass a range of sub-components, which can be structured according to three criteria. Firstly, such components may be located more on the side of “knowledge” or more on the side of beliefs or prescriptive convictions. However, as clear distinctions are often impossible (e.g. Pajares, 1992), we assert a spectrum between knowledge and convictions/beliefs within the notion of professional knowledge. Secondly, following the approach of Shulman (1986), domains of professional knowledge may be distinguished, such as pedagogical knowledge, content matter knowledge or pedagogical content knowledge. Thirdly, professional knowledge can be global, content domain specific, related to a particular content or specific for a particular instructional situation (cf. Törner, 2002; Kuntze & Reiss, 2005). Figure 1 sums up these distinctions, even though we emphasise that the distinctions are less strict than the schematic overview in the figure may suggest.

Even though the relationship between components of professional knowledge and instructional practice is still not completely understood (Tillema, 2000), there are empirical findings suggesting that variables in the domain of professional knowledge play crucial roles for the development of instructional practice (e.g. Putnam & Borko, 1997; Lipowsky, 2004).

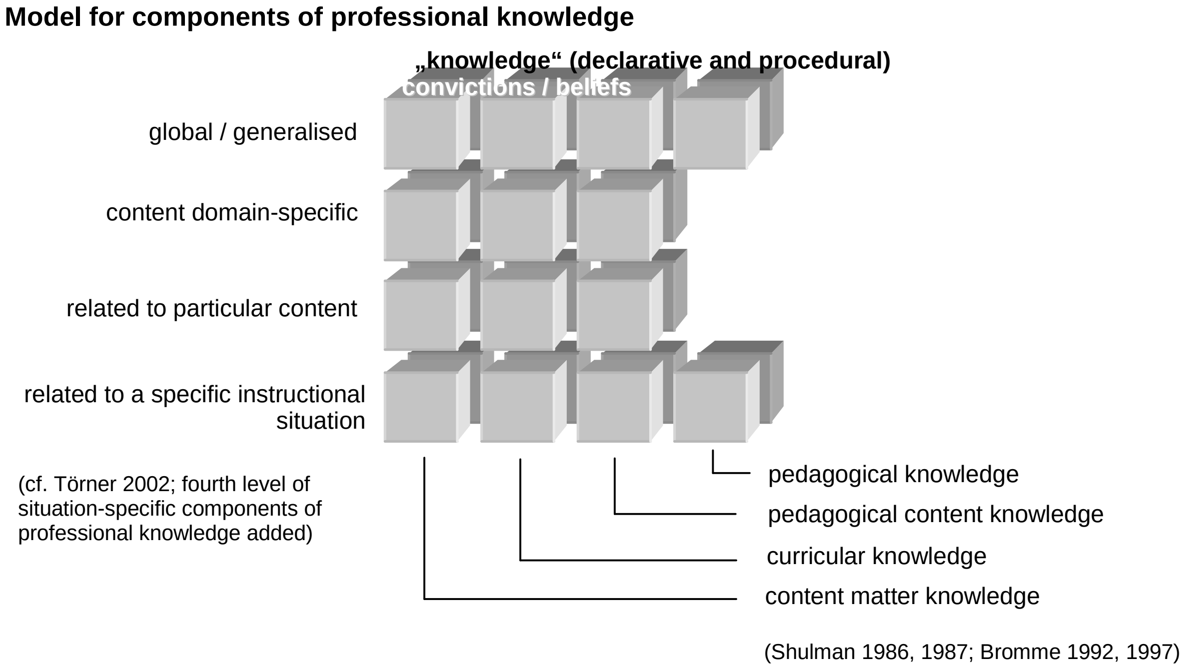


Fig. 1: Model for components of professional knowledge (Kuntze & Zöttl, 2008)

As far as the training for prospective teachers in internship phases is concerned, questions associated with the interplay of instructional practice and professional knowledge are in the centre of interest, because the rather theoretical input of university courses focuses on the development of professional knowledge, which should be linked to instructional practice in internship phases. However, there are inhibiting factors which seem to make it difficult for prospective teachers to establish solid connections between the theoretical input of university courses and practice-related experiences. For example, academic mathematics instruction contents may lack of connections with other theoretical and practice-related knowledge, the contents of university courses may be perceived as unspecific for particular lessons or instructional situations by the prospective teachers and the instructional practice in the internship phases may lack a realistic character as far as circumstances of the teaching profession are concerned.

Consequently, goals for improving school internship phases should focus on the aspects of interconnecting levels of globality in the model of Figure 1, corresponding to theory and content domains on the one hand and to classroom practice in instructional situations on the other. A possibility of facilitating the development of such connections consists in coaching support of the prospective teachers (see section 4 below).

We expect that a focus on strengthening practice-relevant connections in professional knowledge can improve the outcome of internship phases in terms of the prospective teachers' professional growth. As empirical evidence on the effectiveness of intensive internship phases is rare (cf. e.g. Lipowsky, 2004), we will focus on findings concerning the effectiveness of in-service teacher training in the next section, as the integration of instructional practice in the teacher training is a key issue of accompanying internships as well.

3. Effectiveness of teacher training

Teacher training accompanying internship phases should include a focus on the instructional practice

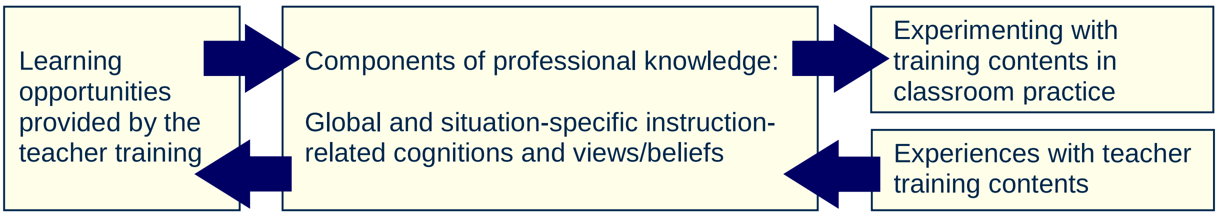


Fig. 2: Model for the implementation cycle of in-service teacher training (cf. Kuntze, 2006)

of the participants, like it is also the case for in-service teacher training (cf. Lipowsky, 2004). Consequently, the design of teacher training accompanying internship phases can profit from empirical findings concerning in-service teacher training. For example, the relationship between learning opportunities provided by the training and instructional practice can be described in a model for the implementation cycle of in-service teacher training (Kuntze, 2006), in which professional knowledge as described above plays a mediating role. The impact of in-service teacher training can be observed empirically on different levels (Lipowsky, 2004; Wade, 1985): The level of impacts reported by the participating teachers themselves, the level of developments in the professional knowledge of the participating teachers, the level of changes in the teachers' classroom practice and the level of competency developments of the students in the teachers' classrooms. According to the overview study by Lipowsky (2004), in-service teachers training programs are effective when they (cf. Garet et al., 2001; Barnett & Sather, 1992; Richardson, 1996; Wade, 1985):

- cover a longer period of time and combine phases of theoretical input, reflection, training and implementation
- support cooperative work in professional communities or teacher teams and enable self-regulated, structured and goal-oriented work
- address convictions and cognitive components of professional knowledge of the teachers and when they have strong connections to mathematics education contents
- are linked to instructional practice, and in particular when they include elements of coaching.

Video-based teacher training can respond to these characteristics: for example, video technology enables participants view instructional situations repeatedly, which can enhance reflection on instruction (Sherin, 2004; Seago, 2004). For this reason, videotaped instructional situations often play a role in coaching projects, too (Sherin & Han, 2003). Lipowsky (2004) identified instruction-related reflection by the participants as a prerequisite of the success of teacher training in the domain of developing professional knowledge. For the case of video-based training, the role of encouraging reflection processes is emphasised e.g. by the findings of Beck, King and Marshall (2002).

4. Coaching

As already reported above, support of teachers by elements of coaching can not only encourage teachers to reflect on instructional quality and on their classroom practice, but it provides teachers with a structured and focused help when they aim at improving and developing their classroom practice (Staub, 2001). The role of the coach can vary: The coach can be an expert, representing the position of research on instructional quality; the coach can be an experienced teacher giving feedback according to the framework of cognitive apprenticeship; the coach can even be a peer sharing the perspective of the learner being coached.

Coaching has shown to be successful when the coaching focused on the professional context and when the coach supported the process rather than taking the role of a problem solver (Joyce & Showers, 1982; Collins et al., 1989; Rauen, 1999).

5. Model of the "in-service teaching" internship and the accompanying course

Against the background of the considerations and empirical findings presented above, we conclude that coaching support can be an effective way of fostering the development of practice-related professional knowledge of prospective teachers, which also conforms with essential training goals of pre-service mathematics education.

Consequently, in a current project, we will integrate coaching components in an accompanying course of an intensive internship phase. This course will take place in the framework of a pilot "in-service teaching" internship programme of Ludwigsburg University of Education. In this pilot programme, prospective teachers spend four days a week during one semester teaching at a school and the fifth day at the University of Education for accompanying courses.

In our project, the development of an accompanying training and its evaluation will be in the focus. The model we use in our project is based on a triple coaching approach, integrating reflections and feedback of researchers, expert practitioners/teachers, and peers. Moreover, the triple coaching approach concerns the three areas of use of materials, representations and tasks, planning of

instruction, and interactions/discourse in the classroom.
Consequently, in the accompanying university course, the focus content areas of tasks/materials, representation/visualisation, and classroom interaction are addressed. The contents and framework of learning opportunities provided by the accompanying course follow the matrix-like structure shown in Figure 3. The cells of this table can be used to give an overview on the different learning domains and coaching activities of the course accompanying the internship.

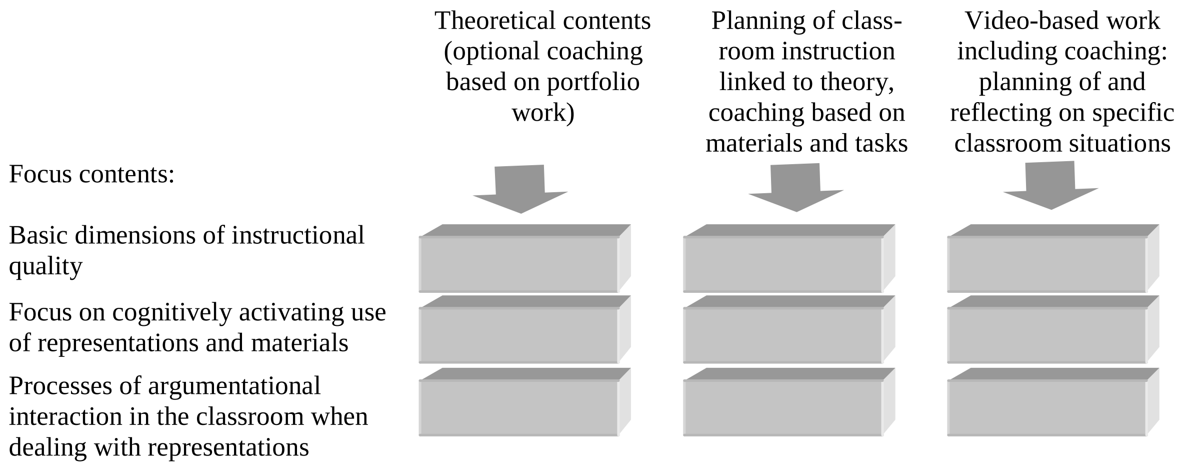


Fig. 3: Structure of learning opportunities in the accompanying university course

6. Evaluation research: Research questions and design of the study

In order to find out about professionalisation processes associated with the internship, the project includes an evaluation research component. The research questions concentrate on the observation of possible developments in the professional knowledge of the participants and on their (self-reported) views on the internship phase.

In the theoretical background section, empirical results concerning in-service teacher training have been asserted to be at least partly valid for the situation of the accompanying course of the intensive internship phase, as prior empirical research in this domain is still relatively rare. This raises interesting additional research questions about the applicability or generalisability of the results cited above. As the internship phase is situated in a relatively early phase of the professionalisation process of the prospective teachers, they might lack of prior instruction-related experiences when being confronted with the learning opportunities of the course. However, an opposite effect might also occur: As there might not yet be very stable classroom routines of the prospective teachers, the internship phase might succeed in the goal of supporting the teachers to build up practice-related professional knowledge and routines coherent with research about instructional quality.

7. Conclusions

On the base of the model for professional knowledge in the theoretical background section and empirical research about the effectiveness of in-service teacher training, elements of coaching are considered as an important support for the professional growth of prospective teachers in the intensive internship semester. Complementary forms of coaching and complementary mathematics education contents are integrated in an accompanying course that aims at establishing strong links between theoretical academic teacher training contents and the instructional practice of the intensive internship phase. These elements have the potential of enhancing professionalisation processes – which sets interesting perspectives for the evaluation research of the project and beyond, e.g. concerning research into impacts of school internships on the professional growth of mathematics teachers at a more general level.

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The influence of localization and materialization of mathematics activities on the indigenous first grade students' learning effects: Two assessment results

Li Tsung Wen KUO, Wei-Hao CHENG and Chih-Chen KAO
The department of early childhood education, National Taitung University,
Taitung, Taiwan. li4135@gmail.com

Abstract

This research aimed to discuss the indigenous students' learning effects of mathematics which was based on the self-designed localization and materialization of mathematics activities and had proceeded for one year. The quasi-experimental method was used in this research. There were 58 indigenous first grade students which were divided into three experimental groups (A, B, C) and one control group (D). Experimental instruments embodied written tests and manipulative tests which were designed by researchers according to the indicators proclaimed by Ministry of Education. The main findings were as followed: (1) The influence of localization and materialization of mathematics activities on the indigenous first grade students' learning effects was limited. (2) According to the result of Paired T-test of the written and manipulative tests, most of scores of manipulative tests were higher written tests.

Research Motivation and Goals

Due to the low socioeconomic status, deficient in family culture, and the inappropriate education strategies of indigenous students who live in remote tribes, many researches showed that indigenous students' academic grades tended to be inferior. Tan (1997, p.37) found out that many researches indicated that the education of indigenous people tended to be low and the ratio of schooling and drop-out of junior high school was higher, therefore, they were in disadvantage to move upward status (e.g., Wang, 1992; Li and Chien, 1992; Li and Hsu, 1984; Tsui, 1983; Tan, 1996). Indigenous students' inferior academic grades and learning difficulty in mathematics were often mentioned in many researches as well (Chen, 1998; Chuan, 2000; Chih, 2001). These descriptions of indigenous students' inferior academic grade and mathematics performance were the spurs for researcher's motivation.

Tan (2002) pointed out that according to the relative researches in indigenous students' learning they prefer dynamic and concrete manipulation of learning (e.g. Liu, 1987; Chu, 1991; Kuo & He 1997; Chih, 1988; Lin, 1998; Kuo, 2001). For the mathematics education in different culture, Bishop and Zaslavsky emphasized to combine learner's cultural background and made mathematics activities useful and successful (Bishop, 1992; Zaslavsky, 1988). Bishop (1992) mentioned that there were different peoples who were frustrated by mathematics or estranged them from mathematics. Chih (2001) in a two-year and half research in indigenous mathematics activity, the greatest contribution was to take cultural context into consideration to students' difficulty in learning. The outcome of the research confirmed the importance of integration of materialization and the understanding of local culture with activities.

Besides, in relative researches, indigenous students preferred materialization and concrete manipulative activities. The learning activities had better to do with living experience and community culture. The mathematics activities which we designed were proceeded in class teaching. Paiwan first grade students in Taitung area were main subjects in this research. The effect of which was compared to the ones of experimental groups and control group.

Research Methodology

Research Design

This research divided subjects into four groups which were A (n=14), B (n=16), C (n=17) and D (n=11). Group A, B and C were experimental groups. Group D was control group. Group mathematics manipulative teaching (10 times * 2 sessions * 40minute = 800 minutes) in the semester and/or five-day summer and winter camp (5 day * 4 session * 40 minute = 800 minutes) were proceeded in experimental groups. There were no treatment for the control group D. 10 mathematics manipulative teaching activities and summer and winter camp were proceeded in experimental group A. 10 mathematics manipulative teaching activities were proceeded in experimental group B. Summer and winter camp were proceeded in experimental group C. There were no teaching activities, summer and winter camp for the control group D. The experiment treatment in this research was based on the result of action research in previous year. We designed "Localization and Materialization of Mathematics Activity" for indigenous first grade students in Paiwan tribe in Taitung, Taiwan. We mainly focused on materialization manipulation in the first semester. Culture was taken into mathematics activities design for second semester.

Research Instruments

According to ability indicators of first grade mathematics in elementary school, researcher edited five written tests (one pretest and four posttests) with similar levels. The researchers read the items to the students in group with one to one narration. Each correct answer of question gained one point. There were 32 points for total. Item difficulty of the written test was p = .42, the index of discrimination D = 49.23, Cronbach α = .908.

Researcher deleted four questions of calculation in above-mentioned written test and turned the above-mentioned written test into manipulation test with concrete items. The researcher worked with one student in the manipulation test. There were 28 points for total. Item difficulty of the written test was p = .53, the index of discrimination D = 52.24, Cronbach α = .875.

Research Result and Discussion

Performance of Mathematics Written Test

For the discussion on students' score in experimental groups (A, B, C) and control group (D) , we could reveal that the mean score (standard deviation) of pretest which was given in the beginning of the first semester in the first year were 12.57(6.86), 12.81(6.94), 13.23(6.66), and 12.09(7.38). The first post-test was given at the end of first semester and the score were 21.27(5.64), 20.12(7.22), 21.62(6.49), and 1.72(6.49). The second post-test was given at the beginning of second semester and the score were 25.28(4.51), 23.75(7.10), 23.41(5.29), and 24.90(7.03). The third post-test was given at the end of second semester and the score were 30.00(2.21), 28.43(4.73), 27.64(4.27), and 28.63(5.98). The fourth post-test was given at the beginning of second year and the score were 27.57(4.81), 26.87(5.80), 26.82(4.53), and 28.91(3.59).

To integrate the above statistics, according to the figure 1, after students in experimental groups accepted the localized and materialized activities, experimental group A had great progress among three experimental groups. The second one is experimental group B. Experimental group C had the least progress. To compare with control group, it had greater progress than experimental B and C after the second post-test. After the summer vacation, the three experimental groups reduced but the control group had progress.

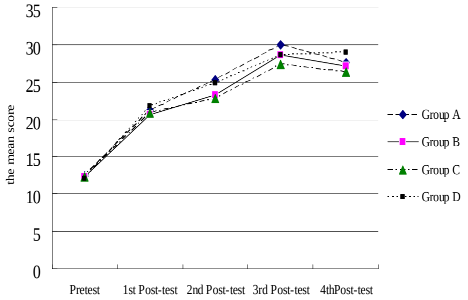

Figure 1. Mean scores of written tests

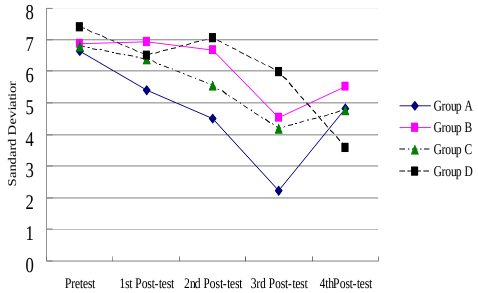

Figure 2. SD of written tests

Table 1
Analysis of Covariance (ANCOVA) of Written Tests

ANCOVA for 1st Post-test	SS	df	MS	F	Sig.
Covariate (Pretest)	659.559	1	659.559	24.371	.000
Between	38.441	3	12.814	.473	.702
Within (error)	1326.092	49	27.063		
ANCOVA for 2nd Post-test					
Covariate (Pretest)	669.980	1	669.980	27.799	.000
Between	53.278	3	17.759	.737	.535
Within (error)	1277.363	53	24.101		
ANCOVA for 3rd Post-test					
Covariate (Pretest)	101.220	1	101.220	5.682	.021
Between	48.367	3	16.122	.905	.445
Within (error)	944.068	53	17.813		
ANCOVA for 4th Post-test					
Covariate (Pretest)	155.490	1	155.490	7.490	.008
Between	44.339	3	14.780	.712	.549
Within (error)	1100.240	53	20.759		

*P<.05

For the discussion on the standard deviation of the four groups, the standard deviation of experimental groups and control group reduced gradually. According to figure 2, experimental group A had lowest standard deviation, and control group had highest one after the third post-test. However, the fourth post-test showed the three experimental groups raised and the control group reduced on SD.

Used the ANCOVA to control the pretest score, according to the table 1, the result showed there were no significant differences between the four groups. The influence of the localized and materialized activities had limitation. One of the main interferences to the whole research process was the control group gave courses review in the two weeks before the second year. From the interview, the control group spent more time on mathematics activities than other groups was another interference.

Performance of Mathematics Manipulative Tests

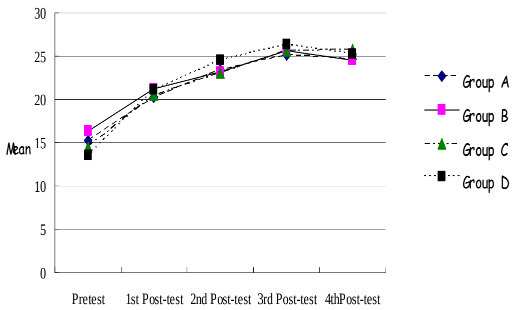

Figure 3. Mean scores of manipulative tests

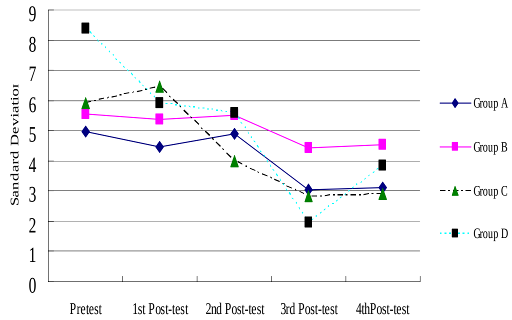

Figure 4. SD of manipulative tests

Table 2

Analysis of Covariance (ANCOVA) for Manipulative Tests

ANCOVA for 1st Post-test	SS	df	MS	F	Sig.
Covariate (Pretest)	595.549	1	595.549	37.195	.000
Between	17.224	3	5.741	.359	.783
Within (error)	784.560	49	16.011		
ANCOVA for 2nd Post-test					
Covariate (Pretest)	467.911	1	467.911	29.191	.000
Between	48.890	3	16.297	1.017	.393
Within (error)	849.544	53	16.029		
ANCOVA for 3rd Post-test					
Covariate (Pretest)	108.507	1	108.507	11.994	.001
Between	19.411	3	6.470	.715	.547
Within (error)	479.479	53	9.047		
ANCOVA for 4th Post-test					
Covariate (Pretest)	129.176	1	129.176	12.603	.001
Between	28.442	3	9.481	.925	.435
Within (error)	543.226	53	10.250		

* $P<.05$.

We could reveal that the mean score (standard deviation) of pretest and post-test of mathematics manipulative test to the four groups (A, B, C, D) were 15.28(5.16), 16.31(5.65), 14.47(5.93), and 13.54(8.38); the score of first post-test were 20.27(4.67), 21.18(5.44), 20.50(5.01), and 21.09(5.92); the score of second post-test were 23.42(4.89), 23.18(5.76), 23.00(3.75), and 24.54(5.59); the score of third post-test were 25.14(3.03), 25.56(4.60), 25.64(2.76), and 26.36(1.96); the score of fourth post-test were 24.64(3.12), 24.71(4.55), 25.28(2.88), and 25.27(3.84).

From the above statistics, it showed that both students in experimental groups and control group had progress in the manipulative test. According to figure 3, the performance of students in control group was much better than the one of students in experimental groups. Moreover, according to figure 4, the standard deviation of control group tended to be lower. It was different from the result of written test before the third post-test. The scores of the four groups were regressed in the fourth post-test.

If we regarded the pretest score of manipulative test as covariance, and analyzed four post-test scores by analysis of covariance, it showed that there were no significant differences of manipulative test between students in experimental groups and control group (as table 2). Therefore, localization and materialization of mathematics activities had limited effects on the mathematical learning.

Comparison between the Performances of Written Tests and Manipulative Tests

The researcher analyzed the data of students' pretest and post-test in written and manipulative test by Paired T-test. According to table 3 and 4, it showed that there were significant differences between written and manipulative test in pretest (T=-5.92, p=.000). For the discussion on the average number of correctness, the number of correctness in manipulative test was higher than the ones in written test. The first and second post-test showed the same results. However, the result showed differently in third post-test. There were no significant differences already (T=-.79, p=.434). The possible reason for this situation might be most of the children had acquired the mathematics concepts for they had learned it for one year. Therefore, there were no significant differences between written test and manipulative test. The interest thing was that there were significant differences between written and manipulative test in fourth post-test (T=-2.60, p=.012). So, the students had better performance in manipulative test with concrete material than the abstract written test.

Table 3

The Average Number of Correctness for Written Test and Manipulative Test

Type of test	Pretest *Mean* *(SD)* n	1st Post-test *Mean* *(SD)* n	2nd Post-test *Mean* *(SD)* n	3rd Post-test *Mean* *(SD)* n	4th Post-test *Mean* *(SD)* n
Written Tests	11.63 (5.65) 58	18.12 (5.24) 54	20.84 (5.06) 58	25.12 (3.81) 58	24.02 (4.30) 58
Manipulative Tests	15.01 (6.14) 58	20.77 (5.13) 54	23.44 (4.89) 58	25.63 (3.26) 58	25.05 (3.51) 58

Table 4

Paired T-test of Written Tests and Manipulative Tests and Deferred tests of Students.

	Paired Variance Deviation				
	M	*SD*	*t*	*df*	Sig.
Pretest					
Written Test – Manipulative Test	-3.36	4.33	-5.92	57	.000*
1st Post-test					
Written Test – Manipulative Test	-2.65	3.02	-6.45	53	.000*
2nd Post-test					
Written Test – Manipulative Test	-2.60	6.56	-3.02	57	.004*
3rd Post-test					
Written Test – Manipulative Test	-.52	5.00	-.79	57	.434
4th Post-test					
Written Test – Manipulative Test	-1.05	3.09	-2.60	57	.012*

Conclusion and Suggestion

There were no significant differences in statistics according to the quantitative data of teaching effects of localization and materialization of mathematics activities. We could just know that the effects of localized and materialized teaching were limited according to the written tests and manipulative tests which were not similar with the other studies. The studies which pointed out combined indigenous culture and live experiences into the teaching activities (You, 2000; Tan, 2002; Tan & Lin, 2002; Bishop, 1992; Zaslavsky, 1988) and manipulative teaching activities to indigenous students (Lin, 2000; Chih, 2001; Tan, 2002) could promote indigenous students' learning effect. The researchers needed to check the research process and extraneous variables and internal validity and find out the possible reasons.

However, according to result of comparison between written tests and manipulative tests, the score of manipulative test was higher than the one written test. We sincerely suggested that teachers used concrete teaching materials much more to help indigenous first grade students' learning of mathematics concepts. In addition, most of activities and teaching materials based on mainstream culture, and that were different from students' living environment and experience very much. Therefore, teachers should endeavor to integrate localization ingredient with activities. The regressive performance which happened after a summer vacation was worth to mindful and make an effort to it.

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Interactive PDF Documents in Math Education Focused on Tests for Differential Equations

Silvie Kuráňová, Mgr.
Science Faculty, Masaryk University
Brno, Czech Republic
kuranova@mail.muni.cz
www.math.muni.cz/~xkuranov/de

Abstract

The progress of blended learning has given rise to the need to prepare quality electronic materials, especially those which use the greatest advantage of an electronic document – its interactivity. This paper presents several types of PDF materials – interactive exercises, tests and games created by LaTeX packages (AcroTeX eDucation Bundle) with a contribution of other supporting instruments (3D graphics, fancytooltips, AcroFLeX). Differential equations, as an important tool of continuous mathematical modeling, have been chosen to demonstrate the still increasing power of PDF documents. This strategy allowed me to introduce innovative approaches in explaining and exercising this part of mathematics at the same time. To create such materials some LaTeX knowledge is needed; nevertheless this article is for all math teachers who are looking for quality interactive materials.

Introduction

As a number of Czech university students increases and a tendency for introducing eLearning (or Blended Learning) grows, there is a big need to develop such electronic learning and training materials, which would maintain the academic level of knowledge and apply the advanced technology efficiently at the same time.

What we can expect from such materials? It is obvious that just digital substitution of a paper, whose only active elements are hypertext references, does not satisfy today needs. The most valuable benefit of these documents should be their interactivity in a sense of requiring an input from a user and providing him an instant reply. Self-testing, exercising, making homeworks, manipulation with graphics (animations, graphs or even 3D pictures), automatic evaluation or immediate feedback – this is just a short list of functionalities that in some way substitute or at least enrich the interacton between a student and a teacher.

For academic purposes there is also a necessity of widely spread document format (interchanging among universities) and, if possible, including free software tools for developing documents. Portable document format (PDF) seems not only to cover all of these requirements but offers much more.

This paper focuses on one possible way of providing interactive PDF documents (tests, exercises, quizzes, games) using AcroTeX eDucational[1] system Tools (AcroTeX). “AeST [i. e. AcroTeX] is a collection of LaTeX packages that work together to provide a comprehensive system for authoring high-quality and visually pleasing interactive digital documents.” [13, p. 2].

Mentioning all the capabilities of AcroTeX would exceed the extend of this article, so the emphasis is given to present AcroTeX exercising and testing facilities and to demonstrate them by practical examples from the field of differential equations. Also other interactive elements included in PDF such as 3D graphs, fancytooltips and AcroFleX graphics are discussed.

PDF – Background

Portable document format (PDF) is appropriate for academic purposes for several reasons. It is reliable and exchangeable format of text documents generally accepted by both dominant operating systems (Windows and Linux). It is also possible to create it by free software (e.g. pdfLaTeX) and finally it brings many new facilities and improvements to its functionality for its constant development (the key functionality being the usage of JavaScript).

1 „The word 'eDucation' is meant to connote 'Electronic Education'. “ [14].

Applying the Acrobat JavaScript brings new opportunities to implement several procedures and objects that, if used correctly, enlarge the power of PDF documents. “You can tie Acrobat JavaScript code to a specific PDF document, a page, field, or button within that document, or a field or button within the PDF file, and even to a user action.” [1]. The latest versions of PDF even support multimedia (audio/video), 3D pictures or annotations, flash animations and more.
All PDF-related features mentioned above bring a surprising interactivity to the document (e. g. you can easily modify the appearance of PDF files, make interactive forms or use form elements in them). AcroTeX user can benefit from these features even without any knowledge of Acrobat JavaScript language.

AcroTeX
“AcroTeX eDucational System provides a unified set of authoring tools” [13, p. 1], which are “designed for educator who wish to write tutorials, topical essays, presentations, exams, quizzes or assignments. The content can be written for the screen, or optionally, for paper.” [13, p. 2].
Author of these macros is university professor D. P. Story[2].
AcroTeX collection of LaTeX packages consists of several components; all helping the author to create a user friendly and edu-focused content, and thus alter white papers into dynamic electronic document. The core is AcroTeX eDucation bundle (AeB) designed for authoring exercises and interactive tests with feedback and evaluation. At this part I aim in the text below.
All informations about AcroTeX along with quality documentation and number of examples are available on the official web site [2] or on the sites of the author, D. P. Story [14].

AeB in practice
This section introduces variety of AeB tools, especially testing facilities, which I tried to demonstrate in this article by PDF materials (shown as screenshots). For my work I have used some exercises from the books [10, 12], a lot of support and inspiration I have found by R. Mařík [8] and by D. P. Story. All the graphs have been created by CAS Maple.

1. Testing facilities (`Exerquiz` package)

1.1. Type of questions
Selective questions
There are two types of questions: single choice (one right answer) and multiple choices (two or more right answers). In order to visually extinguish one type of question from another, we can choose from three types of layouts (see Figure 1). Appropriate seems to be combining circles with check boxes (1+2, Figure 1) or link style with checkboxes[3] (2+3, Figure 1).

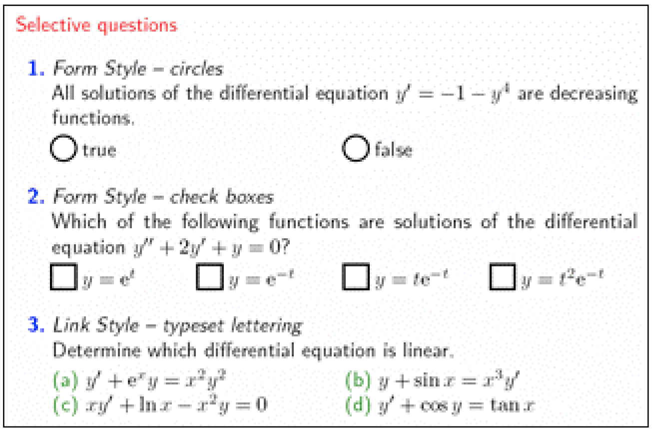

Figure 1: Selective questions.

Objective style questions
a) Text fill-in
If we request an own-words answer we create a text fill-in field using `\RespBoxTxt` command and specify all the acceptable possibilities (questions 1(a) and 1(b), Figure 2). “The underlying JavaScript compares the user's response against acceptable alternatives, as supplied by the author of the question. If there is a match, the response is deemed correct.“ [15, p. 100]. If our question claim more words as an answer (question 1(c), Figure 2) we can also use `\RespBoxTxtPC` command to “create text fill-in questions that awards credit each time one of the key words are found in the

2 The University of Akron (Florida, USA)
3 First approach is possible in the `quiz*` environment, the second in `quiz` environment, more in part 1.2.

student's input string." [15, p. 102].

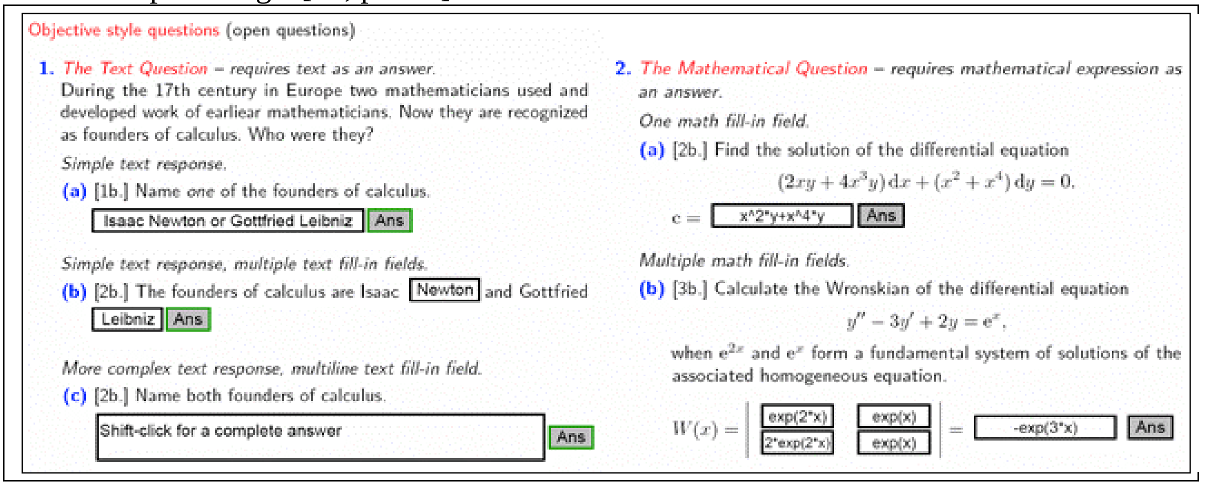

Figure 2: Objective style questions.

b) Math fill-in

AcroTeX command `\RespBoxMath` (question 2(a), Figure 2) enables such questions where mathematical expression as an answer is needed. "The algorithm used for determining the correctness of the answer entered by the user is very simple: The user's answer and the correct answer are evaluated at randomly selected points in an interval, then compared. If any of the comparisons differ by more than a pre-selected amount, an value, if you will, the user's answer is declared incorrect; otherwise, it is considered correct." [15, p. 98].

Questions 1(b) and 2(b) in Figure 2 also demonstrate a possibility of grouping single answers together, which is appropriate where a lot of fill-in answer fields appear. In this case is `\MathGrp` environment used (for both text and math questions) and only one "Ans" button (for displaying a correct response) is generated for the whole group of answer fields.

Multipart questions

For more complicated quizzes where the answer is required in few steps, both above mentioned styles of questions might be decomposed in up to three levels using command `\multipartquestion`.

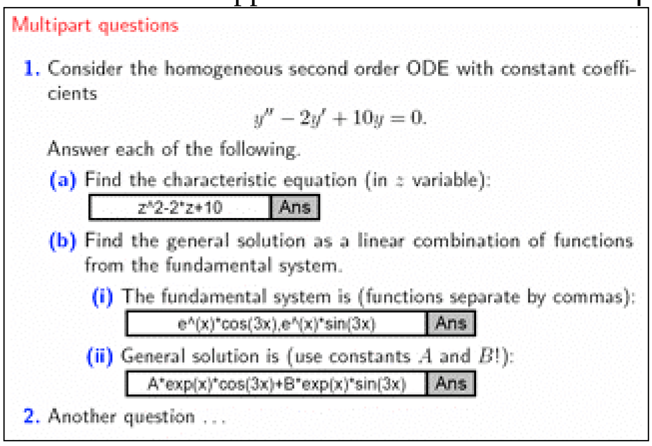

Figure 3: Multipart questions.

1.2. Exercises and tests

`Exerquiz` package provides several types of environments. Questions with solutions are possible to create with `exercise` environment; while for making tests we have more options:

1. `shortquiz` environment for tests with immediate feedback. Several tests of this type for differential equations are made by R. Mařík [8].
2. `quiz` (or `quiz*`) environment for tests with delayed feedback. The test must be initiated by hitting Start button and it is not evaluated until the End button is pressed. "Before completing the quiz, a student can easily change alternatives. This type is more suitable for longer quizzes. The choices the student makes are visually recorded for the student to review and change before clicking on 'End Quiz'." [15, p. 82] After finishing the quiz optionally points, correct answers and solutions are provided.

As an example, see the six-page PDF document *Interactive exercises and quizzes for differential equations* available at [5] with a question (`exercise` environment) and a test with delayed feedback (`quiz` environment), both with solutions at the end of material.

2. Electronic practice cards (eCards)

This tool is appropriate for the type of questions which requires from the student longer own-words answer, complicated calculations or drawing some pictures.

Example 1 – assignment of appropriate type of question:
Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.
All solutions of the differential equation $y' = -1 - y^4$ *are decresing functions.*

Example 2 (figure 4) – demonstration of eCards with a question where sketching direction field of given differential equation is required.

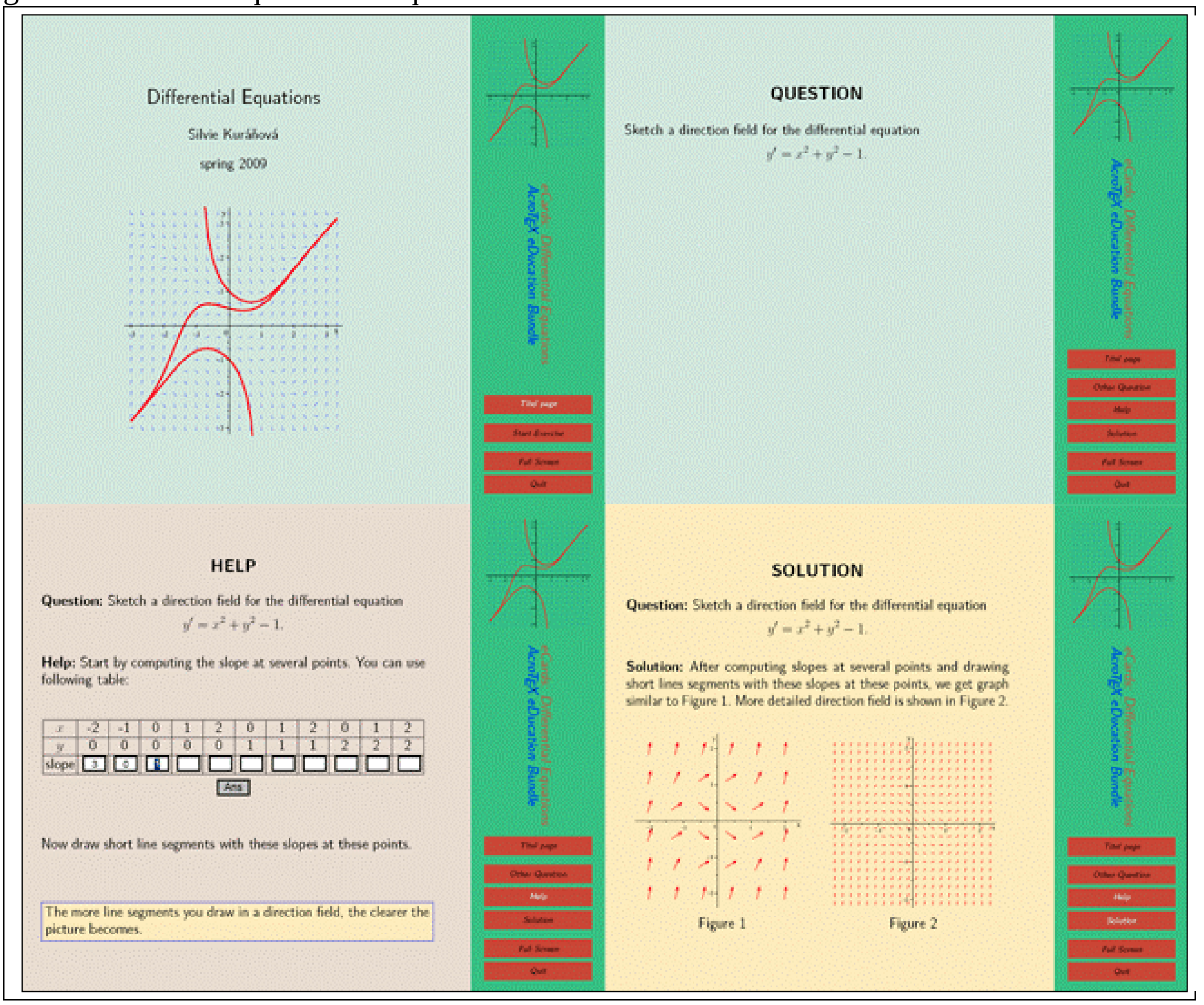

Figure 4: A four-page PDF document, example of eCards.

3. Matching games (Das Puzzle Spiel – DPS)

DPS is one of the AcroTeX game components, easy to create and providing enjoyable study material. "[...] students must match the question posed in one column with its answer in another column. Questions and answers are randomly listed."[13, p. 8]. For inspiration see e. g. matching game *Differential equation with separated variables* in [8].

Graphics in PDF: One picture stands for thousand words

All static graphics have been made by CAS Maple, exported to EPS format and then using `eps2pdf` command transformed to PDF. Incorporating such graphics into my materials required then just a simple `pdfLaTeX` command.
In case of dynamic graphics, Maple offers export to GIF, by which we can get an animated picture ready for posting on web while interlinking with PDF. Generally, dynamic graphics, such as animations or interactive graphs with changeable parameters for drawing, are all very contributory instruments however embedding them directly into PDF document is more complicated. An easy

solution of this challenge provides AcroFLeX.
We can even give PDF document a third dimension by including 3D objects, very useful for visualizing the geometric meaning of mathematical problems, and thanks to JavaScript have an opportunity to manipulate with them (moving, zooming, rotating, changing light etc.).

1. AcroFLeX Graphing Bundle
While AcroTeX is developing since 1999, AcroFLeX is quite new (since 2008). It "is used to create a graphing screen that can be incorporated into a PDF document and viewed within Adobe Reader, version 9.0 or later. The graphing screen can be interactive or non-interactive." [16].

2. Three-dimensional graphics
Adding perspective to simple graphs used in teaching materials can open the doors to a higher imagination and thus understanding of the topic. Concerning this advantage, we have supported the teaching of calculus at Masaryk University by two types of PDF materials with included 3D graphics (see [6 or 7]).
Our 3D graphs have been drawn in CAS Maple, then exported to VRML format and using commercial software Deep Exploration [3] transformed to Universal 3D format[4] (U3D) which is accepted by PDF. Correct viewing of these elements is then assured by using Adobe Reader, version 8.1 and later.

Fancytooltips
Fancytooltips, made by R. Mařík, are very useful for remembering key informations or providing quick connections to the theory within the exercise or quiz. After clicking on the emphasized keyword, a short review (called tooltip) containing important facts, pops up immediately. See an example in Figures 5 and 6 (tooltips are adapted from examples for fancytooltips package [10]). Text, mathematics, graphs or even animations, are possible content of a tooltip. "The buttons are created using `eforms.sty` which is a part of AcroTeX bundle." [9].

Fancytooltips – example

1. *A tooltip with text or static picture.*
   In general, a differential equation is an equation that contains an unknown function and one or more of its derivatives.

2. *Animated tooltip.*
   Solving a differential equation is not an easy matter. Despite the absence of an explicit solution, we can still learn a lot about solution through a graphical approach – direction field. Let us remember a geometric idea which is behind the definition of derivative.

Figure 5: An example of fancytooltips usage 1.
Key words "derivatives" and "geometric idea" are emphasized.

Conclusion
The appropriate use of PDF document in the field of mathematics gives us much more then traditional study materials. The aim of this paper was to introduce advanced technology which enables educators to produce variety of quality interactive materials. The most attention was dedicated to the instruments such as testing facilities, exercises, electronic practice cards or didactic games for training and reflecting the needs of practice, fancytooltips for making connections with theory, 3D graphs for illustrating mathematical problems etc. It is hoped that this type of eLearning support engages students in self-studying (and thus improve their skills in Mathematics) and brings interactivity which is especially helpful to distance students.

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4 For more information about the approach see [11]. The noncommercial alternative is also available, however not yet fully documented, discussed in [4].

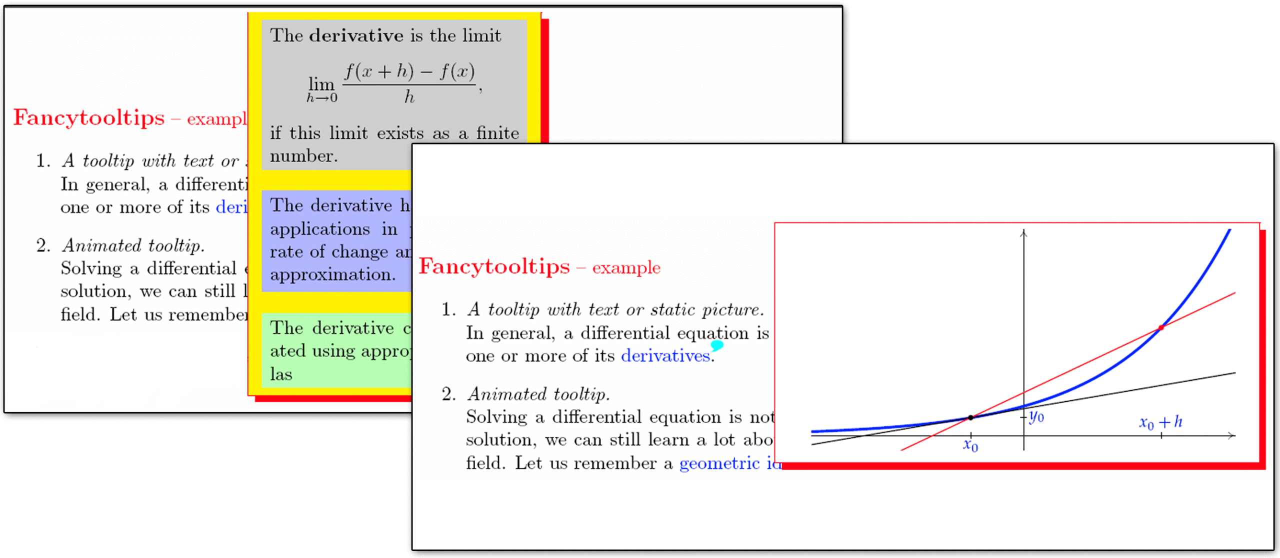

Figure 6: An example of fancytooltips usage 2, screenshots of tooltips for different key word. The higher one for "derivatives", the lower one for "geometric idea".

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[1] Acrobat Developer Center, *JavaScript for Acrobat*. http://www.adobe.com/devnet/acrobat/javascript.html (may 2009).

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Helping a Young Child Connect Fact Family Addition and Subtraction using Tools

Terri L. Kurz, Arizona State University terri.kurz@asu.edu
H. Bahadir Yanik, Anadolu University hbyanik@anadolu.edu.tr
Jorge Garcia, California State University, Channel Islands jorge.garcia@csuci.edu

Abstract:

In order to help children become effective at addition and subtraction, it is important to provide them with an opportunity to investigate and discover the interconnectedness of the two operations. Fact families are one method teachers use to try and help children develop and understand how the operations relate to one another. This paper documents a strategy that was used with a seven year old boy to help him connect addition to subtraction. The strategy incorporated flash card tools to help him create logical problems to discover the mathematical relationship of fact families. With just a few trials, the child was able to create and explain problems that demonstrated the interconnectedness of fact families through addition and subtraction. The model was successful in helping the child advance his understanding. Additionally, it can be extended to more complex addition and subtraction problems as well as multiplication and division fact families.

Introduction

According to the National Council of Teachers of Mathematics (NCTM, 2000), pre-K-2 children need to understand the addition and subtraction of whole numbers including the linkage of the operations. More specifically the expectation states, “understand various meanings of addition and subtraction of whole numbers and the relationship between the two operations” (p. 78). Because addition and subtraction are interrelated, children can use a known addition (or subtraction) fact to find an unknown subtraction (or addition) fact (Baroody, 1990). According to Clements and Sarama (2004), number and operations is one of the most important areas in mathematics education for young children. And Baroody (2004) asserts that numbers and operations constitute an important role in young children’s daily lives and activities. Furthermore, he states: “understanding their applications is a basic survival skill in our highly technological and information-dependent society and, thus a key basis of mathematical literacy, which is now as important as language literacy” (p. 173)

In relation to early number development, there are six growth points in addition and subtraction (Clark et al., 2001) and children progress through these levels. The growth points: “(1) count all (2) count-on (3) count-back/count-down-to/count-up-from (4) basic strategies (5) derived strategies, and (6) extending and applying addition and subtraction using basic, derived and intuitive strategies” (pp. 3-4). Some of the more advanced strategies include derived strategies; the derived strategies contain fact families (Clarke & Cheeseman, 2000).

Schools sometimes teach the interconnectedness of the two operations with fact families (Cobb, 1987). Fact families are four number facts that are connected through two opposite operations such as addition/subtraction or multiplication/division (Peterson, 1990). In a traditional setting, the teacher will lecture and explain fact family problems, providing examples such as 7+4=11, 4+7=11, 11-7=4, 11-4=7. Then, he/she will ask the children to repeat the process with other fact family numbers (Peterson, 1990) with the hope that children will become more efficient with their calculations. Perhaps the teacher will also point out the pattern, stating that the two small numbers have to be added together to make the bigger number and that the bigger number has to be the first number in the subtraction problem. However, children can fail to make the connection, simply going through the motions of creating addition and subtraction problems with the numbers and procedure given without developing the relationship (Baroody, 1990).

Fact families are one of the many strategies that can be investigated by children to help them make advances in number and operations (Clarke & Cheeseman, 2000; Cobb, 1987; NCTM, 2000). The idea is that the fact families allow children to think about part-whole relationships while also helping them realize that subtraction and addition are opposites of one another (Cobb, 1987; Sun & Zhang, 2001; Zhou & Peverly, 2005). In addition, fact families guide children helping them see how an addition fact can be used to find a subtraction fact or vice versa (Sun & Zhang, 2001). Fact families can be taught in first grade (Zhou & Peverly, 2005) and the concept is at or above the average for most children in this grade (Phillips, Leonard, Horton, Wright & Stafford, 2003) while Burns, VanDerHeyden and Jiban (2006) describe single digit fact families as a skill for second grade.

Fact families take time to develop and are somewhat complex for children (Bryant, Bryant, Gersten, Scammacca & Chavez, 2008; Phillips et al. 2003). Children can fail to make the connection, just stating mathematical problems using the four numbers provided. In addition, they may make up problems using all four numbers that simply do not make mathematical sense. The interconnectedness of addition and subtract is difficult for children to understand. In a study by Baroody (1990), first graders who were given extensive training to help them understand the relationship of the two operations yielded disappointing results. Very few children were able to understand the relationship. Noticing the relation between combinations (whole – part 1 = part 2 and whole – part 2 = part 1) may be a key element (p. 170); this concept ties into fact families.

This paper describes the case of a seven year old boy who was taught fact families using a traditional, lecture-based approach. After two years of exposure, he failed to make any connection whatsoever of interrelating fact families with addition and subtraction. Realizing that tools had to be used, simple hand-made cards were created to successfully guide the child though the development of addition/subtraction fact families. This paper describes the model that was used while also extending the model to more complex mathematical ideas. It is the hope that this paper provides a fact family model to help teachers guide children by interconnecting addition and subtraction using the tools described.

The Case

An individual case was examined with a child who failed to learn fact families after numerous attempts over a two year period (grades 1 and 2 (age 6 and 7)). The child was attending a public school in California with the following demographics: 67% Caucasian, 25% Latino/a, 3% African American and 3% Asian. Eighteen percent of the population was deemed economically disadvantaged by the state. Three percent of the school population was English language learners.

The child was taught using the traditional (non-reform based) approach of using fact families to connect addition to subtraction, as previously described. Using this approach, the child was supplied with multiple triangles printed on a worksheet, and each triangle had three numbers in it that could potentially make up an addition/subtraction fact family, see Figure 1. With lecturing and modeling done by the teacher, he was still unable to come up with problems that made sense. Even though he was corrected and the problems were explicitly explained numerous times, the child clearly failed to make the connection. Sometimes, his problems would make mathematical sense (2+5=7) and other times the problems lacked awareness. He would write problems such as 5-2=7, or 7+5=2; he would use the numbers in the triangle and they were in the right form (addition and subtraction problems), but the location of the numbers was unreasonable. When asked why he made a problem such as 5-2=7, his response was that all the numbers had to be used in the making of the fact family problems. If he was pressed that the problem did not make sense, he would say that he had to use all the numbers and that is the only way he could do it using all the numbers. It was hit or miss, just a randomization of placing numbers in the correct blanks (___ + ___ = ___ or ___-___=___) as interpreted by the child from the teachers' explanations.

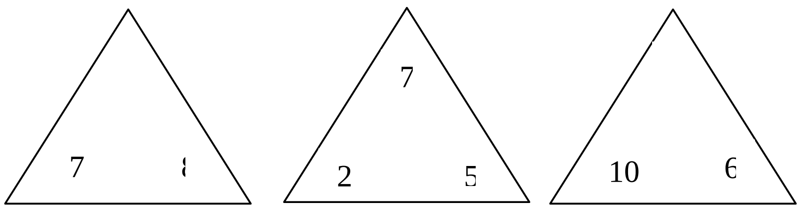

Figure 1. This is a figure of a fact family triangle supplied as a worksheet to the child in a traditional setting.

The Model

It was decided that the child could not learn the fact families without the use of tools to guide his development. He simply could not see the connection and needed to manipulate tools to understand the interconnectedness of addition and subtraction. Through brainstorming, individual tiles as tools with the fact families printed on them were settled on as they could be manipulated and moved to help the child visualize the mathematical problem. Standard, blank, small 3 ½ inch by 5 inch index cards were used. These cards were cut into smaller rectangles, see Figure 2. On each card, a number, operation or equal sign was written. As an alternative, plastic flat squares (inch flats) could have also been used. On each flat, a number or symbol could be written on a sticker or with a dry erase marker. For each fact family, the child was supplied with three rectangles, each with a single number from the fact family. Additionally, the child was supplied with an addition sign, a subtraction sign and an equal sign. The child was also provided with adequate think time; time to develop, build and reflect on the numbers and their relationship to the operations. He was supplied with paper and pencil to keep track of his equations.

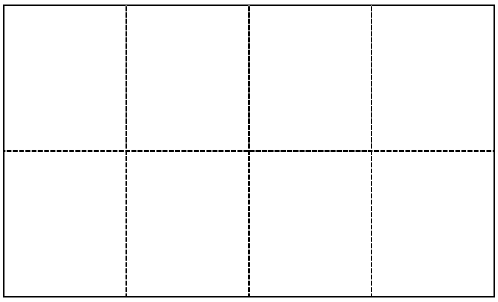

Figure 2. This is a sample showing how to cut the index cards.

The Results

With these six cards (7, 11, 4, +, -, =), the child was asked to make four different mathematics problems that made mathematical sense (see Figure 3). It took awhile; the first attempt at creating the problems took the child about 10 minutes. He created duplicate problems and was guided to find other problems that had not already been discovered. When the child created problems that did not make mathematical sense (7-4=11), the answer was covered up and he was asked to compute the first half of the equation (7-4). When he said the answer was 3, the 11 was then uncovered. He was asked, does what you arranged make sense? Why not? What can you do to fix the problem so that it makes sense? Once the mathematical errors were pointed out, he would rearrange the tiles again to try to find four different problems.

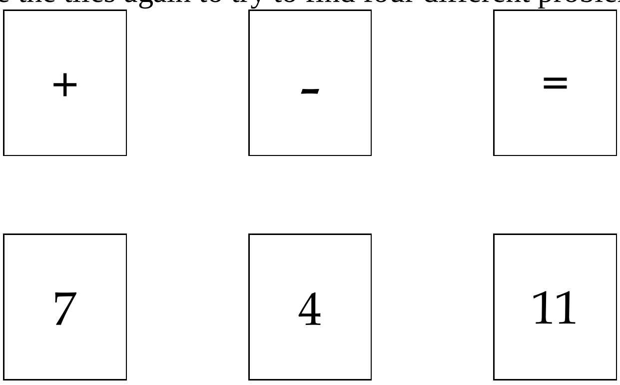

Figure 3. This is a sample showing how the index cards were created.

With each new set of fact family cards, he became quicker and quicker. After he was able to do this with three different sets of fact families and operation cards, he was asked to explain the pattern to making the four problems, as finding patterns may be a key to leading to fact mastery (Baroody, 1990). With this prompting, he was able to explain how the numbers connect to the creation of the problems. He recognized that it was necessary to add the smaller numbers to make an addition problem equal the larger number. As well, he was able to explain that the larger number had to be the first number in the subtraction problem because you are taking amounts away, making a smaller number. He was also able to explain how the operations connected to each other in relation to the fact family. All four numbers made four distinct problems that made sense and connected the two operations.

With the use of flashcard tools and adequate think time, the child was able to grasp the interconnectedness of addition and subtraction after just a few trials with different fact family sets. In addition, he was able to explain, rationalize and justify the problems he created. The knowledge extended beyond the use of the cards as well. The child was able to create the problems without the use of the cards using only fact family numbers after the conceptual understanding was developed. He was able to transfer his knowledge to the original fact family triangles his teachers had started with.

Extending the Model

Linking cubes (such as Omnifix cubes) can also be used to help the child visualize the numbers, see Figure 4. This may be more complex for a child than manipulating number cards, helping them to develop a richer understanding of the interconnectedness of the operations.

Emphasis in textbooks in regards to number and operations is frequently dominated by the use of small numbers (Ashcraft & Christy, 1995). Children are asked to compute small numbers more frequently than they are asked to compute larger numbers, which may lead to the difficulties they have in working with larger numbers (Ashcraft & Christy, 1995). To address this, fact family cards can be extended to more complicated numbers. This model would allow children to extend the idea of fact families to more complex instances in the later elementary grades. For example, children can be supplied with fractional fact families (1 ½ 2 ¾, 4 ¼) decimal fact families (0.15, 1.23, 1.08) and numbers with place values in the tens (28, 11, 17) or hundreds (344, 154, 498).

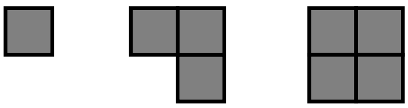

Figure 4. Linking cubes can also be used to help the child visualize the fact families (1, 3, 4).

This index card model can be extended to connect multiplication and division as well. Using blank index cards again, create a set of numbers (3, 4, and 12), the operation cards (multiply (x) and divide (÷)) and the equal sign (=). Have the child come up with four different math problems using the cards you gave him/her (3 x 4 = 12; 4 x 3 = 12; 12 ÷ 4 = 3; 12 ÷ 3 = 4). This investigation can be used to help students discover how the numbers can create distinct problems that relate to one another through multiplication and division. It can also guide them to use a known fact (multiplication problem for example) to find an unknown fact (division problem).

Conclusion

In order for some children to connect addition and subtraction, tools are a necessary building block to facilitate understanding. The child in this case could not make the connection without the tools; the tools were a vital component to advance his understanding. In addition, the think time was essential. Think time provides children with the opportunity to develop strategies and advance ideas (Phillips et al., 2003). With these two components (tools and time), he was able to do in 30 minutes what several hours of classroom time could not do. He was able to explain how numbers relate to one another with respect to addition and subtraction fact families.

This case highlights the importance of teachers taking the extra time to create opportunities for children to discover mathematical ideas with the use of tools. Bryant et al. (2008) pointed out that teaching strategies to children who need intervention is important. In addition, they theorize that concrete, visual representations of number concepts will assist in students' development of mathematical ideas. Specifically, this particular child needed the tools to visualize the connectedness of the numbers in relation to addition and subtraction. The tools made it much easier for him to understand and demonstrate the relationship.

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The van Hiele Phases of Learning in studying Cube Dissection

Shi-Pui Kwan (Mr.) and Ka-Luen Cheung (Dr.)
The Hong Kong Institute of Education

Abstract

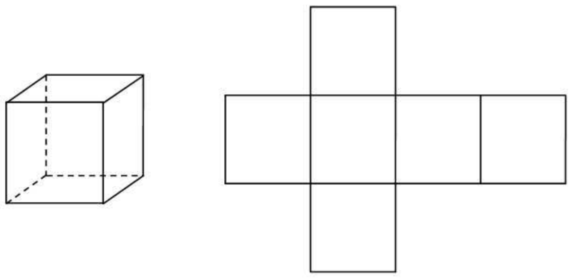
Spatial sense is an important ability in mathematics. Formula application is very different from spatial concept acquisition. But it is often observed that in schools students learn spatial concepts by memorizing instead of understanding.

In the past academic year we had tried out and developed a series of learning activities based on van Hiele's model for guiding learners to explore the cube and its cut sections. The ideas in origami, and mathematical modelling by manipulative as well as mathematical software are integrated into our study.

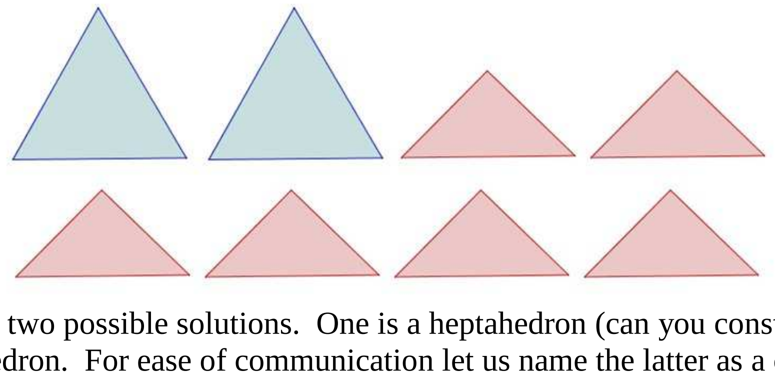
This paper gives a brief account on our works. We start by presenting a sequence of math-rich learning tasks, followed by some related folding ideas and mathematical background analysis. Finally we round up our paper with a concise discussion on some major elements of our design based on the van Hiele learning phases.

An Exploration Learning Sequence

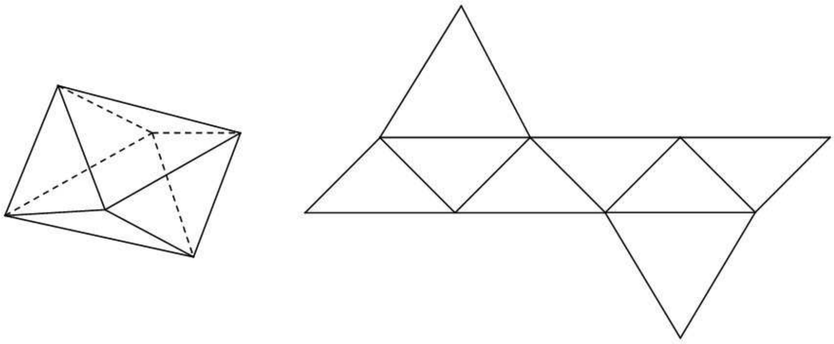
In order to experience in first hand the exploratory processes, we would like to remind readers to spare time and try out the problems instead of reading through the solutions. Now let us begin with a jigsaw puzzle game. Six pieces of identical squares are given. If no piece is allowed to be left behind, what solid(s) can be assembled from them? The solution is trivial. It is a cube and the following net is one of the possible constructions.



Then we proceed on to a more challenging problem. Given 8 pieces of jigsaw puzzles (2 pieces are equilateral triangles with side $\sqrt{2}$ units long and 6 pieces are right angled isosceles triangles with sides 1 and $\sqrt{2}$ unit(s) respectively). No piece is allowed to be left behind, what solid(s) can be assembled from them?

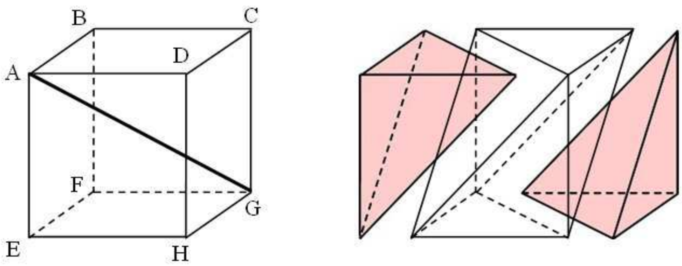


This problem has two possible solutions. One is a heptahedron (can you construct it?) and the other is an octahedron. For ease of communication let us name the latter as a ck-octahedron and denote it by O_{ck}. Examine it closely and you will find that it owes quite a number of symmetrical properties. Can you mention some of them?

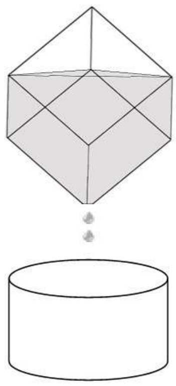


Just like the conical frustum which is part of a cube, the ck-octahedron is actually a part of a very familiar solid. Can you imagine what it is?

It is, in fact, a portion of a cube (a regular hexahedron) obtained by two parallel cross sectional planes perpendicular to a diagonal AG of the cube cutting through the other vertices B, D, E and C, F, H respectively.

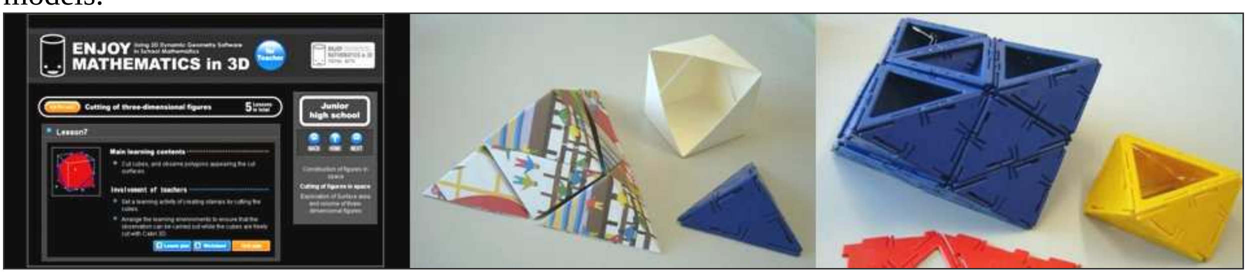


To investigate further the cross sections along this diagonal we carry out the ‘red-wine experiment’ (Blum, W. & Kirsch, A., 1991) and guide students to visualize the various attributes of the cross sections by observing the changes in the liquid surface.



What happens to the surface as the liquid drips down the cylinder? What are the variants and the invariants? How are they related to the three solids so dissected above?

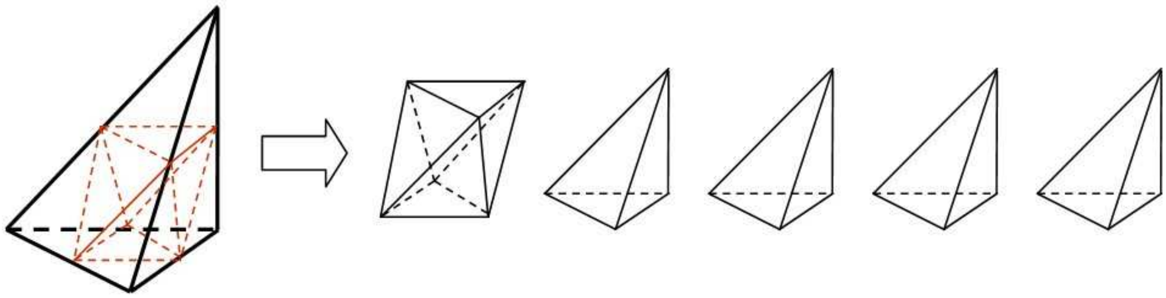
With the rapid advancement of dynamic geometry software, sections of a cube can be modelled effectively with Cabri 3D cg3 files. Some well constructed cg3 files are provided by ‘Enjoy Mathematics in 3D’ in the world wide web and are readily accessible to teachers and students. Besides virtual manipulative, the cube, the two tetrahedrons (let us label them the db-tetrahedrons, T_{db}) and the ck-octahedron in the middle can be constructed by Polydrons (a patent mathematics educational package) as well as paper folded models. Once they are made teachers may guide their students to explore the properties of these solids by concrete models.



Our last dissection activity to mention is an investigation on the volume ratio between a db-tetrahedron and a ck-octahedron.

- Take a db-tetrahedron. Mark the mid-points of all the edges.
- Join the mid-points of the adjacent edges.
- Dissect the db-tetrahedron along these lines into smaller polyhedra.

How many solids are obtained? What are they? In what ways are they alike and in what ways are they different?

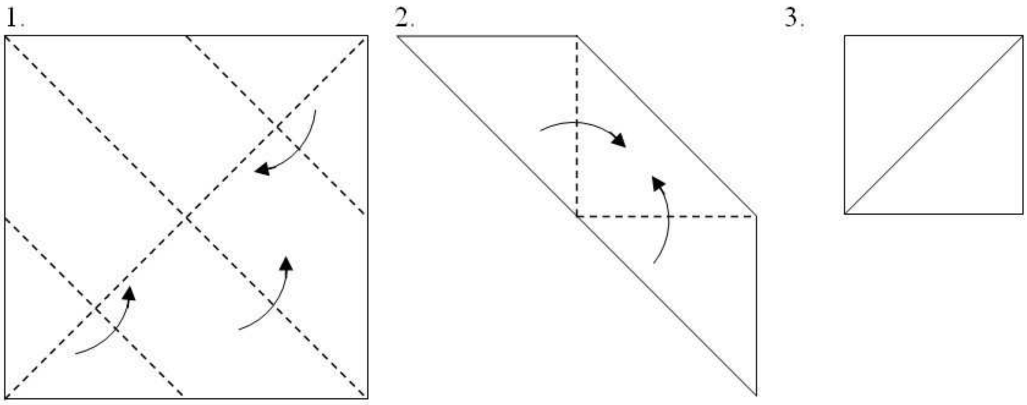


Repeat the same procedure with the ck-octahedron.
What do you notice of? How to determine the ratio $V_{Tdb} : V_{Ock}$?

Connecting Mathematics with Origami

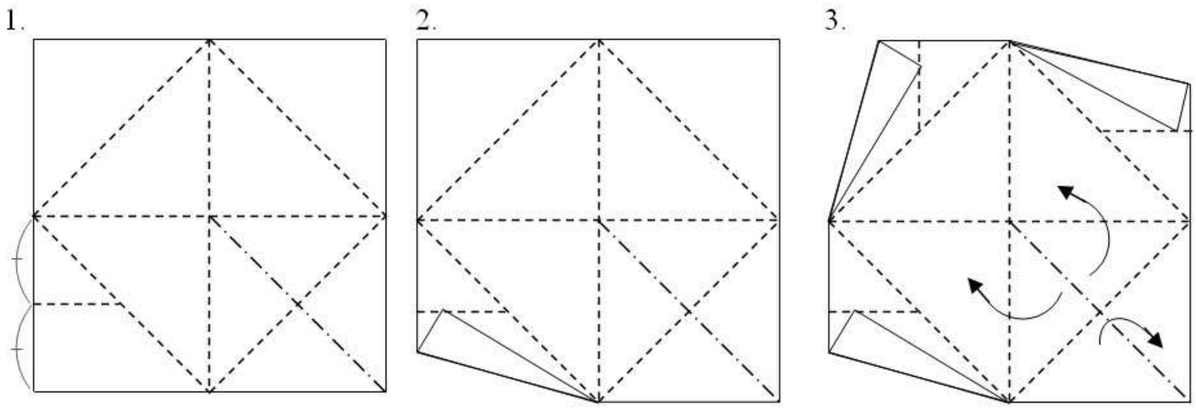
Folded models of a cube and anti-prisms can be found in many origami texts (Mitchell D., 1999 and Fuse T., 1990) and will not be repeated here. Below we introduce our way of folding a ck-octahedron and a db-tetrahedron.

Procedure of folding a ck-octahedron unit:



We need 3 units and assemble them together to form an 'open-through' ck-octahedron. In case you have difficulty in assembling the module, study the net above to look for hint.

Procedure of folding a db-tetrahedron:



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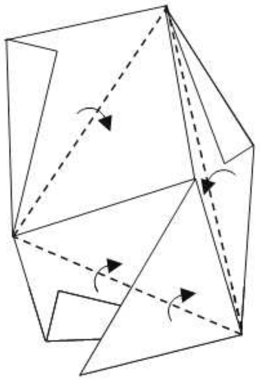

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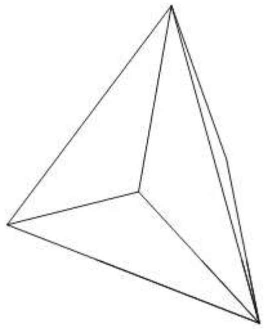

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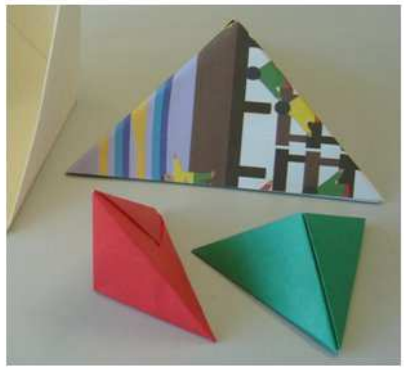
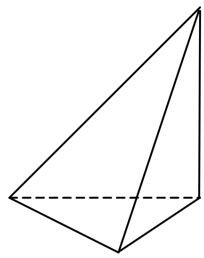


In the above model how is the angle 15° obtained? Why is it mathematically correct?

<u>The Volume Ratio Dissection Problem</u>

There are various approaches in finding the ratio. Three are discussed below.

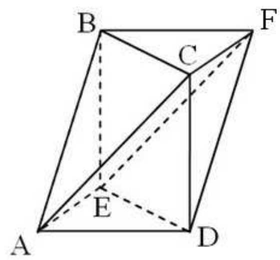


The db-tetrahedron is a triangular pyramid.

Base area = $\frac{1}{2}$

Height of pyramid = 1

Volume of the db-tetrahedron, $V_{Tdb} = (\frac{1}{3})(\frac{1}{2})(1) = \frac{1}{6}$



The volume of a ck-octahedron V_{Ock} can be considered as the sum of two identical pyramids ABCDE and FBCDE.

$AB = AC = BC = ED = \sqrt{2}$

$AE = AD = BE = CD = 1$

Base area BCDE = $(1)(\sqrt{2}) = \sqrt{2}$

Height of pyramid = $\sqrt{1^2 - \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{2}}$

Hence, $V_{Ock} = 2(\frac{1}{3})(\sqrt{2})(\sqrt{\frac{1}{2}}) = \frac{2}{3} = \frac{4}{6}$

Since two db-tetrahedrons and a ck-octahedron combine to form a cube, so alternatively the volume V_{Tdb} can be obtained by $1 - 2(\frac{1}{6}) = \frac{2}{3} = \frac{4}{6}$

$\therefore \quad V_{Tdb} : V_{Ock} = 1 : 4$

Clearly the mastery of the volume formula for a triangular pyramid is required. The base and the height of a triangular pyramid have to be correctly identified or computed. This approach provides ample opportunities for students to practice their skills in applying the formula. However little is done in enhancing their geometrical sense of cube dissection.

Another view of exploring the volume ratio is on the consideration of similar solids. Express the last dissection activity of the db-tetrahedron algebraically we have:

$$V_{Tdb} = \frac{1}{8} V_{Ock} + \frac{1}{8} V_{Tdb} + \frac{1}{8} V_{Tdb} + \frac{1}{8} V_{Tdb} + \frac{1}{8} V_{Tdb}$$

$$V_{Tdb} = \frac{1}{8} V_{Ock} + \frac{4}{8} V_{Tdb}$$

$$\frac{4}{8} V_{Tdb} = \frac{1}{8} V_{Ock}$$

$$\therefore \quad V_{Tdb} : V_{Ock} = 1 : 4$$

Likewise we can form the algebraic expressions for the ck-octahedron dissection. These steps appear to be simple and neat. But we would like to point out that the 1 : 8 volume ratio for similar solids is an abstract mathematical idea. It is usually proved by using similar cubes and the result is then extended to other polyhedra. But in our case for similar db-tetrahedrons and ck-octahedrons this volume ratio is not obvious.

Our third approach in solving the problem is inspired by Liu Hui's Yang-Ma dissection (Shen Kangsheng,1999). As all the edges are bisected and the adjacent mid-points are properly joined together, we can cut the T_{db} and the O_{ck} into smaller T_{db}' and O_{ck}' with edges half as long as the original solids. Please bear in mind that two T_{db} and one O_{ck} of the same dimension can be combined to form a cube of that order.

$$T_{db} = O_{ck}' + 4\,T_{db}' = \text{cube}' + 2\,T_{db}'$$

$$O_{ck} = 6\,O_{ck}' + 8\,T_{db}' = 4\,\text{cube}' + 2\,O_{ck}'$$

By repeating this dissection process the cubes so obtained in any particular order maintain the ratio 1 : 4 while $T_{db} \to T_{db}' \to T_{db}'' \to \ldots\ldots$ and $O_{ck} \to O_{ck}' \to O_{ck}'' \to \ldots\ldots$ These solids are diminishing in size rapidly. Hence we may arrive at the solution $V_{Tdb} : V_{Ock} = 1 : 4$ again. This is a sensible intuitive insight. But how to prove it rigorously while maintaining the geometrical thought? The pre-requisites of this method are few. But it demands a strong geometric sense and a very clear mathematical mind!

Are there still other methods of solution? Can we express the cross-sectional area as a function of its height? Can the volume be found by integration? And what are the values of discussing alternate methods of solution?

<u>The van Hiele's Phases of Learning and Our Exploration Sequence</u>

The van Hiele's geometrical levels of thinking, namely visualization, analysis, informal deduction, formal deduction and rigor are well known to mathematics educators. To facilitate the ascension from one level to the next the van Hieles propose the five phases of learning which provide good guidance for teachers in designing their instructions:

1. Information	2. Bound Orientation	3. Explicitation
4. Free Orientation	5. Integration	(van Hiele, 1986)

Can you tell how these phases are taken care of in our sequence of investigation? And what skills are important for teachers in implementing these learning phases?

We are much grateful to develop a series of interesting learning materials for cube dissections. In retrospection we find that questioning and problem solving skills should be carefully considered in the instructional design. Below are some major elements that we would like to highlight with examples drawn from our study.

• Recalling background knowledge	A review on the net of a solid and a recall of the cube is prior to the study of the ck-octahedron.
• Posing a clear question for exploration	The orientation of investigation has to be made crystal clear. It is the steering star for a yacht setting her sail in a big ocean.
• Asking inverse question	Consider in the learners' perspectives. A good practice is to think in the reverse direction. Instead of taking a cube, slice the two db-tetrahedrons off from it and study the ck-octahedron left behind, we set the eight-piece jigsaw puzzle problem and ask students what can be assembled from them.
• Establishing the language for communication	This is important for explicitation. The name ck-octahedron actually refers to the polyhedron obtained from the dissection of a cube □ cube cut (ck). It should be meaningful to the learners and well related to the study followed.
• Exploring the properties	Questioning and problem solving skills take a prominent role in this process. Take the 'red-wine experiment' as an example.

	What happens to the surface as the wine flows down from the cube? What are the variants and the invariants? How are they related to the three solids so dissected?
• Selecting relevant ideas for inquiry	Open-ended problems are often encountered. To guide our students to move on along the path of exploration making selection is inevitable. Though a heptahedron and an octahedron are both feasible solutions to the eight-piece jigsaw puzzle problem, we concentrate on the ck-octahedron for further study. May be the heptahedron can be recalled again later for free orientation.
• Sequencing and linking up key ideas	Let us look back at our learning sequence: net of a solid → ck-octahedron as a part of a cube → cube dissection along a diagonal → cut sections of a cube → further dissection of the T_{db} and the O_{ck} → volume ratio. Only with key teaching ideas well sequenced and linked learning will not be fragmented into piecemeal.
• Taking records and organizing findings	Students are encouraged to take journal in writing down their own findings and learning which contribute to the reconstruction of geometric concepts. Structure learning is far more valuable that factual memorization.
• Extending the problem for further study	Our dissection method in determining the volume ratio is inspired by Liu Hui. Who is Liu Hui? What is a Yang-Ma and what is a Bienao? How is the volume of a pyramid determined in ancient China? And how it is done in the West? How is it mentioned in the Euclid's Elements? These information/ guidance questions can be left for students to do their own investigation.
• Summarizing and providing feedbacks	Teachers can help students to integrate, to relate and to organize the geometric concepts learned. With reviews and feedbacks students know the way to make improvements.

Though we have only implemented our study under cube dissection, we hold strong belief that these elements are vital not only for this theme but in all geometry learning.

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Elementary Students' Construction of Proportional Reasoning Problems: Using Writing to Generalize Conceptual Understanding in Mathematics

Millard Lamm, M.Ed.
UNC Charlotte, Charlotte, NC, USA
David K. Pugalee, Ph.D.
University of North Carolina at Charlotte, Charlotte, NC, USA
mryamm@aol.com David.Pugalee@uncc.edu

Abstract

This study engaged fourth and fifth graders in solving a set of proportional tasks with focused discussion and concept development by the teacher. In order to understand the students' ability to generalize the concept, they were asked to write problems that reflected the underlying concepts in the tasks and lessons. A qualitative analysis of the student generated problems show that the majority of the students were able to generalize the concepts. The analysis allowed for a discussion of problems solving approaches and a rich description of how students applied multiplicative reasoning in composing mathematics problems. These results are couched in a discussion of how the students solved the proportional reasoning tasks.

Introduction

Proportionality is one of those important mathematical topics that is not clearly defined as a set of ideas that build on each other. Proportional reasoning involves complex thinking involving a sense of co-variation and multiple comparisons and is concerned with inference and prediction involving both qualitative and quantitative methods of thought (Lesh, Post & Behr, 1988). While there is a wide range of studies on rational number, such research does not always emphasize ideas of proportional reasoning that are inherent in the concepts and/or the emphasis is often on the development of 'number sense' without explicit identification of potential ties to of proportional reasoning.

Proportionality permeates mathematics and is often considered as the foundation to abstract mathematical understanding. Analyzing students' thinking relative to their work with problems involving proportions can inform teachers so that their instruction is better suited to promote proportional reasoning. Lesh, Post & Behr (1988) believe that proportional reasoning is the capstone of children's arithmetic thinking and the cornerstone of their ensuing mathematical progress. The influence of instruction on the development of more sophisticated levels of proportional reasoning is well documented in the literature (Steinthorsdottir, 2005; Pittalis, Christou, & Papageorgiou, 2003; Lamon, 1995). Unfortunately, a coherent and well-articulated framework for how such reasoning develops has not been constructed. The lack of such models makes it difficult for teachers to design instruction so that concepts are accessible and students are moved forward in their thinking.

In conclusion, research and related literature on proportional reasoning provide helpful ideas related to problem features and how they relate to solving the tasks while also identifying key components and characteristics of students' thinking related to proportionality. Increasingly complex levels of proportional reasoning require relational understanding (Skemp, 1976) and conceptual knowledge. That is, students must know what to do and why as well as have knowledge of complex mathematical relationships (Hiebert & Lefevre, 1986).

Research Design

The study involved sixth grade students enrolled in a suburban elementary school. Six students were randomly selected from an advanced level mathematics course of 24 students. The stratified random selection allowed for an equal number of boys and girls with five White and one Asian student.

The focus of the classroom-based research project was to explore students' understanding of proportional relationships. Students began with a warm-up problem to get them thinking about proportional relationships. "Yan and David each pay $6 for a pizza. The pizza is cut into six equal slices. How many slices should each receive?"This was followed by two somewhat more difficult problems such as the lottery problem. “Two friends, Anne and John, bought a $5 lottery ticket together. Anne paid $3 and John paid $2. Their ticket won $40. How should they share the money? Show all your work and describe what you did to solve the problem (Peled & Bassan-Cincinatus, 2005).

The focus of this project was to investigate students’ construction of problems as a means of demonstrating whether they generalized their understanding of proportional reasoning. Students were instructed to compose and solve a problem similar to the proportional problems they had solved.

Results

In general, students were constructed problems which reflected that they understood the underlying principles of proportional reasoning problems. Two types of problems were common: percentage applications and ratio problems that did not involve percentages. The majority of students also presented correct solutions for their problems. Half of the students (3 out of 6) used percentages in the solution methods to their problems. This is important to note as the use of percentages is reflective of multiplicative understanding, finding the answer by multiplying the base by the rate or percent.

As students shared their thinking about their problems, they negotiated the shared meaning of proportionality. Their work reinforced the concepts that they had discussed in solving the initial tasks. An analysis of the problem solving discussions showed that the approaches used by the students required a solid understanding of rational number principles and proportional reasoning. The problems clearly indicated an awareness of the multiplicative nature of proportions and did not depend on the use of pattern matching or build-up strategies which are more indicative of additive reasoning (Baxter & Junker, 2001). Four problems are discussed below to demonstrate students’ understanding.

The ‘jawbreaker’ problem was completed by Joe. Joe’s problem involved related rates as he describes a situation comparing number of gumballs to price. He first concludes that each jawbreaker costs 1¢. He presents an interesting way to show the number of gumballs for 1/6, ¼, and 1/8 of the total. Notice that he understands that he can multiply these ratios by a ratio equivalent to one to determine an equivalent ratio showing the number of gumballs out of 48.

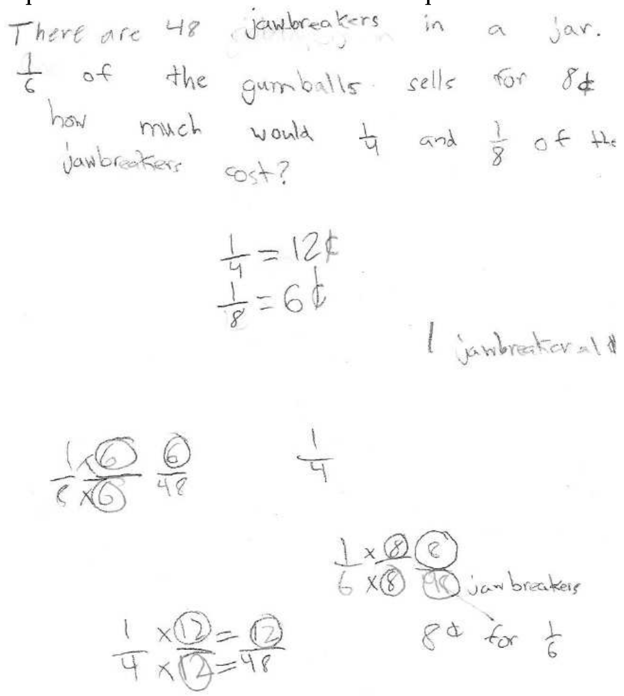

Figure 1. Joe’s Construction of a Ratio Problem

In the skateboard problem, Abbey illustrates how percentages are related to proportional relationships. Notice that she partitioned 60% into more easily manageable components of 50% and 10%. It follows then that 50% of the original price is $12.50 and 10% of the price is $5.00. Comparably, she shows that this process is analogous to finding ½ and 1/10 of the original price. She adds these two amounts to the original price of the skateboard to determine the new price that is 60% more than original.

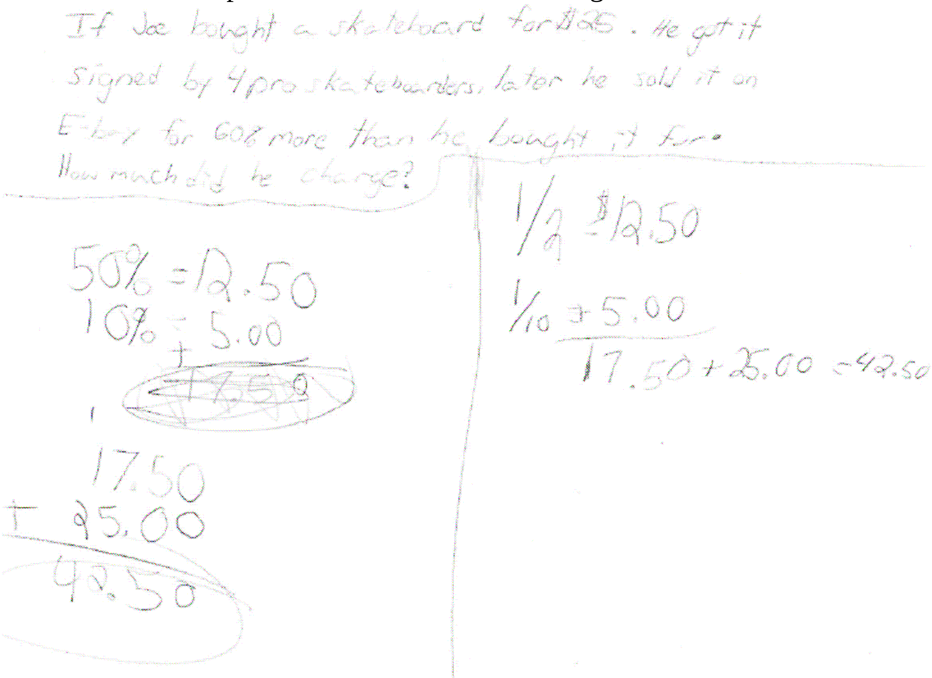

Figure 2. Abbey’s Construction of a Percentage Problem

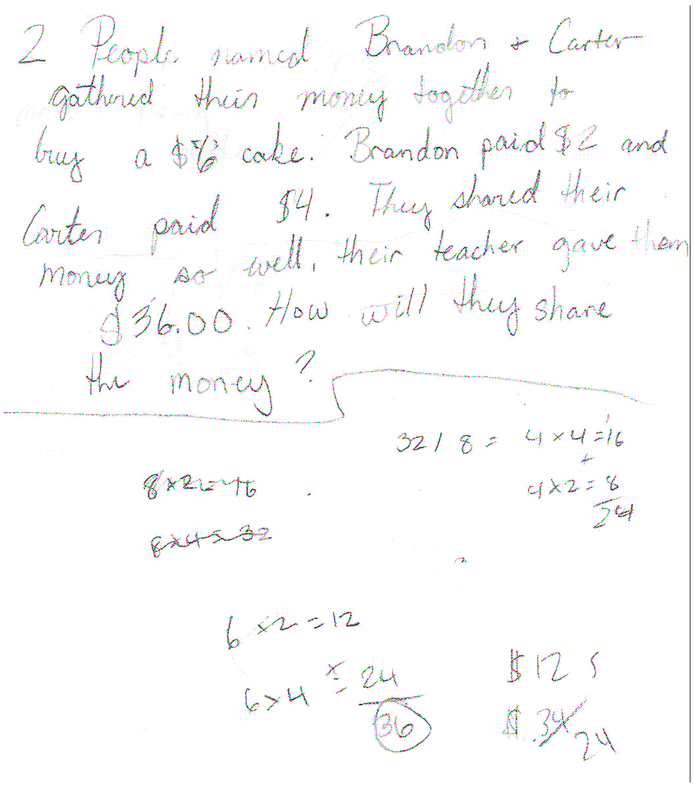

Figure 3. Anna’s Related Rates Problem

In the cake problem, Anna presents a related rate problem that requires multiple comparisons. It is similar in many ways to the lottery problem that was the focus of the initial investigations. Though not explicit in her work, Anna realizes that she can multiply the original cost of the cake by the proportional amounts of each contributor (Brandon with $2 and Carter with $4). This gives us 6 X 2 and 6 X 4. The resulting amounts of 12 and 24 are verified as summing to $36. It isn't clear why Anna initially used 8 but it is obvious that she concluded that this approach did not work (resulting in $24 not the required $36).

Conclusions

The open ended nature of the proportional reasoning tasks allowed the researchers to make inference about students' thinking as they composed and solved problems related to those they had worked on initially as part of the project. Writing, as a generative act, was a powerful way for students to express their understanding and think deeply about the nature of proportional relationships. As they modeled these situations in their solutions to their constructed problems, the multiplicative nature of their proportional reasoning was evident. The analysis of the problems created by the students served as evidence that they had generalized skills in solving proportional problems and could illustrate the underlying relationships of such problems in their own novel applications. Writing and solving problems that reflect important mathematical concepts is a valuable learning tool for students and a powerful means for teachers to assess what their students really know and can do relative to the mathematics as well as a providing direction for additional instruction.

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How to teach modeling in mathematics classrooms? The implementation of modeling tasks. Comparing learning arrangements and teacher methods with respect to student's activities.

Céline Liedmann, TU Dortmund

Abstract

There is a wide consensus that including mathematical modeling into the curricula is an important aim. A lot of attention has been spent on the realistic problems whereas their embedding in a classroom situation is less investigated so far, although the methodical arrangements are of major importance for initiating students' activities. In this paper, the implementation of the modeling task "swimming pool" in mathematics education in two lessons is compared concerning learning arrangement and teaching methods in depth with respect to the students' activities. Two videos about this implementation will be shown and discussed in this workshop. They are supposed to demonstrate in which different ways teachers engage in modeling. The aim is to show teachers, especially those without experience in teaching modeling, how modeling tasks can be implemented.

Introduction

For a certain period of time, there has been an agreement in the pedagogical discussions about the integration of applications and modeling into mathematics curricula (Kaiser-Messmer, 1986(I)) in order to enable students to understand and think critically about their environment, as well as about the usefulness of mathematics in society (Blum & Niss, 1991). Modeling is not only one of the five competences, which are important to teach in German schools. It enables practicing the other competences as well (Maaß, 2006). During modeling the students have to solve problems, discuss, communicate and illustrate their solutions. They can realize and use the relating mathematics in order to cope with the modeling tasks that come from their environment. Hence, the modeling competence is necessary for the students to be able to manage their daily life and to prepare for their later working life.

Project intention

This article is embedded in the project "Developing Quality in Mathematics Education II", called DQME II. A central aim of this project is to link theory and practice. 34 partner institutions like universities, institutions of teacher training and schools are working within this project. The participants (teachers, teacher trainers, researchers) develop material, especially concerning mathematical modeling for teachers in Europe. The developed tasks are evaluated in their own country, edited and internationally spread between the cooperating schools afterwards. Cross-cultural cooperation and exchange of ideas, materials and methods between eleven European countries ensure a successful development of intercultural education.

Methodology

Video-sequences enable to get a better understanding of the development of epistemological aspects of mathematics education. The evaluation of the tasks within this project is videotaped. The video and the modeling task I present in this paper emerged from this project. The original task has been developed and tested in Sweden. From there it has been sent to a participating school in Germany. One of our German teachers modified this task:

Swimming-Pool

Despite the rather cool weather during winter, small outdoor swimming pools are popular among private house owners in Germany. Imagine a swimming pool that is circular with a radius of 2.75 meters and a depth of 1.18 meters. The distance between the water surface and the pool edge is 0.06 meters. Every spring the pool is filled through two water pipes, each of them delivering 20 liters of water per minute. The water cost 2 Euro per cubic meter.

1. Figure out at least 2 meaningful mathematical questions and answer them with a calculation.
2. Find an agreement in your group. Which one of your questions should be written on the blackboard?
3. Hence, solve the questions of the other groups.

Figure 1: The Swimming-pool task

The task in figure 1 was given to five participating teachers in Germany. The requirements were identical. The teachers got the same short instruction: "Please give a lesson with this worksheet!" The implementation, how to teach it, was decision of the teachers. They were allowed to use the teaching methods they preferred. The teachers also did not know how other teachers designed their lessons around the task.

The empirical data emerged from classroom observations in different grades (7-10) in Germany. Every class had between 28 and 30 students, so that we can speak about an authentic classroom situation. The classroom communication was videotaped. The videographers were observers; they were told to take no influence on the classroom communication and on the teacher's way to introduce the concepts. Altogether, five classes were visited and two will be compared here. The qualitative interpretation of data is founded on an ethnomethodological and interactional point of view (cf. Voigt, 1984; Meyer, 2007).

Descriptors

Teachers have to learn how to teach modeling because of the embedding of modeling in mathematics curricula and the necessity of modeling competence. Therefore teachers need to know which abilities students need for modeling. The "modeling descriptors" are developed in DQME II and describe students' abilities (see table 1).

Observation

Two different teachers used the given task in two different ways including different teaching methods. For a better understanding I will use the synonyms teacher A and teacher B. Teacher A gave the students the worksheet and told them that this is their task for this lesson.

In contrast to that, teacher B explained where the task came from, then he made all students read it and afterwards the assignment of the task was summarized by one student.

Learning Objectives		descriptors
Systematisation & Mathematisation	a	Is data needed?
	b	Abstraction
	c	Assigning variables
	d	Making assumptions
	e	Simplifying
	f	Representation(s)
Doing the mathematics	a	Formalising and analysing the mathematics problem
	b	Using data
	c	Approximation and estimation
	d	Use of ICT (software and graphics calculator)
	e	Use algorithms
	f	Mathematical common sense
	g	Proof (validation of the mathematics used)
	h	Use of mathematical representations
Interpretation & Validation	a	Validation of the solution mathematically
	b	Validation of the solution in the 'real world'

Table 1: Some descriptors of modeling processes (Results of the first meeting of the research group, 2007)

The teachers helped their students in different ways to find a solution. Teacher B helped the students with information and explained in detail. Teacher A helped his students to validate their solutions. He was waiting for an explanation of the students results. (Systematisation and Mathematisation a)

student: “That cannot be right. On the blackboard, there is written 18 people and we have 869 people getting the pool to overflow.”
teacher: “869 people get the pool to overflow in your answer?”
student: “Yes!”
teacher: “What measure did you take? How did you calculate that?“
student: “To estimate the people, for an adult, I used your height and the width. 60.”
teacher: “Tell me the volume, which you used.“

Here, the student was doing mathematics (doing mathematics g). He validated the mathematical model he used regarding to the real situation (Interpretation and validation a and b). Teacher B answered the question “Which volume does a person have?” with an action. He tied the cable of the overhead projector around himself and helped the students to measure it. Then he explained the students how to proceed. (Systematization and

Mathematisation a) Teacher A only helped when students ask for. The rest of time the students were working in their groups. Teacher B walked around in classroom, offered help and tried to get an overview on the working process.

The teachers A and B also showed similarities: They chose groups of students for the working process. Teacher B divided the students in groups of 4-5 students and teacher A let the students build groups on their own. The groups were of different sizes (2-5 students).

Figure 2:
Teacher A shows a student solution on the blackboard.

Both took the black board for the result validation (Interpretation and validation a and b). The students wrote their questions on the black board and every group wrote their results under the corresponding question.

Quite contrary was the validation of the results. Teacher A used the students' solutions on the blackboard and discussed them in the plenum. Teacher B on the other hand showed the overview of the solutions on the blackboard and asked for presentation by each group on the overhead projector. So he invested a lot of time in the presentation. Every solution was analyzed by the whole class. They discussed all processes of finding a solution and analyzed the failures. Meanwhile teacher A asked the whole class for the solution and discussed with the wrong answers in the plenum.

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A Collaborative Model for Calculus Reform—A Preliminary Report

Po-Hung Liu, Ching-Ching Lin[1], Tung-Shyan Chen, Yen-Tung Chung,
Chiu-Hsiung Liao, Pi-Chuan Lin, Hwai-En Tseng, Ruey-Maw Chen
General Education Center
National Chin-Yi University of Technology, Taiwan
liuph@ncut.edu.tw

Abstract
For the past two decades, both pros and cons of calculus reform have been discussed. A question often asked is, "Has the calculus reform project improved students' understanding of mathematics?" The advocates of the reform movement claim that reform-based calculus may help students gain an intuitive understanding of mathematical propositions and have a better grasp of the real-world applications. Nonetheless, many still question its effect and argue that calculus reform purges calculus of its mathematical rigor and poorly prepares students for advanced mathematical training. East Asian students often rank in the top 10 of TIMSS and PISA. However, out-performing others in an international comparison may not guarantee their success in the learning of calculus. Taiwanese college students usually have a high failure rate in calculus. The National Science Council of Taiwan therefore initiated several projects in 2008 for improving students' learning in calculus. This paper provides a preliminary report on one of the projects, PLEASE, and discusses how it was planned to respond to the tenets of calculus reform movement.

Introduction
It has been over two decades since the Tulane Conference, the birthplace of calculus reform, was held in 1986. The appeal made by the conference—Toward a Lean and Lively Calculus (Douglas, 1986)—not only initiated the calculus reform movement in the United States, but for the first time ever, motivated numerous research mathematicians to engage in curriculum development. Despite several promising empirical findings having been reported, a widely held conclusion on the effect of calculus reform is still in debate. The equivocal consequence is due to the universal goal of teaching calculus being unattainable and a standard evaluation method is lacking. Calculus curriculum came under scrutiny for several reasons. First, traditional training in calculus, stressing rote calculating and practicing, hinders students from gaining a higher level of conceptual understanding and fails them in studying advanced mathematics course. Second, somewhat related to the first reason, a high failure rate in calculus forces college students to leave engineering or keeps them from choosing a mathematics-related career. Third, the faculty outside mathematics usually complains that students are ill-prepared to apply learned skills and concepts to solve practical problems. Reform effort in calculus curriculum aims to restructure content and develop tools to fix aforementioned pessimistic situations. We will make a brief review of calculus reform projects, followed by a preliminary report on the PLEASE project supported by the National Science Council of Taiwan.

A Brief Review of Calculus Reform
Rooted in its rigorous development in history, traditional instruction in calculus is conducted in logical order in which proving theorems and propositions deductively, based upon definitions and lemmas, plays a critical role; and working exercises with paper and pencil become the dominating mode of learning. This formal approach secures the foundation of calculus, but at the expense of students' intuitive understanding of the discipline. Some mathematicians thus made an urgent call for restructuring calculus curriculum.
Responding to the appeal of the Tulane Conference, Brown, Porta, and Uhl (1990a) reduced calculus curriculum by deleting several topics, such as Roll's Theorem and Riemann sum definition of integral, and integrated technology into the curriculum instead. They claimed that certain topics contained in traditional textbooks are only to fool students into the belief that they have learned something (Brown, Porta, & Uhl, 1990b). Among all reform curricula,

[1] Ching-Ching Lin is a mathematics faculty at National Taipei University of Technology in Taiwan.

Harvard Calculus Consortium (Hughes-Hallett & Gleason *et. al*., 1992, 1994) is the most widely adopted text and has received the greatest attention. Harvard Calculus Consortium reflected the reformed idea of "The Rule of Three" declaring that every topic should be presented in geometrical, numerical, and algebraic ways. Furthermore, it de-emphasized deductive symbolic reasoning by decreasing some sections, and stressed students' ability of application by connecting formal definitions and procedures with practical problems. For instance, *Calculus*: *Early Transcendentals* (Stewart, 1999), one of the best-selling textbooks and regarded as the most traditional textbook at that time, used five sections for discussing the concept of limit, whereas there was only one section on limits in the Harvard Calculus Consortium.

Johnson (1995) reported on the effect of the Harvard Calculus Consortium at Oklahoma State University by comparing students' performance in reformed and traditional calculus. He indicated that reformed classes outperformed traditional classes in Calculus I and II. However, traditional Calculus I students' subsequent performance in mathematics-related courses were better than their counterparts in reformed Calculus I classes. Furthermore, 44% of reformed Calculus I students changed to traditional Calculus II programs and only 18% of traditional Calculus I students shifted to reformed Calculus II. Baxter, Majumdar, & Smith (1998) also surveyed reformed and traditional calculus students' achievement in the Math-ACT and found that traditional Calculus I students' average grade was slightly higher than that in the reformed Calculus I, but only 52% of traditional Calculus I students passed the exam, significantly lower than reformed Calculus I students' passing rate of 64%. As for succeeding performance, reformed Calculus I students surpassed the traditional students in Physics I and Calculus II, yet traditional Calculus I students did better in Physics II and Calculus III. These outcomes seemingly suggest that the effect of the reformed curriculum may decline as the difficulty of the content increases. It appears that the experimental consequences of the reformed curriculum are hard to summarize in a single sentence. Silverberg (1999) proposed that the reformed curriculum may be more effective for those with a weak mathematics background.

Critics and Influence of Calculus Reform

Despite its success in several aspects of helping students to master calculus concepts, some professional mathematicians remain doubtful as to what has really been achieved by reform movements. While responding to Mumford's (1997) arguments, Klein and Rosen (1997) condemned reform supporters by saying that they create a straw man—the traditional calculus curriculum—and blame all faults on it. Reformers put forth various solutions, such as eliminating theories and increasing use of computers, without any scientific evidence. What if they were wrong in identifying the cause of students' failure? In their eyes, traditional calculus actually gives students the opportunity to have a deeper understanding of the subject and reform texts hinder motivated students from developing advanced thinking. Klein and Rosen satirized that calculus reform movements are not for the millions but $millions.

Feffer and Petechuk (2002) took a neutral stance. They agreed that reform curriculum may help students be more capable of connecting mathematics with the real world but, in their eyes, democratizing the curriculum by reducing its rigor actually is just watering it down. Feffer and Petechuk emphasized that calculus should not be expected to apologize for being difficult because skills will help students succeed.

Though calculus reform receives praise as well as criticism, there are some signs revealing its positive influence on textbook development. *Calculus*: *Early Transcendentals* (Stewart, 1999) was once regarded as the representative of the traditional textbook, yet Stewart (2006) claimed in his recent edition that:

> When the first edition of this book appeared eight years ago, a heated debate about calculus reform was taking place. Such issues as the use of technology, the relevance of rigour, and the role of discovery versus that of drill, were causing deep splits in mathematics departments….In this third edition I continue to follow that path by emphasizing conceptual understanding through visual, numerical, and algebraic approaches.

> The principal way in which this book differs from my more traditional calculus textbooks is that it is more streamlined….I don't prove as many theorems….(Stewart, 2006, xiii-xiv).

It appears that, as Stewart (2006) pointed out, both reformers and traditionalists have realized that enabling students to understand and appreciate calculus is their common goal. We are convinced that any reform effort should keep track of this common goal.

PLEASE Project

The PLEASE project adopts a collaborative model consisting of five individual projects conducted by Mathematics and Engineering faculties at two technological universities in Taiwan. In addition to reform calculus curriculum itself, it establishes a system for assisting students' learning not only in pre-calculus, but also in subsequent mathematics-related courses. The title PLEASE stands for six main themes of this integrated project: (1) P—pre-calculus, (2) L—low achievers' learning, (3) E—e-learning, (4) A—assessment, (5) S—statistics and calculus, (6) E—engineering mathematics and calculus. The PLEASE project can be divided into three components: PEA, LEA, and SEA.

<u>PEA component</u>: Technological universities in Taiwan mostly recruit students graduating from vocational high schools, stressing more skill training and practical knowledge. Such an instrumentalist approach may restrict college freshmen's conceptual understanding of fundamental topics in calculus, such as the concept of functions and limits. By following the Rule of Three (every topic should be presented geometrically, numerically, and algebraically), the PEA (pre-calculus, e-learning, and assessment) component combines Calcai, a graphical software, and Mimic Builder, an e-learning device, to develop an e-learning tool for pre-calculus. It enables non-technical users to create the e-learning courses by using the PowerPoint file and working with the assistant Tablet digital pen. The 2-layer slide design also lets the teaching process proceed more smoothly. Furthermore, a web-based test bank is established for assessing college freshmen's concept knowledge in pre-calculus and evaluating the effect of this e-learning tool.

<u>LEA component</u>: High failure rate in calculus is not uncommon in Taiwan's universities. Despite their outstanding performance on international assessments in mathematics, such as TIMSS and PISA, Taiwanese students' attitudes toward mathematics have been reported to be very low (Mullis, Martin, & Foy, 2008). Poor attitudes, to a great extent, weaken these college freshmen's driving force to learn calculus, which is essential for their majors. The LEA (low achievers, e-learning, and assessment) component attempts to construct an auxiliary environment, including computerized adaptive diagnosis evaluation system and teaching assistants, to enhance low achievers' learning. In order to identify difficulties in learning calculus, they are asked to respond to items from the web-based evaluation system. Because the computerized adaptive test is knowledge-structured and hierarchical, we may locate their obstacles in learning calculus and develop a tailored curriculum. Moreover, selected teaching assistants are trained to execute the tailored curriculum and serve as instructors outside the classroom.

<u>SEA component</u>: One of the criticisms of calculus reform is that reformed curriculum may show deficiencies in preparing students to take advanced mathematical courses. The SEA (statistics and calculus, engineering mathematics and calculus, and assessment) component deals with this issue by restructuring calculus curriculum to help students make a connection between calculus and subsequent mathematical courses, such as statistics and engineering mathematics. Several fundamental concepts in statistics (e.g., the expected value of the function of discrete stochastic variable and continuous stochastic variable) require sophisticated understanding of infinite series and definite integral, which are difficult for business majors to figure out. A particular emphasis on these topics will be made to fit their future needs in studying statistics.

Similarly, engineering majors usually have trouble grasping complicated concepts and processes of engineering mathematics such as differential equations, Fourier series, and Laplace transform, all of which entail a solid background in integrals as well as in differentials. In our calculus curriculum, students are trained to construct and solve mathematical models of given realistic problems by introducing to them the solutions of basic types of differential equations, Fourier series, and Laplace transform.

Conclusion

The PLEASE project assumes a collaborative model not only for reforming calculus curriculum itself, but also for establishing an e-learning and assessment platform. Three main components (PEA, LEA, and SEA) cover an extended range of curriculum from pre-calculus to post-calculus courses (Figure 1).

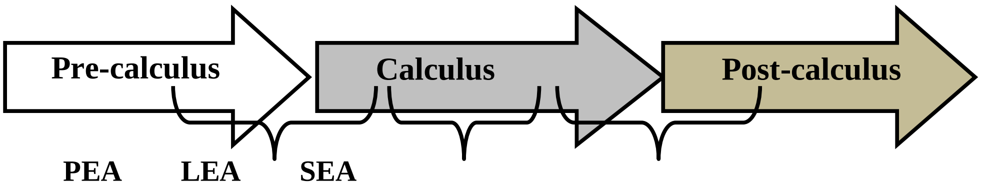

Figure 1

In order to keep away from the debates occurring in calculus reform, we set the issue of rigor aside and take students' learning in subsequent mathematical courses into account by stressing intuitive understanding and application of calculus. This approach, however, may only be conditionally applied. Students in our projects have graduated from vocational high schools and are studying at technological universities, which are usually less theoretical in their professional training. Our chief belief is any curricular reform effort is destined to fail if the curriculum itself is the only concern and an auxiliary system for supporting reform curriculum is lacking. The output of calculus reform should not be a single textbook but a holistic learning platform bridging the gap between preliminary and advanced concepts and knowledge.

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Changes in the North Carolina Mathematics Curriculum: A Comparative Study, 1920s, 1930's with 2003

Corey Lock, Ph.D. Professor of Educational Leadership
David Pugalee, Ph.D. Professor of Mathematics Education,
University of North Carolina Charlotte, Charlotte, North Carolina USA
dkpugale@uncc.edu crlock@uncc.edu

Abstract

The purpose of this paper is to compare curriculum documents for K-12 education from the state of North Carolina from two time periods, 1920s and 2003. The historical development of the mathematics curriculum in North Carolina provides a snapshot of the shifts in mathematics teaching and learning. North Carolina, a state in the southeast of the United States, has had a statewide standard course over a period spanning more than eighty years. A document analysis of printed curriculum standards from allows a description of the mathematics concepts and tasks that were expected of students in those years. The analysis revealed stark contrasts in the focus of mathematics from a very computational emphasis to one of problem solving. The analysis also highlighted the understanding of algebraic concepts and ideas as an essential outcome of current mathematics programs.

Introduction

Curricular documents in the United States, though the country lacks a national curriculum, provide pivotal resources that guide instruction and assessment in schools. The National Council of Teachers of Mathematics provides a comprehensive view of goals and expectations for students relative to five content strands: number and operations, measurement, data analysis and probability, algebra, and geometry (NCTM, 2000). This document also includes five process strands: problem solving, reasoning and proof, communication, connections, and representation. Our discussion will focus on mathematics standards constructed to guide the teaching and learning of mathematics at the state level. While acknowledging the influence of such national efforts, the focus of this study is to conduct a content analysis of curriculum standards for one state to provide a micro-level description of the conceptual and structural differences found in the documents. The results will describe major shifts in mathematics teaching and learning and provide an important historical context for the current priorities.

Research Design

Selection of documents was based primarily upon availability. Though the state of North Carolina has used a standard course since 1899, copies of such documents were not readily available even through a search of state historical archives. The current curriculum document was created in 2003 and though changes are proposed, such changes will likely not become instituted for several years due to financial limitations faced by governmental agencies in the current financial uncertainty.

This study employed qualitative document analysis procedures involving emergent and theoretical sampling of documents, development of protocols for systematic analysis, and constant comparative processes to clarify themes, frames, and discourse (Altheide, Coyle, DeVriese, & Schneider, 2008). This dynamic process allowed the researchers to identify themes and issues in the curricular documents across different periods of time.

Results

Computation as the Fabric of Civilizations

A striking contrast in the documents is the place given to computation in the early curriculum documents. The value and focus given to skills related to computation in the earlier curriculum documents is best illustrated in a discussion of the "Cardinal Principles of Secondary Education," (NC Department of Education, 1935) for principle II "Relation to the Command of Fundamental Processes." The principle includes the "Ability to compute accurately and with reasonable speed. Without this ability on the part of the people generally, civilization would relapse into barbarism" (p. 172).

The 1923 NC document title *Course of Study: Arithmetic* reflects the overarching importance of computation in the mathematics preparation of students in elementary through secondary school. One of the aims and principles of the program of study is "to develop habits of skill and accuracy in computation as well as the power to reason out problems he is apt to meet in everyday life" (p. 293). Initially, the mention of problem solving appears promising, until the next principle suggests that the "knowledge of facts and processes necessary to interpret and solve problems – to apply arithmetical knowledge to the solution of problems of his own everyday experiences as well as the types of problems in ordinary business transactions." Even in grade seven, arithmetic is seen as an "accurate means for measuring, clarifying, and understanding … a vital means for accurately solving problems arising in the daily life" (p. 331). Current standards documents present a divergent view of arithmetic. Current perspectives emphasize the development of 'number sense' though there is still some emphasis on developing fluency (NCDPI, 2003) though the document reflects a movement away from set procedures to an emphasis on understanding multiple strategies. The philosophy reflected in the current standards is that:

The early grades focus on building a strong understanding of number and fluency with mathematics to solve problems. Fundamental to these skills is knowledge of number facts, the computational processes, and the appropriate use of each operation. Together with an emphasis on using mathematics to solve problems, elementary students will build a depth of understanding enabling them to apply the content in a variety of contexts. (NCDPI, 2003, p.4)

The 1930's emphasis on computation is woven throughout the curriculum for students through secondary school. Arithmetical power is identified as one of the objectives for secondary students. Included is the ability "to read and understand numbers, to use accurately and with a moderate degree of speed the fundamental processes with whole numbers, and common and decimal fractions… to determine the number of figures to be retained in a computation, and to estimate the approximate results of a problem" (NCDE, 1935, p. 175). The current standards present a different set of priorities related to number for high school students. Objectives included for developing number sense for the real numbers are defining and using irrational numbers, comparing and ordering, using estimates of irrational numbers in appropriate situations. Also emphasized is developing flexibility in solving problems by selecting strategies, using mental computation, estimation, and calculators or computers as well as paper and pencil (NCDPI, 2003).

Problem Solving: Moving from Procedure to Process

Another major distinction in the documents from the two time periods was the treatment of problem solving. Such differences are highlighted by Hiebert et al. (1997) who argued that reform in mathematics curriculum and instruction should be based on an emphasis of having students problematize the subject. Such an emphasis would emphasize problems solving rather than the mastery and application of skills.

Algebra: The New Milestone for Mathematics Proficiency

The third major finding from the document analysis was the major emphasis on algebra in the current standard course of study and the absence of goals or objectives related to algebraic content in the earlier documents.

The colossal transformation to a mathematics curriculum where algebra holds a place in the core foundation of mathematics is clearly visible in the current report from the National Mathematics Advisory Panel ([NMAP], 2008) who identified algebra as a primary concern in terms of difficulties students experience with learning mathematics. The panel further posits that algebra is the gateway for later mathematics achievement, completion of formal courses in algebra is tied to later individual economic productivity, and that a formal course in algebra should be required of the majority of students before they leave middle school. The NMAP also argued that whole numbers, fractions, and particular aspects of geometry and measurement are the critical foundations for studying algebra and provided a list of topics that should be included in algebra.

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Bridging the gap between technology design and school practice: a specific experiment within the ReMath Project

Laura Maffei
Dipartimento di Scienze Matematiche ed Informatiche, Università di Siena
laura.maffei@unisi.it

Abstract
This contribution describes an experiment carried out by a team within the ReMath (Representing Mathematics with Digital Media) European Project (http://remath.cti.gr). Within this project six digital dynamic artefacts (DDAs) have been developed, thirteen experiments have been planned (Artigue & al., 2007) and carried out, analysis of the collected data are still in progress. In this contribution, we focus on the case of the Aplusix DDA (http://aplusix.imag.fr), from the point in which the designers deliver their product to the team in charge of planning the experiment, up to the point in which the artefact is experimented within the ReMath project.

Introduction
It is well known that the impact of digital media on mathematical teaching and learning in schools at the European level is weak (Artigue, 2007). Although looking into the reasons of that situation is undoubtedly a complex task, it is possible to look for responsibilities to both in the community of researchers in Mathematics Education and in the community of Mathematics teachers. On the one hand, theoretical frameworks emerging from research on teaching and learning mathematics with digital media are fragmented and usually involve assumptions bound to the specific contexts from which they emerged; on the other hand, teachers are often reluctant to use new technologies in class justifying their behaviours in terms of time constraints as well as curricular constraints. New tools ask teachers spend time to familiarize with them, ask them invest time to plan lessons in a new way, ask them to manage time in a different way in classroom. Obviously all these aspects have to be taken into account each time a new tool is integrated in the school practice, but only if considered in a superficial way, the need to change becomes an obstacle impossible to overcome.
Moreover, deeper reasons of the lack of integration of new technologies within the school practice can be found in analysing the tools themselves. In fact, very often, the potentialities of these media are not clearly understandable by teachers; it seems that the design is not immediately transparent to the users.
The ReMath Project, moving from the analysis of the situation both at the design level of a digital medium and at the use level in classroom, proposes a solution 'shortening the distance' among tools' designers, math education researchers and teachers.

A way to make designers, researchers and teachers interacting
According to the assumptions shared by the teams of the ReMath Project, the starting point on which the dialogue between teachers, designers and researchers has to be constructed, consists in identifying the 'didactical functionalities' of a given ICT (information and communication technology) in respect to a stated educational goal.
With didactical functionalities we mean those properties (or characteristics) of a given ICT, and/or its (or their) modalities of employment, which may favor or enhance teaching/learning processes according to a specific educational goal. (Cerulli, Robotti, Pedemonte, 2006, p.1390).
But, who is in charge to identify didactical functionalities (DFs)? Since by definition, they can be identified according to a specific educational goal, either designers, or researchers, or teachers, can propose modalities of employment of ICTs to reach a didactical goal.
The designer who has implemented the piece of software, has been motivated by a (some) particular educational goal (goals), and in designing specific components has been guided by modalities of employment who has hypothesized. As a consequence, there are didactical functionalities that the software has embedded from its origin and which can be called 'design-DFs'.
On the other hand, the 'design-DFs', can be not shared by a researcher who decides to face a research problem, i.e. the researcher can use the software either for a different educational goal or, even sharing the educational goal, the modalities of use can differ. We call 'research-DFs', the didactical functionalities identified by a researcher facing a specific research problem.
The teacher who decides to use a software in class usually after having read the 'design-DFs' described in the manuals of the software, is led to use the software coherently with those. Nevertheless the possibility to attribute new educational goals by a teacher, not belonging to 'design-DFs', can not

be neglected. We call ‘implementation-DFs’, the DFs identified by a teacher, where the word ‘implementation’ stands for ‘implementation in classroom’.
This classification of the different ways of approaching the issue of using a software in the school practice, gives the idea of the complex net of relationships between designers and users with respect to specific educational goals.

The experiment with the Aplusix DDA

Within the ReMath Project, the team of University of Siena, developed a pedagogical plan, that is a teaching/learning sequence aiming at reaching a specific educational goal (Earp & Pozzi, 2006), centred on the use of the Aplusix DDA, by means of a close cooperation with a group of teachers. Referring to the terminology we have introduced, starting from the analysis of the ‘design-DFs’ we can say that an attempt to come to an agreement with ‘research-DFs’, identified by the research team, and ‘implementation-DFs’, identified by the group of teachers, has been done.
Before describing the result of that cooperation, which ended in drawing up a pedagogical plan, called ‘Introduction to algebra: structure sense of algebraic expressions’, the main characteristics of the DDA are reported.

The Aplusix DDA

Aplusix is a computer algebra system which allows students to perform both numerical and literal calculations (Nicaud & al., 2006). One of the main characteristics of Aplusix is the feedback that it provides. The feedback is based on the equivalence between two consequent boxes, each of them containing an algebraic expression. More specifically, different signs show whether the current expression is equivalent/not equivalent to the previous one, or whether it is not well-formed (Fig. 1).

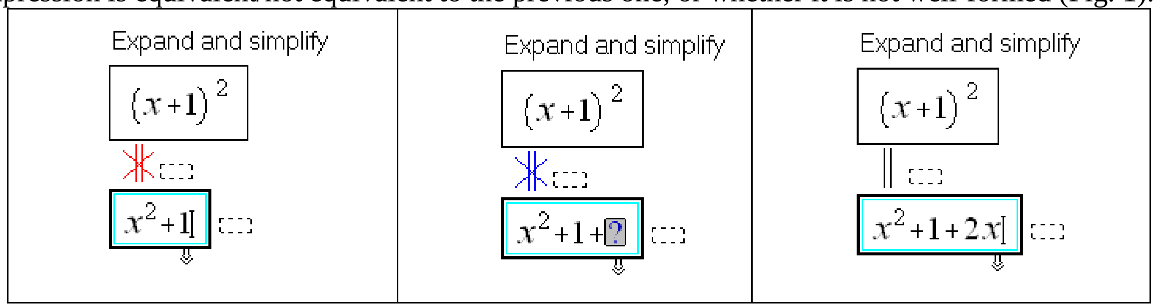


Figure 1. Three different feedbacks provided in Aplusix environment.

The black lines show that the first expression is equivalent to the second, the red crossed lines show that the first expression is not equivalent to the second, and the blue crossed lines show that the expression you are writing is not well-formed (i.e. a plus sign requires an argument).
Another valuable characteristic of the DDA is the possibility to work not only in the usual representation students are used to accomplish calculation in paper and pencil (the standard representation, SR in short), but also in tree representation (TR) (Fig. 2).

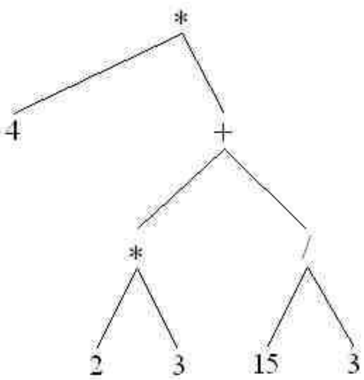


Figure 2. An algebraic expression given in tree representation.

The hypotheses which guided the development of the software are clearly defined by the designers when considering Aplusix a tool able to help students in performing algebraic calculations thanks to the feedback provided. Moreover, the designers state that TR component may constitute an additional support to overcome difficulties in doing algebraic manipulations (Nicaud & al., 2006).

As a consequence, Aplusix seems to be created to help students during training activities. As it will be made explicit in the following, the experiment based on the Aplusix DDA will not be done only in training activities, but in each phase of the mathematical activities in classroom. In this sense, the 'DF-implementation' differ from the 'DFs-design' .

Teaching and learning problems addressed in the experiment

The difficulties encountered by pupils in gaining competencies in algebraic calculation are well-known, and they have been addressed many times and from different point of views (Freudenthal, 1983, Tall & Thomas 1991, Kieran 1992). The introduction of algebra requires the development of a different way of thinking, which cannot be considered as a pure generalization of arithmetic (for an overview of the literature see Mariotti & Cerulli, 2003). Differently from arithmetic, whose main goal is to do calculations so as to have a result, algebra offers an operational language for representing, analyzing and manipulating relations which contain both numbers and letters. This diversity is explained by Sfard (1991) in terms of two different perspectives from which mathematical objects can be conceived: in a structural way (as objects) or in an procedural way (as procedures or processes).

In the school practice, often the rupture between the two different kinds of calculations is not put into evidence as it should be. Many students' difficulties in gaining competencies in algebraic calculations can be explained according to the 'continuity' they established between the two types of calculation.

The pedagogical plan designed with the collaboration of a group of teachers, proposes an introduction to algebraic manipulation starting from the arithmetic field. The school level is the 9th grade, that is the first year of Upper Secondary School in Italy. The teaching/learning sequence has been developed on two basic meanings on which the algebraic calculation is rooted. As a consequence the educational goal is twofold: the equivalence between algebraic expressions and the structure of an expression.

In this contribution we identify DFs related to the first educational goal, that is the equivalence between algebraic expressions. The components of the DDA which are mainly exploited so as to reach this didactical goal consist in the feedback provided and in the TR.

The theoretical framework

The pedagogical plan centred on the use of the Aplusix DDA has been framed by the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008).

This reference frame, drawing from a Vygotskian paradigm, considers learning processes deeply linked to teaching processes, in a social context. It states that the use of artefacts to accomplish a task leads the individual to the construction of *personal meanings* (Vygotsky, 1978), which are related to the actual use of the artefact. At the same time the use of the artefact can be related to specific mathematical meanings. Such double relationship between the artefact and meanings is called semiotic potential of the artefact. Under the guidance of an expert (typically the teacher), students' personal meanings may evolve towards *mathematical meanings*, i.e. meanings coherent with the mathematical theory.

> *Thus any artefact will be referred to as tool of semiotic mediation as long as it is (or it is conceived to be) intentionally used by the teacher to mediate a mathematical content through a designed didactical intervention.* (Bartolini Bussi & Mariotti, 2008, p. 754)

In this perspective, the function of semiotic mediation of an artefact is not automatically activated with the use of the artefact. In order to make meanings emerge it is crucial to identify the relationship, called the *semiotic potential* of the artefact (Bartolini Bussi & Mariotti, 2008), between the use of the artefact and the mathematical knowledge. Awareness of the semiotic potential of an artefact is a requisite for the teacher for developing suitable tasks for making meanings emerge, and for guiding the evolution of students' personal meanings towards mathematical meanings.

The organization of the teaching/learning process according to the Semiotic Mediation Theory is based on three types of activities, differently contributing to make the semiotic potential emerge and develop. These activities are organized in a cyclic structure: the activity with the artefact is followed by the individual semiotic work, the request of a report, which will be discussed and re-elaborated during a collective discussion. This sequence of three types of activities constitutes what is called a 'didactical cycle'**.**

The teaching-learning sequence

Exploiting the interaction between teachers and researchers, the pedagogical plan, entitled '*Introduction to algebra: structure sense of algebraic expressions'*, has been designed. The main interaction between teachers occurs during the planning of the activities to be performed in class.

The pedagogical plan is constituted by a sequence of four didactical cycles. During the activity with the DDA (first type of activity in a didactical cycle) students work in pairs, and as a consequence, they also fill in pairs the worksheets they are asked to complete during the work in Aplusix. Students individually fill the reports they are required to complete after the work in Aplusix (second type of activity in a didactical cycle). Then, under the guide of the teacher, students' productions are collectively discussed (third type of activity in a didactical cycle).
As already said, the pedagogical plan pursues two educational goals: the equivalence between algebraic expressions and the structure of algebraic expressions.
The goal of the first didactical cycle, 'The meaning of equivalence through feedback', consists in making students conscious of the mathematical meanings related to the different signs showing as feedback by the Aplusix DDA.
The second cycle, 'Decomposition of natural numbers', aims at making students re-interpret the validity of the commutative property for some arithmetic operations by means of the 'tree representation'.
As far as concern the third cycle, called 'Syntactical skills', the didactical goal consists in exploiting the potentiality of the tree representation in order to mediate the meaning of structure of an algebraic expression. In fact, on the basis of the comparison between TR and SR, the different affordances of the two different semiotic registers are expected to emerge and become the key point of the discussion
Finally, the last cycle, 'Towards the structure sense', introduces the natural language as an another representation system for algebraic expressions (to put beside SR and TR) to reinforce the stated relationships between the two different representation systems, SR and TR.

The first cycle 'The meaning of equivalence through feedback'

For reasons of brevity in the following we describe only the first didactical cycle which is devoted to the interpretation of the feedback given by Aplusix in respect to the equivalence between algebraic expressions. The duration of the whole cycle should be two hours.
As a first step to plan the cycle, the semiotic potential of the Aplusix feedback has been analysed. Two possible interpretations of the signs given by the DDA between two consequent steps (Fig. 1) are possible (Fig. 3): the first is immediate – correct/incorrect - the second will need the mediation of the teacher to be achieved. To grasp this dichotomy, we called *feedback-signs* the signs and we called *primary interpretation* and *secondary interpretation*, respectively the first and the second interpretation mentioned above.

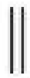
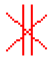
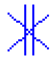
Inscription			
Primary interpretation	correct	incorrect	not specified
Secondary interpretation	equivalent	non equivalent	not well-formed

Figure 3. The possible interpretations of the feedback-signs provided by Aplusix.

The cycle is designed on the hypothesis that the first interpretation emerging should be the primary, and only exploring the different contexts in which the feedback-signs appear, the primary interpretation should be overcome and the secondary interpretation emerge.
In the first phase of the cycle the teacher briefly presents the DDA. In particular, the teacher shows how to log in the system and how to edit an expression in it. After that, students are invited to log in the DDA.
Then, students are provided with worksheets containing instructions both on how log in the Aplusix system (creating an account and a password to be remembered for the consequent access to the

system) and about the tasks to be performed. They are requested to accomplish easy calculations on rational numbers. While performing the task students has to take note of the different signs that Aplusix produces as feedback (feedback-signs). They are also requested to make hypotheses on the meaning of such signs. The primary interpretation is expected to emerge as the requested calculations are easy to accomplish. As a consequence, the students can concentrate more on the feedback-signs than on the completion of the task.

On the base of students' report, the teacher organizes a collective discussion with the objective of making explicit and clarifying the relationship between the interpretation of the feedback-signs in terms of correctness and the interpretation in terms of algebraic equivalence. At the end of the discussion the primary interpretation will have been evolved towards the secondary.

Conclusions

Although the educational goals pursued in the pedagogical plan are different from which envisaged by the designers, they are compatible with them. In describing the design of the pedagogical plan, it has emerged the crucial role played by the, Semiotic Mediation Theory. For this reason, during the interaction with the teachers in building the teaching/learning sequence, the theoretical aspects have been made explicit little by little. Some issues to be more investigated arise. What and in what level of granularity the theoretical framework adopted has to be made explicit between researchers and teachers? Since a software brings in itself a theoretical framework which has inspired its design, could this affect the consequent theoretical choices made by teachers and researchers?

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The Effect of Rephrasing Word Problems on the Achievements of Arab Students in Mathematics

Asad Mahajne, PhD, Ben Gurion University & Beit Berl College, Israel
Mahajne@netvision.net.il
Miriam Amit, Professor of Mathematics Education, Head of Mathematics and Science Education Department, Ben Gurion University, Israel

Abstract

Language is the learning device and the device which forms the student's knowledge in math, his ability to define concepts, express mathematical ideas and solve mathematical problems. Difficulties in the Language are seen more in word problems, clarity and in the way the text is read by the student have a direct effect on the understanding of the problem and therefore, on its solution, could delay the problem solving process. The connection between language and mathematical achievements has a more distinctive significance regarding the Arab student. This is due to the fact that the language which is used in the schools and in textbooks is classical (traditional) Arabic. It is far different than the language used in everyday conversations with family and friends (the spoken Arabic). Our research examine whether rephrasing word problems can affect the achievements of the Arab students in it. The experimental group received mathematics instruction using learning materials of word problems that were rewritten in a “middle language” closer to the students’ everyday language (spoken Arabic), thus keeping the mathematical level of the problems. The research findings showed that students in the experimental group improved their achievements in word and geometric problems significantly more than students from control group.

Introduction

The few studies that have dealt with word problems and geometry problems have focused on difficulties confronting students before the problem-solving stage, or - more specifically – the stage of understanding the problem. During this stage the linguistic and syntactic aspects of the problem are of utmost importance.

Word problems constitute an important part of the general curriculum from 1^{st} to 12^{th} grade. For this reason this study is extremely important. This issue is even more significant in the Arab sector where students speak with their peers and parents in colloquial Arabic, while the mathematics textbook from which they study is written in classical Arabic. For all practical purposes, this constitutes a “second language” for Arab students. Students in this research received mathematics instruction using study units that were rewritten in a “Middle language” – halfway between colloquial Arabic and Classical Arabic. The new rewording of word problems that is proposed in this study is closer to students’ everyday speech and is therefore easier for them to understand.

The objective of this study was to examine the degree of influence that the rewording of word problems and geometry problems has on Arab students’ achievements and upon their attitudes towards these problems.

The research questions were the following:

1. To what degree will rewording of word problems influence Arab students’ achievements in solving this type of problem?
2. To what degree will the rewording of geometry problems influence Arab students’ achievements in this type of problem?
3. To what degree will the rewording of word problems and geometry problems influence Arab students’ attitudes towards these problems?

The Research Population

The research population included 1041 5th and 6th-grade students from the Arab sector in northern and central Israel. This population was made up of three groups:

- **The experimental group** who received instruction and were tested in reworded problems,
- The **first control group** who received instruction and were tested in word problems and geometry problems worded in the traditional manner
- The **second control group** studied word and geometry problems that were written in the traditional manner and were tested using problems written in the new version.

The research variables were the following:

1. improved achievements in word problems (the difference between the scores in the pre-test and post-test)
2. Improvement in achievements in geometry problems (The difference between the scores in the pre-test and post-test)
3. Changes in students' attitudes towards these problems (The difference between Arab students' attitudes before and after the experiment)

The research was performed in five stages:

The first stage: Before the research began interviews were conducted with 30 students from 5th and 6th grade classes. Students were asked to point to the strategies they used while solving word problems or geometry problems. Students appeared to use nine different strategies (that will hereby be referred to as "characteristics."

The Second Stage: In order to examine the validity of these characteristics additional interviews were conducted with other students who were asked to offer their opinion about the characteristics that were found in the first interviews. This stage constitutes an essential prerequisite stage to the writing of the two study units that were based upon syntactic and linguistic difficulties that present an obstacle to students in understanding word and geometry problems.

The Third Stage: All the students filled out a questionnaire to assess their attitudes towards word and geometry problems. In addition, a pre-test on word and geometry problems was given to the entire research population. This pre-test was in the traditional language similar to that written in mathematics textbooks approved by the Ministry of Education.

The Fourth Stage: During the fourth stage the experimental group received mathematics instruction using learning units of word problems as well as an additional unit of geometry problems that were rewritten in a "middle language" closer to the students' everyday language. At the completion of the study students were given a post-test. Students in the experimental group and the second control group were given the same test using the new, rewritten version of the word and geometry problems. The first control group was tested using the same word and geometry problems, but which were worded in the traditional form similar to that in the textbooks. All the students once again filled out the attitude questionnaires that they had filled out during the third stage.

The Fifth Stage: After completing the two study units we interviewed 36 students of both genders from both grades. This constituted the fifth stage of the research. During this stage students were presented with word and geometry problems identical to those in the third stage: once in the traditional form and once in the new, rewritten form. 169 items were received that were connected to solving word and geometry problems. These items were divided into five categories:

a. The words in the word and geometry problems (number of words, and frequency of occurrence)
b. the sentences in the word and geometry problems (long or complex sentences)
c. Hidden or evident clue words
d. The location of the question (at the beginning, middle, or end of the problem)
e. The wording of the problem

Method of Analysis

This research is an integrated quantitative and qualitative research. It was conducted based upon pioneering research results that was conducted in the Arab sector that yielded results that justified this present study.

The research findings showed that students in the experimental group improved their achievements in word problems significantly more significantly than students from the first and second control groups. (The improvement in achievements in word problems among students in the experimental group was 8.96 point while students in the second control group improved their scores by 4.7 points and those in the first control group improved by 2.13 points on a score of 1-100. Additional findings show that students in higher levels of mathematics improved their achievements in solving word problems less than those studying in lower levels. Students whose parents had a higher level of education showed a less significant improvement in achievements in solving word problems. 5th grade students' achievements improved more significantly than 6th graders'. No difference was found in achievements in word problems between the genders, or between students with various levels of Arabic.

Students in the experimental group showed an improvement in solving word and geometry problems of 6.27 points on a scale of 1-100. The degree of improvement in geometry problems among students of the second control group was 3.17 points, while the students in the first control group improved their scores by 1.97 points. 5th grade students improved their achievements in geometry problems significantly more than 6th-graders.

The students studying mathematics on a high level improved their achievements in geometry problems significantly less than students studying at the lower levels. No difference was found between the genders or between the various levels in Arabic from the aspect of improved achievements in geometry problems.

Research results revealed a significant and positive change in students' attitudes towards word problems and geometry problems after the students were exposed to the newly worded study units. This change was significantly greater among boys than girls. Changes in attitudes among students studying higher levels of mathematics were significantly less than among those studying on lower levels, and students whose parents had a higher level of education changed their attitudes less, while students of parents with a lower level of education showed significant, positive changes in their attitude towards these problems.

No significant difference was found beween 5th and 6th grade students, or between students with different levels of Arabic regarding changes of attitude towards word problems and geometry problems. The above conclusions were proven clearly during the interviews that were conducted with students after studying the two units.

OPEN-ENDED APPROACH TO TEACHING AND LEARNING OF HIGH SCHOOL MATHEMATICS.

Radley Kebarapetse Mahlobo MSc
Lecturer: Mathematics Department, Faculty of Applied and Computer Sciences
Vaal University of Technology, South Africa

Abstract

The author shares some of the findings of the research he conducted in 2007 on grade 11 mathematics learners in two schools, one experimental and the other one control. In his study, the author claims that an open-ended approach towards teaching and learning of mathematics enhances understanding of mathematics by the learners. The outcomes of the study can be summarised as follows:

1. In the experimental school, where the author intervened by introducing an open-ended approach to teaching mathematics (by means of giving the learners an open-ended approach compliant worksheet to work on throughout the intervention period), the performance of the learners in the post-test was better than that of the learners from the control school. Both schools were of similar performance in the pre-test. The two schools wrote the same pre-test and same post-test. Both schools were following common work schedule.
2. Within the experimental school, post-test performance of the learners in the class where the intervention was monitored throughout the intervention period (thus ensuring compliance of the teacher to the open-ended approach) out-performed those in which monitoring was less frequent.
3. There was no significant difference in performance between learners from the unmonitored experimental class and those from the control class.

1. Introduction

The kind of teacher envisaged by the New Curriculum Statement (NCS), the curriculum followed in South Africa, includes, among others, being a mediator of learning, and a designer of Learning Programmes and material (DoE, 2002:3). The outcomes specified in the NCS encourage a learner-centred and activity-based approach to education (DoE, 2002:1). According to the Department of Education (DoE) (2008:10), Education and Training in South Africa has 7 critical outcomes and 5 developmental outcomes, which derive from the Constitution. Each of them describes an essential characteristic of the type of South African citizen the education sector hopes to produce. The document further states that these critical outcomes *should be reflected in the teaching approaches and methodologies that mathematics teachers use.*

The above discussion possibiliated the checklist as reflected on the table below.

According to DoE (2005:8), for instance, ‘all learners in Grades 10 – 12 should be given the opportunity of *developing themselves mathematically* .

To measure how well a student performs, teachers have to be able to examine the process of learning, not just the final product (Badger and Thomas, 1992). Such a view of learning and teaching demands an “open-ended” form of teaching, learning and assessment, based on open-ended tasks and questions (Moschkovich, 2004:51-53; Radford, 2001:251; Elbers, 2003:91; Hershkowitz & Schwarz, 1999:150). In her detailed analysis of two United Kingdom schools, Boaler (1997) argues that the school using an open approach to teaching and learning mathematics produced more sustained outcomes in mathematics learning, than the conventional format used by the other school.

The open-ended questions asked are the type of questions asked in a socio-constructivist lesson. The solution path of the learner, rather than that of the teacher, governs the teacher’s intervention.

Critical Outcome	**The teacher's approach:**	**Yes**	**No**
1. Identify and solve problems and make decisions using critical and creative thinking.	1. Makes it possible for learners to have opportunities to make comprehensive use of *their* mathematical knowledge and skills.		
2. Work effectively with others as members of a team, group, organisation and community	2. Encourages an active learner participation in lessons and allow the learners to express *their* ideas more frequently.		
3. Organize and manage themselves and their activities responsibly and effectively.	3. a) Provides every learner with a reasoning experience b) Positions the teacher as the facilitator, and not the source, of learning.		
4. Communicate effectively using visual, symbolic, and/or language skills in various modes.	4. Makes it possible for every learner to respond to the problem in some significant ways of his/her own.		

2. Research Design:

The approach used was intended to answer the following research question: "*What will be the impact of an open-ended approach on the learning of mathematics in grade 11 mathematics classes at the selected experimental school?*" [The full report is a PhD thesis that has just been submitted. For further details consult my promoter, Prof HD Nieuwoudt, at hercules.nieuwoudt@nwu.ac.za.]

The design of the complete research was a mixed-method approach' adapted from Creswell (2003). This report focuses only on the Phase 1 component of the study.

Phase 1 of the investigation was the quantitative part of the investigation where results from experimental and control schools were compared to explore impact of open-ended approach on learning of mathematics. Learners from the experimental school and control school wrote a pre-test at the beginning of the study to establish and compare their pre-requisite knowledge. Analysis of the pre-test results (to be presented at the conference) showed that there was no significant difference in performance between the three categories of learners – the monitored experimental group, the unmonitored experimental group as well as the control group.

The author prepared and gave learners from the experimental school a worksheet covering mathematics topics the two schools – experimental and control – followed. The worksheet asked predominantly open-ended questions and/or tasks for learners to solve. Though questions asked were open-ended, grade 11 mathematics topics used were the traditional topics. The worksheet (also to be shown at the conference) was designed to be used by the learners to solve mathematical problems instead of the learners having to be taught by the teacher from the classroom front. The teacher's role was only to facilitate the learners' attempts to solve the problem. The choice of the topics was based on the common work schedules both schools used.

The learners from both experimental and control schools wrote a post-test at the end of the intervention period, compiled by the author in consultation with the teachers from both schools. The purpose of the post-test was to see if there was any post-intervention difference in performance between the two groups. All the classes that formed part of study covered the same work schedule and wrote the same tests. Care was taken by the research design to limit other possible factors (some of which will be mentioned at the conference) accounting for different behaviours of the groups as far as performance in the tests is concerned.

Regular monitoring of the implementation of the open-ended approach to teaching and learning was done on two of the four experimental classes. The other two were only supplied with the worksheet to use throughout the period of intervention. The design of the investigation was such that the monitored class was the only one in which the teacher's compliance to the open-ended approach was adequately monitored by the author. Because of lack of consistent monitoring on the other classes in the experimental school, there was not enough evidence of compliance or otherwise to the open-ended approach to teaching and learning in those classes. However, a video of the unmonitored class, recorded during the intervention period, pointed to, among others, passiveness among other learners while a *volunteer learner* attempted to solve the problem on the chalkboard. The video of the control school, also taken during the intervention period, showed teacher approach that definitely did not comply with the open-ended approach to teaching and learning. The main focus of monitoring in the two classes was to establish if the type of questions the teacher was asking the learners during the solution process complied with the expectations of the open-ended approach to teaching and learning. The initial briefing of the experimental school teachers by the author at the beginning of the study was intended to explain to the teachers exactly what constituted compliance to the open-ended approach to teaching. However, it subsequently came out that some of the proceedings in the unmonitored classes were not fully compliant to the approach.

3. Phase 1 Results: Pre-test – Post-test

The t-test was used to compare the groups. The difference, if any, of the means of the compared groups was considered to be significant if the *significance level p* was at most 0.05 ($p \leq 0.05$). In cases where significant differences were obtained, the effect size d was calculated to establish the practical significance of the result (Steyn, 2009).

3.1. Pre-test results

The aim of the pre-test was to establish the *prerequisite knowledge* of the learners. The prerequisite knowledge mentioned here was the mathematical knowledge required to facilitate the learners' understanding of the mathematical topics covered during the period of intervention.

3.1.1. Conclusion: Pre-test

There was no significant difference in performance between each pair of the three groups: unmonitored experimental $(N = 73)$, monitored experimental $(N = 93)$ and control $(N = 88)$, as far as their pre-requisite knowledge is concerned. This seems to justify the conclusion that the groups the study investigated were of comparable pre-requisite knowledge.

3.2. Post-test results (These will be presented at the conference)

There were two contexts of looking at the post-test results. The first one was in terms of comparing averages of the post-test marks themselves, while the second was in terms of looking at the post-test marks on a *question-by-question* basis. This report only deals with the post-test averages.

3.2.1. Average post-test performance: Conclusion.

The monitored group outperformed both the control group and the unmonitored experimental group as far as average performance in the post-test was concerned. However, there was no statistical difference in performance between the unmonitored group and the control group.

3.4. Conclusions: Pre-test and Post-test phase

In general, the pre-test – post-test data showed that the monitored group outperformed both the unmonitored group and the control group in the post-intervention test. There was, however, no significant difference between the unmonitored experimental group and control group. This is despite the fact that in the unmonitored experimental school the open-ended approach compliant worksheet was used. With all things – prerequisite knowledge, similar school resource environment, etc. - being the same, one can attribute the difference in results to the role the teacher played during the intervention. If one compares the monitored and unmonitored classes, one realises that both were of similar pre-requisite knowledge, both were of the same school, and both were using the same intervention material. The main difference was in terms of the approaches adopted by the teachers in both classes. Improved performance in the post-test results for the monitored experimental group prompts one to agree with the statement of Hiebert *et al.* (1996) that *w*hen a student learns mathematics through such a problem-based approach, struggling with the difficulties facing him instead of relying on memorisation or any pre-determined rule to search for solutions, it promotes "deep understanding" of the mathematics that is valued. Hodgson and Watland (2004:1), in talking about OEA, said: "Through groups and other learning interactions with their online peers, students acquire deeper understanding because of the opportunities for exposure to multiple perspectives and interpretations".

Mewborn, Lawrence and Leatham (2005:416) also had a positive comment to make about the open-ended approach to teaching and learning:

"I have noted significant improvement in my students' self-confidence and their willingness to share their thinking with others. In fact, they begin to take pride in their explanations and find satisfaction in being able to explain what they are doing and why. They begin to see that there is a point to explaining their thinking. This leads to students feeling more ownership of their mathematical learning"

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Mathematics and Mathematics Education Development in Finland: the impact of curriculum changes on IEA, IMO and PISA results

George Malaty, University of Joensuu, Finland, george.malaty@joensuu.fi

Abstract

Mathematics has got roots in Finland in the last quarter of the 19th century and came to flourish in the first quarter of the next century. In the first quarter of the 20th century, mathematicians were involved in teaching mathematics at schools and writing school textbooks. This involvement decreased and came to an end by the launching of the 'New Math' project. Mathematics education for elite was of positive affect to higher education, and this has changed by the spread of education, the decrease of mathematics teaching hours at schools and the changes in school mathematical curricula. The impact of curriculum changes is evident in Finnish students' performance in the IEA comparative studies, PISA and IMO.

1. Brief and incomplete history of mathematics cultivation in Finland

The University of Helsinki of today is a direct continuation of the first Finnish university established in Turku in 1640. This was the only university Finland had till 1917. Relying on one university, mathematics had got a chance to grow, receive strong roots and to flourish. The strong roots of the Theory of Functions had got its beginning of growth through the five years professorship of the eminent Swedish mathematician *Gösta Mittag-Leffler* (1846-1927) in Finland, from 1876 to 1881.

Two eminent Finnish mathematicians continued *Mittag-Leffler* work, first his student *Robert Hjalmar Mellin* (1854-1933) and then Mellin's illustrious student *Ernst Leonard Lindelöf* (1870 – 1946). Our most well known mathematician, *Rolf Herman Nevanlinna* (1895-1980) was a student of Ernest Lindelöf, who was the cousin of his father. In 1919, at the age of 24, Nevanlinna presented his thesis, but the most important work of Nevanlinna was published in 1925. This includes the invention of harmonic measure and developing the theory of value distribution, named after him "Nevanlinna Theory". After Nevanlinna it is difficult to find a Finnish mathematician of the same international standing. But *Lars Valerian Ahlfors* (1907 – 1996) is such one. He was a student of both Lindelöf and Nevanlinna. At the age of 21, Nevanlinna's teaching gave him inspiration to solve the Denjoy's conjecture problem. This achievement gave him 8 years later, to be one of the two recipients of the First Fields Medal, awarded in 1936 (Kaskimies 1947, Elfving 1981, Ahlfors 1982).

2. Mathematics, mathematicians and mathematics education in Finland

2.1. School teaching and school textbooks before reforms

The cultivation of mathematics in Finland wouldn't have happened without cooperation of eminent European mathematicians. But, it is also true that this cultivation wouldn't have happened unless a success in school teaching of mathematics was achieved. In Finland, it was common to find a distinguished mathematician working as a schoolteacher or writing a school textbook. From the above mentioned eminent mathematicians Mellin and also Nevanlinna had worked as schoolteachers in Helsinki (Elfving 1981, Lehto 2001).

Ernst Bonsdorff (1842-1936), the eminent researcher in the Invariant theory temporarily had held the professorship of mathematics. But, in 1976 Mittag-Leffler was choosen to this professorship vacancy, and not Bonsdorff. This has made a turn in the history of mathematics, and also mathematics education in Finland. Bonsdorff then continued his work as a head teacher of Hämeenlinna Normal Lyceum (Elfving 1981, 58-60). Normal Lyceum is a Secondary Teaching Practice School. Besides his high teaching skills, Bonsdorff wrote some of the best, ever written, Finnish mathematics textbooks, among others Geometry textbook of 1889 (Bonsdorff 1889).

From Rolf Nevanlinna's family both his father Doctor *Otto Wilhelm Nevanlinna* (1867-1927) and his uncle Doctor *Lars Theodor Nevanlinna* (1850-1916) were Secondary School mathematics teachers. They had played a major role in the development of mathematics Education in Finland. Lars Nevanlinna was a teacher for more than 20 years and in 1902 he became the superintendent of Mathematical Subject of the National Board of Education (Elfving 1981). Lars Nevanlinna's textbooks have been used for more than half century (Neovius-Nevanlinna1950).

Last well-known Finnish mathematician, who worked as schoolteacher, was *Kalle Väisälä* (1893-1968). As Rolf Nevanlinna, he was a student of Ernest Lindelöf and at the age of 22 he presented his thesis. This was on Algebra and not on the area of most Lindelöf's students: the Theory of Functions. In 1941/42, the mathematics teacher of Väisälä's son's left the school for a military service and no relevant candidate for substitution was available. This event, besides Väisälä's interest in school mathematics, was a reason to make from a professor of mathematics a schoolteacher. School teaching experience was translated into a series of school textbooks (Kaskimies 1947, 202-203). From 1946 to 1970 Väisälä's textbooks were the most used in Finnish Secondary Schools, among others Algebra textbooks (Väisälä 1963).

2.2. School Spread and mathematics education in the 20th century

After Finland's getting its independency in 1917, the decrease of mathematicians involving in school Education was evident. This was an outcome of two new elements: the establishing of new universities and Higher Education institutes, and the spread of Secondary Schools. Finding qualified teachers for the new vacancies wasn't easy, also for the new universities and higher institutes.

The spread of schools in the 1950s, after the end of wars in 1944, was the greatest in Finnish history. In 1938/39 the number of Secondary School Students was 53 000, but this number became 214 000 in 1960/61 (Halonen 1982, 93). The baby boom and the growth of economy after wars were behind this growth, where the compulsory of Primary School, since 1921 Act (Päivänsalo 1973, 10), had made this growth possible. Further, the spread of Secondary School and the growth of economy gave the base for the establishment of Comprehensive School (Grades 1-9), by the School System Act of 1968. Comprehensive School was established to form the compulsory education in a welfare base. This change has led to the increase of students' numbers at Senior Secondary School and as well in Higher Education.

2.3. The place of mathematics in school curriculum in Finland in the 20th century

In the first decade of the 20th century, the number of mathematics teaching hours was significantly higher than that of any other school subject. The other area of special interest was the study of languages. Among others, in Junior Secondary School (Middle School) four languages were provided, including Finnish. The total number of these four languages' teaching hours was more than twice of that of mathematics. But, the number of teaching hours of Art and Physical Education, all together four subjects, was less than mathematics teaching hours (Halonen 1982, 33).

In 1914 a slight drop had happened to the number of teaching hours of mathematics (Halonen 1982, 33, 51), then again similar drop took place in the years 1918, 1941 and 1948. The most significant drop was that of 1972, at the time of Comprehensive School establishment. The decreasing of mathematics teaching hours, beginning in 1914, has changed mathematics place in our schools. Where the number of mathematics teaching hours before 1914 in the five years of Middle school was 23, the corresponding number in 1972 was 18. In 1986, UNESCO published the results of a survey, in which a comparison of mathematics teaching hours worldwide was provided. Among 27 European countries, the number of mathematics teaching hours in Finland was the lowest and one of the lowest among all of the 94 countries participated in the survey: *"… the range varies from an average of 2.6 hours per week in the case of Finland to an average of 6.1 hours per week in Switzerland – and these average are over 12 years of general education"* (UNESCO 1986, 35). An hour in UNESCO report is of 60 minutes. Here to note that the UNESCO survey aimed to investigate the place of Science and Technology in School Curricula. The survey's data collection had taken place in 1981 (UNESCO 1986, 8). In 1985, slight drop of mathematics teaching hours was again made (Kouluhallitus, 1985, 316-317).

3. Mathematics education reforms, Olympiads and comparative studies

Both, comparative studies on students' achievements, and mathematics curriculum reforms, had started at the end of the 1950s as post war plans in Western world. The lunching of Sputnik in 1957 had made the implementation of these plans urgent (Husén 1967, Vol. I, 25). Also, at that time, the former USSR and other 6 former socialistic countries met for the first time in the mathematical competition known as International Mathematical Olympiads (IMO). Finland was one of the 12 western countries, which participated in the First IEA in

mathematics (FIMS), started at the end of the 1950s. In 1965, Finland became the first Western country to participate in IMO. In addition, Finland participated in the 'New Math' reform project, started at the beginning of the 1960s. This active participation characterizes a general Finnish well.
In the First IEA (FIMS), for Junior Secondary in the case of Population 1a, Finland got the 3rd place among the 10 countries participated, and for population 1b the 4th place among the 12 countries participated. For Senior Secondary School, Finland got the 6th place out of the 10 participated countries, where mathematics was regarded as complementary part (Short courses). But for Senior Secondary School, where mathematics was substantial part (Long Courses) Finland got the 10th place out of the 12. In addition, the standard deviation of Finnish students was the 3rd lowest for Senior Secondary School and the Lowest for Junior Secondary School. For both schools, the more detailed results show that students' scores are around average level, where high scores were missed completely, and quite the same for low scores (Husén 1967, Vol. II, 22-25). These results characterise the education of mathematics in Finnish Schools, up-to-date.
Here to mention to a unique type of consistency in the FIMS Finnish students results. Not only the total sample of Senior Secondary School with 'Long Courses' got the 10th place, but also the upper 4% of this sample got the 10th place among the corresponding samples of other countries (Husén 1967, Vol. 2, 122). The 10th place of the 4% mentioned of 1964 testing of FIMS, and the missing of high scores explains why Finland in 1965 IMO got again the 10th place out of 10 participated countries.
Despite the all-round type of the established Comprehensive School in 1970, and despite the decrease of mathematics teaching hours accompanied, the implementation of the 'New Math' curriculum to all schools, started in the same year, had been of a positive affect on the results of IEA II (SIMS), and as well on IMO results. SIMS testing was in 1981, after 11 years of the beginning of the 'New Math' curriculum implementation. According to the micro study of D.F. Robitaille and A.R. Taylor, Finnish students of both Junior and Senior Secondary Schools had made better in SIMS, more than in FIMS, with the exception of the case of arithmetic test (Robitaille & Taylor 1989, 174, 160-176).
Regards 1981 IMO, which was hold in the year of SIMS testing, Finland made a remarkable progress by getting the 12th place among 27 countries. In the second year 1982, Finland became the 8th among 30 countries, and this is the best result ever for Finland's teams in IMOs. The participated students in IMO 1982 had started schooling in 1970, the year of 'New Math' curriculum implementation for all the 12 years of General Education. In 1983, Finland got the 12th place, among 32 countries, but in the next year, in 1984, Finland got the 29th place among 34 countries. The beginning of the 1980s was the time of replacing the 'New Math' curriculum by the 'Back-to-Basics' one, in all Finnish Schools.

4. Mathematics education changes since 1980

The 'Back-to-Basics' curriculum changes didn't bring back Euclidean structured Geometry and its deduction to Schools. In the 1980s and 1990s emphasis was putting on mastering mathematical skills, especially arithmetical ones. In comprehensive school, arithmetic education became based on drilling. Even rhythmic cassettes were made for learning multiplication tables by heart in the 1980s. This agreed with the emphasis of the 'Guaranteed calculation kills' in the National curriculum of 1985 (Opetushallitus 1985, 147). For Comprehensive School, the study of Algebra became in practice additional arithmetical one (see Malaty 2007). In the case of Senior Secondary School, ready formulas and algorithms were used to solve algebraic problems in mechanical way. Here we present two different examples. The first is for Algebraic teaching (Short Courses), and it shows how mechanical approaches had moved aside logical thinking and its elegancy. To solve the equation $(x+1)^2 = 9$, the only solution provided by a textbook is based on the use of Quadratic Equation formula. In addition, the formula itself is giving as a ready rule (Mäkinen, Sivonen and Rahikka 1995, 199-200). The second example is related to Senior Secondary School matriculation examinations (Long Courses). To solve the equation, $|2x-1| = |3x+2|$, the model solution published in the main Finnish newspaper was starting by writing the equivalency $\left|2x-1\right| = \left|3x+2\right| \Leftrightarrow (2x-1)^2 = (3x+2)^2$ (Helsingin Sanomat 1994). This model does not only represent the use of unneeded long mechanical performance, but how the use of illogical approach had been accepted. The understanding of the concept of absolute value has to lead to the conclusion $2x-1=3x+2 \text{ or } 2x-1=-(3x+2)$, upon which the presented equivalency can be obtained. The proposed model is doing the opposite.

Geometry in Comprehensive School became a calculation of perimeters, areas and volumes, and as well the using of instruments to make constructions with the help of given steps. No justifications for such steps are provided. In Senior Secondary School, teaching Geometry became also related to calculation of areas and volumes with more using of trigonometry. In textbooks, it became difficult to find a model for writing mathematical text, where pure mathematical mistakes became not rare.

Since 1985, besides the 'Back-to-basics' affect mentioned above, special interest arose in 'Problem Solving'. This includes encouraging students from the first Grade of Schooling to solve problems, where the solution is in need of rather common sense, than studying of mathematics. Under the name 'Problem Solving', also puzzles had been offered to children, where they had only to wait until the trick of the solution is presented to them. The so-called 'Ethnomathematics' had been translated into 'Everyday Life Mathematics'. This includes everyday life problems, where among others making diagrams and reading ready ones became a common activity since the beginning of schooling and even in pre-school education (Malaty 2002).

5. The impact of changes since 1980 on TIMSS

The changes we have mentioned are in need for more detailed discussion, but this we leave to another paper. These changes had taken place in our schools in the 1980s and 1990s and this has affected in the results of the two comparative studies, in which Finland took part after SIMS. These studies are the Third IEA study, which turned into TIMSS, and PISA. PISA, by its philosophy is related to Junior Secondary School, but TIMSS is much wider. Finland took part in only the Junior Secondary School part of TIMSS, where testing was in1999.

For Junior Secondary students, where in FIMS and SIMS Finland results in algebra were much better than in arithmetic (Robitaille & Taylor 1989, 174, 160-176), the opposite has happened in TIMSS. Finland had got the 10th place out of 38 countries in the performance of the test on 'Fractions and Number Sense and operations' of TIMSS, but the 20th place in Algebra. Where Finland had got the 18th place in 'Geometry', it got the 15th place in the test of 'Measurement', where calculating areas were common items. The best ordinal rank achieved was the 9th, and this was in the test of 'Representation, Analysis, and Probability', where test items are mainly related to 'Everyday life mathematics' (Mullis et al, 2000).

6. PISA success and the future of Mathematics Education in Finland

PISA tests measure only mathematical literacy, and this is relevant to the Finnish School Curriculum. The over all ordinal rank, for Finland in TIMSS 1999, was the 14th, where performance in 'Representation, Analysis, and Probability' related to everyday life was the best. PISA items are not related to a particular mathematical content, like Algebra or Geometry (OECD 2004, 18). The dispersion of Finnish students' results in PISA is one of the lowest through the participated countries. This was always the case in comparative studies, also before the establishing of Comprehensive Schools. The new in PISA is that high scores are now achieved by reasonable percentage of students. In PISA 2003, where the focus was in mathematics, 6.7% of students achieved the highest level. The percentage of other 6 OECD countries and a Partner Hong Kong-China was higher, leading by Hong Kong (10.5%) and Belgium (9%). But this didn't prevent Finland from getting the First Place among 31 OECD countries, and the second among all the 41 participated countries. The reason is related to the equity principles in Finnish society, according to which special care of students with learning difficulties in mathematics is provided. Therefore, where 26% of all 40 countries participated students were at level on**e** or lower, only 6.8% of Finnish students were of such low achievement. Looking back to PISA tests we find that Finland got the first place in only one test out of four, and this is the test of 'Quantity', where the difference between the mean of Finnish students, in this test, and the mean of each other country has a statistical significance, with the exception of the case of Hong Kong (OECD 2004).

7. Conclusion and future mathematics education development in Finland

Mathematics Education for all after the independency of Finland had met difficulties in offering understanding of mathematics to the level achieved before, where education was for elite and in some cases offered by elite. Therefore in comparative studies, since FIMS, no one from Finnish students got highest scores and, in general, no success was achieved in IMO. But, Finnish teachers have shown high ability in implementing every reform. Even at the time of establishing Comprehensive School, with low number of mathematics teaching hours, SIMS results were better than in FIMS. Success also was achieved in IMO as the 'New Math' had put emphasis in the structure of mathematics. The implementation of 'Back-to-Basics', 'Problem Solving' and 'Everyday Life mathematics' has brought some success in TIMSS and more success in PISA. Here to notice that TIMSS tests are closer to PISA than to FIMS and SIMS. The problem is that we are not now able to get success in IMO and the mathematical level of Secondary School graduated is not satisfactory for higher education. Since 1995 changes are taking place in developing mathematical curricula, where some mathematicians had become involved to some extent in school mathematics education. Also efforts have been made to take care of gifted students. Taking care of gifted students in Physical Education and Art is accepted, but why not in mathematics? This can raise a question about the success of making mathematics understandable and making its beauty and elegance seen. In other words, it is a question of mathematics education success.

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A class practice to improve student's attitude towards mathematics

Maria Flavia Mammana – Mario Pennisi
Dipartimento di Matematica e Informatica
Università di Catania, Italy
fmammana@dmi.unict.it - pennisi@dmi.unict.it

Abstract

For many students, mathematics, traditionally thought to be difficult and dull, is often considered inaccessible, generating a negative attitude towards it.

In order to encourage a positive attitude towards mathematics, we propose class practices that, through research activities, will lead the students to experiment a similar path to the one that has given, as a final product, a structured theory, so as to enhance their self-efficacy, give a correct vision of the discipline and stimulate positive emotions. This can be realized, for example, as a "laboratory activity" in which the students compare ideas, intuitions, arguments, and work together to obtain results, using their critical capabilities in a collaborative learning activity.

A team of university professors and high school teachers has developed a laboratory activity that focuses on some properties of quadrilaterals. The activity has at any rate been experimented in different first biennium classes of some high schools and has obtained very good results.

Introduction

Several national and international studies have focused on learning mathematics at secondary school revealing that students lack both proper knowledge of mathematics itself and the capacity of using mathematical tools to interpret reality. In fact, it is more and more common for students to feel uncomfortable towards mathematics, which they consider an abstract science quite far from their experience and interests, of no use for every-day life, a bulk of disconnected theorems just difficult to demonstrate, a mass of rules and formulas just to remember. A cold and fearsome mountain, too rough to climb, or as project to give up before starting.

In order to handle the difficulties of the students in this discipline the Italian Ministry of the University and of the Scientific Research has assigned resources to projects whose intent is to monitor the phenomenon, to circumscribe it and to reduce it. One of these Projects is the Progetto Lauree Scientifiche - Scientific University Degrees Project (PLS), that aims to improve the relationship between students and basic scientific subjects: chemistry, physics, mathematics and material sciences.

The proposal that we present has been developed within the PLS by a team composed of university professors as well as high school teachers. The group has been working with the purpose of developing in students "*correct attitudes towards mathematics, that is also an adequate vision of the discipline, not just a set of rules to memorize and to apply, but recognized and appreciated as a context to face and to investigate significant problems…*" (ministerial guidelines, 2007) .

The attitude towards mathematics consists of three interacting components: an emotional disposition, a view of mathematics and a sense of self-efficacy [5].

In particular:

the *emotional disposition* is the set of emotions (fear, anxiety, frustration, rage, pride, satisfaction, excitement, joy, to cite some) that are awakened by an activity [13];

the *vision of the mathematics* is the set of beliefs that the person has about it;

the *sense of self-efficacy* is defined as people's beliefs about their capabilities to produce designated levels of performance that exercise influence over events that affect their lives. Self-efficacy beliefs determine how people feel, think, motivate themselves and behave [2].

Such beliefs produce these diverse effects through four major processes. They include cognitive, motivational, affective and selection processes. Precisely:

cognitive processes: Much human behavior, being purposive, is regulated by forethought embodying valued goals. Personal goal setting is influenced by self-appraisal of capabilities. The stronger the perceived self-efficacy, the higher the goal challenges people set for themselves and the firmer is their commitment to them.

motivational processes: Self-beliefs of efficacy play a key role in the self-regulation of motivation. Most human motivation is cognitively generated. People motivate themselves and guide their actions anticipatorily by the exercise of forethought. They form beliefs about what they can do. They anticipate likely outcomes of prospective actions. They set goals for themselves and plan courses of action designed to realize valued futures.

affective processes: The stronger the sense of auto-efficacy the more vigorous people are in facing stressful difficult situations the more they succeed in modifying them. A low level of self-efficacy can generate anxiety as well as depression. Mood and self-efficacy feed each other reciprocally in a bi-directional manner;

selection processes: People are partly the product of their environment. Therefore, beliefs of personal efficacy can shape the course lives take by influencing the types of activities and environments people choose. People avoid activities and situations that they believe exceed their coping capabilities [2].

If emotional disposition, view of mathematics and sense of auto-efficacy manifest themselves in an improper or negative way, they generate in the student a close-mindedness towards the discipline that is impossible to undo. At this point, all intervention of cognitive nature that can be applied are unsuccessful or produce poor results.

The diagnosis of negative attitude has to be, then, a starting point for a intervention that is finalized to modify those components that did not developed and manifest correctly [13].

It is possible to influence this attitude [5] and the most significant mediating factor of the formation of the attitude is constituted by the rule of the teacher. As a coach can influence the self-efficacy and the emotional disposition of his athletes, favouring the consolidation of the future expectation by programming training in which they experience the success of overcoming an obstacle, so the teacher can influence the attitude towards mathematics of his students arranging paths finalized to reach goals that are concrete and that respect the capabilities of the single students and of the whole class.

The activity that we present wants to give the teacher an instrument to favour a positive attitude towards mathematics through a "research activity" proposed to the students in order to enhance their sense of self-efficacy, give a correct vision of the discipline and stimulate positive emotions.

In this paper, after describing the "didactic laboratory of mathematics" and explaining its characterising elements, we propose a laboratory activity on a didactic trail of plane geometry and we describe the results obtained in its experimentation.

The laboratory of mathematics

If it is true that "If I listen I forget, if I see I remember, if I do I understand" (Confuscius) then, in order to influence on the negative attitude towards mathematics from the students, it is opportune to support traditional teaching methodology with new ones that make students builders of their own knowledge.

This may be included, for example, in a *math teaching laboratory*, intended as *"a phenomenological space to teach and learn mathematics developed by means of specific technological tools and structured negotiation processes in which math knowledge is subjected to a new representative, operative and social order to become object of investigation again and be efficaciously taught and learnt"* [4].

The laboratory as a mathematics teaching and learning environment is today often used [1, 6, 7, 8, 11] and also the Italian Mathematics Union, in writing the new curricula, suggests [10]: "*We can imagine the laboratory environment as a Renaissance workshop, in which the apprentices learned by doing, seeing, imitating, communicating with each other, in a word: practising. In the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other, to the interactions between people working together[…] to the communication and sharing of knowledge in the classroom, either working in small groups in a collaborative and cooperative way, or by using the methodological instrument of the mathematic discussion, conveniently lead by the teacher*".

In particular, among others, important elements that characterize an activity in a mathematics laboratory are [12]:

- *A problem to solve*

The proposed activity is a mathematical research activity on a problematic situation to explore which gives significance to the study of mathematics because it makes the students understand what "to do mathematics" means. The aim of the activity is to give the students a versatile working method rather than some specific knowledge. In order to create an atmosphere of research and discovery, the problematic situations proposed to the students must be "new" for them, do not have to bee too easy or too difficult, and their resolutions must require tools that the students have already acquired. We decided to propose an activity on plane geometry in which the students are faced with known and new properties of quadrilaterals [9].

- *Objects/instruments that can be used/manipulated*

The entire activity is based on the use of dynamic type geometric software (we have referenced *Cabri Geometre II Plus*) which are used as research tools. Following the forms, students build with Cabri the figures that they use in their path, and by means of their manipulation discover and verify conjecture and identify their arguments.

- *Working method (relationship-interaction)*

Our proposal is realized as a "laboratory activity" in which the students compare ideas, intuitions, arguments, and work together to obtain results, using their critical capabilities in a collaborative learning activity: the explore and formulate conjectures, they verify them and then give a proof of them. In this way the pupils emerge themselves in a research atmosphere and use its methods. Awakening in them curiosity and initiative, they are given the opportunity to try the relish of a challenge, the joy of a discovery, and the gratification of obtaining the results [8]. The students become researchers and "builders of their own knowledge". We want, in fact, make the student experiment a path similar to the one that has given, as a final product, a structured theory.

- *The role of the expert-coordinator*

It is the job of the teacher to guide the pupils to attain various results by way of trial and error, to direct the students with appropriate suggestions on the path to follow, to question the proposals that still need to be perfected using counter examples, to encourage them to continue, to praise them for every significant result. Moreover, he beats time and create the right atmosphere.

The proposed activity

After having identified the objectives, methodologies and the outline of the theoretical reference the team of professors has traced the path and has written the teaching forms [1]. The teaching forms are a sequence of reflections for the student and help him go over the difficulties that he may meet in his research work.

We decided to choose a geometry trial because it constitutes, with its inexhaustible wealth of results, a privileged field of research and of learning reasoning.

The proposal starts with the problem of how to extend the definitions of median and of height of a triangle to quadrilaterals. In fact, in a triangle we can define the concepts of angle bisector, axis, median and height and the relative notable points. The notion of axis and angle bisector of a triangle can be easily extended to convex quadrilaterals, but concurrency rules are no longer true. The notions of height and median of a triangle cannot immediately be extended to quadrilaterals because in this case there is not an opposite edge to a vertex.

Then, there are the following problems: give "new" definitions of height and median of a quadrilateral and find concurrency conditions.

In the proposal, the students

'**invent**' and define the concepts of *bimedian* and *maltitude* of a quadrilateral – Forms 1-2;

'**discover**', **verify**, **conjecture** and **prove:** *Varignon Theorem* – Form 3; some properties of the maltitudes relative first to trapezium and second to quadrilaterals in general, and then discover and prove a *formula for calculating the area of any quadrilateral* – Forms 4-9; that the axes of a quadrilateral are concurrent if and only if the quadrilateral is cyclic; that the maltitudes of a quadrilateral are concurrent if and only if the quadrilateral is cyclic; that circumcentre and the anticentre of a quadrilateral are symmetric with respect to the centroid of the quadrilateral – Forms 10-13;

reformulate the Brahmagupta theorem in terms of maltitudes and prove it – Form 14.

The experimentation and the results

The experimentation, that has involved 108 students, was done in five classes of four high schools in Eastern Sicily and has been carried out in some cases both during school hours and after-hours, in others only during after-hours, for a total of 20 hours; it was performed in some schools by the teacher of the class in others by an outside teacher. In some classes there were also outside observers (student teachers). The use of the forms has anyway given uniformity to the experimentation.

Students decided to take part of the experimentation themselves on a voluntary basis; they were aware that it was an activity done together with the University and this, rather than discourage them, made them curious and proud to take part in it.

The students were informed of the experimental nature of the activity and of the fact that they would have not been evaluated by the teacher. This permitted even the shyest students, as well as those less capable, to be involved: wrong answers were not negatively evaluated, but actually served as the basis for resulting discussions that would clarify the problem.

In one class, even though it was the first time the students utilized *Cabri*, they were immediately interested.

At the end of the experimentation we thoroughly analysed the teachers' reports and those by outside observers, the student's forms and their final remarks and we found excellent results. Students have worked with enthusiasm and interest, "*stating conjectures and verifying whenever it was asked, posing questions and personal remarks always with more interest, showing maturity and creativity that otherwise would have not come out*" (outside observer).

It is opportune to emphasize that:

- Several times more students gave different proofs for the same theorem, appreciating then their own capabilities, and showing a new enthusiasm and curiosity towards the discipline. This has positively affected their sense of auto-efficacy, their emotional disposition and their vision of the discipline: "*Since I started this* *activity I see geometry in a different way and when I build a figure and I study a figure, I think a lot more and I see all of its aspects"*;

- This positive experience was even more extraordinary for the involvement of some students that, generally, used to attend classes as listeners and did not like to be involved in the educational dialogue: they produced good results and they made their own the method used: "*Since I started this activity, I like geometry and I also work at home*";
- Some difficulties came up in students that were not used to giving proofs and "writing of mathematics". Lots of time was dedicated to the proofs and to writing them. Proofs were first written on the Cabri working area and rewritten on the form only in their final form to which students got helped by the teacher;
- lots of time has been given to the group activity and to the mathematic discussion as an opportunity for comparison both in constructing the knowledge and in presenting the results. These ways of learning, that have created a positive environment of "collaborative competition", were appreciated by the students. "*Group work is enjoyable is beautiful and if you work together you learn better and more easily*"; "*We have learned from this experience that in doing research your individual effort, as well as that of the group is very important*". Every body could say his own opinion without being judged and everyone helped in reaching the final result. The teacher played a fundamental role, checking up on the work done, at time acting confused in order to point out errors or inaccuracies, and praising them for the results obtained. Furthermore, the teacher acted as a moderator when doubts came up;
- the initial skepticism of some teachers was replaced with a great enthusiasm for the activity itself. This helped improve the relationship between the teacher and the students and has favoured a constructive dialogue between them. Also students noticed that "*the activity helped me in the relationship with the professors. I felt important and appreciated*";
- The students appreciated this way of doing mathematics using Cabri, which was new for some of them and was fundamental in the whole activity. In fact Cabri, by simplifying figures construction and allowing their manipulation, helped students to discover and verify conjectures, which was indispensable for their research activity. Cabri was shown to be a versatile and valuable teaching tool for our pupils' mathematic formation: "*You never forget what you have learned with Cabri*"; "*Cabri was really important because perfect figures are helpful*"; "*Moreover we better understood the difference between 'verify' and 'prove', by finding out that not all conjectures were true*".

Mathematics has changed in the student's eyes and it has become an interesting discipline to discover and to investigate: "*It was interesting to know that mathematics is not a dead subject , that not every thing has been written in stone, and to verify personally that still new objects can be investigated, new definitions can be given and new properties can be discovered. Mathematics is alive!* "

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Identifying Modelling Tasks

Stefanie Meier
Technische Universität Dortmund, Germany stefanie.meier@mathematik.tu-dortmund.de

Abstract
The Comenius Network Project "Developing Quality in Mathematics Education II" funded by the European Commission consists of partners from schools, universities and teacher training centres from eleven European countries. One advantage of the project is the mutual exchange between teachers, teacher trainers and researchers in developing learning material. To support the teachers most effectively the researchers asked the teachers what they wanted the researchers to do. The answer was also a question: How can we identify (good) modelling tasks? A discussion ensued in the research group of this project which resulted in a list of descriptors characterising modelling tasks. This paper focuses on the theoretical background of mathematical modelling and will thereby substantiate the list of descriptors for modelling tasks.

Introduction
The work in the Comenius Network "Developing Quality in Mathematics Education II" has one main focus on the development and evaluation of modelling tasks. The idea was that teachers and researchers would develop such tasks in mutual exchange. This is currently taking place. One way of doing this is that the teachers develop tasks and the researchers analyse them theoretically to discuss whether the task is a modelling task or not. To make this easier for the researchers they agreed on a list of descriptors to characterise modelling tasks.
To make the descriptors for the list more explicit different theories about modelling (Blomhøj, Jensen, 2006; Blum, Leiss, Borromeo Ferri, 2006; Greefrath, 2007) which underlie the developed descriptors will be discussed. Further on the descriptors will be compared to lists of modelling competencies from Blum and Kaiser (according to Maaß, 2006) and Ikeda and Stephens (1998), and discussed in this paper.
First three different modelling circles will be described. The last one was the basis for the list of descriptors, thus they will be explained afterwards.
The descriptors will then be compared to different theories about modelling competencies.
In the third part the idea of a checklist for teachers based on the list of descriptors will be presented and discussed. This checklist shall help teachers, especially those at the very beginning of their teaching, to identify and create their own modelling tasks. In the first approach teachers agreed that such a checklist is helpful.

Different models of mathematical modelling

The basis of mathematical modelling is always a real life situation with which pupils have to deal with mathematically. In literature many different models about mathematical modelling can be found.
The first model that will be presented is from Greefrath (2007): It starts with a real situation (Reale Situation). This is not the whole reality from which a situation must be chosen, but an already structured situation from real life (Realität). This should be transformed into a real model (Reales Modell). This real model is a simplified and structured version of the real situation.

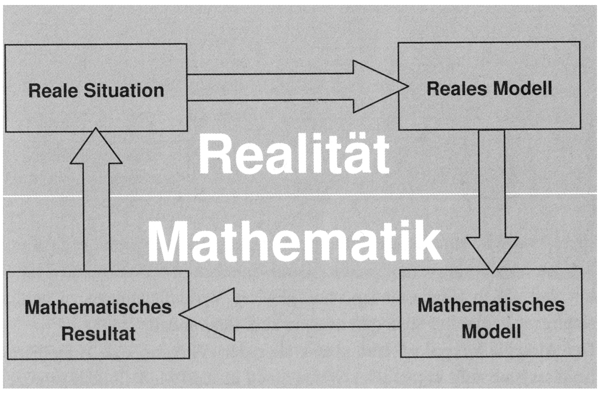
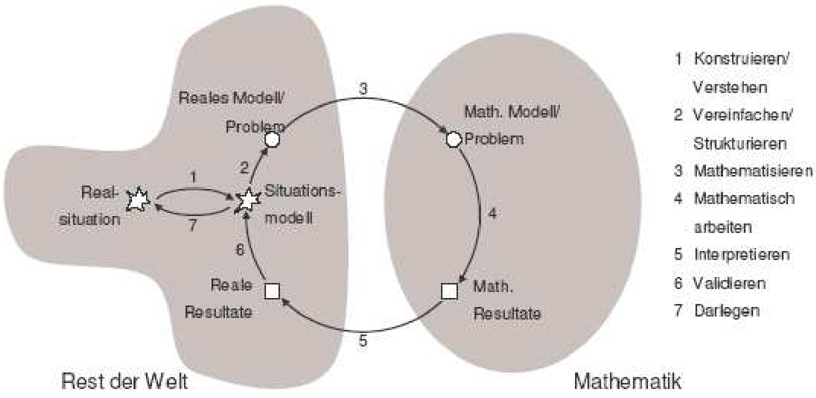


Fig.1: Modelling Circle Greefrath (2007)

This can now be transformed more easily into a mathematical model (Mathematisches Modell) than the initial real situation. The mathematical model should now lead to a mathematical result (Mathematisches Resultat) which has to be set in relation to the real situation. The starting point is a real situation which obviously must be chosen by someone (e.g. the teacher or the pupils) to deal with mathematically. The transformations between the four stages are not named in this model and are unidirectional.
A second model for the mathematical modelling process can be found in Borromeo Ferri, Leiss and Blum (2006). This model is not the first one developed by Blum, however, it is the current one.



This model starts with a real situation (Realsituation) which means the same as the "Reale Situation" in Greefrath's model. From this real life situation, a model (Situationsmodell) results by constructing and understanding real life (1). This model must be structured in the next step to get a simplified model of the real

Fig. 2: Modelling Circle Blum (2006)

situation (Reales Modell).This simplified model can now be mathematised into a mathematical modell (Math Modell). With this step you go from the real world (Rest der Welt) into mathematics. By doing some mathematical calculations a mathematical result will be produced. In the fifth step you have to interpret these results to get real results, which may fit to the starting real life situation. Checking if they really fit to the situation is the next step. In the seventh step the results are presented. This model includes the description of the transformations from one stage to another. The arrows representing the transformation point all in the same direction. What can also be seen very well in this model is that the modelling circle is the connection between the real world (Rest der Welt) and mathematics (Mathematik).
A third model, which was the basis of the first discussion during the first project meeting, is the model of the mathematical modelling process

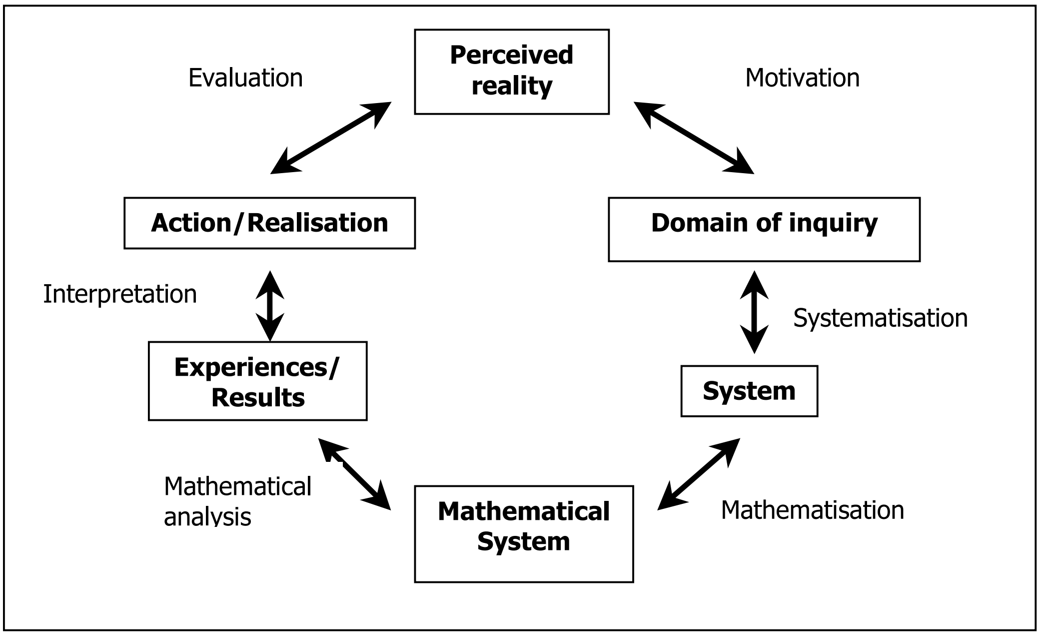


Fig. 3: Modelling Circle Blømhoj & Jensen (2006)

by Blømhoj & Jensen (2006) This model is very similar to that from Blum. The main difference is that the perceived reality (real life) is part of the circle. From this perceived reality, the motivation to deal mathematically with a Domain of Inquiry results. This Domain of Inquiry is comparable to the real situation in both other models. If you have a look at mathematical modelling lessons in school this step has already been done by the teacher. But this must be something the pupils shall learn, too. The following stages in this model of the mathematical modelling process are similar to those of Blum's model. However, at the "end" of the circle there is another difference to Blum's model. Blum includes the presentation of the results, which is not a part of this modelling circle. Another difference is that the arrows in this model point in both directions. This shows what Borromeo Ferri found out in 2006 a bit more clearly: students do not follow modelling circles in a linear way, but you can find all stages in a complete and finished modelling process.
On the basis of the above discussion it can be concluded that the chosen modelling circle from Blømhoj & Jensen is a good basis for developing descriptors for modelling tasks. It will be shown in the following

discussion how this modelling circle was simplified into four stages which could possibly be the descriptions for the arrows in Greefrath's model. Further, the model of Blum is very similar to the model from Blømhoj & Jensen. The only thing missing is the presentation of the results, which is also included in the list of descriptors.

Descriptors for modelling tasks

To make the ideal model of a mathematical modelling process a bit clearer for teachers, it was simplified into four categories:

- Motivation,
- Systematisation and Mathematisation,
- Doing the mathematics and
- Interpretation and Validation.

These resulting topics were then filled with criteria (descriptors), which describe what the learning objectives mean in detail.

Learning objectives	**Descriptors**
Motivation	Engagement (personal and societal)
	Teaching purpose
	Authenticity
	Linking existing mathematical knowledge
	Challenging
Systematisation & Mathematisation	Is data needed?
	Abstraction
	Assigning variables
	Making assumptions
	Simplifying
	Representation(s)
Doing the mathematics	Formalizing and analyzing the math problem
	Using data
	Approximation and estimation
	Use of Information and Communication Technology (ICT)
	Use known algorithms
	Mathematical common sense
	Proof (validation of the math used)
	Use of math. representation(s)
Interpretation & Validation	Validation of the solution mathematically
	Validation of the solution in the 'real world'
	Are the results good enough?
	Or is another cycle needed?

Table 1: Results of the first meeting of the research group_1

In addition to that a list of Learning and Teaching styles, especially communication skills has been developed:

Learning Objectives	**Descriptors**
Group discussion	Justifying
	Discuss and compare different strategies
Presenting the results and process	Oral presentation
	Written presentation
	Posters
	Reflection

Table 2: Results of the first meeting of the research group_2

Although the used modelling circle did not include presenting the results, these learning objectives were also focuses of the discussion to develop a list of descriptors for modelling tasks.

Comparison of the descriptors with theories about modelling competencies

In literature about mathematical modelling, lists about modelling competencies can be found.

Below you find a table which shows the comparison of the descriptors developed in our project with two concepts about mathematical modelling competencies. This comparison shall show that the descriptors not only include already existing descriptions about what modelling is but also expand these descriptions.

DQME II Descriptors	**Modelling competencies by Blum and Kaiser (in: Maaß, 2000)**	**Competencies by Ikeda, Stephens: What are modelling competencies? (in: Galbraith, Blum, Booker and Huntley, 1998)**
Engagement (personal and societal)		
Teaching Purpose		
Authenticity		
Linking with existing mathematical knowledge		
Challenging		
Is data needed?	to look for available information and to differentiate between relevant and irrelevant information	
Abstraction	to mathematise relevant quantities and their relations	
Assigning variables	to recognise quantities that influence the situation, to name them and to identify key variables - to construct relationships between the variables	Were relevant variables correctly identified? (G2) - Did the students identify a principle variable to be analysed? (G4)
Making assumptions	to make assumptions for the problem and simplify the situation	Did the students idealise or simplify the conditions and assumptions? (G3)
Simplifying	to make assumptions for the problem and simplify the situation - to simplify relevant quantities and their relations if necessary, and to reduce their number and complexity	Did the students idealise or simplify the conditions and assumptions? (G3)
Representation(s)	to choose appropriate mathematical notations and to represent situations graphically	
Formalising and analysing the mathematics problem	Doing the maths in common: to use heuristic strategies such as division of the problem into part problems, establishing relations to similar or analog problems, rephrasing the problem, viewing the problem in a different form, varying the quantities or the available data, etc.	Did the students identify the key mathematical focus of the problem? (G1)
Using data		
Approximation and estimation		
Use of ITC (software and graphics calculator)		
Use of algorithms		
Mathematical common sense	to use mathematical knowledge to solve the problem	
Proof (validations of the mathematics used)		
Use of mathematical representations		

Validation of the solution mathematically	to critically check and reflect on found solutions; to review some parts of the model or again go through the modelling process if solutions do not fit the situation; to reflect on other ways of solving the problem or if the solution can be developed differently; in general, to question the model	Did the student successfully analyse the principal variable and arrive at appropriate mathematical conclusions? (G5)
Validation of the solution in the real world	to interpret mathematical results in extra-mathematical contexts; to generalise the solutions that were developed for a special situation	Did the students interpret mathematical conclusions in terms of the situation being modelled? (G6)
Are the results good enough?		
Is another cycle needed?		
Justifying	and/or communicate about the solutions	
Discuss and compare different strategies, Reflection	to view solutions to a problem by using appropriate mathematical language	
Oral presentation, Written presentation, Posters	and/or communicate about the solutions	

Table 3: Comparison of descriptors and competency concepts

What is very noticeable is that the two competency concepts have nothing comparable to the motivation descriptors of the project list. On the one hand this is obvious because the question whether a task is authentic or not has nothing to do with competencies. On the other hand, it is a competency to choose or find authentic tasks for mathematical modelling. And this is not only a competency a teacher shall have, but also the pupils. So tasks shall also support the development of the competency to find mathematics in the real world.

Another difference to both concepts is that the focus “doing the mathematics” is not included in the competencies of Ikeda and Stephens and only included very generally in the concept of Blum and Kaiser. In my opinion “doing the maths” is a necessary competency for mathematical modelling, but it is also nothing characterising mathematical modelling, because it is also needed for example in problem solving.

Both differences found between the existing concepts and the developed descriptors support that the developed descriptors are good characterisations for the mathematical modelling process.

Outlook - Checklist for teachers

On the basis of the above discussion there is a good theoretical background to prove the accuracy and usefulness of the named descriptors in the DQME II list. Furthermore it is an expansion of the already existing descriptions of mathematical modelling. The list of descriptors is used in the project for evaluating the developed tasks. This will be part of the oral presentation of this paper.

Another question I want to follow up on in future is: with the help of these descriptors, can a useful checklist be developed for teachers to identify modelling tasks or maybe some kind of “good” modelling tasks? Not every teacher has a research group to ask if the developed, found or modified task is a modelling task and can support modelling competencies of the pupils. They need a tool to check it themselves since they are used to creating modelling tasks.

A checklist has already been created and will be presented to teachers soon. The checklist and the opinions of the teachers will be discussed during the presentation of this paper.

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Paper&Pencil Skills in the 21st Century, a Dichotomy?

Hartwig Meissner
Professor Emeritus in Mathematics and Mathematics Education
Westfaelische Wilhelms-University Muenster, Germany
Annabella Diephaus
Teacher at the Suente-Rendel-Grundschule Hoerstel-Riesenbeck, Germany

Abstract There is a worldwide development, better to say a non-development: We teach paper & pencil skills in primary schools almost like we did 30 or 50 or 100 years ago. Till today the primary school teachers spend up to more than 100 hours in the class room to teach and to train old fashioned algorithms though in daily life situations and for business purposes everybody uses a calculator. Why do we waste so much time of our children to teach them things which later on they will not need? We see an emotional dichotomy. Despite the research results from many research projects in many countries there still is the fear that the use of calculators in primary grades will harm mental arithmetic and estimation skills. To explain and to overcome that fear we will reflect the nature of number sense and of paper&pencil skills more carefully. We realize that the development of number sense is an intuitive and unconscious mental process while the ability to get an exact calculation result is trained logically and consciously. To overcome the above dichotomy we must solve the hidden dichotomy ***number sense*** versus ***precise calculation result***. We need a new balance. Different types of examples will be given how we can further the development of number sense in a technology dominated curriculum.

A. Analysis

Specialists from many research projects know that the use of calculators in primary grades does not necessarily harm mental arithmetic and estimation skills. But these "logical" arguments do not count. There still remains an emotional component against the calculator use which cannot be eliminated logically. Thus it is too simple just to claim to replace paper&pencil skills through the calculator. We need more than the ability to get a quick and exact calculation result. To remain mentally independent from the calculator we also must concentrate on automatic mental arithmetic and estimation skills. How do these skills develop? And which changes will we get when we change from paper&pencil skills to calculators?

A more profound view of mathematics learning is necessary to identify the nature of number sense and of paper&pencil skills. Learning and understanding mathematics is based on two different types of mental processes, on logical and conscious arguments as well as on intuitive and unconscious mental processes[1]. These two systems of internal representations produce the two interfering concepts ***precise calculation result*** vs. ***number sense.***

1. Precise Calculation Results

There are three techniques to get precise calculation results: Paper&pencil techniques, using a calculator or computer, and mental arithmetic. The teaching and training of paper&pencil skills is time consuming and the results are less safe than via pressing the appropriate calculator keys. It is obvious why the calculator technique dominates outside from school.

2. Mental Arithmetic

Mental arithmetic is a challenge for teachers to "teach" and for students to "learn" because of the two mental modes which are involved. On the one hand the students should be able to explain ***logically and analytically*** how they get the result. But on the other hand we also expect that for specific problems they can ***react immediately*** in a stimulus response style (stimulus response knowledge for e.g. 1+1 table and 1×1 table).

Furthermore we expect such an ***unconscious and intuitive*** stimulus response reaction also when the student gets confronted with computation mistakes. Either he/she spontaneously notices a conflict with his/her intuitive individual stimulus response knowledge or there is a spontaneous reaction like "this is too big" or "this is too small". The latter describes a conflict between the computation result and the individual personal experiences.

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[1] e.g. Vygotzki talks about spontaneous and scientific concepts, Ginsburg compares informal work and written work, or Strauss discusses a common sense knowledge vs. a cultural knowledge. Strauss (1982) especially has pointed out that these two types of knowledge are quite different by nature, that they develop quite differently, and that sometimes they interfere and conflict ("U-shaped" behavior). For more details see also the web site Meissner / Diephaus (2009).

3. Estimation Skills

Estimation is a challenging activity. Before starting computing we ask for the approximate result of a possible solution. Either the computation task already is given in the classical mathematical symbolic notation or we have to solve a word problem. For the first type of problems the estimation result can be found more easily. Here we must round the numbers and compute with rounded numbers. Estimation in this case is a special analytical and logical approximation technique (in German *Ueberschlagen*).

For word problems usually we first analyze the situation described. We then need a modeling process to get a "translation" of the word problem situation into a mathematical notation of a computation problem where we then can get an estimation result via approximation. But there is an alternative strategy to estimate the result for a word problem.

Analyzing a word problem can and should stimulate also subjective domains of individual experiences related to the situation given (Subjektive Erfahrungsbereiche, cf. Bauersfeld 1983). Intuitively and spontaneously non mathematical knowledge and personal experiences get stimulated, too. Estimation then may become a spontaneous and intuitive reaction like "Oh, this must be about ….".

4. Estimation and Sachrechnen

To estimate spontaneously and intuitively an approximate result for a given word problem we need special experiences, environmental and daily life experiences and experiences in comparing and measuring objects. To develop these experiences the German arithmetic curricula include a special topic called *Sachrechnen* (aspects of environmental and domestic sciences). In Sachrechnen we *compare* objects according to their length, time, weight, etc. (*direkter / indirekter Vergleich* in German) and we *measure* objects: Select a unit und try how often that unit fits into the object[2]. Estimation in Sachrechnen then is quite a different mental activity, it is the internalized process of *comparing* or *measuring* (*Schaetzen* in German).

5. Concept of Numbers

In traditional German curricula for primary schools we introduce step by step the "number spaces" [0 - 20], [0 - 100], [0 - 1.000], and [0 - 1.000.000]. Thus also step by step, the "object number" gets reduced into a sequence of digits and the computation with big numbers gets reduced into manipulations with sequences of digits[3]. Outside from school numbers have a different meaning. Here a number mainly is a measurement number (*Groesse* in German) which describes the size (value, magnitude, …) of an object. It consists of two parts, a *quantity number* and the appropriate *unit* like 345 km or 2830 hours or 562048 cents. The quantity number (*Masszahl* in German) tells us how many units we need to represent the size of that object.

6. Number Sense

We have summarized important aspects which are touched when we talk about number sense: "Number sense refers to an intuitive feeling for numbers and their various uses and interpretations; an appreciation for various levels of accuracy when figuring; the ability to detect arithmetical errors, and a common sense approach to using numbers. ... Above all, number sense is characterized by a desire to make sense of numerical situations" (Reys 1991).

Number sense not only refers to numbers but also to both, to conscious and to unconscious techniques to manipulate numbers, and it also includes a feeling about possible outcomes of these techniques. With a good number sense we roughly can predict the result of calculations, sometimes spontaneously (intuitively) and sometimes consciously (by approximating). Number sense also includes an intuitive feeling for additive and multiplicative structures. A central question for future curricula must be, if we can develop a more effective number sense by the use of calculators than we momentarily do in our traditional curricula.

B. Calculators and Arithmetic Learning

We will start with a warning. An unreflecting use of calculators in primary schools may damage some of the traditional goals of arithmetic education. The uncontrolled use might provoke two problems:

- Pressing keys is so easy. Why shall I still learn mental arithmetic?
- Pressing keys is so safe. Why shall I still control my calculator result?

[2] Sachrechnen also includes the topic money and problem solving activities (problems from real life situations like shopping, planning an excursion, constructing a bird-cage, etc.).

[3] Replacing paper&pencil skills through the use of calculators would not change this view.

Of course, a calculator curriculum must face these problems. But in this paper we will not discuss the PROs and CONs about the use of calculators in primary schools and how paper&pencil techniques could be replaced by a calculator use.

Here we will reflect how arithmetic teaching in primary schools may benefit from the use of calculators and how the use of calculators may help to further the traditional mathematical goals. Of course, the calculator is an excellent tool to get quick and safe calculation results. But besides this property it also may serve as a didactical tool to stimulate intuitive and spontaneous ideas and activities in the teaching and learning processes. The possibility to handle a big bunch of quick calculations without any efforts allows a new working style in the class room which was not possible without calculators or computers. We will summarize and analyze some activities.

7. Stimulus Response Learning

Calculators allow and facilitate stimulus response learning. This can be used in competitions to train mental arithmetic. The basic idea is to compute very quickly a given calculation problem to get then an immediate feedback: *correct* or *wrong*. We developed several exercises starting with problems from the 1+1 and 1×1 table:

(a) Individual work sheets, individual training: Type the problem into the calculator and calculate the result in your head. Then press the "=" and see if you were right. If YES write down the result, if NO do the next problem. Later on work on the still open problems.

(b) Competitions mental computation versus calculator, who is first? At the beginning each student wanted to be in the calculator group, later on almost nobody wanted to be there because "I am quicker in head".

(c) Each student gets a work sheet, the use of calculators is allowed. Who has finished the work sheet first? There may be different work sheets according to the students' abilities.

8. Operators

Calculators with a constant facility[4] allow developing a feeling for additive and multiplicative structures. We hide an operator ⊗k and others must find out which operator we hid:

$$X \xrightarrow{\otimes k} Y$$

Select a value for X, press the calculator keys, and interpret Y. Guess what ⊗k might be. If necessary select another value for X, etc. Finally select additional values for X and predict the results.

9. Calculator Games

There are several calculator games which use the constant facility to detect numbers and operations and to develop a feeling for additive or multiplicative structures. We will present an example with the calculator game *Hit the Target*[5].

Example:

$$? \xrightarrow{x\ 17} [800,801]$$

input for z	*display*

Hit the Target

Find via guess and test a number z that z×17 is in the interval [800,801]. Write a protocol of your guesses.

More general: An interval [a,b] is given and a factor k. Find a second number z via guess and test that the product "z×k" is in the interval [a,b].

For primary schools we suggest to concentrate on integers k,z < 100.

Our more than 1000 guess-and-test protocols show that the students after a certain training develop excellent estimation skills. They guess a very good starting number and they develop an excellent proportional feeling. For more details see Meissner 1987.

10. One-Way-Principle

Guess and test or trial and error are not considered to be a valuable mathematical behavior in mathematics education class rooms. But these components are necessary to develop spontaneous and intuitive ideas. Our experiences show that a systematic use of guess and test activities enriches creative and flexible thinking. So we developed a specific teaching method called One-Way-Principle

[4] These calculators can be "programmed" to work as operators "⊗k". ⊗ stands for the four basic operations.

[5] For more details see Lange / Meissner (1980) and Lange (1984).

(Meissner 2003). The One-Way-Principle is a method to use calculators or computers to explore intuitively and/or consciously many functional relationships of the type

$$X \xrightarrow{\sigma} Y \qquad \text{or in case of the four basic operations } \otimes \qquad X \xrightarrow{\otimes k} Y.$$

The basic idea of the One-Way-Principle is not to use reverse functions or algebraic transformations but to see the set of variables as a “unit” which gets explored via guess and test.

Concentrating on the four basic operations in primary schools we can explore with simple calculators additive or multiplicative structures of the type "a ⊗ b = c". Here the One-Way-Principle implies not to switch from addition to subtraction (or vice versa) or from multiplication to division (or vice versa). Instead we have to guess "a" (or "⊗" or b or "⊗b") to use then again the originally given key stroke sequence. Independent which variables are given and which are wanted, there is only the ONE WAY to solve all problems: Always use the same simple key stroke sequence of your calculator. The goal for the learner in the guess and test work is to discover intuitively the hidden relations between the variables and to develop a feeling how to get a good first guess (estimation) and how to reach a given target with only a few more guesses (additive resp. proportional feeling). Thus applying the One-Way-Principle furthers some of the intuitive and unconscious skills described above in no. 2 and 3.

C. Reducing Paper&Pencil Techniques

Again, in this paper we will not discuss how paper&pencil techniques could be replaced by a calculator use. But we will reflect how the traditional teaching and training of paper&pencil skills could be reduced. We think the main question is not how to calculate all possible sequences of digits but to ask first for the importance of each technique.

11. Expanding Mental Arithmetic

Mental computations usually are done with small numbers. We suggest to expand the meaning of “small” and to concentrate the four basic operations "a ⊗ b =" on all a and b where a and b are one-digit- or two-digit-numbers. Adding and subtracting two-digit-numbers already is part of traditional curricula. For the multiplication of two-digit-numbers let the students themselves invent appropriate techniques. Paper and pencil should be allowed to write down results from intermediate steps.

12. Proportional Feeling

In parallel to the conscious techniques from no. 11 the students also should get the opportunity to develop an intuitive feeling for possible results. Playing Hit the Target would be an excellent addendum, see no. 9. The students even might select themselves appropriate numbers for Hit the Target (small or big intervals [a,b], no integer solution for z, …).

13. “Large” Numbers

“Large” numbers in this paper are integers with at least 3 digits. Most of these multi digit numbers are unimportant in daily life because we prefer rounded numbers[6]. Putting important rounded numbers on the number line we do not get an equidistant pattern but a pattern which looks more like a logarithmic pattern. It seems as if we determine the importance of numbers in a similar way as we perceive the intensity of light or of sounds (Weber-Fechner law). This would mean that especially in large number spaces there are only a few “important” numbers. The larger the number space is the more unimportant numbers it will have. Do we still need for all these unimportant numbers the traditional paper & pencil techniques? We suggest to concentrate only on the calculating with “important” numbers.

14. Calculating with Rounded Numbers

Rounded numbers are similar to measurement numbers (*Groessen*, see no. 4 and 5). They consist of two parts, a one or two digit quantity number (*Masszahl*) and a unit (“thousands”, “millions”, etc.). To calculate with rounded numbers we can separate the two parts. We then can calculate with one or two digit numbers and apply techniques about what to do with the units. This approach also furthers Sachrechnen goals:

- For addition and subtraction both numbers must have the same “unit”.

[6] e.g. for the size of a swimming pool or a garbage container, for the distance between two cities or between the earth and the moon, or for the weight of an elephant or a lion, etc.

- Changing the unit implies also to convert the related quantity number[7].
- For multiplication and division there are easy rules how to compute with the units. The students themselves might discover these rules.

15. Number Spaces

Reflecting the topics from above we also should rethink the concept of introducing numbers. It is fine to start in the first grade with [0 - 20] and then [0 - 100]. But when we start using calculators the number space suddenly gets unlimited. We need a spiral approach in which the students themselves can discover numbers and number properties in individual own subjective domains of experiences and where they then can discuss their experiences. A spiral approach also would help to develop a much broader number sense.

16. Decimal Numbers

When we introduce calculators in primary schools we must be aware that the students very soon will discover decimal numbers in the display. But they already have a basic knowledge of writing decimals. According to our experiences they are just happy to learn that 23.5 can be interpreted as 23 cm and 5 mm, or 12.69 as 12 $ and 69 ct or 3.125 as 3 km and 125 m. And when there are more digits behind the "point"? Usually the children accept the simple answer "just ignore those digits" which corresponds to the view from above to distinguish between important and unimportant numbers.

17. FORUM

There is an internet web site to continue the discussion about the future of paper and pencil skills. Those who are interested to offer own contributions to that web site kindly are asked to write an email to Hartwig Meissner (meissne@uni-muenster.de). The FORUM web address is: http://wwwmath.uni-muenster.de/didaktik/u/meissne/WWW/Forum-P&P.htm

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Meissner, H.; Diephaus, A. (2009): The Development of Number Sense. Web: http://wwwmath.uni-muenster.de/didaktik/u/meissne/WWW/mei150.doc

[7] Getting experiences in changing units also may further the development of spontaneous and intuitive reactions as described in no. 3.

HOW TO SOLVE IT

Luigi Menna
Component of G.R.I.M. and PhD student, "Storia e Didattica delle Matematiche, della Fisica e della Chimica"
University of Palermo, Department of Mathematics, Italy menna@math.unipa.it

Abstract
This work is a reflection on the results of an experimentation carried out on secondary school students of between 16 and 18 from various classes. The experimentation aims at identifying the implicit ideas they use when asked to solve a certain mathematical problem. In particular, in giving them these problems an heuristic approach was suggested, and the differences between this and a purely deductive approach were measured. Analyzing the different approaches used by the students and the difficulties they had in distinguishing between argumentative and demonstrative operations has given rise to a reflection on the use of software such as Geogebra and Excel.

1. Introduction

The concept of "rigorous proof", from an educational perspective, is still a problem. Demonstration, meant as the mathematical instrument "par excellence", is often regarded as a hindrance to the development of the intuition and the capacities of exploration (Hanna, 2000).

Several teachers argue that the most convincing approach for a student is the one that requires investigation and exploration, which eventually stimulate intuition. They even maintain that the deductive demonstration may not be taught any longer (Hanna, 2000).

Furthermore, the increasingly frequent use of dynamic educational software pays ever greater attention to exploration. With this type of software it is possible to literally see various different representations of what is being studied in graphs that the students can easily create themselves.

From the point of view of maths teaching, we must underline that all this, as well as naturally encouraging the formulation of hypotheses, also implies the abandonment (intentional or not) of teaching-learning processes involving deductive proof, leaving the experimental approach as the sole mathematical tool brought to the students' knowledge.

The software can examine a huge, or even, if we introduce the concept of continuity, infinite quantity of data (Mason 1991).

In effect, experimental speculation and rigorous mathematical proof probably make up the ideal combination, but I strongly believe that not distinguishing between the two approaches when teaching could be extremely dangerous. A distinction is necessary to be able to outline the contribution each culture may provide (Di Paola&Spagnolo, 2008; Spagnolo, 2005; Spagnolo&Ajello, 2008). This is why I suggest that a serious reflection on the inevitable use of computer sciences in teaching mathematics and physics today, is urgent.

The object of this research is to investigate the implicit ideas used by students when asked to prove a mathematical proposition. We will therefore investigate not only the type of proof used by the students, but also the type of reflections induced by a certain teaching approach and by the use of certain instruments.

The methodology used is "the theory of situations" (Brousseau, 1997, Spagnolo et alii, 2009) both in teaching and in analyzing the data.

Given that in Italy maths and physics are taught by the same teacher, we want to see if this encourages pre-comprehension and/or epistemological obstacles among students regarding the type of proof used in teaching the two different subjects.

2.1. Experimentation: first step

Experimentation was carried out on secondary school students of between 16 and 18 from various classes.

The students interviewed were given two problems:

1. Determine, among all rectangles with the same perimeter length, the one with the maximum area.
2. Determine, among all triangles with a given hypotenuse, the one that has the maximum ratio between the hypotenuse and the sum of the other two sides.

All students know analytical geometry, the concept of place, of points, and the concept of continuity in a function. Only a few of them know the problems regarding derivability.

The students in this experimentation were allowed to use Geogebra and Excel and were asked to write down the reasoning they used to solve the given problems. They were told to write simply and freely about anything that helped them formulate the hypotheses which led to their proof, when they found one. It was clearly stated that

they could use any non-specific terms that came to mind if they felt it necessary. They were also asked to express their impressions and feelings.
In the previous lessons a classroom experimentation on the motion of a mass on an inclined plane had been performed, the experimental data was collected and the curve most likely to describe the event was sought. The students, obviously, were asked to determine unknown quantities and approximate values, to repeat the experiments several times and to draw a conclusion. The teacher who followed them in this experimentation highlighted the fact that even before beginning the experimentation they already had an implicit idea of what should happen given that they had to look for some relation between time and space using certain objects: a trolley, an inclined plane, a position sensor and statistical analysis software. In other words, all they had to do in the lab was to confirm the following hypothesis: the distance covered by a trolley subjected to a constant force is proportional to the square of the time taken to cover this distance.
The problems were tackled by all students using Geogebra. All students were able to make hypotheses. Almost everyone formulated correct hypotheses (in the first problem, the rectangle of maximum area is the one with sides all of the same length; the second solution is that the triangle we are looking for is isosceles and the ratio is $\sqrt{2}$).
As for the proofs only a few students tried to write them down and submitted them.
Among those who tried to demonstrate the proof, a lot were satisfied with the example generated by Geogebra which was fairly clear and meaningful and helped the students in tackling the problem.
After this introductory part, mainly aimed at giving the problem to the students, there began the second part of the experimentation.

2.2. Experimentation: second step

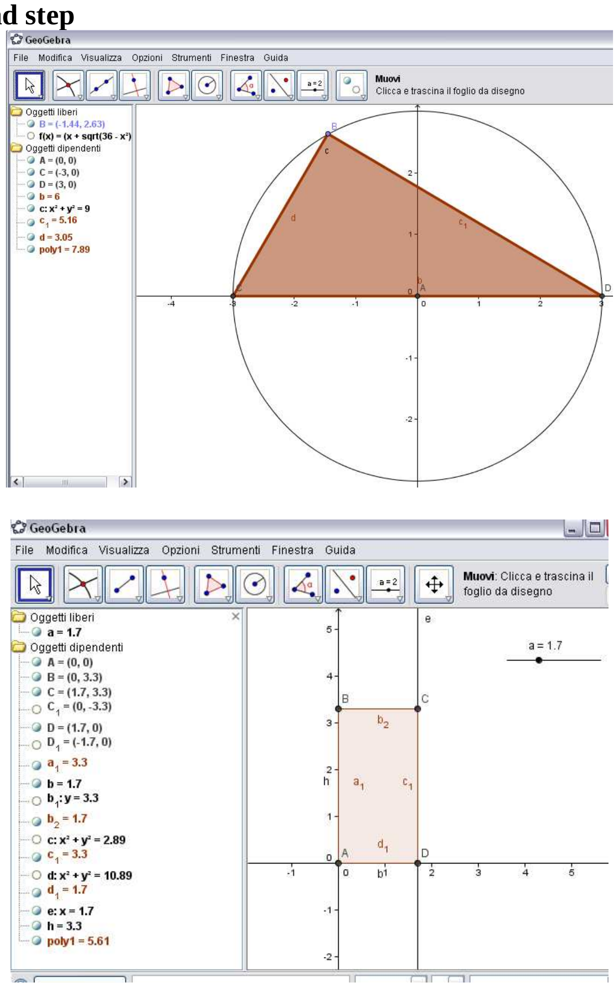

Figure 1

Two different types of strategies were provided to students.

The first strategy presented was a collection of data with the Geogebra, re-elaborated on Excel.
The reasoning processes are very simple since they are based on images generated by Geogebra and modified by "dragging"(Fig. 1). What is important is that this software, in addition to watching what happens moment by moment when the size of a side is modified, shows the dimensions of the segments and areas.
The proposed strategy involves nothing more than reporting the data obtained through real "sampling" in a two column Excel table: the first according to the values of the base, the second to the values of the area (Fig.2). It is a process which, together with measuring physical quantities, sounds familiar to the students because this table is also used to obtain pairs of values from a function using the concepts of a dependent and independent variable (*e.g.*: area and segment respectively). In this way it is easy to obtain points on a Cartesian plane representing not only the experimentation but also a curve which provides an estimated trend of intermediate values which have not been actually investigated. Excel is able to generate a graph connecting the points to give rise to a curve (in the first problem which resembles a parabola in every sense).
If this process was followed by each student we would obtain different graphs all for the same problem applying different values, but which would all concur in identifying the square in the first case and the isosceles triangle (and a ratio of 1.41) in the second.
The methods of solving the two problems are similar. The difference of the second exercise is that it requires the determination of an irrational number that is impossible to identify with absolute precision through data processing techniques.

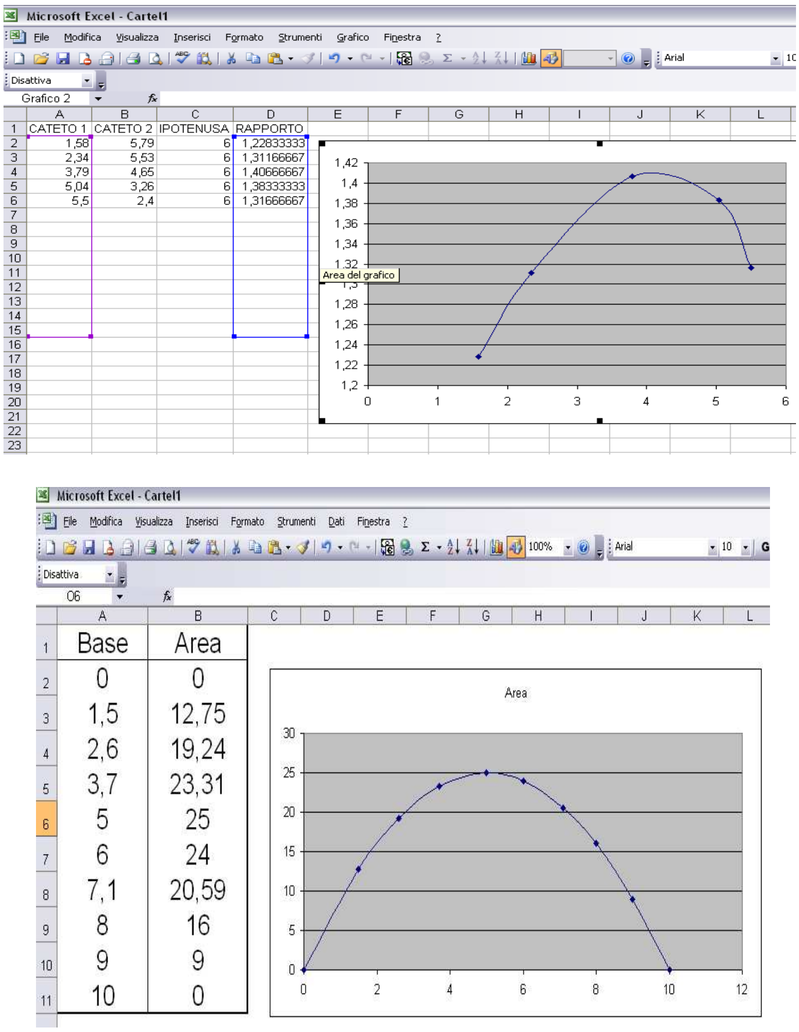

Figure 2

The second strategy is a typical textbook approach, modeling the situation with a function whose analysis provides the result.

2.3. Experimentation: step 3

Each step of this process was followed by personal notes. The students were asked to express their opinions and considerations: which approach did they prefer and why? The analysis of the data shows that most of the students:

- considered the heuristic approach more effective and easy to handle;
- considered the maths proof more elegant and provides the optimum response to the problems given;
- on the other hand, they themselves admitted that they could have never managed to produce a complete proof; the heuristic approach, however, does give the correct answer and lets them create the graph of the function quickly.

As for the comparison between the operating procedures confirming hypotheses on both the physical and mathematical problems:

- few students spoke of the difference between inductive and deductive procedures, thinking that the former is used more in physics and the latter in maths, but without going into details;
- most of the students found that the software gave more rapid solutions to this type of exercise: we can thus conclude that they prefer the inductive method and would like to use it preferentially in class.

3. Conclusions

The main thing that emerges, and on which it would seem necessary to dedicate more attention, is a certain contradiction:

- the software does help students formulate hypotheses;
- all the students found the empiric approach within their understanding, valid and effective;
- the approximations inherent in the use of this software seem acceptable to them (as it was in the physics problem);
- a simple data base like Excel immediately supplies a mathematical model of the solutions.

However, almost all the students found that the deductive procedure:

- is more elegant;
- is easy to apply in general;
- is the “correct” procedure, and would be a Good Thing to learn, develop and use.

Contradiction mentioned above suggest questions about what is the reason why the students consider “correct” and “beautiful” some particular types of reasoning. Therefore, I believe that a further development of the experimentation could be the analysis of the demonstration strategies of students from different cultures.

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Tasks for tests and A-levels using CAS

Heidi Metzger-Schuhäker, Mag. Commercial Highschool HORN, Austria
ACDCA (Austrian Center for Didactics of Computer Algebra) heidi.metzger@hakhorn.ac.at

Abstract
Tasks for different years of the secondary level II are presented on the basis of long lasting experience with computer-assisted mathematics instruction. They include applications of mathematical skills as well as the testing of theoretical knowledge. Finally relevant A-levels tasks are presented that integrate different mathematical contents into every day connections from economy, medical science, sports asf.

0. Explanation

This English text is a short summary from a German paper that was published at the 100th MNU-Congress in Regensburg in April 2009. Main parts concerning the examples of application-oriented test tasks using CAS had to be abbreviated. The whole English version is available at http://www.acdca.ac.at (Website of the Austrian Center for Didactics of Computer Algebra)

1. Initial situation

If mathematics instruction would have just stayed the way it always has been. Simplification of fractions, “wild” compilation of formula, more or less realistic survey tasks, curve sketching “according to recipe”, complex partial fraction extension … Back then everything was just fine. Everywhere you could find a wide range of stereotypical tasks to read, copy or download. These times were over at the latest when the thought achieved to complement this check of automatable skills by a seemingly meaningful approach to mathematics. In the centre of mathematics instruction today should be interpreting, arguing and modelling, whereas the use of computer algebra systems resp. a graphics display pocket calculator seems very helpful in order to reach this aim.

In Austria there have been basic approaches for a certain time as well as efforts by certain groups to support and propagate computer assisted mathematics instruction. As a leader you could name the ACDCA (Austrian Center for Didactics of Computer Algebra).

For the vocational school system – especially in commercial academies, a quinquennial type of school of the secondary level II, comparable with post secondary college with economic focus in Germany – the use of CAS is specified in the curriculum as follows:

MATHEMATICS AND APPLIED MATHEMATICS
Educational and teaching outcomes:
The students are to be

- instructed to independent sense-making
- taking an active studying position
- disputing with economical problems in all years
- gaining insights into the opportunities of the application of mathematical methods in practical operation
- developing a basic understanding for mathematical theories and concepts
- applying mathematical methods to tasks, describing them with appropriate mathematical models, estimating and interpreting solutions
- working independently and in the team
- applying Computer Algebra Systems and/or spreadsheets resp. graphics display pocket calculators in all years and solve mathematical tasks with them

In order not to refuse future trends and break new ground the Commercial Highschool Horn has decided for the introduction of the Computer Algebra System Derive in mathematics instruction more than ten years ago. Other programs like Excel or GeoGebra are used as supporting tools. Meanwhile the HAK Horn only keeps notebook classes whereas the notebooks are bought by the students or their parents in the second semester of the first year. Thanks to this new form of tuition new media can be applied. That proves itself very useful in mathematics when it comes to analyze statistics, check different payment arrangements for internet transactions or as guides for studying in the internet.

2. Tests – a new challenge

The introduction of notebook classes and the resulting change in methods and contents of mathematics instruction we had to face a totally new situation:

- Tasks for tests from CAS-less times had to be adapted resp. canopied totally, hardly one example can be taken one-to-one.

- Real numerical data can be taken. One does not have to manipulate it in order to make the task calculable.
- The elimination of arithmetical difficulties helps to focus on the understanding of mathematical connections.
- Numeral tasks, graphical solutions or mathematical modelling find their place in tests thanks to the use of CAS.
- The preparation of tests takes essentially more time since the teachers have only access to less available material and have to phrase exactly and clearly when it comes to open questions resp. interpretation and argumentation tasks.
- The technical implementation needs an exact plan, a sufficient technical infrastructure, a good teamwork with the appropriate network technicians of the school and permanent control. Nevertheless the risk of a technical breakdown remains.

3. The framework

An important basic consideration about the fact whether the tests should be carried out at the notebooks directly or at other computers in the IT-room is necessary when it comes to written tests in notebook classes.

My personal experiences have shown me that it is much more strenuous to organise an available, working IT-room with a sufficient number of standard computers than to let the students work on their own private machines. Nevertheless it ought to be considered that the control of the students becomes much harder since they have very quick access to their personal files and are able to exchange information at cyberspeed via wireless connection without the teacher realising it. Even if a working monitor-control-software should be at hand, one should consider that the teachers cannot assist the students when it comes to technical breakdowns. Because of this reason I prefer having tests done in an IT-room even it is not always possible due to the schedule.

When preparing tests it has been approved meaningful to think about the following points:

- The functional efficiency of the IT-room has to be checked beforehand in any case.
- Seating arrangements resp. blinds for the neighbour-PCs should be organised (dividers should be used).
- The login at the standard machines is carried out with a particular test profile that should meet the following requirements: no connection to outside, the right to read at one instruction folder, the singular right to write at one solutions folder.
- A “private” collection of Derive-orders (check beforehand!) is allowed in paper.
- Formula worked out together in tuition are being provided in the instruction folder.
- The students each create one local folder on their standard machine wherein they save their tasks.
- Exact instructions concerning the handling ought to be handed out beforehand, maybe even in written form (user name, password, name of the local folder,….)
- The tasks of the test are in any case provided on paper as well.

While executing the test it is beneficial to save the tasks one after the other in the students’ local folders on the standard machine in order to go back to them in case of a technical breakdown. Headlines with name and time in the singular documents help with the tracking of the printouts. If the students make proof prints during the test it is advisable to have them handed out by the teacher in order to prevent disquietness and not to give the student the opportunity of insight to the solutions. The time of release is the time when the task is being transferred into the solutions folder.

After having finished their work the students can check if they have released correctly via the beamer and that they did not hand in an empty folder (have a look at the memory size!)

From personal experience I find it very important to point out that it would be wise to grant oneself the time of more than one lesson for preparation as well as for he wrap up, since not only the login but also the creation of the standards and the printing takes a lot of time.

At this point I have to refer to difficulties that one has to struggle with in the course of so many years when guiding dozens of tests an the computer. The major problems hereby are concerned with data loss, technical breakdown or the area-wide control that seems to be impossible.

At our school we deal with these problems after an agreement and a demand through the state education authority as follows:

- The teacher can be supportive when it comes to CAS problems at their own discretion.

- Data loss (program got stuck) resp. machine breakdown needs the following action:
  - direct elimination of the problem (regular data saving leads to the loss of not more than one task).
  - The working time for the last task is credited.
- When there are general technical problems the test has to be dismantled totally, the solution of the problem has to be awaited and after that the test has to be continued by all students resp. the time of the breakdown has to be credited.
- It would be wise to ensure the presence of a mathematics teacher or the network operator (mobile phone!).

4. Examples of application-oriented test tasks using CAS

As already mentioned in the beginning the expenditure of time for the composition of a test has become much more. Time and time again one has to find the right measure to test arithmetical skills on the one hand and the understanding of mathematical contents on the other. The composition of a test could be as follows:

- Practical arithmetic problem, divided into some tasks, documentation of the steps of calculation, interpretation of the results.
- Mere theoretical example, comprehension questions, elaborate justification, explanations.
- The more complex the tasks, the more theoretical question or open interpretation of results can be integrated into the arithmetic problem and do not have to represent a singular task any more.

5. A-levels – composition of the final examination

Does not every teacher have a high standard when it comes to compose the final examination? You can not start too early with collection appropriate examples. It is important to focus on the personal interest of the students, current topics resp. on tuition focal points. Like other schools we in the Commercial Highschool Horn have decided some years ago to devote the instruction of the A-levels to one common, throughout topic. Thus after the decision for a topic frame an intensive research for already existing tasks (internet, magazines, colleagues) starts.

Criteria for the tasks are:

- Are there similar questions for the preparation in the tuition?
  One of the most difficult challenges for teachers of graduating classes is to find similar complex and extensive examples as in the final examination without anticipating the relevant question in a too concrete way.
- Is a clear question possible?
  Misunderstandings can always happen, since even colleagues teaching the same subject overlook ambiguous phrasing due to their education. It would be beneficial to consult other colleagues (most suitably German teachers) for proofreading.
- Can an open question be weakened by means of canny phrasing?
  The evaluation of open questions is difficult and needs extremely good preparation. To keep the room for the students on the one hand and to remain clearly arranged with the possible ways of solution on the other, it is meaningful not to phrase too generally.
- Are there sheet anchors for the students in the A-levels instruction?
  For an A-levels instruction with a balanced degree of difficulty it is necessary to incorporate examples that can be understood quickly and are solvable for average students. Calculation skills and a sufficient understanding of the mathematical connections shall be tested without special demands.
- Are the examples application-oriented and do they include at least four different areas of the curriculum?
  Especially in their final exam the students should be made aware of the fact that mathematics is present around us in every day life. The different areas are used connected by a common topic (e.g. medicine, sports, stock analysis,…). Independent topics ought to be of equal value, for each of them the same amount of points ought to be reachable.
- Is it possible for the students to use additional material like statistics, newspaper articles, share prices given in the instruction for a more detailed analysis?
  In the course of the following studies it is very important to be able to separate the essential from the inessential information in newspapers, statistics, graphs, asf. For the calculation or presentation of results it is allowed to use different computer programs.

Notenschlüssel:

sehr gut:	91	bis	100
gut:	80	bis	90
befriedigend:	66	bis	79
genügend:	50	bis	65
nicht genügend:		bis	49

Punkteabzüge:

Angabefehler (AF):	2 – 8 Punkte
Rechenfehler (RF):	2 – 4 Punkte
Denkfehler (DF):	3 – 8 Punkte
Formalfehler (FF):	1 – 2 Punkte
Schreibfehler (SF):	1 – 2 Punkte

picture 1

Submission

The instruction is submitted with the consideration of the allowed use of CAS (resp. other programs). An exact grading key and a completely calculated solution are attached. In the case of tasks that allow more than one result and interpretation, only one is carried out exemplarily with consideration to other possible solutions. A penalty on the scores is submitted as well with a little clearance according to the level of difficulty. One possible variation is presented in picture 1.

Correction

As agreed with the state superintendent of schools only the students' printouts are being corrected. In case of doubt the saved solution files may be consulted as decision support. Answers to open questions or theoretical questions are evaluated concerning the understanding of mathematical concepts regarding the qualification of the chosen mathematical models as well as the independent interpretation of the solutions in preferably practical tasks.

6. My personal résumé

After a couple of years' experience with tests and A-levels one could draw the following conclusion:
Students have hardly any problems handling the CAS-programme. They can also deal with the so called arithmetic problems that work a s a kind of sheet anchor and they show good results. The documentation of the steps of calculation takes getting used to. In general there are problems mostly with theoretical questions and understanding questions.

Mathematics is the art to spare oneself calculating. (Bruno Buchberger)

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Models of Mathematics Curriculum Development in Egypt

Fayez M. Mina, MA PhD C.Math FIMA
Emeritus Professor, Department of Curriculum and Instruction, Faculty of Education,
Ain Shams University, Roxy, Heliopolis, Cairo, Egypt fmmina@link.com.eg

Abstract The need for developing mathematics curricula was clarified. Models of mathematics curriculum development in Egypt were identified as: "Temporary Committees" (TC), center of developing curriculum and educational materials (CDCEM), "National conferences" (NC) and "Educational standards" (ES). The advantages and disadvantages of each one of these models were evaluated. Then a new model was suggested covering the whole advantages of these models and avoiding their disadvantages.

Introduction

A mathematics curriculum is a system, its components are: Aims, content, methods of teaching, educational media, educational environment, educational activities and evaluation. The whole mathematics curricula is a sub-system of many wider systems such as; curricula taught, educational system, national culture, regional culture humanistic culture at large. Curriculum development - in general - stems from attempting to cope with changes in these wider systems, differences between educational outputs and the intended ones and in-consistency (ies) between the components of curriculum. So, the need of curriculum development - whether in mathematics or other subjects - seems to be continuous. Normally, there is an attempt to develop mathematics curriculum(1) each 3-7 years with regards to many considerations(2). The process of development itself usually follow - procedures, which can be classified into common ones, referred to as "models".

Models of Mathematics Curriculum Development in Egypt

Models of mathematics curriculum development in Egypt in the past five decades - or so, could be classified as follows(3): Temporary committees (TC), Centre of developing curricula and educational materials (CDCEM), National Conferences (NC) and Educational Standards (ES). The following is an explanation to each model:

1- **Temporary Committees Model (TC):**
A committee to develop a mathematical curriculum is formulated by a ministerial decree from some professors of mathematics in universities, some professors of mathematics education as well as some mathematics educators(4). The committee hold some meetings and write a report with the suggested changes - almost in the content of this curriculum. Once the report is accepted by the minister, the committee finishes its work.

2- **The Centre of Developing Curricula and Educational Materials Model (CDCEM):**
This centre was established in 1990, to be affiliated with the office of the Minister of Education(5). The major task of the centre is issuing school text-books. The normal procedures were as follows:

   a) Writing a scope and sequence for mathematics covering the whole basic education stage (1-8 grades). Writers of a particular text-book are supposed to consider what is taught in the corresponding grade.
   b) Formulating a team of writers(6). The team put a plan to accomplish the task, with individual assignments to its members.
   c) The products of the team should be seen - and are subject to modification, by an editor.
   d) The team writes a teachers' guide for that text-book.
   e) Experimenting - at least - a sample of topics and their guides in different parts of the country.
   f) Introducing necessary changes in the text-book and the teachers' guide.
   g) Putting a plan for introducing the text-book and the teachers' guide to schools. This plan normally includes in-service teacher education sessions(7).

3- **National Conferences (NC):**
Some conferences are held at the national level representing university professors - including professors of education and some professors of educational research centers, educational administrators, teachers, students, parents, political powers and so on. They are hold with participation of non - governmental organizations(8). Two ideas were stressed in these conferences; The first is that education is much more effective in the

life of society than a group of people decide on its matters only, while the second is that the importance of education is well recognized[9]. Two conferences were devoted to develop two educational stages[10].

4- **Educational Standards (ES):**
In 2003 the educational system in Egypt has joined the movement of "National Standards". A three volumes book included "National Standards for Education in Egypt" was issued with six major documents, one of them is devoted to curriculum[11]. This document includes a part for mathematics curriculum. In the academic year 2006/2007 the process of developing mathematics education according to the suggested standards was initiated. The year 2006/2007 witnessed the writing of mathematics text-books and teachers' guides for grades 1-3 primary in a national competition among Egyptian publishers according to the educational standards, while the year 2007/2008 witnessed almost the same thing for the fourth primary and the first preparatory[12].

Advantages and Disadvantages of the Existing Models of Developing Mathematics Curricula.

Before proceeding, a reference should be mentioned to hinders of changing outcomes of mathematics education development in Egypt. Some of these hinders are: The system of evaluation, the difficulty in introducing radical changes into methods of teaching, the use of educational media and educational activities, and the prevailing environment of education.

It must be noted that almost all attempts to develop curricula – and mathematics curricula took place in primary and preparatory education, without having secondary education[13].

Keeping in mind this, the major advantage of TC is that it is easy to formulate and to work out. On the other side the major disadvantage that they are almost confined to the content, without dealing – or attempting to deal with – the content, without dealing – or attempting to deal with – the other components of curriculum.

The model of CDCEM has many advantages such as representation of almost all concerned people, editing the work, having a teachers' guide, experimentation and in-service teacher education. The major disadvantages are that it is based on the "scope and sequence" which has become rather historical and the hesitation to introduce major changes in the content of mathematics curricula[14].

The assumptions underlying NC, can constitute their basic advantages, but disadvantages can be seen in the weak representation of different sectors as well as the formality that their conclusions are dealt with[15].

Although the writer has some reservations on the movement of standards, it could help in authoring school text-books. The major advantages of ES are that it allows teaching advanced topics from primary education –eg probability, and that it gives much more interest to some neglected topics in primary education - eg estimation and mental arithmetic[16].

The suggested model

Needless to say, the temporary model is easily rejected. Keeping in mind a lesson from NC that different concerned groups must be represented as well as thinkers all over the country and NGO's in the process of curriculum development at any stage, the suggested model can be a mixture of CDCEM and ES. A rather detailed description to the suggested model is presented as follows:

1- Identifying standards and indicators (aims)
2- Drafting a text-book and a teachers guide. (whether by charging some experts or a competition).
3- Introducing applications in other disciplines and possibilities of integrated topics.
4- Modifying the drafts according to the last point.
5- Holding in-service education sessions for teachers.
6- Experimenting the text-book and the teachers guide, including all the sub-systems of curriculum.
7- Evaluating of the new materials by samples taken from all concerned people.
8- Re-modifying the new materials according to the results of evaluation.

9- Checking regularly the new materials according the needs for curriculum development (mentioned above).

10- Introducing necessary changes, with a possibility of a total or a radical changes.

Notes

(1) Apart from partial modifications.

(2) Such as, availability of needed budget, public opinion views- whether teachers or parents or others, plans and possibility of in- service teacher education… etc.

(3) Procedures might differ within the same model. Actually, a model is an abstraction of the common procedures taken in that model.

(4) In some cases, the findings of these committees are discussed with some teachers, though no changes obvious have been reported as a result.

(5) The writers mean by this models the products of this centre within the period 1990-1993, which includes mathematics text- books for the first four grades of primary education. Since 1993, the centre concentrates on editing and typing the text-books which won in public competitions.

(6) Including some professors of mathematics education, some mathematics inspectors and a teacher of mathematics teaching the same grade.

(7) These training sessions are hold to - at least - trainers of trainers, with regard to the great numbers of teachers of primary education.

(8) Especially the Egyptian Society for Development and Childhood.

(9) Mrs. President chaired all these conferences and some of the best thinkers participate in them.

(10) In particular; The National Conference for Developing Curricula of the Primary Stage (1993) and the National Conference for Developing Preparatory Education (1994), in addition to other national conferences in education.

(11) Other documents include the standards of: An effective school, teacher, distinguished administration, societal participation and learner.

(12) The plan of the Ministry of Education in Egypt includes that mathematics text-books took place in 2008/2009 for the 5^{th} grade primary and the 2^{nd} grade preparatory, while in the academic year 2009/2010 covering all the primary and preparatory grades.

(13) Recently (in 2008) , there was a plan to change the structure of secondary education, its curricula, admission to higher education as well as the process of evaluation, to be administered by the year 2013.

(14) E.g. teaching probability and the use of calculators in primary education.

(15) A hint should be given to the facts that the major slogan of the conference about primary education is that pupils are asked to study basically “activities” and the slogan of the conference concerning preparatory stage is to “empower students to get keys of knowledge”. In practice, no procedures have been taken to put these slogans into action.

(16) Nevertheless, the written document about the characteristics of the subject matter included – sometimes - vague phrases. Further, in-service education sessions were limited and teachers guides were printed late, so, the new text-books were badly received by some teachers, parents as well as pupils.

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TO TEACH COMBINATORICS, USING SELECTED PROBLEMS

Laurenţiu Modan

Abstract: In 1972, professor *Grigore Moisil*, the most famous Romanian academician for *Mathematics*, said about *Combinatorics*, that it is "an opportunity of a renewed gladness", because "each problem in the domain asks for its solving, an expenditure without any economy of the human intelligence". More, the research methods, used in *Combinatorics*, are different from a problem to the other! This is the explanation for the existence of my actual paper, in which I propose *to teach Combinatorics, using selected problems*. **MS classification**: 05A05, 97D50.

INTRODUCTION

Combinatorics, a distinct and special branch of *Mathematics*, appeared as a consequence to solve *counting problems*. The reasoning, used in it, is less analytic, because this is made in a primordial way, like as in *Metalogic*.

We explained in [2] and [3], and now we shall reiterate, that a rigorous and educated thinking in *Mathematics* must begin learning *Enumerative Combinatorics* and then continuing with *Combinatorial Probabilities*, because they represent together, the simplest, but the very refined description of the daily life. More, we must not forget that today, by its new branches as: *Graph Theory, Matroid Theory, Code Theory* etc., *Combinatorics* is the most dynamic domain in *Mathematics*, yearly having the biggest number of proposed and solved conjectures.

In the actual Romania, *Discrete Mathematics,* and therefore *Combinatorics,* are ignored effectively. So, the precarious education of the students, in this fundamental domain, is reduced only to wield simple algebraic relations, usually known as *"formulas"* and which never permit them, to use correctly, the *arrangements*, the *combinations*, the *permutations* or the *number of the functions* defined on finite integer sets.

We remember here (see [2], [7]), that for a given set $S=\{x_1, x_2, \ldots, x_n\}$, with the *cardinal* $|S| = n$, we have:

i) the *permutations* are different between themselves, *only by the order* of the elements, their number being $n!$;

ii) the *arrangements* of the groups with *m* elements from *n*, are different between themselves, *by the order and the type* of the elements, their number being $A_n^m, n \geq m$;

iii) the *combinations* as the *number of the subsets in S*, having *m* elements, are different themselves *only by the type* of the elements, their number being $C_n^m = \binom{n}{m}$, $n \geq m$;

iv) the *number, of the functions* $f : S =\{x_1, x_2, x_3,\ldots, x_n\} \rightarrow T =\{y_1, y_2, y_3,\ldots, y_m\}$, is m^n.

When we begin to teach *Combinatorics*, we must insist in front of our students, about the *essential role of the words used in the texts* of this domain. Only in this way, we realize the achievement of the best knowledge in *Mathematics*!

For our study in teaching *Combinatorics,* the start point was relating to every student, even if in the high-school course, who might wield without no difficulty, all the notions of this domain. To arrive to our aim, I used a very popular Romanian *Collection Book for Algebra* (see [1]), which in the last 30 years, is the most recommended to the students in the high-school course. Although its exercises and problems are chosen to train the students in any branch of *Algebra*, I noticed astonishingly, that for *Combinatorics,* there are a lot of problems, bad enounced or bad solved …

RESULTS

At the 10-th chapter of [1], in *Problem 25,* we found the same error as in 1981 edition (appeared at „Didactică şi Pedagogică" Printing House). The text of the quoted problem is the next:

(1) "For a game, 3 boys and 5 girls are organized in 2 teams, heaving 4 persons each. In how many manners are organized these teams ? Find the number of the situations, if in each team with 4 persons, we must have only a boy".

We notice for the solution, that 2 teams heaving 4 persons each, will use the subsets with 4 elements, from a set with cardinal 8. So, the first team is chosen in C_8^4 ways and the second in C_4^4 ways. It follows, that for our 2 teams hawing 4 persons each, there are $C_8^4 \cdot C_4^4 = 70$ possibilities.

Normally, the second part of the problem (1) is a *nonsense*! Indeed, if in each team there is only 1 boy, for the first team we need 1 boy and 3 girls, and for the second we have 1 girl less, respecting the given conditions! From here, we state that for the second part of (1), the correct enunciation must be:

(2) "In how many manners are organized 2 teams having 4 persons each, if in any of them, it must be at least, 1 boy ?"

The first team, containing 3 girls and 1 boy, will be chosen in $C_5^3 \cdot C_3^1 = 30$ manners and the second team, hawing 2 girls and 2 boys, will be chosen in $C_2^2 \cdot C_2^2 = 1$ ways. Finally, we decide that the solution of (2) is given by the product $\left(C_5^3 \cdot C_3^1\right)\left(C_2^2 \cdot C_2^2\right) = 30$.

In [4], I proved as a generalization of (2), that $2n-1$ boys and $2n+1$ girls can form in $(4n)!/((2n)!)^2$ possibilities, 2 teams having each, $2n$ persons. If we impose that in each team to be at least 1 boy, the possibilities are given by:

$$\sum_{k=1}^{n-1} C_{2n-1}^{k} C_{2n+1}^{2n-k} = \left(C_{4n}^{2n}/2\right) - (2n+1), \tag{3}$$

a sum which asks a *strong work* to compute it!

We shall go on, our incursion in teaching *Combinatorics*, correcting the solution of *Problem 7* of the 10-th chapter from [1], in which the number m^n, of the functions *f : S→ T,* is badly used! The text of this quoted problem is:

(4) "Find the cardinal of the set containing 5 digit numbers which could appear using the even digits from the set $\{0,2,4,6,8\}$."

The solution of (4) is based on the following two cardinals:

$$\begin{aligned} &\left|\{g/g : \{d_1, d_2, d_3, d_4, d_5\} \to \{0,2,4,6,8\}\right| = 5^5, \\ &\left|\{h/h : \{d_2, d_3, d_4, d_5\} \to \{0,2,4,6,8\}\right| = 5^4, \end{aligned} \tag{5}$$

where d_1, d_2, d_3, d_4, d_5 are digits and where, for the functions *h*, we must exclude the situations which begin by $d_1 = 0$. Now, the number asked in (4), is $5^5 - 5^4 = 4 \cdot 5^4 = 2500$.

I wanted to correct the error for the solution of (4), appeared in [1], because through this manner, I understand to sustain one of the first *counting principle* appeared in *Mathematics* and having a very strong importance in *Algebra*, *Combinatorics* and *Discrete Probabilities*. Emphasizing the great utility of this *counting principle* which must remain *well stamped upon the mind* of every person interested in *Mathematics*, I proposed in [5], as a good training, the following generalization of (4):

(6)

"i) Find the cardinal $E(n), n \in \mathrm{N}^*$, of the numbers with *n* digits, formed only with the even digits.

ii) Find the cardinal $O(n), n \in \mathrm{N}^*$, of the numbers with *n* digits, formed only with the odd digits.

iii) Find $n \in \mathrm{N}^*$, so that $E(n) + O(n)$ would be a perfect square.

iv) For the cardinal $N(n), n \in \mathrm{N}^*$, of the number with *n* digits, formed with all digits from $\{0,1,\mathrm{K},9\}$, find $(N(n), O(n) + E(n))$.

v) Find $n \in \mathrm{N}^*$, so that $\dfrac{N(n)}{O(n) - E(n)} \equiv 0$ (mod 144)."

For the interested reader, we note here, the answers of (6):

(7) $E(n)=4\cdot 5^{n-1}$, $O(n)=5^n$, $E(n)+O(n)$ is a perfect square if $n=2k+1, k\in \mathrm{N}$, $(N(n), E(n)+O(n))=2^{n-1}, \dfrac{N(n)}{O(n)-N(n)}\equiv 0$ (mod 144) only if $n=4k+1$, $k\in \mathrm{N}$.

In the end of this proposed way *to teach Combinatorics using different problems*, I come with another example (see [6]) relating to the number of the functions $f:S\to T$:

(8) "i) Find the set $A=\left\{ n\in \mathrm{N}^* \middle/ \dfrac{n}{n-2001}\in \mathrm{N}\right\}$.

ii) Find the number of the functions, $f: B\to B$ where : $B=\left\{ n\in \mathrm{N} \middle/ \dfrac{n^{2001}}{n-2001}\in \mathrm{N}\right\}$."

The solution of (8) is based on:

(9) $A=\{1972,1978,1998,2000,2002,2004,2024,2030,4002\}$,

and on the fact that for B, after the division, we must impose the conditions:

(10)
$$\frac{2001^{2001}}{\mathrm{n}-2001}=\frac{3^{2001}\cdot 23^{2001}\cdot 29^{2001}}{\mathrm{n}-2001}\in \mathrm{N}\Leftrightarrow \mathrm{n}-2001=3^{\alpha}\cdot 23^{\beta}\cdot 29^{\delta}\Leftrightarrow$$
$$\Leftrightarrow \alpha,\beta,\gamma\in\{0,1,\mathrm{K},2001\}\Leftrightarrow|\mathrm{B}|=2002^3.$$

It occurs that the number, of the functions $f: B\to B$, is : $2002^{3\bullet 2002^3}$

We invite the interested reader to see [2], for other properties and problems about the *surjective* or *injective* functions defined on discrete and finite sets.

CONCLUSIONS

Each notion in *Combinatorics* must be taught by the intermediate of a well chosen problem, exactly as I tried accomplishing in the anterior paragraph. This is the unique way in which we can incite the curiosity of the students, either in the high-school course, or in the university course, all for a difficult domain, which is very different from the others in *Mathematics*. Finally, I underline that immediately, after the moment when a good understanding was established in the mind of our students, for these new notions, which must be well fixed, we also apply for … *selected problems in Combinatorics* …

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Prof. univ. dr. Laurențiu Modan, Department of Mathematics, Faculty of Computer Science, Bucharest
modan_laurent@yahoo.fr

Modelling Geometric Concepts Via Pop-Up Engineering

Vivekanand Mohan-Ram BSc, Dip.Ed, MEd
Senior Lecturer, Mathematics Education, Faculty of Education, Health and Science
Charles Darwin University, Northern Territory, Australia. vmo63973@bigpond.net.au

Abstract
The main purpose of this workshop is to focus upon a complementary approach to the study of, and the investigation into, concepts related to Geometry- Space Strand. It ought to benefit educators especially those who prepare teachers for the primary/elementary schools. Participants in this workshop will initially learn the skills needed in Pop-Up Engineering to produce 'hole' 3-D paper models which illustrate some particular geometric concepts. The process of the construction of these models allows for building imagery, testing predictions, arousing and satisfying curiosity, connecting to Geometric concepts and most of all motivating and holding interest. It is envisaged that this approach to the teaching and learning of geometric concepts will provide grounds for discussion, enrichment, exploration, clarification of and ownership of ideas, and cross curriculum integration. It has the potential to reduce the apparent difficulty students experience with the study of geometric concepts.

Introduction:

Over the years children and adults have maintained a fascination with pop-up books and cards. It is, I imagine, the thrill of experiencing the unexpected objects popping up as the books and/or cards are opened, and seeing two dimensional objects transformed into three dimensional figures. This too may have contributed to the children's curiosity. It is this intrigue which has led me to explore the possibility of utilizing the mechanism of paper engineering (as it is called) to construct 'hole' geometric figures, and thus introduce children to geometric concepts. I am of the opinion that this type of visual experience, complemented with intrigue and problem solving will enhance the children's thinking and understanding in the study of geometry.

Historically, since the 18th century, some amusing children's books contained, flaps to turn, peep-holes, and cut-outs. In 1855, Dean of London published the first pop-up book called *Little Red Riding Hood* (Hiner, 1985). Today we see many children's books and cards which are produced with a pop-up mechanism. The mechanism used produces a whole range of exciting possibilities and movements such as translation, rotation, reflection, enlargement etc. Hiner (1985) suggested that 'by choosing the right mechanism and then adapting and developing it to suit your particular need, you can express an idea a mood far more vividly than any static picture ever can.'

Skills required for this workshop are i) Cutting, ii) Folding iii) Gluing and iv) Scoring.

I will assume that you are sufficiently experienced with numbers i) to iii) above but 'scoring' may be new to you. For this you will need an empty ball point pen or any such implement which will 'mark' the paper without tearing it. Scoring then refers to marking an impression on the paper in such a way that a 'sharp' line is formed but does not cut through the paper. This scored line produces a well defined line and/or a hinge. It is one of the important mechanisms in paper engineering. It adds to the quality of the hinge, and it gives a professional look to the work when completed.

Choosing the appropriate quality paper will enhance the model produced. The recommended quality paper is manila 100 - 120 g/m^2.

This leaflet was typed on 80 g/m^2 paper (A4 size). This is regarded to be too thin and will not usually suffice. Manila paper of 100 - 120 g/m^2 is recommended especially if the fibres in the paper could make a good crease when it is scored and folded.

Very thick paper usually is not ideal for pop-ups. The size you choose for your pop-ups is rather important. An A4 size paper folded in half along the shorter length is usually convenient for the students to use with these activities.

Other important materials for the workshop are a sharp pencil, white and colour A4 sheet of paper, a pair of scissors, a ruler (metal edge), a craft knife, a pair of compasses, a cutting board or heavy card board, a tube of paste, a protractor, and 1 cm graph paper.

Related mathematics topics:

The following list suggests some of the topics and processes which may emerge from the various activities. You may be able to add others to this list.

- Accurate construction of geometrical figures
- Angles, parallel lines and planes
- Classification of 2D shapes and 3D figures
- Polygons and their properties
- Perimeter, area, volume.
- Transformation:- translation, reflection, rotation, enlargement and reduction of shapes/figures
- Making predictions and justifying these orally
- Two and three-dimensional representation of objects.
- The language of Mathematics and
- Connecting to other curriculum areas.

With the time at our disposal we will try to complete five activities.

Each activity is subdivided into a five stages which you should follow through to fully benefit from the experience.

Stage 1. Instructs you **to fold and cut** the paper along specific guide lines.

Stage 2. Invites you **to imagine** a particular process associated with the first stage and asks you to answer related questions.

Stage 3. Invites you **to predict** a result if you had correctly followed the above set of instructions.

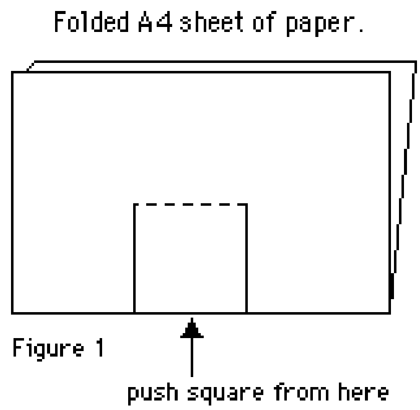
Stage 4. Invites you **to carry out** the instructions and to test your imagery and predictions.

Stage 5. Invites you **to discuss** your out comes, respond to the questions and compare your 'model' with that of your colleagues.

ACTIVITY 1(a)

Stage 1.

- Fold an A4 size paper in half by bringing together the shorter ends. Crease the fold
- Construct/Draw a square on the folded paper such that a side of the square lies on the fold. (please see Figure 1 below)
- Cut the sides of the square which are perpendicular to the fold.
- Score the side of the square which is parallel to the fold.(the dotted line)



- Opening the folded paper only slightly, push the folded edge of the square through to the inside, until the scored line prevents further movement.

Crease along the scored line.

Stage 2. *Imagine* the folded paper being slowly opened.
What figure will emerge?

Stage 3. *Predict* what figure will result when the angle of the fold is 90°**.**

Stage 4.

- Proceed to carry out the actions described above and test your imagery and prediction.
- Were you surprised at your creation?
- Did the creation coincide with your imagery and prediction?
- Slowly close the paper and describe the transformation of the figure.

Stage 5. Discussion:

- The place of this activity in a regular mathematics classroom.
- What specific geometric concept can be illustrated using this model?
- Suggest a title for this activity.
- Any other?.

ACTIVITY 1(b)

In this activity you are asked to use your acquired skill and produce a model which shows three cubes adjacent to each other and of different sizes (please see figure 2 below).

You should sketch a diagram before proceeding with the construction. Also follow the stages as in activity 1(a) above.

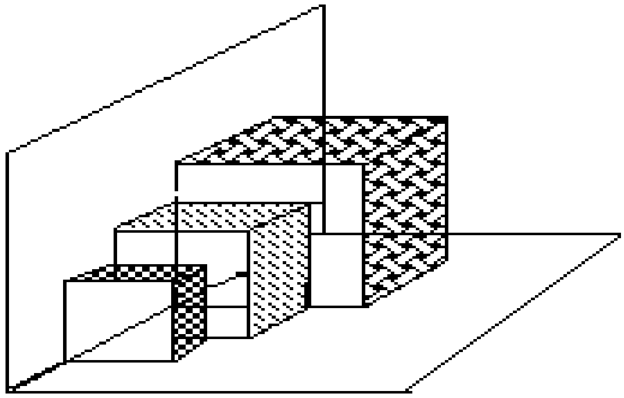


Figure 2

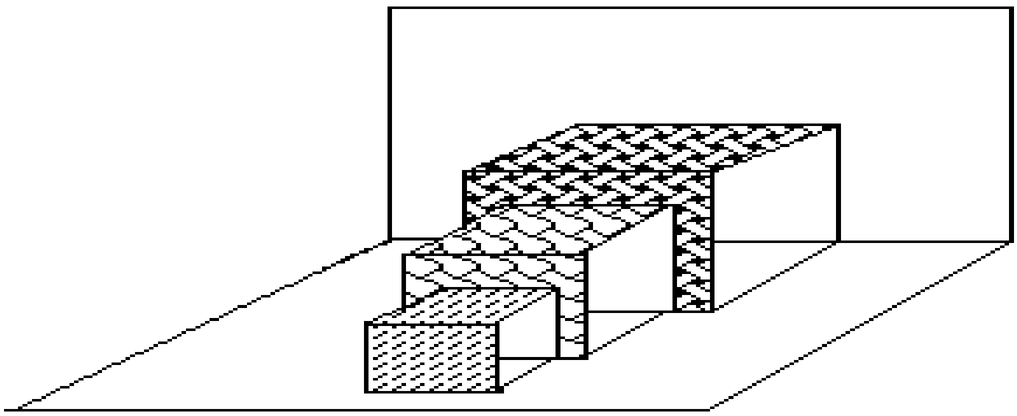


Figure 3

ACTIVITY 1(c)
In this activity using your acquired skill produce a model which shows three cubes one infront of each other, in the formation of a stair and the cubes increase in size (please see figure 3 above)
ACTIVITY 1 (d)
In this activity you are asked, without the aid of a cue, to construct a cube with in a cube with in a cube. In other words three cubes of different sizes, one within the other.
ACTIVITY 2
Stage 1
Fold an A4 size sheet of paper as in Activity 1.
- Draw one of the rectangles as shown below in Figure 4
- Cut, crease and fold as in Activity 1.

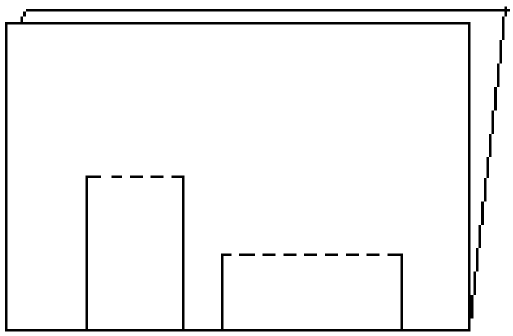


Figure 4

Stage 2.*Imagine* the folded paper being slowly opened. What figure will emerge?
Imagine the figure which would emerge if you had chosen to construct the other rectangle in Fig. 4.
Describe how these two figures are different.
Stage 3.*Predict* what figure will result when the angle of the fold is 90°.
Stage 4. Proceed to carry out the actions as in Activity 1 and test your imagery and prediction. How could the Activities above be used in the classroom to explore mathematical content ?
Stage 5. Discussion:
As above.Suggest a title for this activity.
Should time permit we will explore the construction of other activities. Etc Construction of a set of horizontal parallel planes. Construction of a set of vertical parallel planes. See more models on display.
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Exploring the mathematics that children read in the world: A case study of Grade 8 learners in a South African School

Lesego Brenda Mokotedi
Mathematics Education, PhD student, Tshwane University of Technology, South Africa
brendamokotedi@msn.com

Abstract

This paper presents a qualitative study in which an attempt was made to extend the debate surrounding the use of real life contexts to make mathematics more meaningful and real. The study investigated Grade 8 learners' knowledge of number, understanding of number concepts and the kinds of connections they make between number and the context in which number is used. An important aspect of the study's methodological approach involved an examination of the comments that learners made about what they said they know about number. A response to the question: "Why is the number in the picture?" provided a framework for establishing how learners saw relationships between number and the context in which numbers are used. A face scenario with four questions was given to learners to elicit these relationships. Results pointed to the usefulness of real life contexts as tools that have a central role in uncovering what learners know about number and how they use that knowledge to understand situations that call for proficiency in mathematics.

Introduction

In the physical world humans come across situations in which their knowledge of different symbols, words, number and diagrams is put into practice so that they can look at and understand what they see. Gutstein (2003:41) refers to this practice as making sense of the world in which we live. To understand the physical, social and mental world implies that our proficiency in literacy and numeracy are essential tools that help us to make sense of new experiences. Recent suggestions regarding approaches to school mathematics place a high premium on the use of contexts and situations to develop mathematical concepts and procedures, contextually-driven justifications for conjectures obtained through inductive generalizations, to demonstrate and induct learners to the field of mathematical applications and modeling (Julie, 2006:49). This is because real life contexts are believed to be platforms on which learners can use their school-learned mathematics to understand real-life situations. But how often are real life contexts used to explore learners' understanding of numbers and number concepts? This question formed the central concern of the study. Learners were presented with an everyday situation based on which they were asked the following question: "what do you know about the number in the picture?"

Design of the Study

The study involved 49 Grade 8 learners from a Middle School in a semi-rural Mafikeng, South Africa. The dynamics of the learning environment were characterized by varying intellectual abilities, varying learning styles, diverse cultural backgrounds and different socio-economic status of learners. These dynamics made the classroom a melting pot of all kinds of learning diversities. The aspect of socio-economic background is critical from Cooper's (1998:512) point of view. Cooper argues that the relationship between socio-economic status, culture and cognition is important since the use of certain items may lead to an underestimation of the mathematical capacities of children from different social backgrounds. The study therefore involved a diverse group of mathematics learners not only in terms of performance in Mathematics but also in thinking and learning
styles. An important aspect of the study's methodological approach concerned learners views and understanding of number when embedded in a real life context. Learners were presented with a real life situation in which they were expected to identify and explain in writing the mathematics which they thought was demonstrated in the scenario (see figure below).

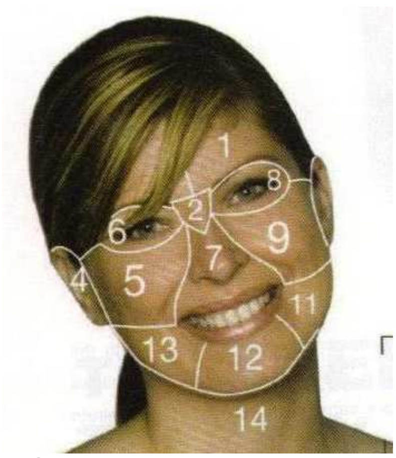


Figure 1: The face scenario (New Visions Magazine, vol. 22)

Why use a picture of a numbered face in particular? The National Curriculum Statement (NCS)

(Department of Education, 2003:7) places strong emphasis on the importance of selecting contexts inwhich learners can engage description of numbers, various representation of numbers, how different numbers can be thought about, and building learners number sense which is the foundation for further study in mathematics. It is further articulated in the curriculum that learners should use number sense and numeration to develop an understanding of the multiple use of numbers in the real world, to communicate mathematically and the use of number in the development of mathematical ideas (Department of Education, 2002: 11). The face scenario and the four questions that were based on the scenario were considered appropriate considering what is contained in the mathematics curriculum policy about reading, seeing, and writing mathematics at Grade 8 level. The study focused on four key issues namely, what mathematics learners saw in the scenario; what shaped the understanding of what they saw; how learners used their school-learned mathematics to understand what they saw, and lastly, what connections they made between the numbers that they chose and the specific part of the face in the scenario.

Data analysis

The analysis of the qualitative data was in the three stages: (1) familiarisation and organisation, (2) coding and recording and (3) summarising and interpreting the data (Ary et.al, 2006:490). As Ary et. al (2006:490) recommends, qualitative data in this study was directly transcribed to avoid potential bias in interpretation that may come with summarising. The primary analysis technique was reading the data, that is, learners' written responses. This was necessary to familiarise the researcher with the data in order to get an overall idea of what it was offering. In general the researcher was looking for trends when grouping and summarising data. Views were then established and interpretable chunks of data across the four scenario questions were highlighted for analysis to see what issues characterised them. Some responses were unclear due to serious language difficulties. This was considered inevitable. In the analysis of learners' responses the researcher sought to identify what was the predominant view that learners held for the fourth question which was stated as: "why is that number on the picture?" Responses to this question formed the central part of the inquiry process as it was related to the use of numbers in this particular context (face scenario). An example of a typical response of learners to the fourth question was as follows:

> On the above picture I see a girl's face with many numbers on it. And I `ve noticed that there are numbers that are not there which are three and ten. When I look at the face I see mathematics and calculations. I see numbers which are like puzzles and there are shorting puzzles that has to be on the face or I would call it a face puzzle, cause its like a face puzzle with shorting puzzles (Learner L23).

Results and observations

The first question which the study set out to answer was: *What mathematics do learners see in the scenario?* Grade 8 learners were able to identify numbers in the given scenario. Approximately two fifths of the sample was able to specify the different types of numbers that they saw. Learners mentioned concepts such as prime numbers, composite numbers, odd and even numbers. The nature of comments they made about the mathematics that they saw revealed that learners' responses were mainly at a surface level meaning that they were able to provide interpretations about numbers which one can say were anticipated. It became evident that what learners had acquired in their mathematics classroom had a significant influence on what they were seeing, because the mathematical concepts that they mentioned in their responses to the four question were acquired in the mathematics classroom. Learners' understanding of what they saw was shaped by what was already known to them. This was evidenced by the manner in which the observables were: (i) communicated (e.g. numbers were listed numerically), (ii) categorised (e.g. odd numbers were separated from even numbers), (iii) number concepts were mentioned (e.g. integers, odd numbers, even numbers, composite numbers and prime numbers) . This indicated that they were using their school-learned knowledge to make sense of what they were seeing. It also indicated that learners did not only see numbers but they were able to say something "deeper" about what they were looking at. All learners except one (L6) could only recognise the mathematics that was specific to Learning Outcome 1, i.e. number and number relationships. L6 took a geometric view of the picture when he associated picture with the concept of polygon. The concept polygon is specific to Learning Outcome 3 (Space & Shape) in the NCS (Department of Education, 2002)..

The second central question was: b) *How do learners explain in writing the purpose of numbers as they appear in the scenario*? The scenario provided learners with the opportunity to recognise the use of numbers in a realistic situation that was presented to them. Two learners made precise descriptions that were consistent with the philosophy behind "face mapping" in relation to the numbers that they chose. Face mapping has a background from from skin care processes (http://www.worldwideshoppingmall.co.uk/body-beauty). A total of 8 learners succeeded in identifying relationships between specific numbers and specific parts of the face. These learners explained that the numbers that they chose were there to identify specific parts of the face in the scenario. However, the scenario unveiled constraints of working with contexts to enhance the teaching and learning of mathematics related to number. This was evidenced by the difficulties encountered by 22 learners when they were asked to explain what the numbers they have chosen stand for in the picture. Although these learners were able to identify numbers from the scenario, they find it difficult to explain why the numbers are there.

The third question was: *What is the effectiveness of the scenario as a tool for exploring Grade 8 learners' knowledge and understanding of number*? The scenario served as a tool that uncovered learners' thinking about natural numbers. The scenario was an enabling tool for some learners while to others it could be considered a constraining tool. It was through the scenario that learner's misconceptions, misinterpretations, poor assumptions about numbers, little understanding of natural numbers and rich connections were revealed. The scenario was therefore instrumental in revealing the learners' surface and deep understanding of number concepts. A key finding related to this study was the uncovering of a more comprehensive portrayal of the kinds of connections some learners (particularly L6 and L30) were able to make. Two of these responses are quoted below:

The scenario was an opportunity for learners to use their knowledge of number to interpret a part of reality presented to them. In conventional mathematics classrooms, learners' content knowledge related to number concepts is tested through the use of formal tasks which are later assessed quantitatively. Such tasks provide little opportunity for learners to express their knowledge of number for teachers to diagnose misconceptions, levels of understanding, contradictions and thick descriptions about learners' knowledge which are rich and pedagogically useful. As an educator who spent two years working with these learners, the scenario afforded me the opportunity to diagnose my learners' uncertainties in their knowledge and competency about number. Having spent two years teaching these learners it became clearer to me that the analysis of their responses to the scenario provided a framework of reference to begin to reflect on my own teaching and knowledge of number concepts. In particular it is critical to consider the notion of reading the world using mathematics. The fact that most of the learners could not explain why numbers are in the picture suggests that their knowledge of number is something that may have "divorced" from their everyday experiences. Learners encountered difficulties in transferring their knowledge of number to a different situation. This lack of transfer may be attributed to traditional mathematics practices which are textbook-bound and involve little integration across concepts in the mathematical domain. Taking into consideration these learners' responses to this activity, the scenario may be considered an effective tool to explore Grade 8 learners' understanding of the number concepts.

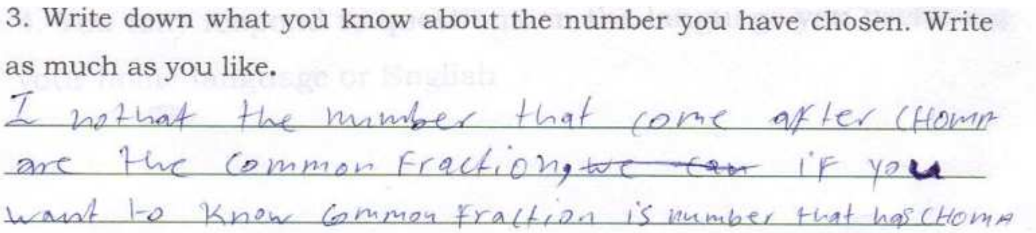
Findings

The analysis of the data revealed four important fings that pointed to the usefulness of context in the teaching of mathematics in relation to number concepts.

Finding 1 was related to the question: *What does the nature of comments reveal about learners' knowledge and understanding of number*? Kilpatrick, Swafford and Findell (2001:117) indicate that how learners represent and connect pieces of knowledge is a key factor in whether they will understand it deeply and can use it for problem solving. The degree of learners' conceptual understanding is related to the richness and extent of the connections they have made. This was evident in L6 and L30's responses. L6 and L30'S responses are (respectively) to the second instrument question which was: Write down what you know about the number you have chosen.

> I like number 5 because it easy to multiply with and 5 is a prime number and odd number. 5 is half of 10 and ¼ of 20.
>
> I know that five is a natural and a prime number because the factors of five are 1,5. And that five can also be expressed as a hole number. And that -5 is smaller than 5.

These learners were able to provide multiple representations of number, insights on how the number 5 can be thought about e.g 5 is ¼ of 20 and it is half of 10. These two learners descriptions of what they know about number is a portrayal of mathematics as aliving body of knowledge. Their comments revealed some understanding of what they know about the number 5. This can also be seen in L12's response to the third question as follows:


3. Write down what you know about the number you have chosen. Write as much as you like.

I nothat the number that come after CHOMA are the common Fraction, ~~we can~~ if you want to know common fraction is number that has CHOMA

Learners were requested to respond to the second question which was: "From the picture above choose the number you like and say why you like the number". Instead of choosing a number L12 listed all the numbers that he saw in the numbers that he saw in the scenario as it can be seen in his response below. What this learner said he knew about the numbers he rote was clearly a misconception. According to this learner when these numbers are placed behind each other with a decimal sign in between then they can be identified as common fraction

> 1,2,7,8,5,6,12,11,3,4,9,13,I like this number because the number that come after comma are the common fractions .

Finding 2 was as follows: Learners' responses unveiled possible constraints in representing in words what they saw. This phenomenon was attributed to language difficulties. It was clear that most learners encountered difficulties in communicating what they saw in the scenario. The instances of breakdown in communication were notable in learners' responses to questions. This was evidenced by the prevalence of spelling mistakes and direct translations that L1 made, for example. The difficulties were attributed to language inability to express in written form what the learner needed to communicate. Language is seen as a tool for communicating and thinking (Setati, 2005:94). Here the possibility of second language being a barrier to communication is explored.

Finding 3: Learners descriptions of what they saw in the scenario suggested that they had various interpretations of the same picture (face scenario). This was an indication that the picture appeared differently to these learners. L30 saw fractional divisions on the picture. In L22's view the lines on the face formed pieces, while to L15 the lines appeared to have formed blocks. L23 saw puzzles and that some puzzles were missing. L6 said that he saw polygons. L30 like L6 used mathematical concepts to interpret what he saw. A possible explanation for the diverse interpretations provided by these learners could be linked to the notion of "situation specificity" as articulated in Anderson, Reder and Simon (1996:6). From this perspective it acknowledged that learning is not wholly tied to a specific context, and mathematical competence is not always contextually bound. But what does this claim imply for this study? The concept of polygon, as it emerged in the analysis of L5's response for example, was learned in the mathematics classroom. It seems that this concept now became a tool that is available to this learner to make sense of what he is able to see in a different situation (the face scenario in this case). Similarly L30 who related the divisions in the face scenario with fractions, reminds us that what she learnt in the classroom about fractions was not tied to that classroom context. The following quotation by6 Cohen et al (2003:13) could help us better understand these learners varying perspectives and diverse interpretations of the same phenomenon.

> Concepts enable us to impose some sort of meaning on the world; through them reality is given sense, order and coherence. They are the means by which we are able to come to terms with our experiences. How we perceive the world then, is highly dependant on the repertoire of concepts we can command. The more we have the more sense data we can pick up and surer will be our perceptual and cognitive grasp of whatever is 'out there'. If our perceptions of the world are determined by the concepts available to us, it follows that people with differing sets of concepts will tend to view the 'same' objective reality differently.

Finding 4 was linked to the question: What is the reason for learners' choosing numbers but not indicating the reason for their appearance in the picture. Considering the learners (8) who made comments that are explicit about what they thought numbers represent in the picture, we are able to see that the use of contexts in mathematics is desirable. These learners used their mathematics school-learned knowledge to understand what they were seeing. So what does this mean to twenty two (22) learners who were unable to "see" the utility of numbers in the scenario? What does it suggest about the classroom practice to which these learners are adapted? Adler et al (2000:12) comment that there is no doubt that integration within mathematics is desirable but this approach will place new demands on teachers.

Conclusion

The use of contexts to connect mathematics to the experiences of learners is desirable and more pronounced in the National Curriculum Statement. The face scenario which was used in this study exhibited certain key features of this approach. A critical examination of issues that emerged in the qualitative analysis of data in this study seems to suggest that, in the real world mathematics plays a completely different role. There is therefore a need to bridge gaps between school mathematics and mathematics in real life with an eye towards extending research in the direction of usage of number to express phenomena. The idea that knowledge is situated should be a point of departure. The study demonstrated that learners should be encouraged to view mathematics as an interesting, powerful tool that enables them to understand and express phenomena. This can be achieved through a suitable classroom practice.

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On Teaching Quality Improvement of a Mathematical Topic Using Artificial Neural Networks Modeling (With a Case Study)

Hassan. M. Mustafa
Educational Technology Department-Faculty of Specified Education-Banha University Egypt
Currently With Arab Open University (K.S.A. Branch, IT & Computer Department)
E-Mail:hmustafa@arabou.edu.sa
Ayoub Al-Hamadi
Institute for Electronics, Signal Processing and Communications (IESK)
Otto-von-Guericke-University Magdeburg
E-Mail: Ayoub.Al-Hamadi@ovgu.de

Abstract
This paper inspired by simulation by Artificial Neural Networks (ANNs) applied recently for evaluation of phonics methodology to teach children "how to read". A novel approach for teaching a mathematical topic using a computer aided learning (CAL) package applied at educational field (a children classroom). Interesting practical results obtained after field application of suggested CAL package with and without associated teacher's voice. Presented study highly recommends application of a novel teaching trend based on behaviorism and individuals' learning styles. That is to improve quality of children mathematical learning performance.
Keywords: Artificial Neural Networks, Learning Performance Evaluation, Computer Aided Learning, Long Division Process, Associative Memory.

1. Introduction

It is announced (in U.S.A.) that last decade (1990-2000) named as Decade of the brain [1]. Accordingly, neural network theorists as well as neurobiologists and educationalists have focused their attention on making interdisciplinary contributions to investigate essential brain functions (learning and memory). Recently, Artificial Neural Networks (ANNs) combined with neuroscience considered as an interdisciplinary research direction for optimal teaching children methodology how to read. This direction motivated by a great debate given at, [2] as researches at fields of psychology and linguistic were continuously cooperating in searching for optimal methodology which supported by educational field results. Nevertheless, during last decade phonics methodology is replaced –at many schools in U.S.A.- by other guided reading methods that performed by literature based activities [3] . More recently, promising field results are obtained [4] that support optimality of phonics methodology for solving learning/teaching issue "how to read?" [5-6]. Additionally, recent mathematical modeling for phonics methodology has been presented in details, [7]. Herein, this optimal approach adopted for improving teaching/ learning performance of a mathematical topic to children of about 11 years age. The suggested mathematical topic is teaching children algorithmic process for performing long division. Specifically for two arbitrary integer numbers chosen in a random manner (each composed of some number of digits). By detail, adopted principal algorithm for applied Computer Aided Learning (CAL) package consisted of five steps follows. Divide, Multiply, Subtract, Bring Down, and repeat (if necessary), [8]. For more details about recent view concerned with the effect of information technology computer (ITC) on mathematical education, it is advised referring to, [9]. The rest of this paper is organized as follows. At next section, a basic interactive educational model is presented with a generalized block diagram. Obtained results after application of suggested CAL package at the case study at next third section, in addition obtained simulation results. At the last forth section some interesting conclusions in addition to suggestions for future work are presented. Finally, an Appendix is given for simplified flow chart of adopted CAL package.

2. Basic Leaning/Teaching Model

Generally, practical performing of learning process - from neurophysiologic P.O.V. - utilises two basic and essential cognitive functions. Both functions are required to perform efficiently learning / teaching interactive process in accordance with behaviourism, [10-12], as follows. Firstly, pattern classification/recognition function based on visual/audible interactive signals stimulated by CAL packages. Secondly, associative memory function is used which is originally based on classical conditioning motivated by Hebbian learning rule. Referring to Fig.1 shown in below, the illustrated teaching model is well qualified to perform simulation of above mentioned brain functions. Inputs to the neural network learning model at that Figure, are provided by environmental stimuli (unsupervised learning).The correction signal for the case of learning with a teacher is given by responses outputs of the model will be evaluated by either the environmental conditions (unsupervised learning) or by the teacher. Finally, the tutor plays a role in improving the input data (stimulating learning pattern), by reducing noise and redundancy of model pattern input. That is according to tutor’s experience, he provides the model with clear data by maximizing its signal to noise ratio. However, that is not our case which is based upon unsupervised Hebbian self-organized (autonomous) learning, [13-14]. Details of mathematical formulation describing memory association between auditory and visual signals are given at,[7].

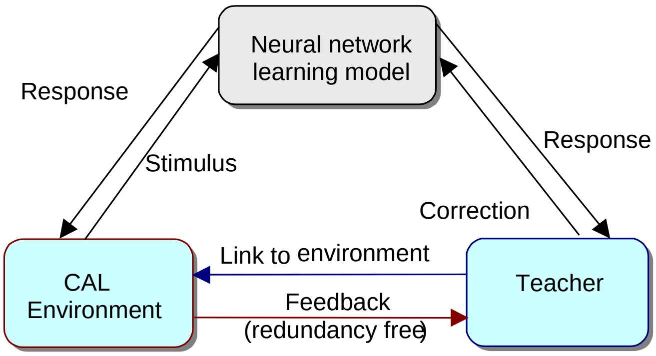

Fig.1: Illustrates a general view for interactive educational process, adapted from [6].

3. Results

The results obtained after performing practical experimental work in classroom (case study) is shown at next subsection 3.1. Additionally, at subsection 3.2 realistic simulation results are introduced. Interestingly, it is clear that both obtained results (practical and simulation) are well in agreement and supporting each other.

3.1 Case Study Results

A learning style is a relatively stable and consistent set of strategies that an individual prefers to use when engaged in learning [15-16]. Herein, our practical application (case study) adopts one of these strategies namely acquiring learning information through two sensory organs (student eyes and ears). In other words, seen and heard (visual and audible) interactive signals are acquired by student's sensory organs either through his teacher or considering CAL packages (with or without teacher's voice). Practically, children are classified in three groups in according to their diverse learning styles (preferences).

The two tables (Table.1 & Table.2) given in below illustrate obtained practical results after performing three different learning experiments. At table.1, illustrated results are classified in accordance with different students' learning styles following three teaching methodologies. Firstly, the classical learning style is carried out by students-teacher interactive in the classroom. Secondly, learning is taken place using a suggested software learning package without teacher’s voice association. The last experiment is carried out using CAL package that is associated with teacher's voice. This table gives children's achievements (obtained marks) considering that maximum mark is 100. The statistical analysis of all three experimental marking results is given in details at Table.2 shown in below.

Table.1: Illustrates students’ marks after performing three educational experiments.

Classical Learning	**35**	**43**	**29**	**50**	**37**	**17**	**10**	**60**	**20**	**48**	**15**	**55**	**40**	**8**	**20**
CAL without Voice	**39**	**29**	**52**	**60**	**50**	**68**	**62**	**30**	**55**	**42**	**40**	**59**	**48**	**70**	**2**
CAL with Voice	**65**	**70**	**50**	**75**	**45**	**50**	**62**	**90**	**85**	**50**	**80**	**90**	**58**	**55**	**60**

Teaching Methodology	Students' average Achievement score (M)	Variance σ	Standard deviation $\sqrt{\sigma}$	Coefficient of variation $\rho = \sqrt{\sigma} / M$	Improvement of teaching Quality
Classical	32.46	265.32	16.28	0.50	-
CAL (without tutor's voice)	46.80	297.49	17.24	0.36	44.1%
CAL (with tutor's voice)	64.33	283.42	16.83	0.26	98.2%

Table.2: Illustrates statistical analysis of above obtained children's marks.

3.2 Simulation Results

The suggested ANN model adapted from realistic learning simulation model given at [6] with considering various learning rate values. It is worthy to note that learning rate value associated to CAL with teacher's voice proved to be higher than CAL without voice. Simulation curves at Fig.2 illustrate statistical comparison for two learning processes with two different learning rates. The lower learning rate (η = 0.1) may be relevant for simulating classical learning process. However, higher learning rate (η = 0.5) could be analogously considered to indicate (approximately) the case of CAL process applied without teacher's voice.

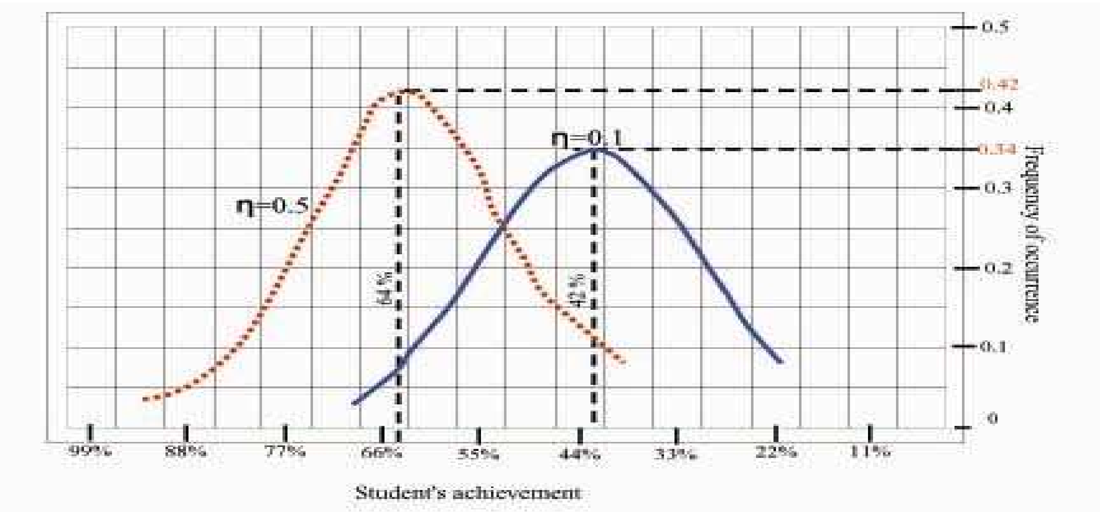

Fig.2 Illustrates Simulation results presented by statistical distribution for children's (students) achievements versus the frequency of occurrence for various achievements values, at different learning rate values (η = 0.1& η = 0.5).

Learning Rate value	Children's average Achievement score (M)	Variance σ	Standard deviation $\sqrt{\sigma}$	Coefficient of variation $\rho = \sqrt{\sigma} / M$	Improvement of teaching Quality
η=0.1	42	428.5	20.7	0.61	-
η=0.5	64	918.1	30.3	0.47	66%

Table 3: Illustration of simulation results for different learning rate values η.

4. Conclusion

This paper comes to two interesting conclusive remarks given as follows:

- Evaluation of any CAL package quality is measured after statistical analysis of educational field results. So, above suggested strategy provides specialists in educational field with fair unbiased judgment for any CAL package. That is by comparing statistical analysis of simulation results with natural analysis of individual differences obtained in by practice.

- After practical application of our suggested multimedia CAL package (case study), interesting results obtained considering diverse individuals' learning styles. Obtained results are depending only upon two cognitive sensory systems (visual and/or audible) while performing learning process.
- Consequently, by future application of virtual realty technique in learning process will add one more sensory system (tactile) contributing in learning process. So, adding of the third sensory (tactile system) means being more promising for giving more additive value for learning/teaching effectiveness. Finally, for future modification of suggested CAL package measurement of time learning parameter will be promising for more elaborate measurement of learning performance in practical educational field (classroom) application. This parameter is recommended for educational field practice, [17] as well as for recently suggested measuring of e-learning systems performance [18].

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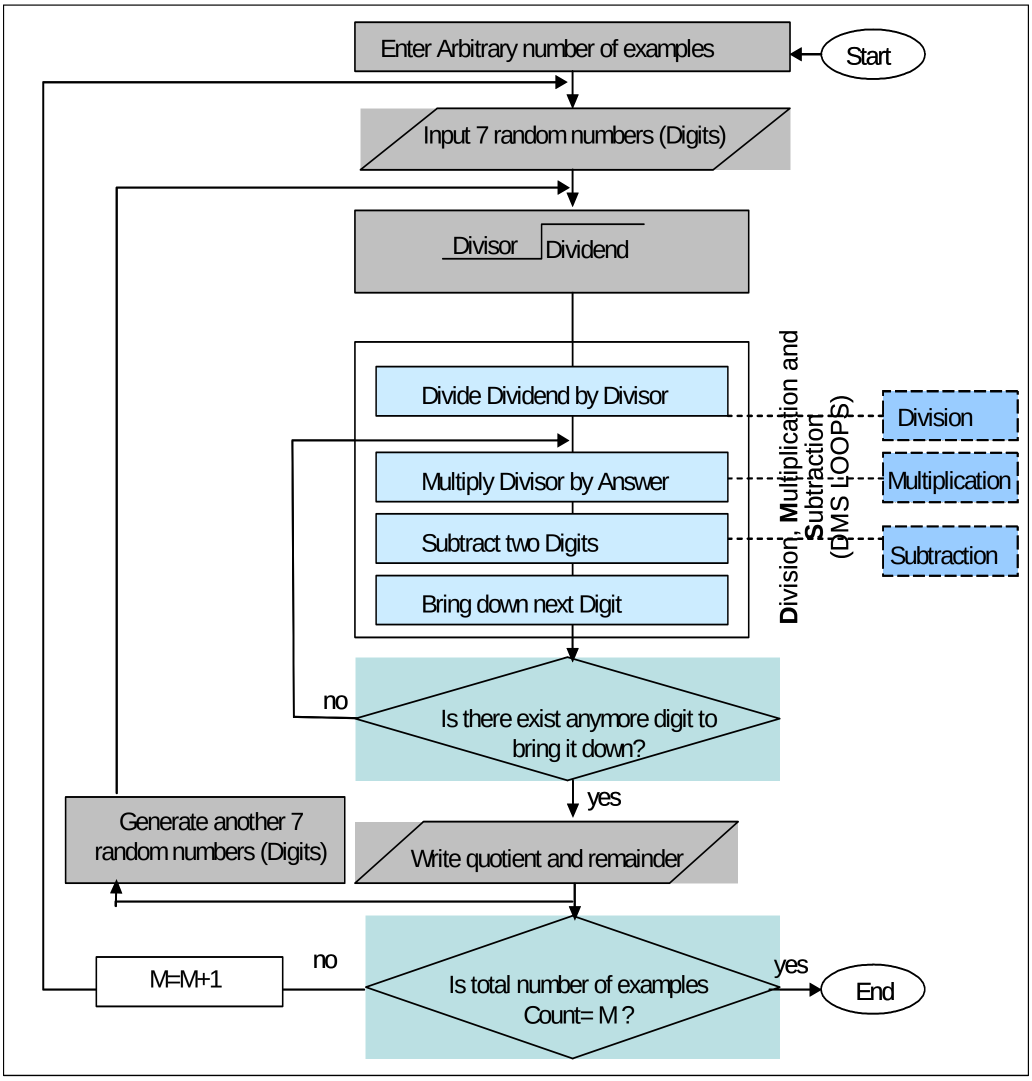
APPENDIX



The shown figure in the above illustrates a simplified macro-flowchart describing briefly algorithmic steps for CAL package designed under supervision of this manuscript's authors,(by Eng. Mohammed H. Kortam). That is according to above long division principles given at above reference [8].

New Forms of Assessment in the South African Curriculum Assessment Guidelines: What Powers do Teachers Hold?

Willy Mwakapenda
Professor of Mathematics Education
Tshwane University of Technology, South Africa mwakapendawwj@tut.ac.za

Abstract
This article opens up a discussion on the power that teachers have in mathematics curriculum at the Further Education and Training level. It is related to the general question: who holds the power in school mathematics education in South Africa? To what extent is the teacher given an opportunity to exercise their power in mathematics assessment? If the teacher is given power, what does that power allow teachers to do, and under what conditions does this happen? The case of mathematics is presented here to illustrate the above complex questions of teacher power in new forms of assessment in the curriculum.

Introduction

From the vantage point of new forms of assessment, this article is an attempt to unpack the question of teacher power by looking at how teachers are positioned in the National Curriculum Statement (NCS) Assessment Guidelines for Mathematics (Grades 10-12, Department of Education (DoE), 2005). I focus on the new assessment guidelines for two reasons. Firstly it is because it is widely recognised that assessment is the engine of education systems. Conceiving assessment as an engine is a powerful way of thinking about education. Stated more practically, when we look at assessment, we look at an engine: what drives education systems. Education systems run on the fuel of assessment. The engine-power of assessment can be seen for example, in South Africa, in how the outcomes of assessments are not only celebrated, but also how under-performing schools and their administrators are perceived by society. Focusing on assessment is consistent with the view that that assessment is an "integral part of teaching and learning. For this reason, assessment should be part of every lesson and teachers should plan assessment activities to complement learning activities" (DoE, 2005, p. 1)

. The DoE (2005) states that guidelines assist teachers in the teaching. Teachers are encouraged to use these guidelines as they prepare to teach the National Curriculum Statement. The assessment guidelines are conceived as a critical resource that should be able to assist teachers in their teaching of mathematics in accordance with the national policies. Viewing assessment guidelines as a resource i.e. as tools for looking into learning systems (Davis & Simmt, 2003) and what becomes of learning draws us to a key conceptual backbone of educational thinking in our context of education in South Africa in relation to resources and tools for mathematics education. Adler (1999) points out that "access to a practice requires its resources to be 'transparent'" (p. 48). Adler also introduces the notion of "visibility and invisibility" in relation to "transparency in the practice of teaching mathematics" and argues that "Resources need to be seen to be used. They also need to be invisible to illuminate aspects of practice. For talk to be a resource for mathematics learning it needs to be transparent; learners must be able to see it and use it" (p. 63). From a perspective of transparency, I analyse assessment guidelines which mathematics teachers are called upon to use as resources in their work. This analysis attends to the complexity of assessment policies and the legitimating power that they are intended to give to mathematics teachers. I describe two key aspects that are constituents of the engine of assessment guidelines, namely "daily assessments" and "programme of assessment".

Daily assessment

Two forms of assessment have been proposed in the NCS: continuous assessment and external assessment. Continuous assessment is a form of assessment, which when used jointly with "informal daily assessment" and "formal programme of assessment" (p. 1) is instrumental for: the development of "learners' knowledge, skills and values", and the identification of "learners' strengths and weaknesses". As it stands, continuous assessment should have a significant role to play in shaping learners' learning and "proficiencies" in mathematics. However, given that this form of assessment only "counts 25%" of the final mark at Grade 12, does that not mean that there is less recognition at the policy level of the significance of continuous assessment?

A key component of continuous assessment is "daily assessment" (p. 2). According to the DoE (2005), this kind of assessment is essentially formative as it occurs "during learning activities" where the aim is for the teacher to *monitor* learner progress. Furthermore, it is stated that this monitoring by the teacher "*can* be done through question and answer sessions; short assessment tasks completed during the lesson by individuals, pairs or groups or *homework* exercises" (p. 2, emphasis added). The marking of these assessments has a powerful pedagogical dimension. According to the DoE (2005, p. 2),

> Individual learners, groups of learners *or* teachers *can* mark these assessment tasks. Self-assessment, peer assessment and group assessment actively involves learners in assessment. This … allows learners to *learn* from and *reflect* on their own performance (emphasis added).

The DoE states that "the results of the informal daily assessment tasks are not formally recorded *unless the teacher wishes to do so*" (p. 2, emphasis added). Nevertheless, there is importance attached to these assessments because "teachers *may* use the learners' performance in these assessment tasks to provide verbal or written feedback to learners, the School Management Team and *parents*". .However, given that "the results of these assessment tasks are not taken into account for promotion and certification purposes" puts into question the significance of these assessments.

One might consider these assessment proposals as liberating given that: a) a range of strategies, not just a single one, are suggested for monitoring learner progress; b) the teacher or learner can mark these assessments, so it does not matter who marks them; c) there is a taken-for-granted assumption that learners should learn from and reflect on their performance as they engage with assessment tasks; and d) "The results of the informal daily assessment tasks are not formally recorded *unless the teacher wishes to do so*". With respect to (a), we need to ask the question: how do teachers decide what form of assessment task should be given to learners and *when* should this happen? If teachers decide to give learners "homework exercises", how do they *decide* which form of tasks should be allocated for homework? Therefore, while we are told: "teachers' lesson planning should consider which assessment task will be used to informally assess learner progress", it is not clear how the teacher needs to select or plan for these tasks particularly given that there are several forms of regulatory tasks that are seemingly transparently available and made known to teachers. With respect to (b), it is important to ask the question: how are teachers able to decide which tasks should be marked by learners, and which ones can only be marked by teachers? With respect to (c), we need to ask the question: what "opportunities to learn" (Weber, Maher, Powell & Lee, 2008) mathematics are presented in the tasks and learners' performance in these? How are these learning opportunities evident in tasks, and can teachers anticipate these? In what ways can teachers be able to think about the nature of these opportunities and at what time they might arise? A similar question needing to be asked with respect to (d) is the following: how do teachers decide which assessment results are useful to record and which ones are not? In all these questions lie tensions and dilemmas which undermine the power of teacher decision making because of the contradictory nature in which opportunities to make decisions are framed.

Of pedagogical importance in the NCS guidelines is the importance of feedback. It is stated that "teachers *may* use the learners' performance in these assessment tasks to provide verbal or written feedback to learners, the School Management Team and *parents*. This is particularly important if barriers to learning or poor levels of participation are encountered". Aside from the question of what kind of feedback is more appropriate and for what purposes, there needs to be engagement with the issue of what kind of feedback needs to be given to parents. In relation to this, how do teachers decide to use verbal rather than written feedback? If written feedback is given to parents particularly the kind of feedback that is consistent with the taxonomy and rating scales proposed (see p. 6 in the NCS mathematics assessment guidelines), how do teachers ensure that parents are able to understand what the feedback means? I ask this question while acknowledging the fact that there does seem to have been a paradigm shift in assessment in South African education that is resonant with the widespread wave of reform that is shaping current theoretical thinking in assessment (Davis & Simmt, 2003).

It seems quite clear here that teachers have a considerable amount of flexibility in the nature and extent of the assessments that should constitute "daily assessment". However, it is surprising that these daily assessments are accorded very little importance if any at all. According to the DoE, "the results of these assessment tasks are *not taken into account for promotion and certification purposes*" (p. 2). Why should teachers take daily assessments seriously when little value has been placed upon these?

Program of assessment

On the other hand, there is assessment that appears to fall under what is called "Program of assessment" which seems to be more valued than daily assessment.

> Teachers should develop a year-long formal Programme of Assessment for each subject and grade. In Grades 10 and 11 the Programme of Assessment consists of tasks undertaken during the school year and an end-of-year examination. The marks allocated to assessment tasks completed during the school year will be 25%, and the end-of-year examination mark will be 75% of the total mark (DoE, 2005, p. 2).

What is entailed in "tasks undertaken during the school year"? How much control does the teacher have in the nature of what these tasks look like? How are these tasks different from "daily assessment" tasks? Whatever these tasks are, it is clear here that because they are developed by the teacher, the teacher has a fair amount of control over how these need to look like. In fact, because assessment of these tasks "counts 25% of the final grade or year mark", it means that the teacher should take these more seriously than the daily assessments. However, it appears that the teacher has *little control* over the *number* of assessments of this form (Morais, 2002). This is because, according to the DoE (2005, p. 3, emphasis added), "If a teacher wishes to add to the number of assessment tasks, he or she *must* motivate the changes to the head of department and the principal of the school". In addition, "The teacher must provide the Programme of Assessment to the subject head and School Management Team before the start of the school year". The latter point means that once the teacher has developed the program of assessment, that program is no longer in their control, given that they need to provide a motivation for changing *their own* plan of assessments" once submitted to school management, learners and parents (p. 3).

From the above, there seems to be an emphasis on the "number of assessment tasks" in the Program of assessment, rather than on the nature of those assessments. What is the main reason for asking teachers to submit a plan of assessment to the subject head and the school management team? It is obviously clear that the aim in the NCS guidelines is to ensure that there is a regulatory mechanism that should guide the instrumentation of assessment in schools. However, to what extent does this regulatory mechanism address issues of quality in the way it has been stated? And how would the school management team, learners and parents judge the quality of these assessments? An interesting development in the NCS assessment guidelines is the fact that there is an attempt to move away from tests and examinations as providing the only means of providing feedback on learners' progress.

> The remainder of the assessment tasks should not be tests or examinations. They should be carefully designed tasks, which give learners opportunities to research and explore the subject in exciting and varied ways. Examples of assessment forms are debates, presentations, projects, simulations, literary essays, written reports, practical tasks, performances, exhibitions and research projects (DoE, 2005, pp. 3-4).

We see here that opportunities are being created, as learners engage with assessments, to "research" and "explore" mathematics as a discipline: what it means, and perhaps how it applies to learners' everyday lives. However, while opportunities are being opened up for assessment, it is not clear what these proposals mean for schools and learners who come from disadvantaged contexts. So the power question here concerns research for what purposes (Murray, 2002) and who benefits from such research.

One clearly robust ways in which mathematics can be excitingly explored is to involve learners in technological contexts. For example, one assessment standard in Learning Outcome 2 states that we know that learners are able to investigate, analyse, describe and represent a wide range of functions and solve related problems when they are able to "Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by *available technology*, to make and test conjectures about the effect of the parameters k, p, a and q for functions including: y = sin(kx)" (DoE, 2005, p. 16)

In Learning Outcome 4 (data handling), one of the contexts requires learners to calculate "the variance and standard deviation of sets of data manually (for small sets of data) and using *available technology* (for larger sets of data), and representing results graphically using histograms and frequency polygons" (p. 21). Learners are also required to "use available technology to calculate the regression function which best fits a given set of bivariate numerical data" (p. 24). Given the flexibility and efficiency of technologies such as handheld graphing calculators, the proposals being suggested in the curriculum guidelines are commendable given that they have the potential to allow learners to work efficiently with mathematical ideas and computations involving these. However, while the teacher might plan his/her assessment in keeping with these technological opportunities, one needs to recognise whether in disadvantaged contexts such as rural township schools would be able to afford these. In such a case, the *choices* for the teachers are further limited in terms of their selection of assessment tasks and tools that could be used to enhance learners' engagement in these. While technological tools may add a conceptually and didactically powerful dimension to teaching, when the conditions in which teachers teach mathematics are hostile, the power of teaching tools becomes limited.

Emerging contradictions

The above analysis of the assessment guidelines has indicated that teachers are given some power and flexibility over what goes on in the daily assessments that learners engage with in their mathematics activities. The teacher is given power to choose from a range of strategies for monitoring learner progress. Once assessment tasks have been undertaken by learners, the teacher can decide whether to mark them or whether learners should mark their own written work. Particularly interesting, the teacher can choose whether to record the results of the assessments or not. “The results of the informal daily assessment tasks are not formally recorded unless the teacher wishes to do so”. While it appears that the teacher is given power over assessment at the informal daily level, this power is highly limited for two reasons. First, the results that emerge from the teacher’s exercise of such power over assessment are not given much political significance. Secondly, it is not clear how the teacher is to exercise such power. Because of these reasons, I propose that while the intention of the NCS is to allow teachers freedom to work in ways they find themselves in their contexts, such freedom is only an imagination. The question then becomes, why should the NCS provide these opportunities for teachers to exercise their freedom or power over assessment when in fact the same NCS knows that teachers will eventually have limited power? What is the aim of the NCS in having such proposals? I suggest that the NCS finds itself in this predicament because of an attempt to align itself, as can be expected, to the principles of outcomes-based education, OBE.

According to Spady (1998), there are three key assumptions to OBE. “All students can learn and succeed, but not on the same day in the same way; successful learning promotes even more successful learning; and *schools control the conditions that directly affect successful school learning*” (emphasis added). It is the third assumption that is more pertinent to “blind spots” (le Grange, 2004) and the closed assessment box I am opening here. It seems that the NCS is attempting to give teachers more power over daily assessment because teachers, as critical constitutive agents of schools, control the conditions that directly affect successful school learning. We are talking here about the day-to-day work of teachers as learning managers in their own classrooms. It is the centrality of the teacher that the NCS seems to be rightly uplifting here. According to Todd and Mason (2005), “The most effective factors [for improved learning] depend on the teacher, and other distal variables have an impact to the extent that the teacher exploits their potential in enhancing learning” (p. 229). Todd and Mason continue to suggest that “The challenge for South African teachers is to maximize these proximal factors that have been identified in the research, in spite of the difficulties they face because important distal variables remain unsatisfied”. Is the way the NCS assessment guidelines are stated an attempt to satisfy the “proximal” factors associated with effective learning to which the teacher is a central part? The analysis presented above points to the affirmative. I suggest here that a further elaboration of the rationale and conceptualisation of daily assessments is necessary in order for South African education policy to “maximize the ability of teachers to exploit… proximal factors” which according to Hattie (1999, in Todd and Mason, 2005, p. 227) are concerned with teachers coming to “know what our students are thinking so that we can provide more feedback…and develop deep understanding”. The key issue centres on recognising the need to have “teachers who understand their discipline well, and who care about their students and what they know”. For it is such teachers who “will be better able to set challenging goals and to provide well-directed feedback” (Todd & Mason, 2005, p. 227). I posit that mathematics education in South Africa can only be able to obtain such kind of teachers if policies are developed and implemented in such a way that they recognise the power that teachers have over daily assessments in addition to, and more importantly, sensibly recognising the value of these assessments..

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Investigating Elementary Teachers' Mathematical Knowledge for Teaching Geometry: The Case of Classification of Quadrilaterals

Dicky Ng, EdD
Assistant Professor, Mathematics Education, School of Teacher Education and Leasdership
Utah State University, Logan, Utah 84321 ngdicky@bu.edu

Abstract
This paper examines the mathematical knowledge for teaching (MKT) in Indonesia, specifically in school geometry content. A translated and adapted version of the MKT measures developed by the Learning Mathematics for Teaching (LMT) project was administered to 210 Indonesian primary and junior high teachers. Psychometric analyses revealed that items related to classification of quadrilaterals were difficult for these teachers. Further interactions with teachers in a professional development setting confirmed that teachers held a set of exclusive definitions of quadrilaterals.

Introduction

New direction on the study of teacher's knowledge of mathematics has received tremendous attention since the Shulman's (1986) seminal work of identifying pedagogical content knowledge (PCK) as the missing link in the knowledge teachers need to bridge between subject matter knowledge and pedagogical knowledge. Ball and her colleagues (2001) further refine our understanding of this knowledge by introducing mathematical knowledge for teaching (MKT) as a specialized knowledge of the content that is situated in the context of teaching (Figure 1). This construct underscores that the knowledge required for teaching is determined by the mathematical work of teaching (Ball, 1999). Four domains of mathematical knowledge are hypothesized in the U.S. construct of MKT: common content knowledge (CCK), specialized content knowledge (SCK), knowledge of content and students (KCS) and knowledge of content and teaching or KCT (Ball, Thames, & Phelps, 2008).

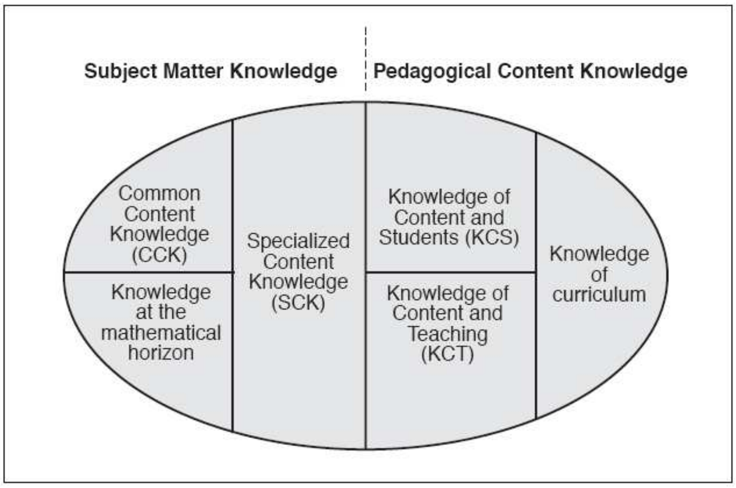

Figure 1. Domains of Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008, p. 403).

Although this construct is developed based on U.S. based teaching practices, recent evidence suggests that it may translate to other cultures (Delaney, Ball, Hill, Schilling, & Zopf, 2008). However, despite commonalities in teaching practices, teaching is a cultural activity (Stigler & Hiebert, 1999), and thus, has to be approached with caution by taking into account differences that may impact understanding of particular topics.

This study examined Indonesian elementary teachers' mathematical knowledge for teaching specifically on school geometry and addressed the following research questions:

- What elementary school geometry topics did Indonesian elementary teachers found to be difficult and why were these topics difficult for them?

Method

Subjects consisted of 210 Indonesian elementary teachers participating in professional development programs focused on mathematics and science between July and November 2007. An adapted version of the Mathematical Knowledge for Teaching (MKT) geometry measures, originally developed by the Learning Mathematics for Teaching project at the University of Michigan, were administered to measure the teachers' mathematical knowledge for teaching geometry. Figure 2 shows a sample from the released items (Hill, Schilling, & Ball, 2004). Detailed analyses of the challenges and issues in translating and adapting the measures are found elsewhere (Ng, 2009).

Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
35 x 25 125 +75 875	35 x25 175 +700 875	35 x 25 25 150 100 +600 875

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

	Method would work for all whole numbers	Method would NOT work for all whole numbers	I'm not sure
a) Method A	1	2	3
b) Method B	1	2	3
c) Method C	1	2	3

Figure 2. Released Item from SII/LMT (Hill, Schilling, & Ball, 2004)

Two psychometric analyses were conducted to assess the performance of the MKT measures: comparing the point-biserial correlation estimates between the U.S. and Indonesian measures and evaluating the relative item difficulties using a one-parameter Item Response Theory (IRT) model between the two countries. The point biserial estimates provide information on how the items are correlated with one another. Higher point biserial correlation indicates stronger relationship among the items and the construct being measured (Delaney et al., 2008). Interactions with the teachers during the professional development program provided further anecdotal evidence on teachers' MKT.

Results

The correlation between the Indonesian and U.S. point-biserial was moderate (Figure 2, r = 0.369). One item in the Indonesian version had negative point-biserial correlations of -0.045 (Item 3c, Table 1), indicating that respondents who scored well on other items in this test were more likely to get this item wrong than right. This item asked the teachers if it was possible for a parallelogram to have congruent diagonals. A possible hypothesis to explain this difference relates to the way parallelograms are represented in Indonesian curriculum (Departemen Pendidikan Nasional, 2003). Parallelograms are depicted as a quadrilateral with two pairs of parallel and congruent sides. However, there is no indication in Indonesian textbooks or curriculum on considering, say, a rectangle as a special type of parallelogram.

Table 1. Point biserial correlations estimates between U.S. and Indonesian measures.

Item	Indonesian point biserial estimates	U.S. point biserial estimates
1a	0.419	0.561
1b	0.100	0.565
1c	0.325	0.634
1d	0.357	0.715
1e	0.186	0.539
2a	0.524	0.590
2b	0.420	0.606
2c	0.400	0.444
2d	0.476	0.696
2e	0.371	0.620
3a	0.179	0.628
3b	0.491	0.689
3c	-0.045	0.592
3d	0.274	0.650
4	0.121	0.502
5	0.325	0.415
6	0.227	0.694
7	0.412	0.683
8	0.305	0.438

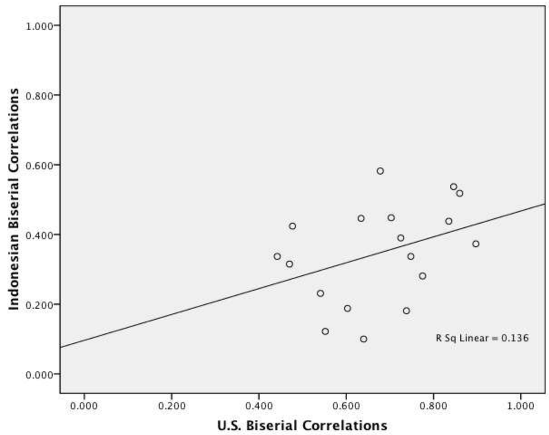

Figure 2. A regression line fitted to a scatter plot of the U.S. and Indonesian biserial correlations.

Such distinct treatment of shapes, in this case quadrilaterals, made items that required the teachers to examine the relationship between classes of shapes, such as item 1b, 3a, 4 had much higher level of difficulties of about two standard deviations compared to U.S. difficulties (Table 2). As mentioned before, shapes such as square, rectangle, and parallelogram are treated as distinct entities in the Indonesian curriculum. Interestingly, the Indonesian term for rectangle is literally “long square”, showing a relationship between the two. However, due to instructional treatment in the textbook, teachers lacked understanding of how these shapes were related. On the other hand, referring to a rectangle as a long square may also prevent students from recognizing that a rectangle can be a square since one pair of sides of a rectangle “has to be” longer than the other pair.

During the professional development program, the teachers worked in groups to explore characteristics of each of the quadrilaterals: square, rectangle, parallelogram, rhombus, kite, and trapezoid. Next, they were asked to compare the shapes in term of their characteristics, looking at their sides, angles, diagonals, reflective, and rotational symmetries. They were then asked to create a hierarchical classification of the quadrilaterals. Only two out of the six groups were able to come up with the correct classification. Even after these activities, teachers had difficulty to evaluate correctly whether the statement such as “No rectangle is a rhombus” is always, sometimes, or never true.

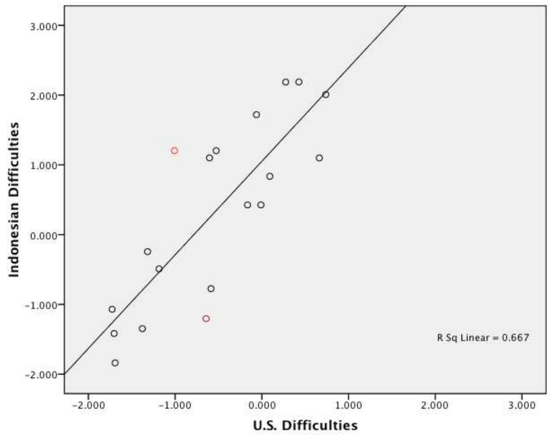

Figure 3. A regression line fitted to a scatter plot of the relative difficulties of items in the Indonesian and U.S. versions of the MKT measures.

Table 2. Comparison of the difficulty estimates for MKT geometry measures between U.S. and Indonesian.

Item	Indonesian difficulties (SE)	U.S. difficulties (SE)
1a	-0.491 (0.236)	-1.185 (0.068)
1b	2.186 (0.277)	0.275 (0.057)
1c	0.836 (0.242)	0.092 (0.057)
1d	0.425 (0.237)	-0.165 (0.056)
1e	1.719 (0.261)	-0.062 (0.056)
2a	-1.347 (0.281)	-1.378 (0.073)
2b	-1.838 (0.316)	-1.692 (0.084)
2c	-0.774 (0.244)	-0.587 (0.059)
2d	1.099 (0.257)	0.663 (0.061)
2e	-1.417 (0.270)	-1.704 (0.084)
3a	1.203 (0.246)	-1.008 (0.064)
3b	-1.070 (0.245)	-1.727 (0.085)
3c	-0.243 (0.241)	-1.318 (0.071)
3d	-1.204 (0.249)	-0.643 (0.059)
4	2.188 (0.291)	0.427 (0.059)
5	2.007 (0.281)	0.737 (0.062)
6	1.203 (0.242)	-0.527 (0.058)
7	0.426 (0.248)	-0.011 (0.056)
8	1.100 (0.258)	-0.604 (0.059)
	0.316 (0.259)	-0.548 (0.065)

Discussion

Results from psychometric analyses revealed that questions that asked the teachers to relate different quadrilaterals were relatively more difficult for the Indonesian teachers compared to the sample from the U.S. teachers. One reason for this difference is the way the topic is treated in the two countries. Delaney and colleagues (2008) point out that school curriculum is an important factor in determining mathematical knowledge. Although curricula from two countries contain the same topics, the treatment of these topics may vary by country. The Principles and Standards for School Mathematics document states the expectation for grades 3-5 students should "identify, compare, and analyze attributes of two- and three-dimensional shapes and develop vocabulary to describe the attributes; and classify two- and three-dimensional shapes according to their properties and develop definitions of classes of shapes such as triangles and pyramids" (NCTM, 2000, p. 164). In contrast, the Indonesian curriculum document states that one of the objectives for geometry is for students to "identify two- and three-dimensional shapes based on their properties, characteristics, or similarities" (Departemen Pendidikan Nasional, 2003, authors' translation). The treatment of two- and three-dimensional shapes in the Indonesian curriculum is thus different; shapes are introduced as distinct objects and no efforts to relate them can be found in the standards and textbooks.

Usiskin and Griffin (2008) examined 101 high school geometry textbooks used in the United States, 15 of which were textbooks for preservice elementary school teachers, and found varying definitions of the quadrilaterals. For instance, among these textbooks there were inclusive and exclusive definitions of a trapezoid. The inclusive definition states that "a trapezoid is a four-sided closed figure with a pair of parallel sides" whereas an exclusive one states that "a trapezoid is a four-sided closed figure with a exactly one pair of parallel sides". Thus, someone who holds an exclusive view will not consider a parallelogram to be a trapezoid. There is possibility that Indonesian teachers held exclusive definitions of quadrilaterals.

This study has many limitations. First, no follow interviews were conducted with the teachers to assess in-depth their understanding of definitions of quadrilaterals and why they found this topic to be difficult. Second, the MKT measures did not cover every geometry topics in the curriculum, for instance there were no questions related to nets of three-dimensional shapes among the MKT measures. Finally, textbooks for preservice teachers were not examined to assess whether indeed exclusive definitions of quadrilaterals were used. This study is only a beginning effort in understanding what specific school geometry topics teachers find difficult and how definitions and treatment of quadrilaterals may affect teachers' understanding. Many questions remain unanswered which require further research:

- How does adhering to exclusive definitions of quadrilaterals differ in terms of instructional practices compared to inclusive definitions?
- How may that in turn affect students' understanding especially when they begin formal study of geometry in high school?

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Mathematics Teacher TPACK Standards and Revising Teacher Preparation

Margaret Niess
Professor Emeritus of Mathematics Education, Science and Mathematics Education, Oregon State University, Corvallis, Oregon, 97331 USA niessm@onid.orst.edu

Abstract
What knowledge do teachers need for integrating appropriate digital technologies in teaching mathematics? An overarching construct called TPACK is proposed as the interconnection and intersection of knowledge among technology, pedagogy, and content and is referred to as the total knowledge package for teaching mathematics with technology. Five stages in the process of developing TPACK - *recognizing, accepting, adapting, exploring,* and *adapting* – describe the process of teachers' learning to integrate technology. Teachers learn to teach mathematics from their own learning – K-12 mathematics - collegiate mathematics coursework, teacher preparation program, field experiences and professional development as they teach mathematics. The challenge is to identify appropriate experiences to guide this integration of technology in teaching mathematics in ways that develop TPACK. A framework for these experiences directs attention to emergent social and psychological perspectives.

Introduction
Technology tools like spreadsheets and calculators are typically viewed as computational tools for arithmetic computations. Yet, these tools are more accurately described as dynamic, algebraic reasoning tools. While many teachers fear that these tools rob students of "doing the math," those who understand the tools' affordances can revolutionize their students' learning of mathematics. Rather than a rote-, algorithmic-, and answer-driven experience, students can be engaged in problem extension, asking "what if" questions of the problems, modeling different views of problems, working with open-ended problems, and encouraging question-posing to reveal the mathematics behind solutions to problems. Student can model mathematical systems rather than being restricted to multiple, repetitive symbolic manipulations for each variable change in a problem. In the process they investigate concepts of variables and covariation and are engaged in important mathematical processes, such as problem solving and mathematical modeling to analyze changes in various contexts (NCTM, 2000). The result is that they develop a more robust understanding of the content and processes of mathematics.

This vision of spreadsheets and calculators as algebraic reasoning tools rather than arithmetic computational tools, as tools to think mathematically, and as tools to learn mathematics suggests a significantly different mathematics curriculum and instruction than evidenced in the previous century. The problem is that today's teachers learned mathematics in the past and typically view that mathematics must be learned the way they learned. In their teacher preparation programs, they developed general pedagogical methods and strategies with little attention to teaching mathematics with digital technologies. They may have had a technology course focused on the affordances and constraints of the technologies but little if any focus was on the integration of the technologies in teaching and learning mathematics. What happens when they enter the classroom to teach mathematics? They are more apt to teach mathematics the way they learned it – without integrating digital technologies that afford new ways of thinking in mathematics, technologies with the potential for engaging students in higher order thinking and reasoning to support them in learning mathematics.

Technology, Pedagogy, and Content Knowledge (TPACK)
Shulman (1986) launched a new way of thinking about the knowledge teachers need for teaching with a construct that he called pedagogical content knowledge (PCK). This new way of thinking about teacher knowledge called for the integration of content knowledge (the knowledge previously considered the primary knowledge domain for teachers) and pedagogical knowledge (the knowledge about teaching and learning). The revolution was that a teacher's success in teaching relied on the knowledge from the intersection of these two knowledge bases. PCK was

described as the way of representing and formulating subject matter knowledge that makes those ideas comprehensible to learners.

This PCK lens redirected the efforts of educational researchers investigating the knowledge that teachers needed for teaching with technology. Earle (2002) explains why teachers' knowledge is seen as the key variable in teaching with technology:

> Integrating technology is not about technology – it is primarily about content and effective instructional practices. Technology involves the tools with which we deliver content and implement practices in better ways. Its focus must be on curriculum and learning. Integration is defined not by the amount or type of technology used, but by how and why it is used. (p. 8)

This research direction ushered the description of an overarching knowledge construct as the interconnection and intersection of Technology, Pedagogy And Content Knowledge, called TPACK, the total knowledge package for teaching subject matter content with technology (Margerum-Leys & Marx, 2002; Mishra & Koehler, 2006; Niess, 2005; Pierson, 2001). Basically, these researchers defined TPACK as the knowledge that teachers need to teach with and about technology in their assigned subject areas and grade levels. The vision of TPACK has developed to the point that the American Association of Colleges of Teacher Education directed a collaboration of multiple TPACK authors in *The Handbook of Technological Pedagogical Content Knowledge for Educators* (Silverman, 2008).

Developing a Mathematics TPACK

What does TPACK knowledge mean for mathematics teachers? Niess (2005) adapted Grossman's (1989, 1990) four components of PCK to describe teachers' knowledge of incorporating technology in teaching mathematics as the knowledge and beliefs teachers demonstrate that are consistent with:

- An overarching conception about the purposes for incorporating technology in teaching mathematics;
- Knowledge of students' understandings, thinking, and learning of mathematics with technology;
- Knowledge of curriculum and curricular materials that integrate technology in learning and teaching mathematics;
- Knowledge of instructional strategies and representations for teaching and learning mathematics with technologies.

Do teachers either have or not have TPACK? Niess, Sadri, and Lee (2007) proposed a developmental model for TPACK based on Rogers' (1995) five-stage process by which a person makes a decision to adopt or reject a new innovation. Over a four-year period, Niess, et al. observed teachers as they learned about spreadsheets and integrating spreadsheets as learning tools in their mathematics classrooms. Analysis of these observations described teachers at five stages:

1. *Recognizing* (knowledge) where teachers are able to use the technology and recognize the alignment of the technology with mathematics content, yet are not willing to integrate the technology in teaching mathematics in their classrooms.
2. *Accepting* (persuasion) where teachers may attempt to engage their students in learning mathematics with an appropriate technology as part of the process of determining if they have a favorable or unfavorable disposition toward incorporating the technology in their classrooms.
3. *Adapting* (decision) where teachers engage their students in activities in teaching and learning mathematics with an appropriate technology.
4. *Exploring* (implementation) where teachers actively integrate teaching and learning of mathematics with an appropriate technology.
5. *Advancing* (confirmation) where teachers evaluate the results of the decision to integrate teaching and learning mathematics with an appropriate technology.

An important consideration in these levels is that teachers may or may not traverse linearly through them; they may traverse through the early levels again for new and emerging technologies as they consider their usefulness in teaching mathematics. Each TPACK level was then expanded using the lens of the four conceptions of the knowledge and beliefs teachers demonstrate in TPACK (Niess, 2007).

The Association of Mathematics Teacher Educator's (AMTE) Technology Committee then developed a visual description of the TPACK levels (see Figure 1). On the left side, the figure highlights PCK as the intersection of pedagogy and content. As knowledge of technology intersects with pedagogical and content knowledge, the teachers' knowledge base emerges as the knowledge described as TPACK – where teachers actively engage in guiding student learning of mathematics with technologies.

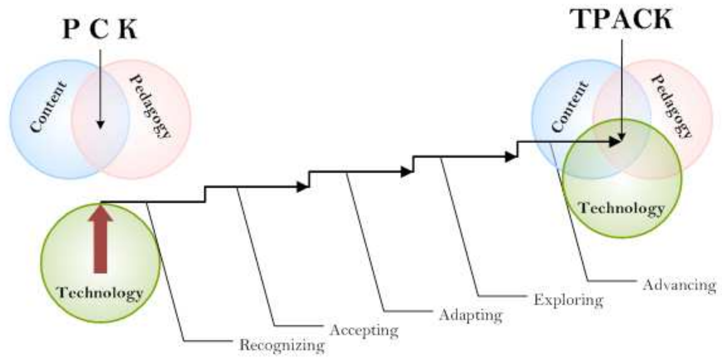

Figure 1. Teacher knowledge as their thinking and understanding merge toward TPACK.

Preparing Mathematics Teachers to Develop TPACK

Teachers' knowledge for teaching mathematics is a construction that emerges from their early mathematical learning experiences, collegiate mathematical learning, teacher preparation programs, and professional development experiences as they are actively engaged in teaching mathematics. Many digital technologies (calculators, spreadsheets, applets of virtual manipulatives, dynamic geometry tools, and computer algebra systems) offer a broad spectrum of mathematical capabilities. Even more technologies are emerging and becoming accessible in schools for students in learning mathematics. The key challenge is for mathematics teacher educators to design, implement, and evaluate new teacher preparation programs that provide experiences that support the development of the knowledge, skills, and dispositions in TPACK for teaching mathematics.

Table 1. A Framework for Learning to Teach with Technology

Professional Expertise	Social Perspective	Psychological Perspective
Professional Identity	Pedagogical Social Norms	Pre-service teacher's beliefs about their own role, others' role and the general nature of technology
Technology Specific Pedagogy	Norms of Pedagogical Reasoning about Technology	Pre-service teacher's overarching conception of teaching mathematics with technology
		Pre-service teacher's knowledge of student understandings, thinking, and learning mathematics with technology
Content Knowledge	Classroom Pedagogical Practices With Technology	Pre-service teacher's knowledge of instructional strategies and representations for teaching with technologies
		Pre-service teacher's knowledge of curriculum and curricular materials

How mathematics teachers learn, with whom they learn, and the context in which they learn are fundamental to *what* they learn (Greeno et al., 1996). Harrington (2008) notes that learning to teach mathematics with technology must be viewed from both social and psychological perspectives. She describes TPACK from an emergent perspective that takes into account what is known about TPACK from the psychological perspective focused on the four components of TPACK (Niess, 2005) and includes what needs to be known from a social perspective as shown in Table 1. The framework guides the educational experiences for teachers to teach mathematics where they gain professional identity, technology specific pedagogy and content knowledge.

Mathematics pre-service teachers' development of TPACK depends on many factors, including experiences that use appropriate technologies as they learn mathematics at the collegiate level. Their content learning environments must go beyond simply expecting them to mimic experiences modeled as they learned mathematics. From social and psychological perspectives, these pre-service teachers need to engage in more experiences than simply learning the mathematics; they need to engage in analyzing (1) the affordances/constraints of using a certain technology to teach particular mathematics content, (2) teaching the content changes as a result of using the technology, (3) creating appropriate assessments that include the use of technology, (4) posing questions that enhance and extend students' learning of mathematics while using technology, and (5) developing their knowledge about technologies that exist for teaching and learning specific mathematics concepts. Not all collegiate mathematics content faculties have an advanced level of TPACK. Yet, current thinking suggests that they need to guide pre-service teachers' thinking about what is taught with the technology, why the technology is chosen for the particular mathematics concept, the affordances and constraints of using the technology to teach the particular mathematics concept, why certain questions are posed in scaffolding instruction, and why the assessment is chosen.

Teacher education courses, in particular the mathematics methods courses, have potential for impacting pre-service teachers development of mathematics TPACK. As in the recognition of the importance of PCK for teacher knowledge, teacher preparation programs must integrate learning and teaching with and about technology from the content – the mathematics- and the pedagogy perspectives. Pre-service programs need to engage future teachers in aligning concepts and skills with appropriate technologies with national/state content standards, forcing them to reflect on how and why the technology should be used in mathematics instruction (Niess, 2005). They need to learn how to reason pedagogically about technologies as well as in their practices in a mathematics classroom. Developing these habits of mind in social situations are critical if teachers are to develop the psychological perspectives of professional identity, technology specific, pedagogy and content knowledge. Further research is needed to develop a model of mathematics teacher education that takes into account structures of the coursework, approaches to technology education, and activities pre-service teachers experience.

Field experiences are essential teacher preparation experience, providing authentic contexts for thinking about, designing, implementing, and assessing the impact of integrating technologies in learning mathematics. As an alternative, Bullough et al. (2003) found that pre-service teachers were able to synthesize their coursework and field experiences more effectively and their relationships with the cooperating teacher were more collaborative in nature when they were placed in classrooms with peers. Harrington (2008) captured pre-service teachers TPACK development in a collaborative field experience where they: (1) offered ideas during team lesson planning, (2) justified their thinking to peers, instructors and cooperating teachers, and (3) when making choices during their own teaching. Taken together these opportunities define *patterns of participation* across the learning contexts of peer interaction, coursework and field experiences as they learn to teach with technology (Peressini et al., 2004).

Pre-service teachers need multiple opportunities and contexts in which to develop their TPACK and this development takes time. Their development is influenced by past experiences and formal ideas begin to surface during their coursework. Formal ideas are enacted as pre-service teachers begin their field experiences. Important research questions need to be framed and studied to guide the development of programs to provide pre-service and in-service teachers with opportunities to develop and display their TPACK. What are the situations that facilitate this development? What happens beyond the traditional models of teacher preparation? Research is definitely needed.

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Balancing the Use of Technology and Traditional Approaches in Teaching Mathematics within Business Courses

Mehryar Nooriafshar
mehryar@usq.edu.au University of Southern Queensland, Toowoomba, Australia

Abstract
Technologies associated with modern computing are being commonly used in education. Over the past few years, the usage has increased considerably. This increase is also attributed to the availability of more improved technology products and services at much lower costs.
As a result, many successful educational multimedia products have been developed which have made significant contributions to learning and teaching mathematics at various levels. However, it is not always clear what exactly the position of technology in education is. In other words, to what extent does the technology-aided means of learning enhance learning and add value to the conventional materials? How are they supposed to supersede or excel the learning effectiveness of traditional methods of teaching?
This paper explores the possibilities of utilizing the latest technologies such as Virtual Reality (VR) environments and Tablet PCs in conjunction with the traditional approaches and concepts in creating a balanced and more effective learning and teaching conditions. It also demonstrates how the creation of a situation where 'one cannot see the wood for the trees' can be avoided by striking the right balance. **Key words:** Technology, Quantitative, Teaching, Traditional

Introduction

This paper explores the role of the technology in creating better and further opportunities for quantitative techniques via the use of multiple senses. Before presenting the research findings, let us discuss some of the main teaching/learning principles used in designing effective learning resources for the purposes of teaching mathematics and related topics.

The technology will make it possible for us to simulate some of the teacher-learner interactions too. Imagine a Powerpoint presentation with the lecturer's narration recorded as a voiceover with various marking such as highlighting sections recorded at the same time. The final result will be an ideal record of an interactive face to face session captured as it takes place in a classroom. This method is very much suited to teaching mathematically oriented subjects. The saved file in various formats such as Flash or compatible MS Media Player can be distributed to all students for further and future reference.

The teaching materials and approaches, regardless of the mode of their delivery mode, must be based on certain established learning principles. For example, the learners' modal preferences should be taken into consideration so that they can have a choice for learning via their preferred styles and senses. Different people learn in different ways. For instance, some prefer listening; some people like reading and others prefer seeing how things are done. It does not necessarily mean that each person must have only one preferred way. Often people have more than one preference. It is a good idea for any leaner to find out about their dominant learning style.

Learning approaches such as learning by association (attaching a memory handle for recalling and remembering) and learning by understanding (building on learners' existing knowledge) are some of the important and effective learning methods. The following sections will discuss these approaches.

Let us take a brief look at a comparative study of employers' and students' needs and expectations in terms of learning and teaching in the next sections.

The Needs and Expectations of the Employers

It can be argued that the employers are the ultimate customers of the educational institutes. Hence, their needs and expectations should be taken into consideration in designing teaching materials.

As an initial study, in 2005, a sample of 50 organisations representing both goods producing and service providing industries were randomly selected from the Darling Downs Region (in and around Toowoomba) of Queensland in Australia. Most (90%) of these industries employed less than 100 employees. Data collection was carried out by telephone and a specially designed brief questionnaire was completed during each call. The questions aimed to identify the applicability of quantitative Production and Operations Management techniques favoured and utilized by these industries.

It is interesting to note that 50% of the surveyed employers believe that university graduates do not posses the necessary practical skills to undertake tasks within industries.

These findings can help course improvements with a view to catering for the needs of the industries. The next section investigates students' learning needs and preferences with a view to linking them with the employers' requirements. Hence, improvements in conveying the underlying messages and concepts to the students can be a basis for addressing the employers' problem.

The Learning Needs and Preferences of Students

A group of twenty first-year undergraduate students were selected for the purposes of an experiment on the effectiveness of teaching basic mathematics concepts via practical teaching aids. These students were from different mathematical backgrounds and the majority did not have a very strong background in quantitative fields.

These students were taught the basic principles of identifying and plotting graphs of polynomial equations of different degrees. It should be mentioned that these basic skills form the foundations of understanding, learning and using more advanced techniques in quantitative subjects. Curve fitting, regression, linear programming and its derivatives are some of the examples. The students were taught the main concepts in a very practical manner by shaping and positioning the flexi-curve on the axes drawn on a whiteboard. The basic scientific calculator was used to work out angles associated with the slopes. The protractor was used to measure and mark the angles on the whiteboard. The main purpose was to equip the students with the ability to recognize and visualize the general shape of a polynomial equation by simply looking at its main components such as the coefficients, powers and constant values.

The equipment used included basic scientific calculators, protractors and a flexi-curve. This experiment was based on the idea of guiding the students towards finding the answers instead of simply giving them the information. It also placed an emphasis on the visual aspects of teaching and learning methods.

The effectiveness of the above-mentioned approach (teaching basic mathematics concepts via practical teaching aids) was tested by identifying and measuring students' performance and learning preferences. A comparison between students from different mathematical backgrounds was also made.

The findings of this study suggest that students, regardless of their background in mathematics, have a preference for visual methods of learning mathematical concepts. It was also demonstrated that most students who participated in the study, enjoyed learning mathematics and believed that they would benefit from it in their future studies and career.

Finally, about 10 students were selected randomly and then were tested on the concepts provided to them. The performance of these students was quite satisfactory. The marks considered by the author ranged between 70% and 95%. This was achieved by asking students different questions based on the materials presented to them. The above findings illustrate that students needs and preferences are not very much different from the employers. Both parties recognize the importance of the quantitative fields and have a preference for practically oriented approaches.

The Latest Technologies and Approaches in Learning and Teaching

The clicker technology has successfully been used in classroom teaching. As reported by Hafner (2004) Paul Caron uses Classroom Performance System (CPS) in his law classes at the University of Cincinnati to break through the "cone of silence". In 2005, the author devised a method of utilizing the technology in such a way that incorporated the established and traditional methods and concepts in learning and teaching. Hence, the system was adopted and used in an interactive and constructivist manner. It is noteworthy to mention that it was the first time at a Queensland university that this kind of technology adopted in the classroom. The outcome of this experiment was very encouraging as both formal and informal feedback by students confirmed their interest in attending all classes and learning much more quickly.

They also suggested that this way of learning made the lectures interesting and exciting (Nooriafshar, 2005).

The applicability of Virtual Reality in teaching was tested by the author in 2007 by carrying out a comparative study (Nooriafshar, 2007). In this study, the visually rich multimedia ideas were taken a step further by enhancing them so that the learner can interact with the subject in a more realistic manner.

As an extension of the application of VR in quantitative subjects, a latest 3D development environment called *VirtualStage* by Dakine Wave Limited (http://www.dakinewave.com/) was adopted to create simulations of classroom sessions in a realistic manner. As part of this project, various learning situations were created and produced as virtual reality productions. Learning and teaching methods which were adopted in these developments are based on established concepts such as learning by guidance.

A series of 3D presentations were developed in VirtualStage. The topics included Decision Theory basics and Introduction to Goal Programming. These presentations demonstrate how the Socratic method of teaching, which usually takes place in a face to face situation, can be simulated, created and captured for replay. The findings of the VR research demonstrated that:

Visually rich 3D presentations can provide effective teaching and learning environments. A virtual reality multimedia can even further enhance learning by incorporating more realistic images, visual features and dialogue. This combination would lead to a situation where the learners could immerse themselves in the environment and interact with objects and scenarios in a dynamic manner.

Another latest technology which is worth a mention is the Tablet PC or graphically enabled computers. The University of Southern Queensland is now using tablet PCs across faculties, in a coordinated approach funded through a university Learning & Teaching fellowship. These devices are ideal for teaching mathematical topics to distance education students. A constructivist approach which was successfully tested by the author is as follows:

- The teacher sets some questions for the learners
- The learner receives the questions via email.
- The learner attempts solving the problems using either a Tablet PC or a digital notepad.
- The learner sends the attempted solution to the teacher via email.
- The teacher corrects the submitted solutions as one does on paper and in hand writing.
- The digitised file, with corrections, is sent back to the learner.
- The cycle can repeat until satisfactory results are achieved.

The Tablet PC is an excellent tool for explaining mathematical concepts and procedures, in a symbolic manner, in the class and also recording the session for off campus students.

Although not a highly technical method, Mind Mapping is an excellent way of teaching, learning and revising mathematical concepts. The original idea of Mind Mapping goes back to the 70s when Tony Buzan developed this very useful, practical and natural way of representing ideas. Originally, Mind Maps were developed manually using, ideally, colours to stimulate Right Brain activity. Nowadays, comprehensive computer software programs can assist with drawing, enhancing, storage and distribution of these learning resources. As suggested by Buzan Online Limited (2006-2009), the iMindMap can be utilized for the purposes of Planning, Organising, Creating and Innovating. Teaching and learning mathematical topics will certainly be assisted by using the iMindMap software and mind mapping in general. The latest versions and related environments such as MindGenius (2009) can also add interactivity for exploring and carrying out what-if analysis.

The following mind map shown as Figure 1 was developed by the author using the iMindMap software, for teaching how to choose the most appropriate Forecasting technique to use under different circumstances.

After showing the Mind Map to the students in the class, the lecturer can then present various possibilities to the them. Some examples are as follows:

- If Data Size is Small and Type is Stationary then apply Simple Exponential-Smoothing.
- If Data Size is Large and Type is Non-Stationary and Cost is Low then apply Regression.

- If Data Size is Large and Type is Non-Stationary and Forecast Needed is Short Term then apply Double Exponential-Smoothing.
- If Forecast Needed is Long Term then apply Judgemental Forecasting.

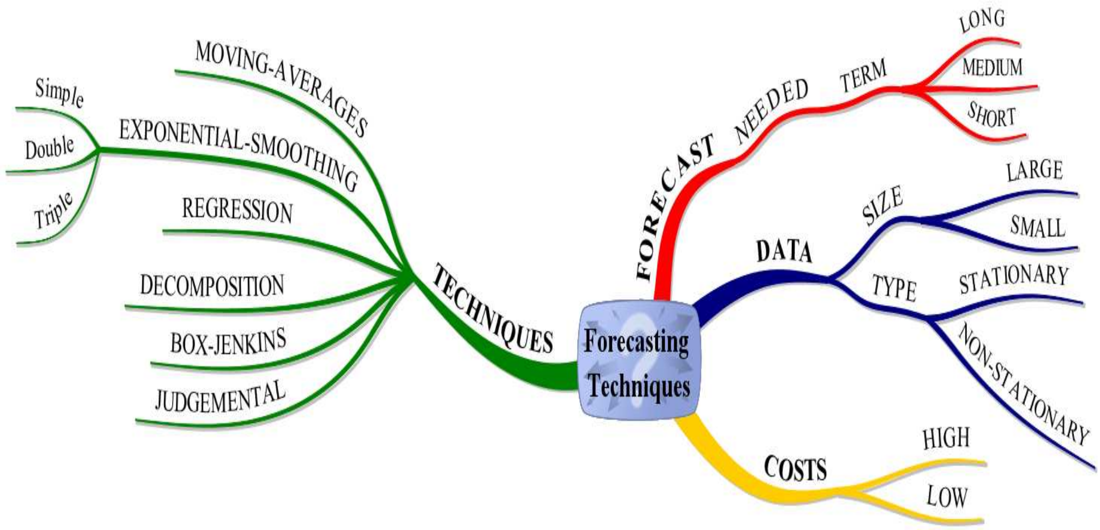

Figure 1 – Mind Map Representing Forecasting Techniques

It should be mentioned that as additional enhancements, various graphics and image-associations along the branches, can be incorporated into the mind map.

A commonality of the way that the above-mentioned technologies were used by the author is the consideration of established learning and teaching concepts. In other words, the technology has not been simply used to substitute the traditional and successfully tested and established methods. In developing technologically based teaching materials, means of incorporating students' learning preferences, allowing them to build new knowledge based on what they already know and learning by association were always considered a priority.

Conclusions

It was demonstrated that teaching approaches have a significant effect on students' learning and meeting their future employers' needs. It was shown how to link both the students' and employers' needs through effective methods of designing teaching materials. Hence, the main purpose is to make it possible for the learner to build new meanings without simply memorising pieces of information received from the teacher.

A number of methods and means of utilizing the latest technologies suitable to mathematics education were presented in the paper. An important message and finding was that in spite of the fact that the technology plays an important role in modern day education, it must not be regarded as a substitute for the established methods and concepts. Having a balanced approach in design of the technologically based learning and teaching materials will certainly help with meeting the needs of the ultimate customers.

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Chapter-spanning Review: Teaching Method for Networking in Math Lessons

Swetlana Nordheimer
research partner, Department of Didactics and Mathematics, Mathematic Institute
Humbodt-Universität zu Berlin, Germany nordheim@mathematik.hu-berlin.de

Abstract
Central to this article is networking in math lessons, whereby concentration is placed on the construction of a student-focused teaching method for the networking of mathematical knowledge in the lower secondary. Firstly, normative standards and descriptive results will be compared. Secondly, several already existing teaching methods for networking in math lessons will be added to the method of „chapter-spanning task variation". Using this method, attention is be placed on the integration of mathematical content and specific social netowrk-form (e.g. teacher led classes, group-work etc.). This paper will be concluded with the presentation of the testing of the method in the school context).

Introduction

With respect to networking in math lessons in secondary education I, there is a gap between the normative standards and the descriptive results. I will present the normative standards according to the concept of so called „basic experience" (Winter, Baptist 2001) versus the results of older and newer empirical studies (Bauer 1988, Baumert, Klieme 2001). Many pupil do not regard math lessons to be a "universe with a maximum level of inner (deductive) networking and openness toward new orders and relationships" (Winter, Baptist 2001) as described in the first of these so called „basic experiences", but as a collection of "incoherent materials neighboring each other" (Bauer 1988) which „are not in a sufficient manner" (Baumert, Klieme 2001) connected with each other. In stead of a „reservoir of models suited to rational interpretation or to the systematic organization of the following of operations" (Winter, Baptist 2001) mentioned in the second of the „basic experiences", many pupil experience mathematics as „a self-sufficient structure which has little contact with other areas of perception" (Bauer 1988). Consequently, difficulties come up for the third „basic experience", in which mathematics appears as the „practice field for heuristic and analytical thinking" (Winter, Baptist 2001) because many pupil are not successful in translating knowledge learned in math lessons into the „processing of complex questions" (Baumert, Klieme 2001).

In order to counteract the problem described, the *chapter-spanning review* will be introduced as a teaching method for the stimulation of networking in math lessons. Based on the third „basic experience" (Winter, Baptist 2001) not only the processing, but also the development of complex questions stands, by way of the pupil, at the centre of this teaching method.

1. Networking Concept

Brinkmann's dissertation (2002) constructs the theoretical basis of the teaching method. Accordingly, networking will be understood as the process and result of the relational situation of mathematical content and application on the level of the teaching materials as well as the cognitive level of the student. According to Brinkmann it is possible to categorize networking as being outer and inner mathematical in the same manner at both levels.

In the frame of this paper the level of the teaching material and the cognitive level of the student will be assessed on the epistemic level according to Brinkmann. The term *epistemic,* introduced by myself here, should serve to emphasize the significance of knowledge in both cases. I will add to this idea of „level" by observation of networking on the social level. The latter should bring the potential of the social structure of the study group for the development of networking in math lessons to fruition.

2. Design of the Teaching Method

There are various suggestions for the stimulation of networking in math lessons to be found in the didactics of mathematics, especially on the epistemic level. For example Vollrath (2001) suggests making the *topic threads (central themes and terms)* of mathematics visible to pupil with the help of tables of contents. He also suggests designing transitions between various textbook chapters through *themes* (inter-mathematical, chapter-spanning contexts) and *groups of themes* (application oriented, chapter-spanning problems). pupil will also be given the possibility of continually working on mathematical problems and studying relationships through self-productions using *study diaries* (Gallin, Ruf 1998). Brinkmann (2003) suggests the usage of *mind-maps* and *concept-maps* in lessons in order to encourage networking.

The social level of networking, and especially the role of social networks in the construction of knowledge, is particularly thematised in general pedagogics (vgl. Fischer 2001). For example, e*xpert groups* and *learn by teaching* are suggested and observed in order to encourage the development of knowledge networking.

One of the few approaches in the didactics of mathematics, in which the epistemic and the social level of mathematical learning are interlocked with concrete exercise examples, is the method of student-centered exercise variation (Schupp 2003).
In the following this method is transformed into the *chapter-spanning review* method with the goal of placing a focus on the networking of teaching material; unlike Schupp, who focused on the discovery of problem-solving strategies. In addition in my opinion, the variation of the exercise should become more strongly connected to the teaching plan and the textbook. On that account the student-centered exercise variation will be synthesized with the above mentioned method for the promotion of networking in lessons taken from the didactics of mathematics and general pedagogics.
The segmenting of the teaching material and a student's mathematical knowledge into categories will be kept as part of the segmenting of the class in the phase of experimental training. As a result the pupil will be able to discover differences and similarities between the chapters of the book as well as connecting central themes and central terms and treads with the help of a content-oriented index. This develops through the modification of the table of contents of the textbook or notebook by placing the chapter and subchapter titles in a left-hand column and placing the exercise names in the header. By doing this, it is possible for the pupil to say which skills are connected to which exercises by ticking of the corresponding content skills.
The individual phases of the method will now be introduced in the following in connection with this design.
Preparation: To begin the pupil solve an introductory exercise with their classmates. This implies a cooperative context. The number of the textbook chapters of the initial exercises of school year are presented to the pupil.
Expert training: Each student chooses an initial exercise. Pupil with the same exercise work together in a group to solve it and prepare the presentation of the exercise. Subsequently, each group determines which field of skill the exercise belongs to by filling in the skill table included in the table of contents of the textbook.
Expert round: The groups are reorganized. Now experts from each initial group will meet together in one group. The goal of this phase is to have each group, using the skills from the initial exercises, create at least one chapter-spanning exercise, write down the solution, and determine the skill field
of the exercise.
Plenum: The exercises and solutions of the groups are summarized in a notebook. The notebook will provide a table, in which the exercises are paired with the respective skills.
Both levels are integrated with one another in the expert round. In addition the realization of the epistemic level and social level are thereby brought together. In doing this the pupil can independently discover themes and groups of themes and formulate chapter-spanning exercises.

3. Testing at a Grammar School

For the testing of the teaching method six initial exercises from various subject areas using the topic tangram-puzzle as a connecting element were developed and applied in an 8^{th} grade class in 2007/08. The skill table and the initial exercises were derived from the teaching exercises, as the lesson was structured according to the textbook. In the beginning phase there were only three hours of class time available to be used. To get started the pupil were challenged along with their classmates to form a square using seven tangram-stones. The pupil were subsequently introduced to the following initial exercises.
Exercise A : „Table“ In order to build a tangram table out of wood, the whole tangram diagram is enlarged. At the same time the longest side of the smallest triangle (ca. 5.5cm) lengthens by **x** cm. How does the area of the whole diagram and of the individual pieces change?
Exercise C: „Functions“ Sketch your whole quadratic solution diagram in a coordinate system. Which function graphs can you find in this sketch? Construct the respective function equation.
Exercise D: „System of Linear Equations“ Sketch the whole solution diagram in a coordinate system. Construct linear systems of equations with two equations and two variables, whose solutions correspond with the vertices of the tangram stones.
Exercise E: „Symmetry“ Sketch the whole solution diagram. Which symmetrical tangram stones do you find in the sketch? If necessary describe the kinds of symmetry and sketch the axis of symmetry and the centers of symmetry. Explain.
Exercise F: „Darts“ There is a new magnetic dart game in the market, that looks exactly like a quadratic tangram puzzle. How great are the chances of hitting a parallelogram or a triangle?
Since difficulty levels are often felt to be subject-dependent and various, the adaption of the difficulty level to the exercises was by-passed in this case in order to counteract the problem by means of a measure of self-assesment. This was achieved by presenting all exercises to the pupil at the same time and allowing them to

choose one. With this method six groups were formed within five minutes. The observed collaboration as well as the thoroughness of the solutions varied from group to group, but all were in a position to prepare a presentation.
In the next lesson the pupil had the chance to ask questions about the exercises and complete the skill table. The skill table was subsequently analyzed doing the discussion of the lesson.
The pupil developed fifteen exercises, of which only two can be presented as examples here. In the following exercise a pupil on a purely mathematical basis combines the two large chapters on symmetry and functions (of Brinkmann's term „inner-mathematical”).

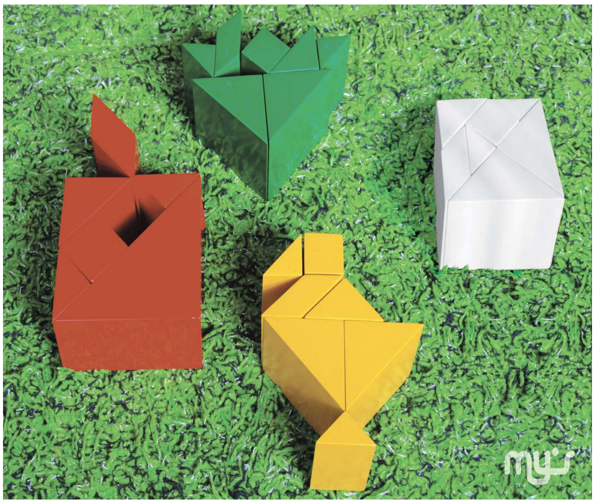
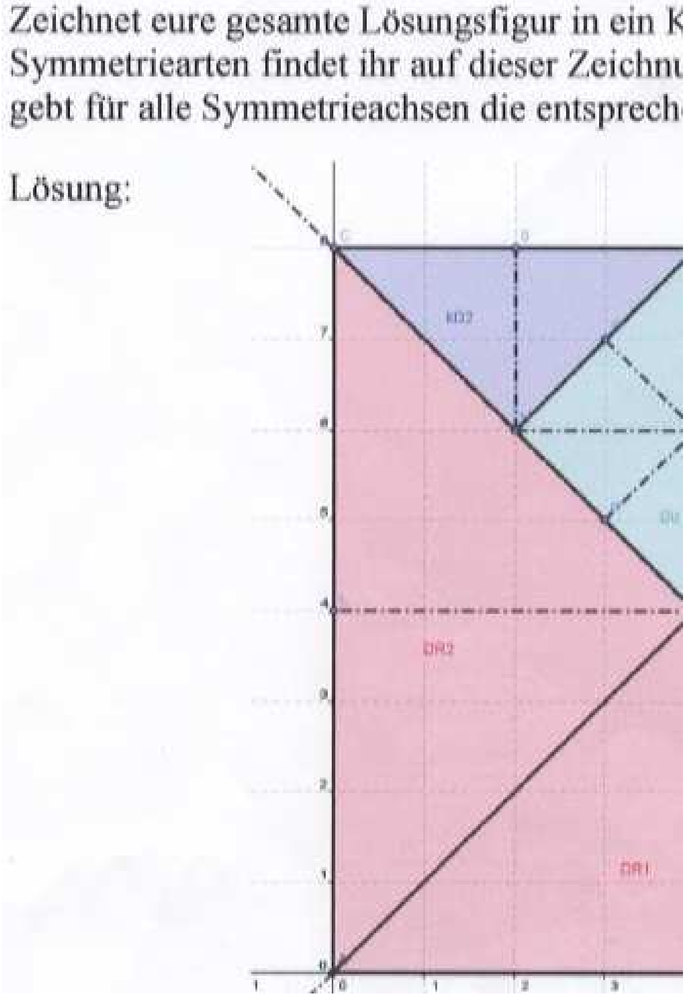


Image 1

Exercises B: „Cubes“ In the picture you see cube-formed tangram-games. The stones are made of tin and are hollow inside. How many



do you need for a cube-formed tangram-game if you use the dimensions of the wood tangram?

The axis of symmetry is not only to be sketched, but also described with function equations. The main exercise here is to discover the that connect the parts. In this way terms from different themes and exercises appear together meaningfully in one and the same sentence. With minimal change to the mathematical references as well as to the grammatical structure of the initial exercise a theme appears, which can serve as a bridge between the taught units of symmetry and linear functions (Vollrath 2001).

The next exercise (see image 3) was developed by two pupil. With an introductory text the reader is placed in a room in a Hollywood world. The text also contains the most important dimensions of the room. The following tasks are related to the tangram. It is asked how many tangram squares fit onto the surface and if spaces are thereby left uncovered. Also it is asked of how many squares are laid after 330 secs. While solving the problem, the pupil tried to use different means of presentation. They presented their calculations in graphs and tables.

Zeichnet eure gesamte Lösungsfigur in ein Koordinatesystem. Welche Symmetriearten findet ihr auf dieser Zeichnung? Nennt die Symmetriearten und gebt für alle Symmetrieachsen die entsprechende Funktionsgleichung an.

Lösung:

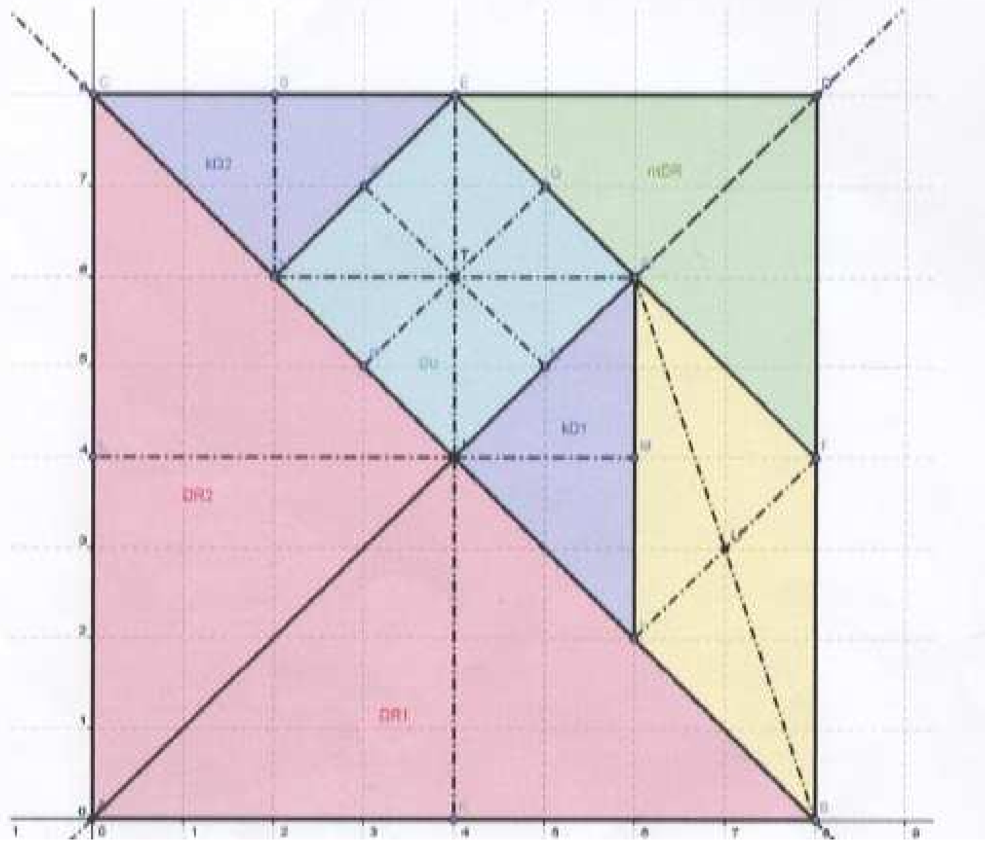
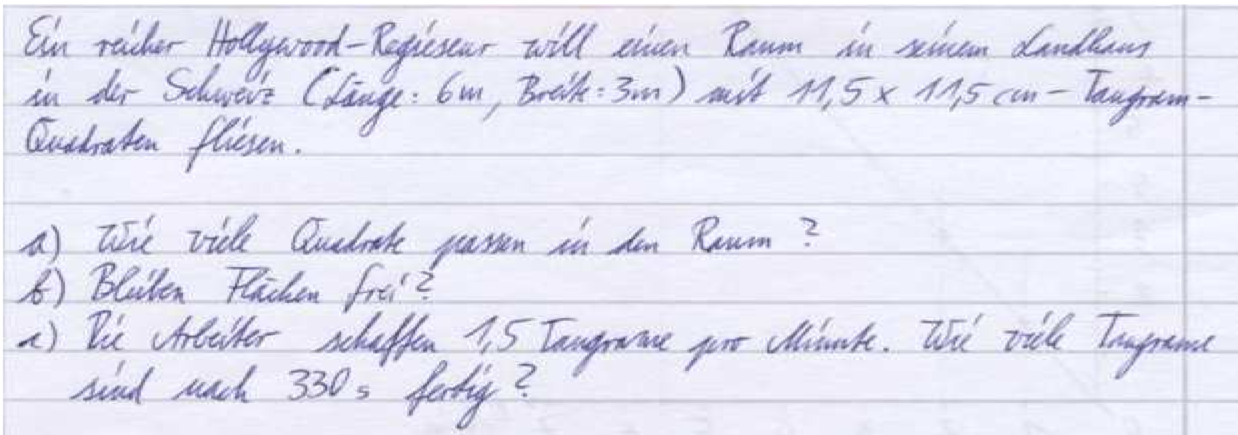


Image 2



Ein reicher Hollywood-Regisseur will einen Raum in seinem Landhaus in der Schweiz (Länge: 6m, Breite: 3m) mit 11,5 x 11,5 cm – Tangram-Quadraten fliesen.

a) Wie viele Quadrate passen in den Raum?
b) Bleiben Flächen frei?
c) Die Arbeiter schaffen 1,5 Tangrams pro Minute. Wie viele Tangrams sind nach 330 s fertig?

Image 3

With this exercise it became obvious that it is not simple to develop an authentic exercise with reference to reality while aiming to combine as much content (here proportional functions and area calculation) as possible together. The connection is extra-mathematical and makes reference to a concrete situation, which is supposed to be modeled through mathematical means.

However, if one looks at the challenge which pupil face with the development of networking exercises in the chapter-spanning review, one notices that the real problem for the pupil is not the modeling of an extra-mathematical situation. The real problem is in presenting mathematical content, which is represented by the headings in the textbook, in context. Consequently, one can term the exercise at hand as an „inverse modeling exercise". Such exercises are known in the teaching methodology as „pseudo-authentic"-exercises. The neologism by the author „inverse modeling" bases itself on the negative coloration of the term „pseudo" and denotes here an exercise, in which mathematics is consciously translated into the extra-mathematical in order to shed light on mathematical content. By doing this, reality is not modeled through mathematics, rather mathematics is modeled through reality with the goal of networking mathematical issues in the perception of those thinking and learning (compare Jahnke 2001). The question of reality or otherwiseof the Hollywood context appears in a different light with this background.

The diversity and quality of the resulting exercises is reason enough to show that the class time used for the testing of the method was effectively used. The time needed for correcting and feedback could nevertheless be seen by teachers as a problem. It is possible to extend the method over six class sessions in order to shift the correcting and feedback out of the preparation time and into the class time. The correcting can thus be divided up amongst the pupil. The feedback is, as a result, also much quicker.

4. Conclusion

"Mathematics as an ideal practice field for heuristic and analytic thinking that seizes up everyday life and talks it up in a specific way" (Winter, Baptist 2001) is seen as the „basic experience" in math lessons. The complexity of the requirements increases if various mathematical categories need to be used in order to solve a problem. As shown above, at least a share of pupil can, through preparation in expert training, be placed in a position to network mathematical content both inner- and extra-mathematically. There is reason to accept that the self-produced exercises are more appealing to the rest of the class than complex exercises out of a textbook. Thus pupil are at least given the experience of mathematics as an exercise field for heuristic thinking while solving such exercises. It can be assumed that an elaboration of the same and the ascertainment of the suitable method for further school grade will contribute to the issue of the networking of mathematics in the lower secondary.

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Students' knowledge of Application of Mathematics – From Diagnostics to Innovations

Reinhard Oldenburg
Professor of Didactics of Mathematics and Computer Science
Goethe University, Frankfurt, Germany, oldenbur@math.uni-frankfurt.de

Abstract

The results of a questionnaire that should reveal students' knowledge about the use of computers in mathematics and the relevance of applications of mathematics in our society clearly show that current math teaching does not provide adequate ideas about the importance of computers. We describe the results and give examples of mathematical activities that are suitable to both foster mathematical concepts and widen the mathematical view. Possible changes in the curriculum are discussed.

Introduction

Computers are no longer a new ingredient in math education. The majority of German students have at least some experience with learning mathematics with computers in the classroom. However, the impact of computers on our society is much broader than just as a tool for learning. It is a tool for calculating things that would not (or not so easily) be computable without and this turns more and more disciplines into computing disciplines. Do students in school learn anything about this?

The questionnaire study, its Interpretation and fundamental ideas

In joint work with Markus Vogel (University of Education, Heidelberg) we investigated computer related knowledge of teacher students. The students were in the second half of their second year.

Most of them (86%) reported that computers were used in math lessons back when they were in school themselves. The software used was mainly spreadsheets (81%) but also computer algebra (27%) and geometry systems.

When asked for their attitudes and believes about computers the results were as might be expected after this: Almost all of them said that computers should be used and math education and most of them (68%) were sure that computers have a positive influence on the students' motivation.

Given this, the answers to the question „Give examples were computers are useful for mathematics in general or for applications of mathematics" were a bit surprising: 16% gave no answer and 19% mentioned doing fast calculations, but without examples. In the next lesson I took the opportunity to ask the students what calculations they had in mind. The only answer was that "fast computers have the advantage that the addition is done quickly when buying a lot of products in the super market." Well, it is an interesting exercise to figure out if spending a billion dollar at a super market makes up a calculation that takes more than 1 second on a modern computer. But to be serious again, this answer shows that even students who have seen the use of computers in math lessons have no idea why computers boost the influence of mathematics in the tremendous way it does. Another 11% of the students mentioned that computers can be used to calculate complex formula, but again they had no idea about what these formulas could be or where they might occur.

Similarly disappointing were the answers to the question "Did you ever feel that you had a gain by doing math on the computers.": 65% no, 8% yes, during a course on numerical analysis, 8% yes, in the geometry lectures.

Summing up these findings one may wonder if it is inconsistent that so many students had used math software in math lessons and nevertheless had no real idea about why math+computers are such a powerful combination. However, there is an easy explanation: Most of the time computers are used in math lessons, their purpose is to support mathematical activities that were invented before the use of computers. A typical example is given by dynamic geometry systems applied to the problem to find a line tangent to two circles. This is a problem that was historically solved by ruler and compass construction. There are other ways to solve this problem, e.g. calculating. Calculating was made very easy by computers and thus solving the problem by calculation can be considered to be the natural way to use a computer on this problem. However, in math lessons, we hide these calculations deep inside the geometry program and work on the level of synthetic geometry. Much of computer use simply has the aim to illustrate ideas and techniques that were adequate in the pre-computer era. This view is supported by the data from the survey: 85% suggested that computers are useful for mathematics because they allow to visualize graphs or (to a lesser extent) other mathematical objects.

To prevent misunderstanding: I do not say that using dynamic geometry systems or using a computer to visualize graphs is a bad idea. Not at all! We all know how much beautiful mathematics can be done

that way and that many students appreciate it. The point is that this is important, but it is not the whole story. If we restrict computer use to that kind of activity, students are actively guided to a problematic view about the relationship between mathematics and computers.
What is needed is a modification of the curriculum that gives students insight into the modern use of computers as specific mathematical tool. Of course, the subject is extremely broad ranging from cryptography to weather forecast. Therefore one should identify some fundamental ideas of computer based mathematics (CBM). They may mediate in some sense between fundamental ideas of mathematics and of computer science. Here is an ad hoc collected list:

- Discretization with linearization: Especially in time based simulations, the continuum of time (and eventually space) is divided in small pieces so that change between them can be viewed to be linear. Governed by this idea are (among others) solution to differential equations and numerical integration.
- Search guided by cost function
- Symbolizing: Problems can often be reduced by casting them into a symbolic form.
- Simulation can compensate for a lack of theory

Given the broad field of applications this is a surprisingly short list. In fact many methods rest on common principles. This is good news for the attempts to bring these ideas to school.

Examples of Tasks that promote the fundamental ideas of CBM

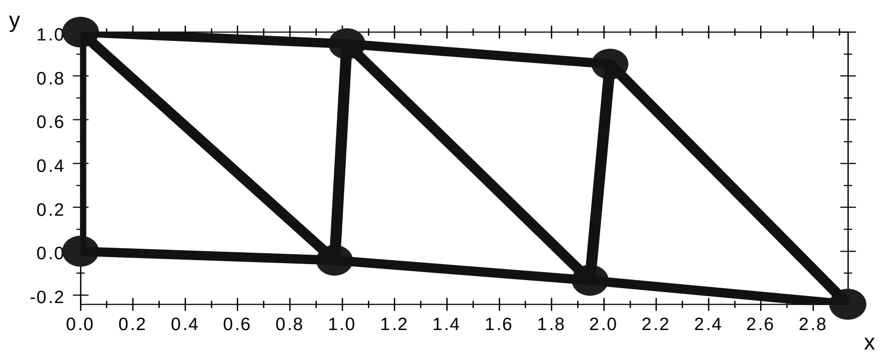
The following examples are selected in way that prototypically shows how the fundamental ideas of CBM mentioned above can be brought to the classroom.

Optimization

In school mathematics finding extremal values is linked very closely to calculus. I don't want to argue that this link is unimportant, but I think it should be complemented by the fundamental idea of search. For example, given a function in two variables like $f(x,y)=x^2+2x+y^2+5y+xy/2+1$ it is very easy to calculate function values. And then, it is easy to look for values of x and y that yield small function values. Students doing this with f defined in a program or on a calculator so that evaluation is fast for them will almost for sure discover some search strategy how to make the function value smaller, and they will – without doubt – understand that a computer can do this search for them. After half an hour they have an idea about what numerical multi-dimensional minimization is and then they can use it to model situations by defining functions to be minimized. The technological basis can either be a computer algebra system (e.g. using the lbfgs package in the free Maxima system), a programming language with minimization code (e.g. Python with the Scipy library (both free as well)) or a spreadsheet like Excel. Excel contains the solver utility, that allows to minimize the value of calculated cell very easily. One simply has to select the cells that may be changed and the cell to be minimized.
With this technology at hand students can answer questions like this:

What parameter values in a function should be used to fit given data. For example, students can fit a circle $(x-x_0)^2+(y-y_0)^2=r^2$ to a number of points by minimizing the sum of defect squares.
What is the shape of a fast slide between two points (the brachistochrone problem)?

Fig. 1: A crane bended by a heavy load



What configuration will a set of springs take at rest when a given force is applied. A real world application is the deformation of a crane under a load (see Fig. 1).

The brachistochrone example can be solved in a computer algebra system by the following small code. The slide is approximated by a polygon line with 50 points:

```
n:=50                               // Number of points
ya:=0.0                             // Start in point (0,ya)
xe:=3; ye:=-1                       // End in point (xe,ye)
v:= y ->  sqrt(-2*9.81*y)           // Velocity in height y
ttime:=sum( 1/v(0.5*(y[i]+y[i+1]))*
             sqrt((y[i]-y[i+1])^2+(xe/n)^2),  i=0..n-1)
// ttime=total time= Sum over all times in the intervals t= 1/v * s
f:=subs(ttime, y[0]=ya, y[n]=ye) // incorporate end point values
ys:=NEWTON(term,[y[1],…,y[n]],[-1,…,-1])  // find optimum
```

The result plotted as line is shown in Fig. 2.

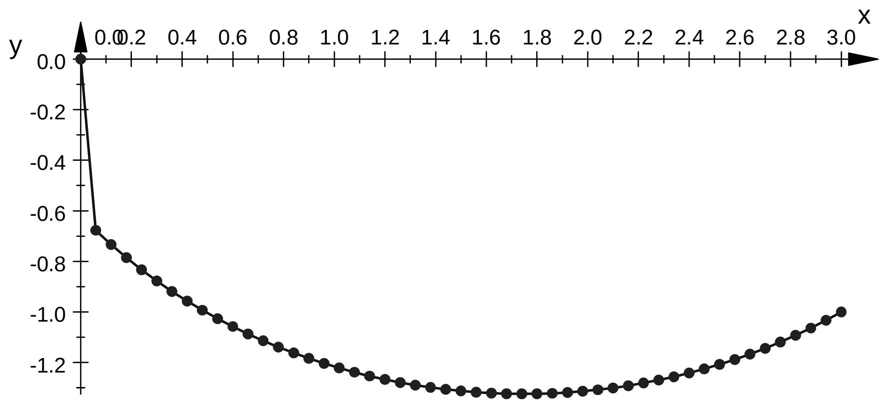


Fig 2: A numerical approximation of a brachistochrone

A slight generalization is constraint optimization. With this technique, only a subspace that is defined by a set of equations is searched for the minimum. A simple example is that of hanging chain.

Again, the method of discretization is important: The chain is approximated by points $(x_i, y_i), i = 0..n$. The quantity to minimize is its energy which is the sum of the energy of its segments and the energy of each segment is proportional to its length (which is in turn proportional to its mass) and its middle height: $E = \sum_{i=0}^{n-1} \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} \cdot \frac{y_i + y_{i+1}}{2}$

A constraint results because the chain has a fixed length: $L = \sum_{i=0}^{n-1} \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2}$.

Either one uses a specializes optimization algorithm for constrained problems or one adds a penalty term to the objective function (with μ a “large” number, say 100) and minimizes the following function:

$$\tilde{f} = \sum_{i=0}^{n-1} \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} \cdot \frac{y_i + y_{i+1}}{2} + \mu \cdot \left(L - \sum_{i=0}^{n-1} \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} \right)^2$$

The result, as shown in Fig.3 clearly shows that a hanging chain is not a parabola.

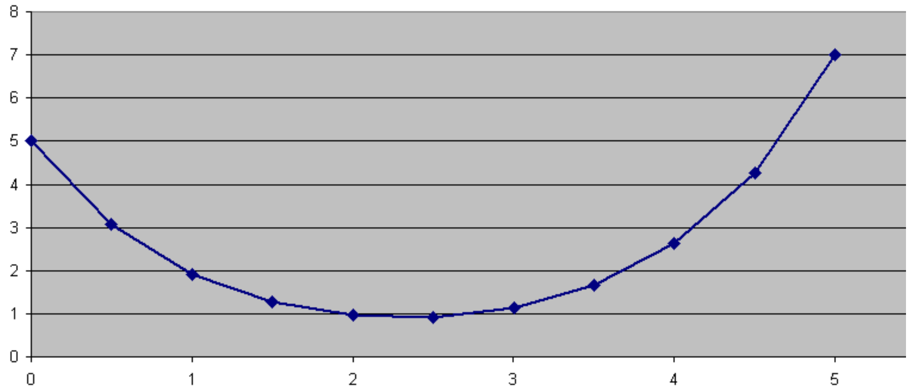


Fig 3: A hanging chain

Summing up, we see that working with optimization algorithms gives students opportunities to work with expressions, functions, equations, to model real world situations, to apply the fundamental ideas of applied mathematics and to see how computers' calculation power can be used to solve nontrivial problems.

Simulating Heat propagation with a spreadsheet

In this section we shift from optimization to differential equations but in a form that takes out most of the analytic obstacles.

A simple experiment shows that the temperature of some hot object decreases and its temperature approaches the temperature of the environment. Data can easily be collected and entered into a spreadsheet. While Temperature is a function of the continuous independent variable time t, measurement introduces a discretization and this can be carried over to the model. Between points in time t_{i+1} and t_i the temperature decreases: $T_{i+1} = T_i - \text{cooling down}$. The cooling down will be large if the time step is large and if the difference to the temperature of the environment T_E is large: $T_{i+1} = T_i - \Delta t \cdot (T_i - T_E) \cdot c$

The parameter is chosen so that the calculated temperature fits the experimental data well.

Now we shift interest from one body to two connected bodies, one hot, the other cool. Similar consideration as above show that now the temperature difference to the neighbor is relevant: $T_{\text{neq}} = T_{old} + (T_{\text{neighbor}} - T_{\text{old}}) \cdot k$. Again, this can be calculated easily in a spreadsheet or any programming language. And it opens the possibility to look at a really nontrivial example: The heat propagation in a rod, e.g. the metal of a pan standing on a hot oven.

Now space is discretized, i.e. the rod is made up of cells that interchange energy with its neighbors:

T1	T2	T3	...	Tn

The change of energy of the *i*-th segment is: $\Delta E_i = k \cdot (T_{i+1} - T_i) \cdot \Delta t + k \cdot (T_{i-1} - T_i) \cdot \Delta t$.

The development in time is described by a second index *j*. As above: $T_{i,j+1} = T_{i,j} + \frac{1}{C} \cdot \Delta E_i$. Putting this together and setting $c = \frac{k}{C} \cdot \Delta t$ yields $T_{i,j+1} - T_{i,j} = c \cdot \left(T_{i+1,j} - 2T_{i,j} + T_{i-1,j}\right)$ which is easy to calculate. At time j=0 all initial temperatures T_{i0} must be set and at the left and right border the temperature must be pre-described for all time steps (e.g. a hot and a cold end of the rod). Then all middle values can be calculated uniquely.

This example shows that with rather basic calculations a nontrivial example of a process in space-time can be calculated. This provides students with a prototypical idea about how similar processes can be calculated, an example being weather forecast. It is obvious, that a weather model must include much more details (three-dimensional space, not only temperature is important, not only heat transfer but also convection, etc.) but nevertheless the fundamental ideas of CBM are the same.

Digital Image processing

Digital images are encoded as a matrix of pixels. For gray scale images each pixel is characterized by a single number, its brightness. Thus a gray scale digital image is exactly the same as a matrix in mathematics. One may apply a function to each entry to increase or decrease brightness, contrast or

even to turn an image to its negative. This involves only easy algebra. However, usually one has to write programs to do this. To eliminate this difficulty I wrote some Java applets that are accessible on my webpage (http://www.math.uni-frankfurtde/~oldenbur (only in German)) overcome this problem. Students can perform various operations by specifying the mathematics transformation functions. Examples are shown below in Fig. 4 and 5.
Studying these image processing algorithms introduces students to computer based applications of mathematics that brings out all the fundamental ideas described above.

Fig 4: An Applet to transform the brightness of pixels according to a function

Fig 5: Example of a calculated local displacement

Conclusion

The examples presented here can only give a rough idea about what can be done in math education in high school when the computational power of today’s computers is utilized as a problem solving tool. Almost all applications can be boiled down to simple forms that illustrate the principles and fundamental ideas of CBM. We believe that this should give students better insight into the role of math and computers in our modern society and we expect that this influence could be demonstrated empirically by observing a change in the students’ mathematical belief systems. Although this sketches a plan for future research, it should be pointed out that the topics given here and some others have already been taught successfully in high school.

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From Physical Model To Proof For Understanding Via DGS: Interplay Among Environments

Iman M. Osta, Doctorat in Didactics of Math
Associate Professor of Math Education, Department of Education, Lebanese American University, Beirut, Lebanon Iman.osta@lau.edu.lb

Abstract

The widespread use of Dynamic Geometry Software (DGS) is raising many interesting questions and discussions as to the necessity, usefulness and meaning of proof in school mathematics. With these questions in mind, a didactical sequence on the topic “Conics” was developed in a teacher education course tailored for pre-service secondary math methods course. The idea of the didactical sequence is to introduce “Conics” using a concrete manipulative approach (paper folding) then an explorative DGS-based construction activity embedding the need for a proof. For that purpose, the DGS software serves as an intermediary tool, used to bridge the gap between the physical model and the formal symbolic system of proof. The paper will present an analysis of participants’ geometric thinking strategies, featuring proof as an embedded process in geometric construction situations.

Introduction

Mathematical proof has been a focus of reflection throughout the various stages that mathematics education has undergone. With each new wave of school mathematics the place, status, importance and even format of proof were subject to debates (Hanna, 2000; Hanna & Jahnke, 1996). For many years now, the widespread use of Dynamic Geometry Software (DGS) has been raising many new interesting questions. Laborde (2000) discussed major concerns identified in the literature, which she condensed in one disturbing question: “Is proof activity in danger with the use of dynamic geometry systems?” An extensive body of discussions and debates is found as to the necessity, usefulness, meaning, and types of proof in school mathematics (e.g. Arzarello, Micheletti, 1998; Christou, Mousoulides, Pittalis & Pitta-Pantazi, 2004; Hoyles & Healy, 1999; Leung & Lopez-Real, 2002), particularly when it is so easy to move elements of a geometric figure and observe many examples that support a certain conjecture. Pandiscio (2002) reports that pre-service teachers expressed their concern that with the use of DGS, students might believe that formal proofs are unnecessary; although they still believed that a formal proof is different from a proof by many examples. Others contend, on the other hand, that DGS can be used to help students see the need for deductive reasoning (Wares, 2004).

Since most of the frames of thought agree to the fact that proof is at the heart of mathematics (Knuth, 2002), the literature in math education is witnessing, since the 1990s (Hoyles & Healy, 1999, Jones, 2000; Laborde, 2000; Mariotti, 2000), numerous attempts to investigate and engineer teaching strategies and learning situations whereby DGS is used for enhancing proving abilities. “Quasi-empirical investigations” (Connor & Moss, 2007; de Villiers, 2004) are acquiring more and more importance, highlighting functions of proof that were traditionally undermined. Examples of such functions are explanation, understanding, insight, validation and discovery. These non-deductive methods of investigation, which rely on experimental, intuitive and inductive reasoning (de Villiers, 2004), are seen to provide more meaningful contexts for teaching-learning geometry with DGS than the classical approach of proof as a way of obtaining certainty. According to de Villiers (2007), the latter approach “stems largely from a narrow formalist view that the only function of proof is the verification of the correctness of mathematical statements”.

The present paper was motivated by the following assumptions: a) the development and widespread of DGS use are changing the way geometry is taught and learned, b) the tools available to geometry teachers and students in the classroom affect the nature of geometry in school math curricula as well as its teaching approaches, and c) learning does not happen automatically when students use DGS. Tasks should be reflectively designed to incite knowledge construction (Laborde, 2001). This paper reports a study attempting to contribute to the discussions about proof in Dynamic Geometry Environments (DGEs) and to suggest a type of situations whereby proof emerges naturally to fulfill a need felt by the learner, instead of being required by an external authority (the teacher or the curriculum). The experimented problem-situation would raise a genuine need for proof as a way to foster understanding of unforeseen mathematical relationships. The idea of the situation is based on a belief that in a DGE, construction tasks are full-fledged problem solving situations in which instances of proof are needed while the construction process is taking place. A special feature of the suggested situation is that proving is not explicitly required per se, but the need for a proof emerges naturally while participants try to find a point with specific properties.

Context

Not long ago, geometry in the Lebanese curriculum was for a long time taught in an abstract way. Most geometric objects were introduced with formal definitions followed by properties and theorems generated deductively. Very little attention was given to intuition, perception, or figure construction. Particularly the topic *Conics* was taught with an algebraic approach, with almost no geometrical connection at the beginning. The very first contact that students have with the topic is made through the following definition at the beginning of the chapter in the math textbook:

> A fixed straight line (D) and a fixed point F being given in a plane, we call *conic* of *focus* F and *directrix* (D) the set of the points M of the plane, the ratio of whose distances from F and from (D) is equal to a given positive number e. The number e is called the *eccentricity* of the conic.

Beside the complexity of language and unfamiliar phrases used in this definition (e.g. the ratio of whose distances from F and from (D)), four new terms (conic, focus, directrix and eccentricity) are introduced for the first time. No visual representation of parabola, ellipse or hyperbola is shown until the fifth page of the chapter. After the definition, the ratio relationship is quickly transformed into a complex algebraic equation involving two variables x and y, coordinates of point M in a presumed coordinate system, and a parameter e named eccentricity.

Once the three types of conics are distinguished (by name and not by shape) depending on the different values of eccentricity, their algebraic equations get treated separately, as three independent entities, with no geometric connection between them. The geometrical definitions and properties of the three types of conics are only presented at a later stage, after the long algebraic work.

The study

A didactical sequence on the topic "Conics" was developed in a teacher education course tailored for pre-service secondary math methods course. Many of the teachers participating in this course have learned geometry under the old curriculum, in an abstract and formal way. The sequence is designed to put them in a learning situation fostering metacognitive dialogues. It is composed of several problem-situations, connected in a way to gradually offer new tools to promote learning. In the limited scope of this paper, I will only present two phases of the sequence:

Phase 1: Producing conics through paper folding (See Fig.1)

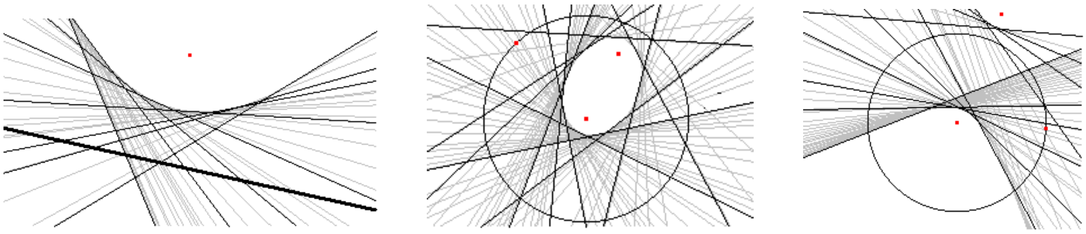
Task1: Having, on wax paper, a straight line and a point not on the line, make many folds of the paper, by overlapping the point with random points of the line. Describe the result.

Task2: Having, on wax paper, a circle and a point inside the circle, make many folds of the paper, by overlapping the point with random points of the circle. Describe the result.

Task3: Having, on wax paper, a circle and a point outside the circle, make many folds of the paper, by overlapping the point with arbitrary points of the circle. Describe the result.

Phase 2: DGS model of the paper folding tasks

Construct using Cabri-Géomètre a model of the paper folding activity. In this DGS model, the three conics are just visually perceived (envelopes of the sets of straight lines) but are not actually constructed (drawn). The task is to physically construct the conics by finding, constructing and tracing the point of tangency between the presumed conic and the moving straight line.



Outcome of Task1 Outcome of Task2 Outcome of Task3

Fig.1. The outcomes of the three paper folding tasks

Method

Data from participants' work were collected over six semesters (an average of six student teachers enrolled every semester). The aim was to investigate participants' thinking and proving strategies and to identify what they consider to be an acceptable, sufficient, plausible proof. Following are the data collection techniques:

- Observation: Observation notes were taken during participants' work.
- Audio taping: Working in pairs or in groups of three, the participants were encouraged to discuss their reasoning and think aloud about their strategies. These dialogues were audio-recorded, transcribed and analyzed.
- Record of computer files: Participants' work on the computer was saved every few minutes in different consecutive Cabri files, which allowed a follow-up of the construction and proving attempts, through a follow-up of their figure constructions and manipulations.

Some results

Within the scope of this paper I will present some of the global reflections and conclusions that the empirical data raised. The above problem situations evolved into an alternative use of proof and path to it, in a dynamic geometry environment. While DGS is most commonly used for exploring geometric figures, formulating conjectures, verifying conjectures or properties, the present problem situations propose a different context. DGS's function here is not to create or to confirm a conviction about a geometric property or relationship. Such conviction was already created by a different artifact, the paper folding model. It is this latter model which has lead to the conviction that, by considering the family of perpendicular bisectors of a moving segment, a conic (or a silhouette of it) is created. This compelling observation raised the need for a proof that would explain "why" the three forms are generated.

The status of the proof in this situation is rather explanatory, aiming at connecting the concrete model to the abstract properties of the geometrical figures and at understanding "why", by the same paper folding process on three figures, one can generate those three geometric objects that participants studied previously as geometrically separate and different. Without being asked, participants started looking for an explanation, trying to connect the concrete model to what they knew about the more formal properties of the geometric objects involved: the parabola as the set of points equidistant from a point and a straight line, the ellipse as the set of points whose sum of distances from two fixed points is constant, and the hyperbola as the set of points whose difference of distances from two fixed points is constant.

As the connection turned to be a too complex task, the second problem was proposed, whereby Cabri is used to bridge the gap between the physical model and the formal symbolic system of geometric proof. The function of DGS here is to explore and facilitate the explanation of an uprising, obviously valid fact, rather than to verify or validate a less compelling observed one. Thus the situation shifted from the more common use: *drag to find a pattern, state a conjecture and verify it*, to: *we know the result to be true from concrete experimental investigation. Let us try* *to explain why it is true in terms of other well-known geometric properties; in other words, how it is a logical consequence of these other properties.*

The record of observations revealed the progress toward deductive proof through the processes of interplay between three models of the same problem: At a first stage, between the paper folding model and the DGS model, and at a later stage between DGS and paper-and-pencil model, in a proof-for-understanding situation. An interesting phenomenon was observed by which, after hasty and random manipulations of the DGS geometric figures, experimentation gradually became more rational. Participants started thinking about the possibility of a point to be the required one before moving it to check its trace. They tried to deductively check the validity of the constant sum property before venturing to move the point. Most groups resorted to sketching the figure or parts of it on paper while looking for a point with the required property. For most of the groups, the final solution of the problem was achieved when they worked on paper, after extensive explorations with the DGS figure.

In the case of the ellipse, for example, a group of participants started dragging and tracing the perpendicular bisector in an attempt to visualize the possible position of the point of tangency. For them, point I seemed to be the solution (see Fig.2).

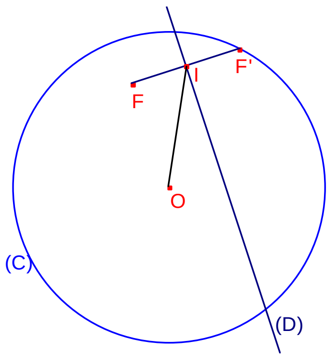


Fig. 2. Attempt to prove that IO + IF constant

Despite the fact that dragging and tracing the point did not yield an ellipse, participants still tried to deductively investigate whether it satisfies the constant sum property. "*No, it doesn't work… We want to find something constant… a constant sum… I think it should be the radius. Can we prove that IO+IF=R?*"

This reflects interplay between perceptive, empirical and deductive evidence, in the context of DGS. The empirical evidence was undeniable, point I does not generate the ellipse. But the perceptive impression, and the way the three points F, I and O are relatively positioned, incited the participants to look for a formal relationship that would make point I the searched point, despite empirical evidence.

This is an example that goes against common cases where learners don't see the relevance of verbal deductive proofs because perceptive and empirical evidence is to them enough of a conviction. Participants were convinced that if they succeed to deductively prove that point I satisfies the constant sum property, they would have a more valid reason than empirical evidence to say that I is the point of tangency.

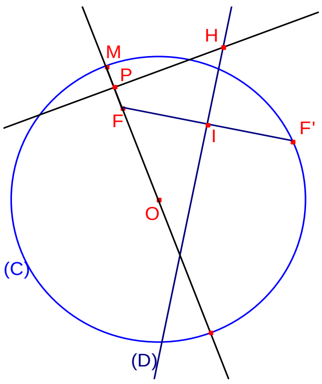


Fig.3. Many points were dragged

Then they tried to select several "plausibly" selected points and trace them to check if they produce the ellipse. Those selected points are mainly points of intersection between significant objects in the figure: I, midpoint of FF', P, midpoint of FM, M, intersection of OF with the circle, H, intersection of the respective perpendicular bisectors of FF' and FM (see Fig.3), and other points created by the participants through joining points, extending segments, constructing other perpendicular bisectors, then considering intersections.

Throughout the above exploratory stage, interesting instances of attempting deductive proof started to appear, then to take more and more place in the process, as the figure became more complex and the exploration more tedious.

Conclusion

The dynamic geometry tool provided, in the analyzed situation, a mid-way representation of the problem, between the physical model, which provided the convincing evidence, and the more abstract deductive thinking (proof). Insight and progress toward deductive proof (finally conducted on a figure sketched on paper and not on the DGS figure) were fostered by two roles of the DGS software:

- An active positive role, by which manipulation of the figure leads to better understanding the geometric relationships
- A passive negative role, by which the DGS figures act as an obstacle that learners should overcome by resorting to deductive proof. Indeed, creating new points and objects, moving basic elements, “messing-up” with the figure, make it so complex, “fluid” and “evasive” that learners would need to sketch what they consider relevant parts of the figure on a solid support, on paper.

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Using ClassPad-technology in the education of students of electrical engineering (Fourier- and Laplace-Transformation)

Ludwig Paditz, University of Applied Sciences Dresden, Germany
paditz@informatik.htw-dresden.de

Abstract:
By the help of several examples the interactive work with the ClassPad330 is considered.
The student can solve difficult exercises of practical applications step by step using the symbolic calculation and the graphic possibilities of the calculator. Sometimes several fields of mathematics are combined to solve a problem.
Let us consider the ClassPad330 (with the actual operating system OS 03.03) and discuss on some new exercises in analysis, e.g. solving a linear differential equation by the help of the **Laplace transformation** and using the **inverse Laplace transformation** or considering the Fourier transformation in discrete time (the **Fast Fourier Transformation FFT** and the **inverse FFT**). We use the **FFT**- and **IFFT**-function to study periodic signals, if we only have a sequence generated by sampling the time signal.
We know several ways to get a solution. The techniques for studying practical applications fall into the following three categories: ***analytic***, ***graphic*** and ***numeric***. We can use the Classpad software in the handheld or in the PC (ClassPad emulator version of the handheld).

References:

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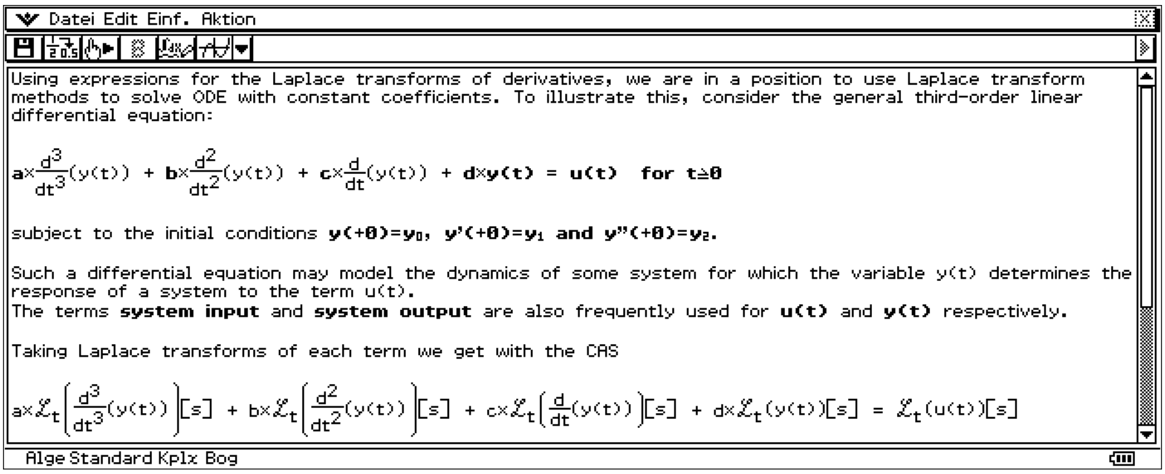
Example of solving a linear ODE with initial condition, several ways of solution:



The last input line yields in the CAS software:

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$$a\cdot\int_0^\infty \frac{d^3}{dt^3}(y(t))\cdot e^{-s\cdot t}dt+b\cdot\int_0^\infty \frac{d^2}{dt^2}(y(t))\cdot e^{-s\cdot t}dt+c\cdot\int_0^\infty \frac{d}{dt}(y(t))\cdot e^{-s\cdot t}dt+d\cdot\int_0^\infty e^{-s\cdot t}\cdot y(t)dt=\int_0^\infty e^{-s\cdot t}\cdot u(t)dt$$

Using the **laplace**-function of the ClassPad we get the complete transformation:

laplace(a×y''' + b×y'' + c×y' + d×y,t,y,s)

$$-a\cdot\left(s^2\cdot y(0)+s\cdot y'(0)+y''(0)-Lp\cdot s^3\right)-b\cdot\left(s\cdot y(0)+y'(0)-Lp\cdot s^2\right)-c\cdot(y(0)-Lp\cdot s)+Lp\cdot d=0$$

Thus the full transformation of the equation is

$$-a\cdot\left(s^2\cdot y(0)+s\cdot y'(0)+y''(0)-Lp\cdot s^3\right)-b\cdot\left(s\cdot y(0)+y'(0)-Lp\cdot s^2\right)-c\cdot(y(0)-Lp\cdot s)+Lp\cdot d=\int_0^\infty e^{-s\cdot t}\cdot u(t)dt$$

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In the last line we have with **Lp** the Laplace transformation of y(t), sometimes we denote **Lp** by Y(s). Furthermore we discover the initial conditions y(0), y'(0), y''(0).
We denote the right hand side of the last equation by U(s) and solve this equation for Y(s):

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$-a\cdot\left(s^2\cdot y(0)+s\cdot y'(0)+y''(0)-Lp\cdot s^3\right)-b\cdot\left(s\cdot y(0)+y'(0)-Lp\cdot s^2\right)-c\cdot(y(0)-Lp\cdot s)+Lp\cdot d=U(s)\,|\,Lp=Y$

$$-a\cdot\left(s^2\cdot y(0)+s\cdot y'(0)+y''(0)-s^3\cdot Y\right)-b\cdot\left(s\cdot y(0)+y'(0)-s^2\cdot Y\right)-c\cdot(y(0)-s\cdot Y)+d\cdot Y=U(s)$$

solve(ans,Y)

$$\left\{Y=\frac{a\cdot s^2\cdot y(0)+a\cdot s\cdot y'(0)+b\cdot s\cdot y(0)+a\cdot y''(0)+b\cdot y'(0)+c\cdot y(0)+U(s)}{a\cdot s^3+b\cdot s^2+c\cdot s+d}\right\}$$

Using the initial conditions we get:

ans|{y(0)=y_0,y'(0)=y_1,y''(0)=y_2}

$$\left\{Y=\frac{U(s)+a\cdot s^2\cdot y_0+a\cdot s\cdot y_1+b\cdot s\cdot y_0+a\cdot y_2+b\cdot y_1+c\cdot y_0}{a\cdot s^3+b\cdot s^2+c\cdot s+d}\right\}$$

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Now we consider the following example: a = 1, b = -3, c = 3, d = -1, u(t) = cos(2t)*e^(-t) and y(0) = 0, y'(0) = 1/32, y''(0) = -3/16.

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ans|$\left\{a=1,b=-3,c=3,d=-1,y_0=0,y_1=1/32,y_2=-3/16,U(s)=\mathcal{L}_t\left(\cos(2t)\times e^{-t}\right)[s]\right\}$

$$\frac{\frac{s}{32}+\frac{s+1}{(s+1)^2+4}-\frac{9}{32}}{s^3-3\cdot s^2+3\cdot s-1}$$

expand(ans,s)

$$\frac{s+3}{32\cdot\left(s^2+2\cdot s+5\right)}-\frac{1}{32\cdot(s-1)}+\frac{1}{32\cdot(s-1)^2}$$

y(t)=invLaplace(ans,s,t)

$$y(t)=\frac{t\cdot e^t}{32}-\frac{e^t}{32}+\frac{\cos(2\cdot t)\cdot e^{-t}}{32}+\frac{\sin(2\cdot t)\cdot e^{-t}}{32}$$

$\mathcal{L}^{-1}_s\left(\frac{s+3}{32\cdot\left(s^2+2\cdot s+5\right)}\right)[t]$

$$\frac{\cos(2\cdot t)\cdot e^{-t}}{32}+\frac{\sin(2\cdot t)\cdot e^{-t}}{32}$$

$\mathcal{L}^{-1}_s\left(-\frac{1}{32\cdot(s-1)}\right)[t]$

$$\frac{-e^t}{32}$$

$\mathcal{L}^{-1}_s\left(\frac{1}{32\cdot(s-1)^2}\right)[t]$

$$\frac{t\cdot e^t}{32}$$

Alge Standard Kplx Bog

Finally we see the inverse transformation of the several summands.
The direct way of solution consists in using the dSolve-function:

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dSolve(y'''−3y''+3y'−y=cos(2t)×e^{-t},t,y)

$$\left\{y=t^2\cdot e^t\cdot \text{const}(3)+t\cdot e^t\cdot \text{const}(2)+e^t\cdot \text{const}(1)+\frac{\cos(2\cdot t)\cdot e^{-t}}{32}+\frac{\sin(2\cdot t)\cdot e^{-t}}{32}\right\}$$

Define $y(t)=t^2\cdot e^t\cdot C3+t\cdot e^t\cdot C2+e^t\cdot C1+\frac{\cos(2\cdot t)\cdot e^{-t}}{32}+\frac{\sin(2\cdot t)\cdot e^{-t}}{32}$

done

$$\left\{\begin{array}{l} y(0)=0 \\ \frac{d}{dt}(y(t))=1/32\,|\,t=0 \\ \frac{d^2}{dt^2}(y(t))=-3/16\,|\,t=0 \end{array}\right|_{C1,C2,C3}$$

$$\left\{C1=-\frac{1}{32},C2=\frac{1}{32},C3=0\right\}$$

$y(t)\,|\left\{C1=-\frac{1}{32},C2=\frac{1}{32},C3=0\right\}$

$$\frac{t\cdot e^t}{32}-\frac{e^t}{32}+\frac{\cos(2\cdot t)\cdot e^{-t}}{32}+\frac{\sin(2\cdot t)\cdot e^{-t}}{32}$$

Using the initial conditions we get in one step:

dSolve(y'''−3y''+3y'−y=cos(2t)×e^{-t},t,y,t=0,y=0,t=0,y'=1/32,t=0,y''=−3/16)

$$\left\{y=\frac{t\cdot e^t}{32}-\frac{e^t}{32}+\frac{\cos(2\cdot t)\cdot e^{-t}}{32}+\frac{\sin(2\cdot t)\cdot e^{-t}}{32}\right\}$$

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This exercise shows how to work with the CAS to support the learning process of our students. We can see step by step in the handheld what happens to solve a given problem.

Example of computing a FFT of a sequence generated by sampling the time signal:

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The Fast Fourier transform (FFT) is an efficient algorithm for computing the Discrete Fourier Transform (DFT).

Consider the 2π-periodic function
f(x)=x, 0≤x<π/2, and **f(x)=π−x**, π/2≤x<π, and **f(x)=0** for π≤x<2π.

the delayed triangular pulse

Using N=32 samples at intervals T=2π/N:

At first we define the 2π-periodic right-continuous function **f(x)=x**, $0\le x<\frac{\pi}{2}$, and **f(x)=−x+π**, $\frac{\pi}{2}\le x<2\pi$.

Define $y1(x)=\frac{\pi}{2}-\left|x-\frac{\pi}{2}-2\cdot\pi\cdot \text{intg}\left(\frac{x}{2\cdot\pi}\right)\right|$

done

By the help of the **heaviside function** we get now the delayed triangular pulse:

Define $y2(x)=y1(x)\cdot H(y1(x))$

done

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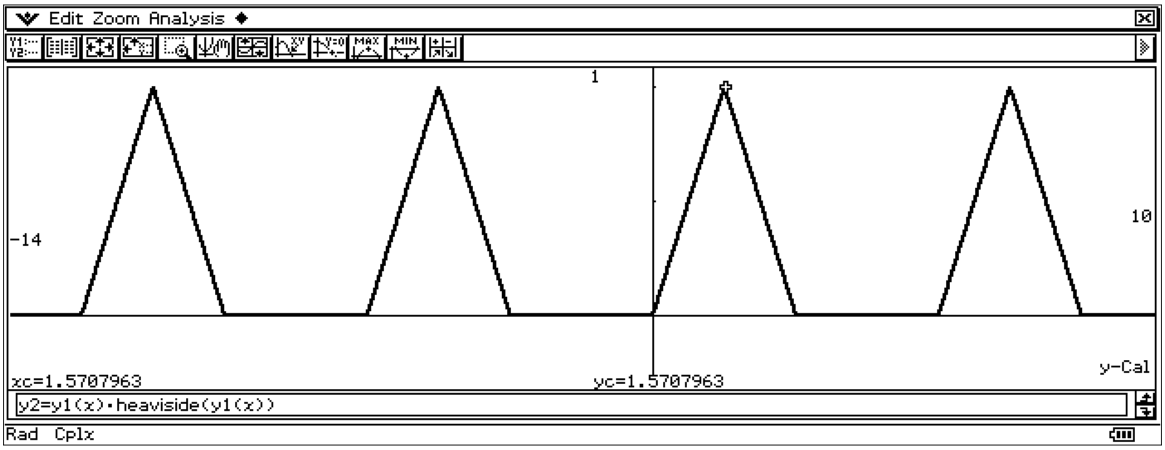
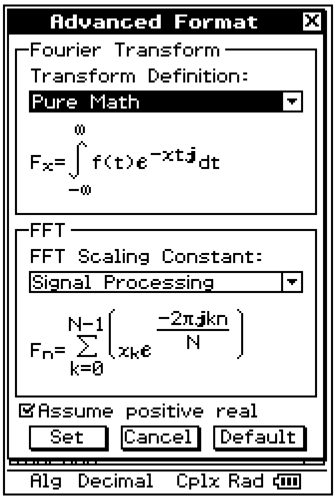
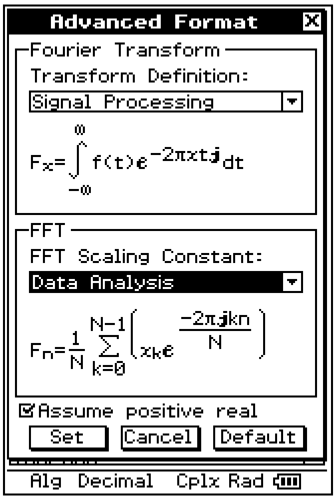


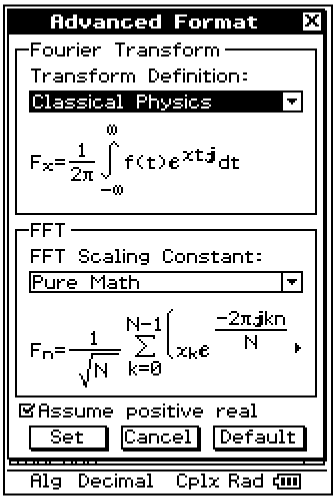
Now we have to setup the **Advanced Format** to compute the results in the wished manner:



or

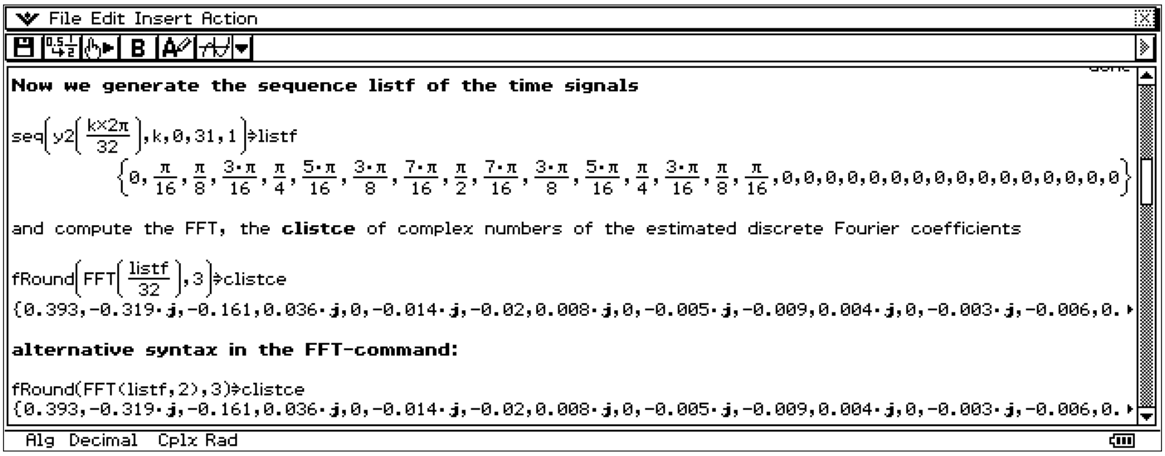


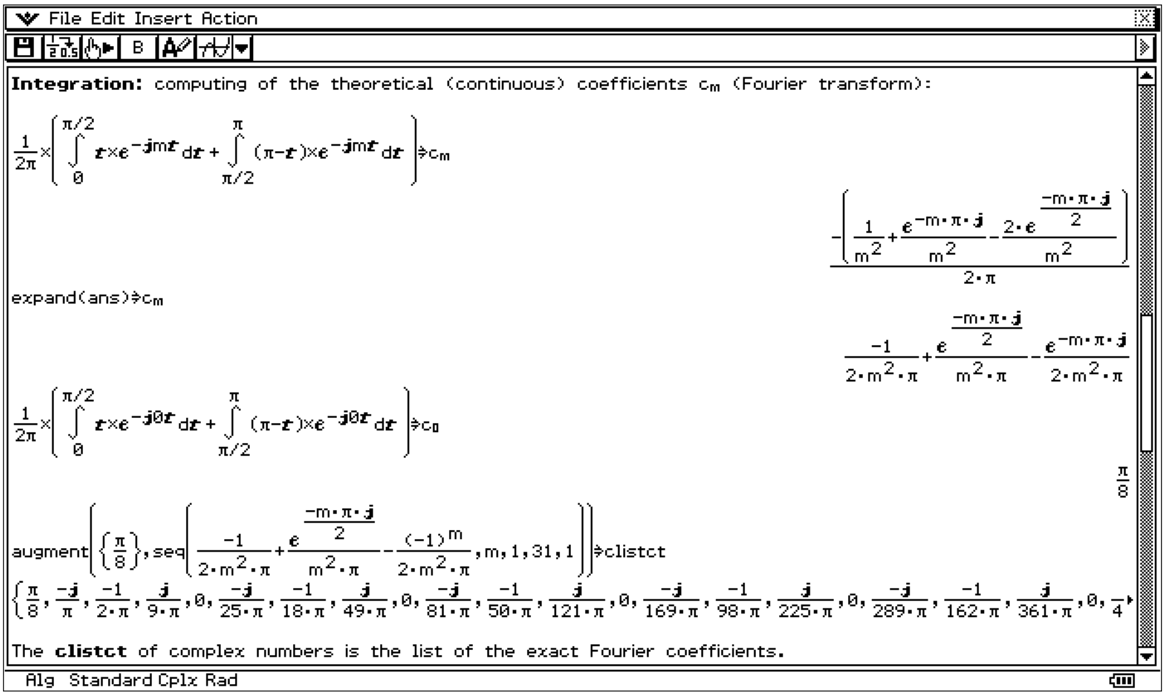
or

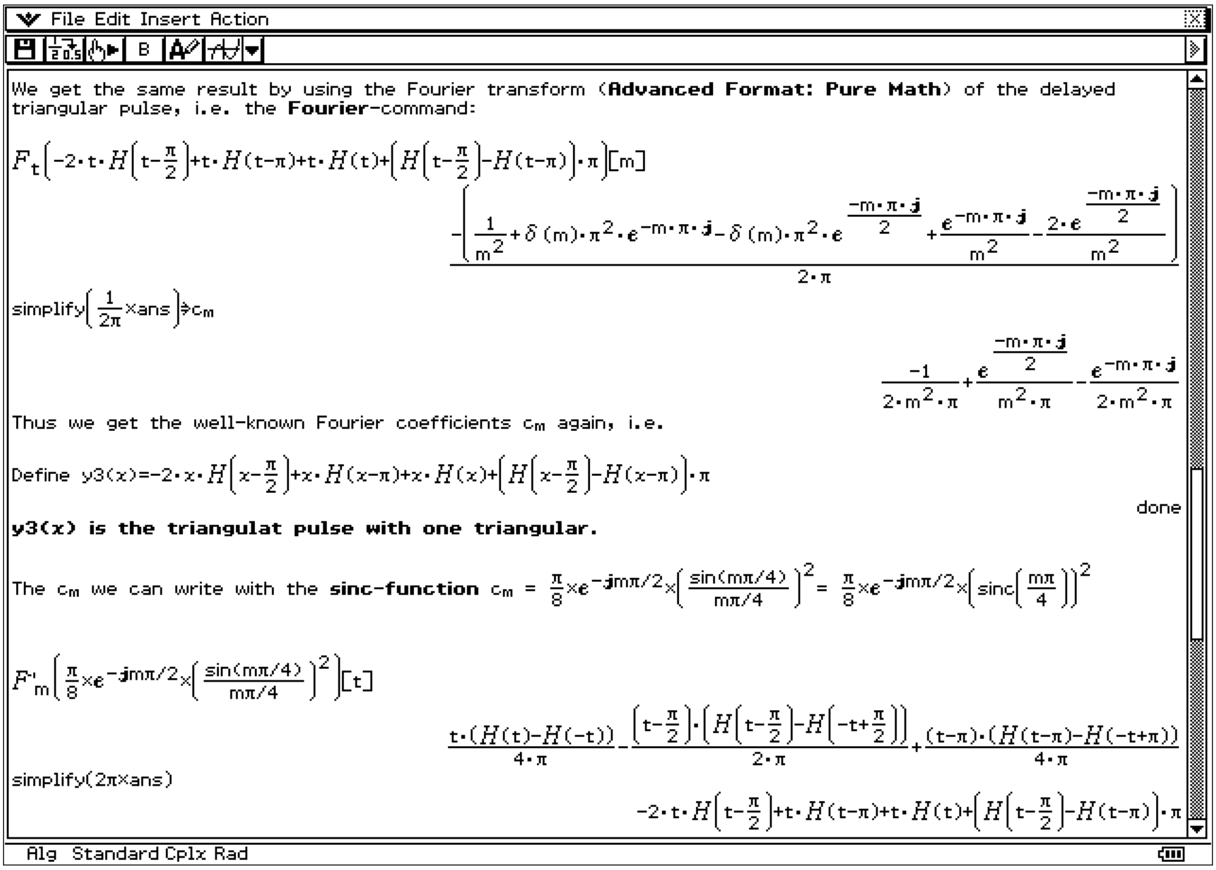


…

We use the Advanced Format for FFT with Signal Processing and divide the listf by N or we use the optional Parameter 2 (=Data Analysis) without division the listf by N:







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We get the same result by using the Fourier transform (**Advanced Format: Pure Math**) of the delayed triangular pulse, i.e. the **Fourier**-command:

$$F_t\left(-2\cdot t\cdot H\left(t-\frac{\pi}{2}\right)+t\cdot H(t-\pi)+t\cdot H(t)+\left(H\left(t-\frac{\pi}{2}\right)-H(t-\pi)\right)\cdot\pi\right)[m]$$

$$-\frac{\left(\frac{1}{m^2}+\delta(m)\cdot\pi^2\cdot e^{-m\cdot\pi\cdot j}-\delta(m)\cdot\pi^2\cdot e^{\frac{-m\cdot\pi\cdot j}{2}}+\frac{e^{-m\cdot\pi\cdot j}}{m^2}-\frac{2\cdot e^{\frac{-m\cdot\pi\cdot j}{2}}}{m^2}\right)}{2\cdot\pi}$$

$$\mathrm{simplify}\left(\frac{1}{2\pi}\times ans\right)\Rightarrow c_m$$

$$\frac{-1}{2\cdot m^2\cdot\pi}+\frac{e^{\frac{-m\cdot\pi\cdot j}{2}}}{m^2\cdot\pi}-\frac{e^{-m\cdot\pi\cdot j}}{2\cdot m^2\cdot\pi}$$

Thus we get the well-known Fourier coefficients c_m again, i.e.

Define $y3(x)=-2\cdot x\cdot H\left(x-\frac{\pi}{2}\right)+x\cdot H(x-\pi)+x\cdot H(x)+\left(H\left(x-\frac{\pi}{2}\right)-H(x-\pi)\right)\cdot\pi$

done

y3(x) is the triangulat pulse with one triangular.

The c_m we can write with the **sinc-function** $c_m=\frac{\pi}{8}\times e^{-jm\pi/2}\times\left(\frac{\sin(m\pi/4)}{m\pi/4}\right)^2=\frac{\pi}{8}\times e^{-jm\pi/2}\times\left(\mathrm{sinc}\left(\frac{m\pi}{4}\right)\right)^2$

$$F'_m\left(\frac{\pi}{8}\times e^{-jm\pi/2}\times\left(\frac{\sin(m\pi/4)}{m\pi/4}\right)^2\right)[t]$$

$$\frac{t\cdot(H(t)-H(-t))}{4\cdot\pi}-\frac{\left(t-\frac{\pi}{2}\right)\cdot\left(H\left(t-\frac{\pi}{2}\right)-H\left(-t+\frac{\pi}{2}\right)\right)}{2\cdot\pi}+\frac{(t-\pi)\cdot(H(t-\pi)-H(-t+\pi))}{4\cdot\pi}$$

simplify($2\pi\times$ans)

$$-2\cdot t\cdot H\left(t-\frac{\pi}{2}\right)+t\cdot H(t-\pi)+t\cdot H(t)+\left(H\left(t-\frac{\pi}{2}\right)-H(t-\pi)\right)\cdot\pi$$

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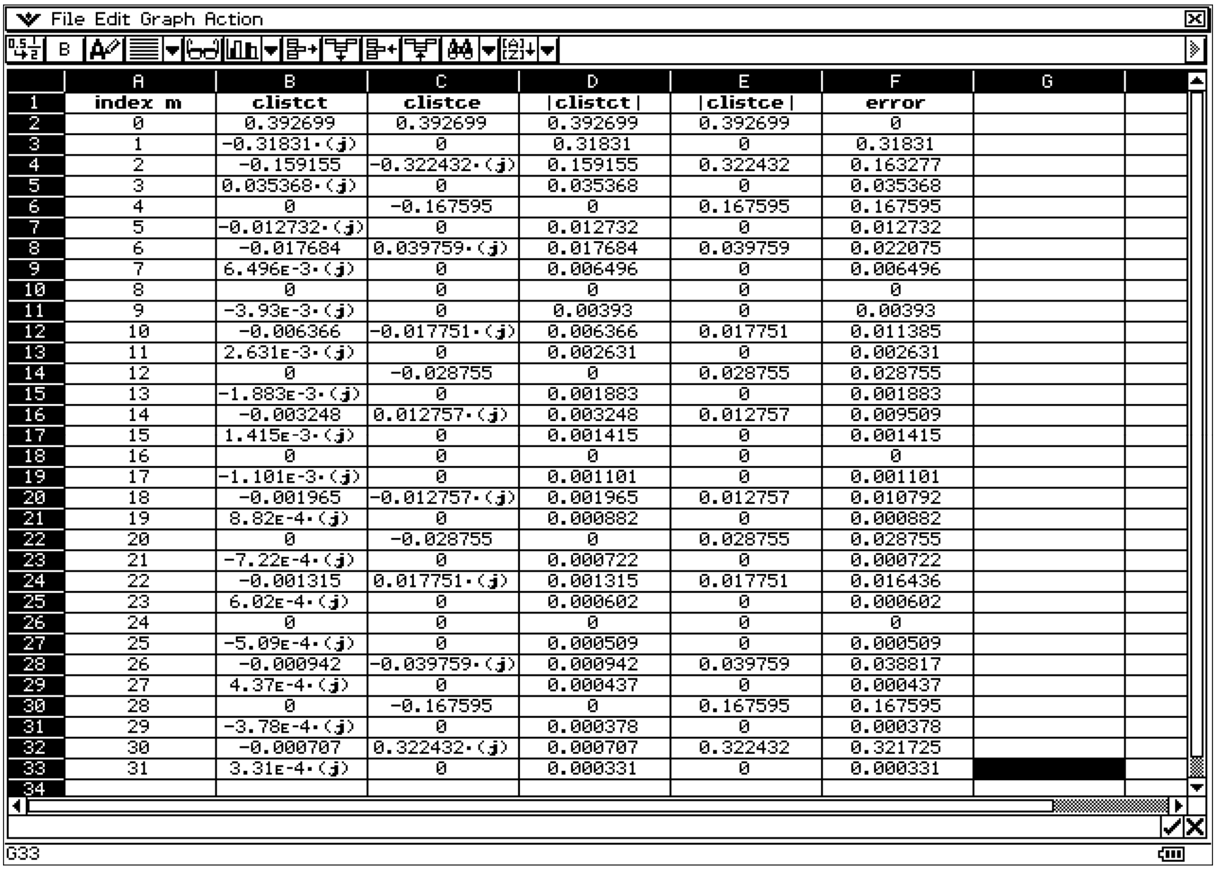
Have a look into the lists (spreadsheet-application)



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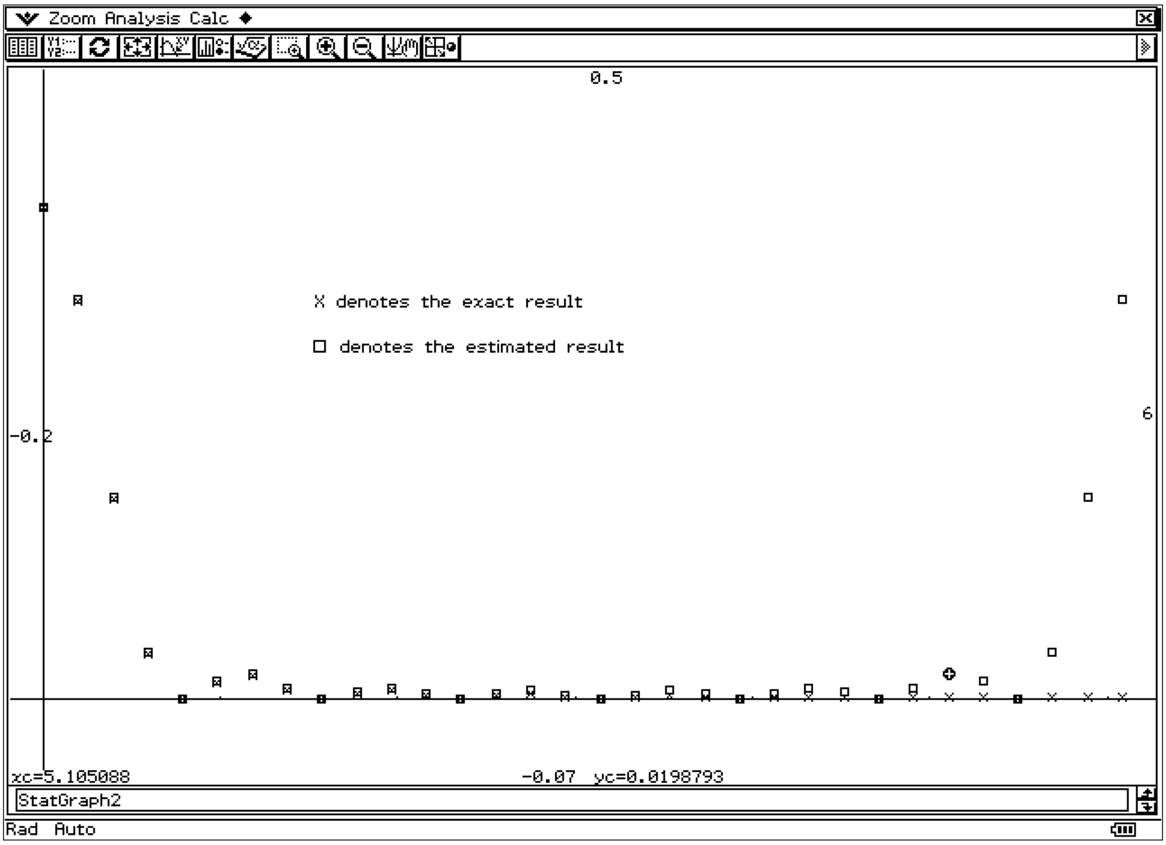
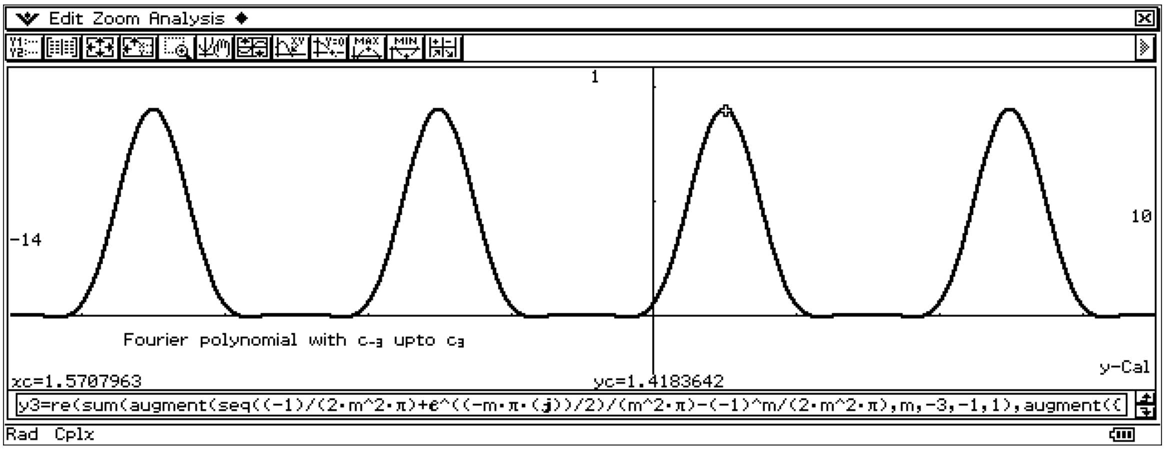
	A	B	C	D	E	F	G
1	**index m**	**clistct**	**clistce**	**|clistct|**	**|clistce|**	**error**	
2	0	0.392699	0.392699	0.392699	0.392699	0	
3	1	−0.31831·(j)	0	0.31831	0	0.31831	
4	2	−0.159155	−0.322432·(j)	0.159155	0.322432	0.163277	
5	3	0.035368·(j)	0	0.035368	0	0.035368	
6	4	0	−0.167595	0	0.167595	0.167595	
7	5	−0.012732·(j)	0	0.012732	0	0.012732	
8	6	−0.017684	0.039759·(j)	0.017684	0.039759	0.022075	
9	7	6.496E-3·(j)	0	0.006496	0	0.006496	
10	8	0	0	0	0	0	
11	9	−3.93E-3·(j)	0	0.00393	0	0.00393	
12	10	−0.006366	−0.017751·(j)	0.006366	0.017751	0.011385	
13	11	2.631E-3·(j)	0	0.002631	0	0.002631	
14	12	0	−0.028755	0	0.028755	0.028755	
15	13	−1.883E-3·(j)	0	0.001883	0	0.001883	
16	14	−0.003248	0.012757·(j)	0.003248	0.012757	0.009509	
17	15	1.415E-3·(j)	0	0.001415	0	0.001415	
18	16	0	0	0	0	0	
19	17	−1.101E-3·(j)	0	0.001101	0	0.001101	
20	18	−0.001965	−0.012757·(j)	0.001965	0.012757	0.010792	
21	19	8.82E-4·(j)	0	0.000882	0	0.000882	
22	20	0	−0.028755	0	0.028755	0.028755	
23	21	−7.22E-4·(j)	0	0.000722	0	0.000722	
24	22	−0.001315	0.017751·(j)	0.001315	0.017751	0.016436	
25	23	6.02E-4·(j)	0	0.000602	0	0.000602	
26	24	0	0	0	0	0	
27	25	−5.09E-4·(j)	0	0.000509	0	0.000509	
28	26	−0.000942	−0.039759·(j)	0.000942	0.039759	0.038817	
29	27	4.37E-4·(j)	0	0.000437	0	0.000437	
30	28	0	−0.167595	0	0.167595	0.167595	
31	29	−3.78E-4·(j)	0	0.000378	0	0.000378	
32	30	−0.000707	0.322432·(j)	0.000707	0.322432	0.321725	
33	31	3.31E-4·(j)	0	0.000331	0	0.000331	
34							

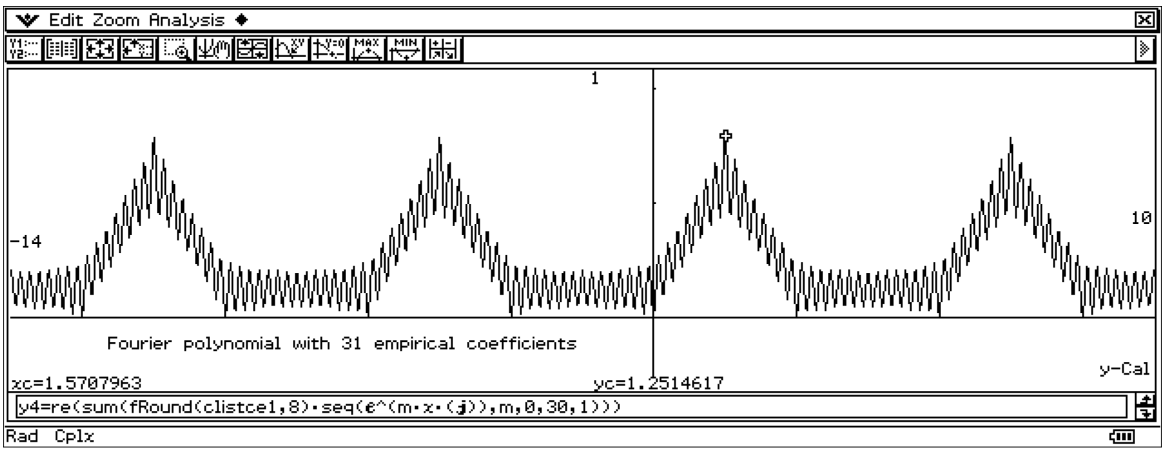
G33

Consider graphical representations of the absolute values of coefficients:



Graphical representations of the Fourier polynomial:





Reflective practices – a means to instil a deep learning approach to mathematics or another time consuming fad?

Work-in-progress

Mandy Parnell BEdHons(LdshipScMathEd), Embury Institute for Teacher Education
Durban, KZN, South Africa foxyp@telkomsa.net

Abstract

In this presentation I will report on a work-in-progress study that I am presently undertaking with second year General Mathematics undergraduate B.Ed students at a private institution for teacher education in South Africa. I first implemented the idea of reflective writing informally with scholars and then later for assessment purposes with undergraduate students. These tasks provoked very different responses from the scholars and university students, both positive and negative and prompted informal research by myself to ascertain how reflective practices can be incorporated into the mathematics curriculum. My primary objective is to investigate how I, as a lecturer can encourage and motivate students to engage in and make reflective practices an integral part of both their learning and understanding of mathematics and their teaching practice. Thus the focus of my presentation will be on different types of reflective practices and how they can be incorporated into a higher education mathematics programme.

Introduction

I have always held the opinion that reflective writing is a useful aspect of learning. Therefore when I saw the difficulty that some students had with the general mathematics course I was teaching, I saw reflective writing as a means to encourage them to engage with the course work and hopefully adopt a deeper approach to learning which in turn would develop a better understanding of the work. Furthermore I am of the opinion that pre-service teachers need to "own the knowledge" in order to be able to explain a concept to their own learners in the future. I felt that my learners would be more honest in expressing their feelings, frustrations and achievements in a written journal as opposed to volunteering their opinions in the lecture situation. In addition, the assignment provided me with the opportunity of looking into the window of their understanding of the content and provided an insight into the affective side of learning.

I implemented the reflective writing assignment in 2008 to a group of second year general mathematics undergraduate student teachers. The students were a diverse group with varying mathematical abilities and experiences. Although the feedback that I gleaned from the students, by means of a questionnaire, indicated that the reflective writing assignment had been beneficial the students themselves did not always see the value nor did they perceive reflective writing to be an appropriate part of a mathematics curriculum. The students also expressed the opinion that they had found it to be time consuming.

The purpose of implementing the reflective tasks in 2009 as part of my lecturing strategy is primarily to try and encourage students to engage with the lecture content. The rationale in setting a variety of reflective tasks as part of my lecture is to adopt a less time consuming approach that will encourage the students to think about what they have learnt and understood during the lecture with regard to:

- the development of their ideas as they interacted with different mathematical concepts.
- difficulties that they encountered during the learning and reflective process.
- the strategies they acquired and developed to overcome the above difficulties.
- their evaluation of their performance, identifying any gaps in their learning, and progress that then enabled them to plan their studies.
- as a means of reviewing and consolidating their learning.

This I believe will develop in the students, the capacity to evaluate their learning and make judgements regarding the way forward.

In addition, I hope to promote the following qualities and skills, namely

- a deep approach to learning
- confidence in their ability
- a positive attitude to mathematics
- critical thinking skills
- the ability to explain a mathematical process confidently, clearly and effectively.

I believe that reflective practices will encourage a deep approach to learning in that not only is independent thought promoted but students are given a voice to express their opinions, fears and questions. This will result in an increase in self confidence and awareness. It is my hope that students will be compelled by the tasks and activities to move beyond a surface approach to learning (passive assimilation of knowledge) to a deep approach to learning (enquiry)

In discussion with these students, it became apparent that for most of them the concept of reflective writing in mathematics was a huge paradigm shift and many were not convinced of the benefits of reflective writing. It is in anticipation that I look forward to feedback from the students at the end of the module. According to

Di Biase (1998), Mezirow's theory of transformative learning provides for change in attitudes, beliefs, practices and perceptions. It is my expectation that a number of students will have come to see the value of reflecting on course material and appreciate the benefits that the resulting deep approach to learning has had, for them.

Description of the model for solution/innovation

As this model is being implemented with new students who have little or no prior exposure to reflective writing in a mathematic curriculum I decided to start by giving them a reflective writing assignment. This would make the students conscious of the idea of reflective learning and allow them to make an informed comparison between reflective writing, which previous students have found to be time consuming, and the alternative reflective practices to consider which they have found to be the most useful and which they are most likely to implement.

The reflective writing assignment required the students to complete three reflective journal entries per week over a two week period. An affective/attitudinal entry; a mathematical content (What is it about?) and a process entry. The journal prompts related directly to the content covered during lectures the previous week. The intention of the journal prompts was to supplement the lecture process by helping the student to firstly think critically about the prompt and how they could make sense of it and secondly make connections between the theory and the constructs that they had learnt formally during the lecture. Since reflection is the process that gets us from just experiencing an activity to actually understanding the activity, in answering a prompt question the students have the capacity to draw on the past (their lectures and any prior knowledge that they have of the topic) and the present (their textbook and whatever other resources they have at hand) to direct themselves into a better future which in this case was understanding of the topic (Hinett, 2002). By having to work through the journal prompts on their own the students had to learn to think independently. The students were aware that they would not fail the task if they got the incorrect answer but that they needed to discuss their process and identify areas of difficulty, if any, and what steps they took to overcome these. The students were encouraged to help one another. I also made suggestions as to what other resources might help them understand and integrate the new knowledge with their existing knowledge. I believe that when students realized that there were content areas that they were experiencing difficulty with, they had to make a decision to either seek help or to do nothing, which would then impact on their test and exam results. Watson says "Reflection can happen through writing, speaking, listening, reading, drawing, acting, and any other way you can imagine" (2001, p. 1). Since my intention is to implement reflective tasks that develop the skills and benefits of reflective writing but not in such a time consuming manner that puts undue stress on the students, then next step in my research was to implement different activities and tasks both individual and collaborative in nature during each lecture session. The intention of these activities was to enable the students to link the subject content from lectures to the reflective practice tasks, reflect on the process and ascertain what they have learnt and how they link this to the knowledge that they already have. Furthermore, I would like students to reflect on how they could implement this type of reflective task into their teaching practice. These tasks have been designed to encourage reflective practice and accountability for one's learning. At the following lecture I would gather feedback from the students to ascertain whether or not they had indeed done what they had said they would do to understand the material better and adopt a deeper approach to their learning.

Description/evidence of the extent to which the model was successful with respect to the targeted problem/obstacle

I will draw to the attention of the audience, both the positive and negative responses to the implementation of these activities. In addition, I will discuss and address problems or areas of difficulty that arose with suggested possible solutions

Possibilities for transfer of the model to different environments.

Reflective writing currently plays a large role in student teacher practicum. I am of the opinion that reflective practices can be implemented within any curriculum since, if undertaken effectively; it has the potential to develop a deep approach to learning.

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A Discussion of different teaching strategies adopted during a Statistics tutorial

Vasos Pavlika, B.Sc M.Sc PGCE Ph.D C.Math FIMA CPhys CSci MInstP CITP MBCS FIAP MIET MIEE Senior Lecturer, Information and Software Systems Department, University of Westminster, Harrow, Middlesex, United Kingdom V.L.Pavlika@wmin.ac.uk

Abstract

In this discusses four different approaches used during a statistics tutorial of a group of first year undergraduates studying computer science related degrees at the University of Westminster UK. The four approaches were each implemented in an attempt to keep the students interested in the statistics topics delivered. It was found that "Chalk and Talk" (i.e. board work) was not the best form of imparting knowledge to the students of the group as determined by student analysing feedback forms and generally observing student behaviour and listening to student comments over a number of years delivering statistics topics. The duration of each tutorial was two hours.

The teaching strategies adopted were:

a) A class quiz.
b) Group explanation of material to members of the individual's group.
c) Group explanation of material to members of the entire class.
d) Students teaching at the front of the class.

Each of the methods will now be discussed with the relative merits and defects included for a comparison. It was found that each method worked better at the end of each module when the students were more familiar with the topics introduced on the module.

Introduction

This part of the paper discusses the teaching methods that were used in the session. The rationale for choosing the methods came from observations over many years of classes delivered by the author. It became clear that certain characteristics permeated each tutorial, these were

i) who students preferred to sit with
ii) students insistence on wanting a break at the end of the first hour of a two hour tutorial session
iii) noise level increasing later in the course, this may be due to the students becoming familiar with each other and more comfortable with the tutor. Further research into causes of this are a subject of future research.

Class Quiz

In order to determine members of each group the author nominated four so-called captains and each captain took it in turns to choose a member for his/her group. To determine which captain started first a coin was tossed. During the tutorials it was found that there were at most four groups with four members in each group.

Once the groups were determined each group had to prepare two questions for each of the other groups to pose them as exercises. The questions could be taken from the class notes or any of the recommended text books of the module. Both the ***Posing Team*** (the team setting the questions) and the ***Attempting Team*** (the team answering the questions) were permitted to use the class notes. The scoring for the quiz was as follows:

Rules of the Quiz

- If the posing team could solve the particular problem they were awarded one mark.
- If the team attempting the question could also solve the problem then the Attempting Team was awarded one point (the Posing Team was still awarded one point).
- If the team attempting the question could not solve the problem then the Posing Team was awarded two points and the Attempting Team no points.
- For a question to be allowed, the posing team had to be able to solve the question set, this was determined before the quiz began by the author.

Below is an actual set of questions set by a particular team with the respective scoring also shown.

Question 1

The weight of a sample of patients being treated were measured and found to be:
103, 127, 96, 110, 115, 72, 97, 134.

i) Find the sample mean and standard deviation
ii) Find a 95% confidence for the mean μ
iii) Find a 90% confidence for the mean μ
iv) Compare the widths of these intervals
v) Do the confidence intervals of ii) and iii) contain the value 100?

Question 2

The scores for three tests undertaken by a year 5 group are presented in the table below:

	Test 1 (score out of 20)	Test 2 (score out of 50)	Test 3 (score out of 100)
Lowest score	5	12	20

Highest score	19	43	75
Lower quartile	9	19	35
Upper quartile	15	35	70

Which of the following statements are true?

a) The highest and lowest marks for the year group declined over the three tests
b) There was no change in the year group's performance over the three tests
c) Marks for the year group improved over the three tests.

Convert the results into percentages, and draw Box and whisker diagrams on the same scale to compare the tests.

Question 3

Using the Normal Distribution solver found at http://davidmlane.com/hyperstat/z_table.html, solve the following:
The mean mass of 500 male students at a college is 68.6 kg and the standard deviation is 6.8kg. Assuming that the masses are normally distributed, estimate how many students have a mass

a) between 54.5kg and 70.5kg
b) more than 84 kg

Question 4

Derive from first principles the ordinary least square coefficients given by:

$$a_0 = \frac{(\sum y)(\sum x^2) - (\sum x)(\sum xy)}{N(\sum x^2) - (\sum x)^2}; \; a_1 = \frac{N(\sum xy) - (\sum x)(\sum y)}{N(\sum x^2) - (\sum x)^2}$$

it was noticed in this question that the students relied heavily on Spiegel [7] to state the required formulae which was permitted by the author for this question.

Question 5

Find the least square line for the following data:

Height x	70	63	72	60	66	70	74	65	62	67	65	68
Weight y	155	150	180	135	156	168	178	160	132	145	139	152

Question 6

Prove that

$$s = \sqrt{\frac{\sum x^2}{N} - \left(\frac{\sum x}{N}\right)^2} = \sqrt{\overline{x^2} - \overline{x}^2}$$

where symbols have their usual meanings. Again it was noticed that the students referred to one of the course text books to state this result, namely Clarke [3] but this was again permitted by the author. It was found that this method of learning was immensely popular with the students instilling a sense of teamwork. The author had to ensure that student enthusiasm during the competitions was within an acceptable level, as student debates became extremely animated. This method of learning allowed the author to determine the level of each team and to get “a feel” for what parts of the module the students found difficult as these were the questions that tended to be set by the Posing teams.

Sample Results

Below are a set of results conducted during an actual tutorial for teams labelled A, B, C and D. Team A was the Posing Team.

Posing Team A	Team A	Team B	Team C	Team D
Question 1	2	0		
Question 2	1	1		
Question 3	1		1	
Question 4	1		1	
Question 5	1			1
Question 6	2			0
Totals	8	1	2	1

For the sample results Team A was posing the questions and thus had the greatest total. Each team took turns to pose questions and the overall totals were determined.

Group Explanation of material to Members of the individuals group

Using this approach the author put the students into groups and asked members of the group to nominate another member of the same group to discuss and explain parts of the lecture that they had attended during the week or previous weeks. This worked well with the members of the group that were comfortable reporting back to their peers. The author went round to as many groups as possible during the session to listen to explanations given by individuals of each group.

Group explanation of material to members of the entire class

Using this approach members of a group explained a particular topic to the entire class, the students did not approach the front of the class but remained at their desks and a "joint" discussion of a particular mathematical topic was given. This turned out to be extremely fruitful as even the apparently weaker students were able to make a contribution during the feedback period.

Students teaching from the front of the class

Using this approach allowed for volunteers to come to the front of the class to deliver/explain part of the previous lecture or lectures of previous weeks to the remainder of the class. This worked well with those students that were comfortable talking at the front of the class and of course with those students that were sufficiently competent/knowledgeable with the material delivered in the lecture. On no occasion was a student invited to the front of the class by the author, only students who wanted to come to the front did so. Topics that seemed to be popular were:

- discussions on the normal distribution.
- discussions on measures of dispersion.
- discussions on measures of central tendency.
- discussions on methods of presenting data.

Topics that were not popular and in which very few volunteers approached the front of the class were:

- discussions on other probability distributions
- discussions on OLS.
- discussions on skewness.
- discussions on moments.

Encouraging this style of teaching enabled the author to determine the level of those students that chose to report back to the group. Of course it also became clear which topics were the most challenging for the students, as already stated, for the unpopular topics, on some occasions there were no volunteers to report back to the group.

Discussions

Four methods of teaching have been discussed in this article. Each method had definite positive and negative aspects. These will be summarised below.

Class Quiz	Positives	Negatives
	Attempting Team worked well together to solve problems set.	The Setting Team generally had all questions set by the most able of the group, the less able students did not participate.
	Knowledge shared by members in both the Setting and Posing Teams.	Students occasionally wanted "Chalk and Talk" sessions from the lecturer especially when a new and difficult topic had been discussed in the previous lecturer. Thus students thought more specific learning took place during a "Chalk and Talk" session.
	Able to look over all module notes and thus continually revise material.	

Group explanation of material to members of the individual's group	Positives	Negatives
	Reporters demonstrated their knowledge and enhanced their own as well as their colleagues knowledge.	The more reserved and shy students did not participate in this approach and were often left feeling left out.
	Knowledge shared by an able member of the team.	An exercise in which the more able students excel and the less able feel intimidated.

Group explanation of material to members of the entire class	Positives	Negatives
	Even less able students reported back to the class.	More dominant personalities overwhelmed the less outgoing

		members of the group.
	Knowledge shared by many members of the tutorial group.	Less able and shy students felt left out.

STUDENTS TEACHING AT THE FRONT OF THE CLASS	POSITIVES	NEGATIVES
	Able to determine the exact level of an individual member of the group.	Appropriate to only the outgoing members of the class.
	Knowledge shared predominantly by the most able student of the class to his/her peers.	Not many students volunteered to report back to the class from the front of the class, hence this teaching approach could not be used for the entire two hours.

Conclusions

In this article different teaching strategies utilised during tutorial sessions have been discussed. Due to the very nature of the approaches adopted no one approach was exclusively used during any particular session. It was found that in most cases:

1) Group explanation of material to members of the individual's group
2) Group explanation of material to members of the entire class.

could be combined in a tutorial session and was found to work well. *The Class Quiz* could be used for the entire two hours. The method of *Students teaching from the front of the* class was generally adopted to implement a different teaching strategy in the tutorial and to break up routine, especially during sessions in which students attempted questions individually at their desks. Other approaches to teaching in tutorials sessions are now being considered by the author and is area of considerable research in the education sector. These techniques include (but not exhaustively) the following:

i) mini-lecture (lasting about fifteen minutes) followed by student centered work
ii) controlled group discussion in which a different statistical topic written on pieces of paper and placed in a sealed envelope by the are placed one on each desk, a small group of students on opening the envelope then discuss the topic on the card.
iii) Buzz groups: a small statistical is given to each student group, typically to prepare a discussion to be given to the entire group on a statistical subject designated by the author.
iv) Mini debates, this typically consisted of two groups uniting to oppose two other groups in the tutorial to debate which and when statistical tests were appropriate to use e.g. the student t-test versus the Normal Distribution.
v) Student presentations, more confident and able students were asked (if they wanted to of course) to give a discourse von a particular statistical topic.

Many other teaching styles were used with a varying degree of success, theses included: brainstorming, i.e. asking students to give their comprehension of a particular statistical concept in class feedback session and using formal and informal teaching styles as discussed in Gibbs [4]. It was found that the students much preferred an informal relaxed teaching atmosphere, class room management has been a previous research topic of the author as described in Pavlika [5] and the results ascertained further illustrate the methods discussed in that article.
Other interesting teaching strategies can be found in Ashcroft [1], Rogers [6] and Brown [2].

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The Learning of Mathematics for Limited English Proficient Learners: Preparation of Doctoral Level Candidates

Theresa Perez (tperez@uncc.edu)
David K. Pugalee (David.Pugalee@uncc.edu)
University of North Carolina at Charlotte
Charlotte, NC, USA

Abstract

Across the United States, there is a growing number of students for whom English is not their first language. These students experience many challenges adjusting to new educational environments. These students are often denied access to the full curriculum in mathematics (Reyes & Fletcher, 2003) and the resulting opportunities for higher level educational experiences in mathematics and the resulting higher economic employment options. Educators need support in understanding and responding to the linguistic and cultural challenges that these students face in learning mathematics. A course entitled Language, Culture, Mathematics and the LEP Learner is part of the doctoral courses available to Curriculum and Instruction students at UNC Charlotte. The course focuses on theoretical and applied models of teaching and learning mathematics for English as Second Language Learners. Research and current practice are reviewed with an emphasis on the design, implementation, and assessment of instruction for this population of learners. A qualitative analysis of students' final research projects using narrative analysis methodologies showed that students (1) position issues within a larger sociocultural framework (2) advocate for the negotiation of pedagogical principles that blend language learning strategies with effective mathematics pedagogy and (3) identify assessment policies and processes that are supportive and limiting for these learners.

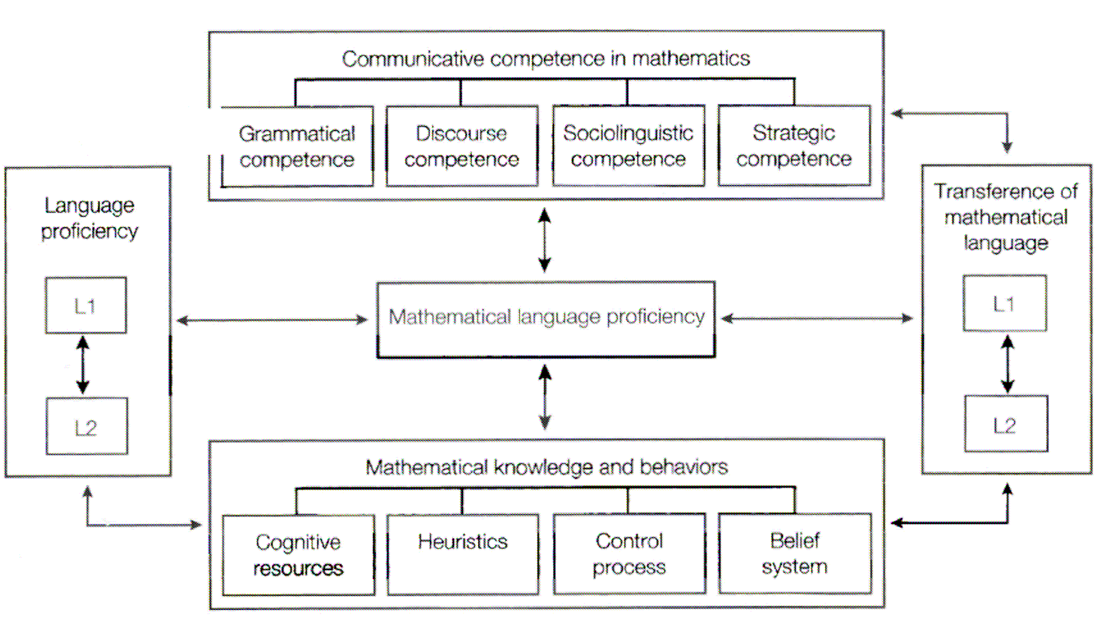
Introduction

Language and culture provide a dynamic system which influences teaching and learning. Learning mathematics requires multiple and complex linguistic skills that second language learners may not have mastered (Cuevas, 1984). An emphasis on language in the teaching and learning of mathematics is essential if English Language Learners (ELL) is to have access to the technical careers that require a solid background in mathematics and science. In today's mathematics classrooms, students must deal with communication demands (oral and written) that require participation in mathematical practices such as explaining solution processing, making and describing conjectures, proving conclusions and presenting arguments and justifications. These processes are in addition to those related to acquiring technical vocabulary, developing comprehension skills necessary to read and understand various mathematics texts, or in solving 'word' problems (Moschkovich, 2002).

The complexity of the relationship between language and mathematics learning becomes evident through a situated sociocultural lens. Moschkovich (2002) identified several communication components that only become visible through such lens: (1) Participation in mathematical discourse moves beyond learning vocabulary to participating in the use of discourse practices such as using representations to support claims; (2) Students may use gestures, objects, everyday experiences, first language, code switching and multiple mathematical representations; (3) There are multiple uses of bilingual conversations between students such as labeling objects or explaining a concept, justifying an answer, or describing a mathematical situation; (4) Students bring varied competencies into the classroom and may be proficient at presenting clear arguments or using mathematical constructions though their vocabulary may be inadequate.

Olivares (1996) identified three characteristics, for non-native speakers of the language, in which communication in mathematics differs from everyday communication. First, students are required to work with abstractions and symbols that do not typically facilitate comprehension in everyday speech. Second, each element of a proposition is essential for understanding the entire proposition. Understanding or making inferences without fully understanding each part is practically impossible. Third, elements of mathematics propositions often have such specificity that they cannot be rearranged. Olivares' model of communicative competence in mathematics emphasizes the complexity of the language-mathematics connection.

Figure 1. Communicative Competence in Mathematics (Olivares, 1996, p. 221)



The importance of language in mathematical discourse is evident from the above discussion. Assessment becomes an additional component that requires a command of academic English and the register for mathematics. Consider the following example from a state high school exit exam (Filmore, 2002, p.3).

If **x** is always positive and y is always negative, then **xy** is always negative. Based on the given information, which of the following conjectures is valid?

A. $x^n y^n$, where n is an odd natural number will always be negative.

B $x^n y^n$, where n is an even natural number, will always be negative.

C. $x^n y^{m,}$ where n and m *are* distinct odd natural numbers, will always be positive.

D. $x^n y^m$ where n and m are distinct even natural numbers will always be negative.

What does success with such an item require? Students must be competent in dealing with exponents and multiplication of integers; use logical reasoning; be familiar with the structure of conditional sentences; know the meaning of technical terms such as negative, positive, natural, odd, and even in relation to discussions about numbers; and know frequently used words such as if, always, then, where, based on, given information, the following, conjecture, distinct, and valid.

Description of the Course

The course, EDCI 8020: Language, Culture, Mathematics and the LEP Learner, focused on theoretical and applied models of teaching and learning mathematics for English as Second Language Learners. Research and current practices served as a foundation for discussions and readings. Research and theoretical perspectives were reviewed with an emphasis on the design, implementation, and assessment of instruction for this population of learners. The major goals of the course were to

1. Trace the legal, historical, and political context of ESL in the United States.
2. Describe the theoretical underpinnings of ESL and the language and mathematics connection.
3. Identify best instructional practices for ESL mathematics learners based on current research and curriculum theory

4. Describe and analyze assessment practices and issues in mathematics related to ESL learners.
5. Develop an instructional intervention for ESL learners, including a paper framing the intervention in a theoretical and research base.

Participants

The participants were six doctoral students enrolled in a curriculum and instruction program with one specializing in urban education, three in literacy education, and two in mathematics education. The five females and six males had a wide range of teaching experiences at both the elementary and secondary level ranging from four years to more than twenty years of classroom experience. Two of the participants were administrators in school districts, one worked for a city agency, and one was a lecturer at the university.

Research Design

Narrative text analysis of documents was selected as the appropriate methodology to understand students' perceptions and applications related to the course content (Qualitative methods using content analysis of student papers to identify patterns, core constructs, and themes related to student's projects was the overarching method. The researchers agreed that the sentence would be the primary unit of analysis though several sentences might be chunked if appropriate to preserve meaning. The segmentation procedure was therefore focused on units of meaning whether they were partial, complete or multiple sentences that represented a consistent idea, argument chain or discussion topic (Chi, 1997). The units were then categorized into pedagogical principles, philosophical and theoretical concepts, tasks related to instruction, assessment or student learning. Once the data from the students' papers were analyzed and coded, the researchers met to debrief about the process and to resolve issues related to the categorization of the segmented units. The resulting data provided the students' perceptions and hopes relative to the teaching and learning of mathematics for limited English proficient learners.

Results

The results of the narrative text analysis provided descriptions of the students' thinking relative to three themes: (1) the sociocultural nature of language issues (2) knowledge of pedagogical principles that blend language learning strategies with effective mathematics pedagogy and (3) an awareness of how assessment policies both support and hinder ELL learners.

Sociocultural Nature of Language Issues

The first results discussed demonstrate that participants in the study, position issues within a larger sociocultural framework. There was agreement among all of the participants that beginning teachers are not given adequate training prior to teaching and support while teaching, to be able to accommodate ESL students in U.S. classrooms. The notion that teachers can be outstanding academics in their area of expertise, but lack an understanding and training in the emerging linguistic needs of their ESL students was also evident in the responses that we received. A recurring theme among participants was that the United States employs a large immigrant population to sustain their economic prosperity, and thus is obliged to educate the children of immigrants.

The students' responses consistently reflected and discussed constructivist ideas and philosophies. Such learning theories were directly related to classroom practices.

> The constructivist learning environment can be better suited for supporting the ELL by incorporating the following strategies: bilingual instruction, access to opportunities where ELLs can share their home culture, the allowance of ESL students process new ideas in their home language, using resources that increase the dialogue between the teacher and the ESL student, utilizing culturally responsive instructional methods, diagnostic reform, and consciously planning instruction using Vygotsky's Zone of Proximal Development.

An emphasis on classroom communication must be "continuous and ongoing". LEP students often face barriers to participation in constructivist environments. Mathematics is a unique language that is highly symbolic and abstract requiring students to master vocabulary, sentence structure, and interpretation of illustrations (see Usiskin, 1996)

The importance of sociocultural awareness as an integral part of an effective teacher preparation program was mentioned in all the papers acknowledging a problem or promoting more focused preparation. One high school teacher offered, “While a novice teacher at the high school level, I ascertained quickly that high school teachers were generally stellar academics in their areas of expertise, but many lacked an understanding of, and any training in, how to reach students’ developmental academic levels and emerging linguistic levels.” Another student suggested that all pre-service and in-service teachers receive ELL training. She proposed an emphasis on coursework and field work in language acquisition, language development, cultural diversity, and methodology including possibilities of study abroad programs, participation in community service, and learning a second language.

Pedagogical Principles Blending Language with Mathematics Pedagogy

Students offered multiple approaches to effectively deal with language issues in mathematics such as collaborative learning, contextual assignments, reading mathematics, problem-solving, teacher collaboration, and the use of effective reading programs and expressive gestures. There was also an elucidation on the importance of the teacher being involved in classroom discourse by offering opportunities to discuss strategies and sharing of ideas, both between the teacher and students but also among the students. Such practices were viewed as being central to the development of metacognition which was identified as an essential cognitive process to LEP students to develop if they are to be “full participants in the complex communicative environment of the mathematics classroom”. A second cognitive principle that appeared in several papers was the importance of scaffolding coupled with explicit strategy instruction for ELL students.

A common thread among the papers was the power of such practices to transform the learning opportunities not only for ELL students but for all students. The following excerpt from one of the students is indicative of how all the students viewed the pedagogical principles identified as effective with ELL learners.

> It seems the strategies discussed for ELL learners cannot be simply implied for ELL students. They are strategies that any teacher can use to be an effective classroom teacher and are simply strategies of good teaching. “Good teaching is teaching for all. These strategies will help ELLs, but they will help typical learners as well” (Drucker, 2003, p. 22). By imposing these strategies on traditional English speaking students, they will not become less educated, but they will become better learners. Popkewitz (2004) implies that there is no magic solution to teaching ELL students, but it is a mixture of many important things. The most important ingredient of all: Good teachers who communicate to all students that they care.

Assessment Policies and ELL Learners

Participants identified assessment policies and processes that are supportive as well as well as those that are limiting for ESL learners. Referring to the size of the ESL population and the lack of teacher training in U.S. public schools, one participant stated that, the sheer size of the school population, “Should drive policy reform at the national and state levels to include intensive coursework in English as a Second Language for all current and prospective teachers.” Another set of solutions referred to the assessment practices of educators and the need to teach test-taking strategies to ESL students. One participant summed it up by saying, “In an era of high stakes testing, we do students a disservice unless we explicitly teach them how to take tests. These strategies should focus on larger areas of cognition, language function and higher order thinking, which will not only improve performance on standardized assessments, but also serve the student as a lifelong learner.” A theme that ran through most of the papers was the notion that we test immigrant children on what is valued in American schools, and ignore the learning they bring from their home country, which may reflect different values. Participants stated that these results do not necessarily indicate a lack of proficiency on the part of the non-English speaking child. One participant stated that not accounting for knowledge of other languages and cultures when being tested, “Results in an improper diagnosis of deficient learning. Diagnostic practices need to change to reflect a better assessment of the cognitive abilities of all immigrant children."

The students believed that policy reform at both the national and state levels was necessary to address the needs of this "significant segment of the school population". One underlying problem that was identified was the lack of assessments that allow ELL students to demonstrate their cognitive abilities. Such lack of appropriate assessments may result in "ESL children tend[ing] to be placed in classes that focus on developing computational skills and place little emphasis on problem solving strategies even though their overall cognitive abilities may be higher" (see Chamot, 1992). Such practices present a "deficit model" of cognition for many immigrant students.

Conclusions

There are numerous and complex issues related to the teaching and learning of mathematics for ELL students. This course provides a crucial link to address access issues for students who do not speak the native language of instruction. As one teacher in the class confided:

> As a high school classroom teacher I can admit that, until the writing of this paper, I was not aware of the myriad of issues facing multilingual students, especially in the mathematics classroom. In my opinion the mindset that mathematics is universal is no longer true (and perhaps it never was).

But it is about more than raising awareness. There must be deliberate and research-based efforts to address the problems that the students face when struggling with language issues as they learn mathematics. As pointed out by the students in this course, such practices are just 'good teaching' and hold potential not only to positively affect ELL learners but all students. As teachers, administrators, and other professionals become aware of the sociocultural nature of mathematics teaching and learning and struggle with the pedagogical and assessment issues related to ELL learners, there is a nucleus of awareness raising that has the potential to begin a wave of reform to change our approaches and our beliefs in providing all students with opportunities to develop their full mathematical potential.

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Cooperative Learning and Peer Tutoring to Promote Students' Mathematics Education

Angela Pesci
Associate Professor, Department of Mathematics, University of Pavia, Italy
angela.pesci@unipv.it

Abstract
On the basis of experiences and studies developed in the last ten years, the contribution aims to discuss some different peculiarities between Cooperative Learning and Peer Tutoring models in Mathematics lesson. These models are specific interpretations of a way of conducting Mathematics lessons which requires the activity of students, their personal participation in the construction of knowledge. In the description of the two teaching-learning models, the analysis will deal in particular with the social aspects these models involve. Describing these two modalities of cooperation, also the importance of the care for the choice of suitable mathematical tasks and for different pedagogical setting they require will appear clearly. The issues described, together with the analogies and differences between the two models, could contribute to suggest more adequate didactical projects for teachers and deeper studies about students' collaboration based models for researchers.

The Cooperative Learning Model
Starting from the conviction that it is necessary to redefine the didactic system (teacher – student – knowledge – environment, Brousseau, 1997) in the terms of a more global interpretation of the personal relationships which intertwine in it (Pesci, 2002), I studied and put in practice with a group of teachers the model of "cooperative learning". It is well known that it is an educational strategy based on social mediation: the resources for the construction of knowledge are the students, who are called upon both to accomplish a disciplinary task and to develop social abilities and the role of the teacher remains fundamental, being the organizer and facilitator of the entire process. In the quoted work I described an historical itinerary of the development of this teaching-learning model in the last 30 years, detailing the general principals which the model is based on: in synthesis, *positive interdependence*, which is reached when the members of the group understand that collaboration is necessary and that individual success cannot exist without collective success; furthermore, it is fundamental the definition and *assignment of roles* to each component of the cooperative group. The division of social and disciplinary skills amongst the members of the group encourages collaboration and interdependence, assures that individual abilities are utilized for the common work and reduces the possibility that someone refuses to cooperate or tends to dominate the others.

Another essential component regards *social abilities*: an efficient management of interpersonal relationships requires that the students know how to sustain a leadership role within the group, take decisions, express themselves and listen, ask and give information, stimulate discussions, know how to mediate and to share, know how to encourage and to help, facilitate communication, create a climate of trust and resolve possible conflicts. These abilities must be taught with the same awareness and care with which disciplinary abilities are taught.

Without entering in other details (for more pragmatic aspects I remind to Pesci, 2002, 2004), I believe important for the aims of this contribution to see the five *roles* in each cooperative group, according to the translation of this model done in the practice of our experiences.

the orientated to the task: this is the student who must make sure that his group reaches the best result in relation to the mathematical task assigned. He has to translate the task into an appropriate work plan, making sure that no-one is lost in secondary aspects of the problem, making the point of the situation and urging the group to take decisions;

the orientated to the group: this is the student who is responsible for the communicative climate. He must, therefore, make sure that everyone participates positively in the solution of the task, encouraging anyone who seems to be in difficulty, making sure that the interventions are balanced in times and ways and playing down any possible conflicts;

the memory: he is responsible for the verbalization of the results of the group. During the work, he repeats the shared decisions, asks for the confirmation of partial formulations of the

results and of the final report, agreeing with all of the components of the group, but overall with *the speaker*;
the speaker: he is the manager, for the group, of the oral report on the results of the collaborative work carried out. He arranges, with *the memory*, the final version of the results reached and reads them to the entire class in the final presentation phase;
the observer: he is responsible for the observation of the interactive process of the group. He observes whether or not each one carries out the task actively and appropriately, for example without predominating, whether or not each one suitable performs correctly his role. He takes notes on what he has observed and communicates them to the entire class in the final discussion phase.

It is essential that there is the rotation of these *roles* for the students: each one must have the possibility to live every *role*, making experience of different duties to accomplish: only in this way all his/her resources could emerge and develop.

The role foreseen for the teacher is that of *supervisor.* Beyond the organization of the work outside of the class (choice of the disciplinary task, choice of the criteria for the formation of the group, preparation of the didactic material), in class, during the cooperative work, he must not give suggestions relative to the solution of the task assigned but be particularly attentive to the interrelational processes.

At the end of the group work there is a class discussion in which all the results obtained are shared, as well as any possible unresolved problems. This final phase foresees the presentation of the *speakers* and then the presentation of the *observers*. At this point the discussion is opened up to the whole class and the teacher is responsible for a fruitful debate, both on the results of the disciplinary task assigned and on possible problems emerged during the assumption of the roles. It is, therefore, evident that opportunities for reflection, both on the discipline and on the interpersonal relationships, are continuously offered to the class.

At the conclusion of all the work, it is important that the students are invited to express their *evaluation* of the work done, for example, on a form prepared by the teacher, structured with precise questions or open for freer observations by the students.

In relation to the cooperative experiences already carried out and analyzed it must be stressed that positive results have been reached both on a disciplinary and on a relational level (Baldrighi et al., 2003, 2004, 2007). The most significant attitudes and behaviors observed by teachers have been the following.

On the disciplinary level, the pupils learn to collaborate in order to get a good result, thus reducing the anxiety often connected to individual tasks and facilitating the learning process; they get more easily involved in the development of concepts and show that they feel protagonists and active players in the discoveries and elaborations carried out; they become aware that collaboration is much more rewarding than competition and gives better results; they show greater motivation in learning and they put more trust in their own possibilities (this is particularly true for students of an intermediate/low level), spontaneously expressing their ideas and thus voluntarily bringing themselves into play.

The positive value of cooperative experiences refers also to interpersonal relationships: students more easily develop or consolidate friendship; they realize that it is not necessary to be “the best” of the class in order to be accepted into it and give useful contributions for all; they show a better willingness in following the teacher's didactical project.

The Peer Tutoring Model

After some years of practice based on the *Cooperative Learning* model, we developed also experiences in secondary school through *Peer Tutoring*, aiming at both the disciplinary support for pupils in difficulties and the involvement of students (whether or not in difficulties) in the same project (Torresani, 2008). Assuming that usual support classes do not always give satisfying results and are quite disappointing for teachers too (Torresani, 2008, Cusi, 2007), it seemed a better choice to organize activities involving the entire class with the aim of recovering disciplinary abilities of students in difficulties and at the same time of strengthening those of the others.

After a suitable preparation of the class, the teacher usually proceeds to the organization of couples or groups of three formed by students with significant differences in terms of school results and where it seems most likely for interpersonal relationships to be consolidated or created from the beginning. The teacher then arranges with the class a schedule for the project which organizes the timing of the subjects. The teacher must also arrange the necessary material (papers with activities of gradual difficulty taken from books or elaborated for the purpose). During the tutoring activity the teacher must always be available as an expert to resolve possible doubts to *tutors* and as supervisor of the activity.

The student involved in the role of *tutor* must use many cognitive abilities: he will have to give suggestions and provide explanations, manage the material and select the subjects which his student has to reinforce through exercises, check and report the results. Among the requisites necessary for the success of the tutoring process we must not forget the social abilities essential for the creation of a good relationship putting the pupil at ease and allowing the sharing of the proposed objectives.

The systematic monitoring of the results carried out by the *tutors* is often helped by a form which has to be filled-in by the student. This form is provided at each meeting and suggests a reflection on the difficulties and the particular abilities of the pupil.

The *pupil* will have to keep a copybook dedicated to this experience, which will relate both the activities carried out in class and the homework assigned by the *tutor*.

Tutoring experiences on a chosen specific disciplinary subject are usually kept in two or three meetings with a final classwork. While the *pupil* is developing his disciplinary task, the *tutor* elaborates a tutoring report and, in some cases, he also carries out a part of the task. This report is a significant tool not only in relation to the activity of disciplinary recovery but also for the metacognitive reflection on one's own work: in fact, in elaborating this report, the *tutor* pupil is called to give previsions referring to the possible results of his tutored *pupil* (also considering the will for work showed by the pupil) through questions like the following:

1. Among the subjects discussed (*here follows a list of the subjects*) mark with a + the subject you think your pupil is most prepared on and with a – that on which you think your pupil is still in difficulties.
2. How would you evaluate the work of your pupil in carrying out the activities developed in class?
   Scarce Discontinuous Diligent Constant Remarkable
3. How would you evaluate the work of your pupil in carrying out the homework?
   Scarce Discontinuous Diligent Constant Remarkable
4. Did you have any trouble in your relationship with your pupil? If so, what kind of trouble?
5. What did you find to be the most difficult thing in teaching?
   a) Give explanations about processes
   b) Manage the material
   c) Understand your pupil's difficulties
   d) Manage the relationship with the pupil (be patient, gain his trust,...)
   e) Other ……………………………………….
6. When the communication with the pupil was successful, this was because:
   a) You used simpler words than those on the book or those used by the teacher
   b) Between classmates there is no fear in expressing perplexities and reservations
   c) You understood what your pupil was most skilled in and, starting from there, you succeeded in making him progress
   d) …………………………………………………

With reference to the question n. 5, the great majority of the *tutor* pupils chooses the following aspects: give explanations about processes and understand the pupil's difficulties. These aspects are taken by many *tutors* as challenging tasks as they greatly enhance the pride for their personal abilities. If we consider the teachers' observations and the students' final reports, we can see that the awareness of the particular difficulties of the process of teaching/learning gives origin, in *tutors*, to a sort of solidarity towards their teacher, thus improving the relationship with the latter.

From a cognitive point of view, this learning process is useful both for the *tutor*, who consolidates his knowledge (teaching is like "learning two times"), and for the *pupil*, who

receives an individualized lesson.
Considering the results collected by students, we can say that dialogue between peers provides a greater freedom and spontaneity, eliminating the tension and uneasiness often perceived by pupils in their relation with the teacher. In fact, the majority of students usually indicates in the fact that between classmates there is no fear in expressing perplexities and reservation the main cause underlying the success of the communication between pupil and *tutor* (answer b) to question n. 6). Another important aspect is usually seen in the fact that simpler words than those on the book or those used by the teacher are being used (answer a) to question n. 6).
The dialogue between peers thus becomes a tool facilitating the sharing of the objectives and the students' awareness of the attitudes leading to a failure, which must then be recognized and elaborated to overcome difficulties.
This is indeed a promising strategy because it encourages students to take charge of the problem of disciplinary recovery; it favors the assumption of responsibility in learning and reduces to a minimum that fatalist attitude which is often related to one's own failures in mathematics.

A sort of comparison between the two models

The main idea which underlies both *Peer Tutoring* and *Cooperative Learning* is the conviction that the assumption of roles in a group or a couple makes the pupils responsible for their own learning process, significantly favoring it at the same time. The goal of an educator, not only of the teacher, is that of succeeding in making *the other* assume the responsibility of a task.
The description of the two models in the previous sections has put in evidence that both the *Cooperative Learning* and the *Peer Tutoring* models aim to develop students' competence in assuming specific roles, balanced between the attention for disciplinary tasks and the care for interpersonal positive relations. Both the models, furthermore, suggest the teachers have a specific care for planning *metacognitive* tasks, which require students' reflection on what they did, what they are doing and how they could do better. It is also important that in any case students could have trace of the whole activity done: a sort of personal collection of works, of results, of reflections.
On the basis of our studies on experiences accomplished in secondary school, it seems that, as far as the *mathematical task* is concerned, *Cooperative Learning* Model is more adequate when

- the subject to work about is new and open to different paths of inquiry; for instance, when a new (i.e. not yet presented by teacher) demonstration of a theorem is required (Baldrighi, Fattori, Pesci, 2004) or when a complex problem situation has to be explored (Euler's formula for faces, corners and vertices of a polyhedra, Angelini & al., 2007);
- the task is quite difficult and the cognitive resources of the whole group of students are necessary; for instance when the problem is not familiar for students and a good expertise of the contents involved is required (Baldrighi, Bellinzona, 2004, Baldrighi, Bellinzona, Pesci, 2007)
- a subject studied a long time before has to be reminded; for instance, when the teacher plans to develop a mathematical content faced by students one year before or more, instead of asking directly to the students if they remember something or not, it is more fruitful prepare a task to accomplish in cooperative groups: it happens that in group is easier to recall appropriate terms, definitions, properties and the successive work of the teacher becomes more adequate.

Peer Tutoring model seems appropriate when a specific mathematical content has to be re-covered only by a part of students, who showed difficulties. An inquiry activity based on peers' reflection on their own errors (wrong strategies or misconceptions) is very relevant for this model: in this case, mates' cognitive resources and peers' language could be fruitfully exploited. Experience makes it clear that not only students in difficulties are more likely able to recover (Torresani, 2008) but also that *tutor* pupils strengthen their competence in mathematics, their language abilities and their skills in interpersonal communication.
As far as the *social* and *relational* aspects are concerned, the difference is in the typologies of roles assumed by students in developing their collaboration:

- in the *Cooperative Learning* model, the students assume roles which are different at the social level (*the oriented to the task, the oriented to the group, the memory, the speaker* and *the* *observer*) but they are involved in the same way in facing the problem posed;

- in the *Peer Tutoring* model, the difference is both at the level of roles assumed (*tutor* and *student*) and at the level of the way of facing the problem posed: in this case it is obvious that the *tutor* knows the problem and therefore his/her cognitive effort is more oriented to understand the strategies and difficulties of his/her *student*, rather then to propose personal solutions.

When a teacher is able to intertwine, during the educational process, both activities in cooperative groups and of peer tutoring, all the roles can be assumed by the students, and this is a very rich occasion for personal improvement, for all the students.

In conclusion, it seems relevant to recognize the peculiarities of these two teaching learning models, together with their analogies and differences, at least for two main reasons. Firstly, if a teacher has this awareness, his/her competence in planning didactical interventions could be more suitable and fruitful for improving students' education, both in mathematics and in interpersonal relationships. In addition, I believe that the same awareness could suggest to researchers in mathematics education deeper analysis of these two models and other points of view for further explorations.

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Building leadership capacity in the development and sharing of mathematics learning resources, across disciplines, across universities.

Anne L. Porter BCom DipEd, DipCompSc MSc PhD
Associate Professor Statistics, School of Mathematics and Applied Statistics,
University of Wollongong, NSW, Australia alp@uow.edu.au

Abstract

In this paper we examine an Australian project in which we seek to develop leadership capacity in staff and students throughout the country, such that they may contribute to and lead others to contribute to the development and sharing of learning support resources for mathematics and statistics across disciplines and universities. One of the tangible outputs is a set of video based learning support resources that can be embedded in subjects across disciplines and shared across institutions. However the guiding aim is to develop leadership capacity, in its simplest form leading others to lead others to contribute to the project. Leadership may also be developed and exercised across different aspects of the project whether it be mapping needs, drawing together disciplines groups, finding ways to recognise and reward those engaged in the process, developing resources and the associated skills, ensuring copyright adherence, creating learning designs for optimal use of resources, evaluating the impact on student outcomes, peer review and the dissemination of findings.

Introduction

Several issues underpin and drive this Australian project, *Building leadership capacity in the development and sharing of mathematics learning resources across disciplines and universities*. The first set of issues relate to the need for resources, while the second relates to the process through which resources are developed.

The need for services and support and more specifically for the development and sharing of learning resources in mathematics at the tertiary level in Australia, can be attributed to the decline in levels of mathematics subjects taken by high school students. This decline is evident worldwide (Luk, 2005; Williamson et al, 2003). Declines in mathematics skills brought about by a lowering of contact hours and entry standard, student ability, lack of engagement (National Symposium on Mathematics Education for 21st Century Engineering 2007) pose challenges for teaching and learning mathematics in higher education institutions. Declining skills impact on students in a diverse range of disciplines, the biological and social sciences (Wilson, 2007), mathematics (Luk, 2005), the physical sciences (Gill 1998) and engineering and science (Williamson et al, 2003).

While there has been a decline in the mathematics skills of Australian students (Barrington & Brown, 2006), there has been no such decline in the role of fundamental sciences such as mathematics. Australian National priorities for research funding recognise that '*Technological advances are often unexpected and a strong foundation in mathematics and the fundamental sciences will provide an environment that fosters creativity and innovation.*' (National Priorities, 2009). Participants at an Australian Symposium on *Learning Support for Mathematics and Statistics* also recognize, *'the critical roles of mathematical and statistical skills in underpinning student success in many courses'*. Associated with this participants recognised: *The need to care for students entering with diverse mathematics backgrounds and skills* and a *need to provide a range of services tailored for needs of relevant courses, circumstances and cohorts (Gadsden, online 1/4/08).*

This project's main goal is to create leadership opportunities for staff to create and share sustainable mathematics and statistics resources to support student learning. This endeavour is too large for a small group of individuals so to adequately address the needs of students, opportunities are provided for academics to share knowledge, skills and experiences through support and regular meetings.

The emphasis on sustainability arises not only from the magnitude of the task but from the tenuous nature of the provision of mathematics learning support in Australian institutions. An examination of this provision in Australia shows a range of approaches (Wilson, 2007). In the Australian context, several universities have no mathematics learning support centre,

some have lone workers, while a few Universities have a solid core of mathematics learning support personnel. Characteristic of many universities that have mathematics learning support are restrictions in terms of access. Resources may be limited to bridging courses for 100 level students or to small group work as opposed to individual tuition. Diversity in the levels of support has been identified and is often closely related to the strategic priorities given to mathematics support in universities. At the lead institution the need for mathematics support is pervasive across many disciplines. In the search for a model for supporting students, the initial focus has turned to the provision of electronic forms of support. This has been predicated on several initiatives, 2004-2008 which focussed on improving learning outcomes (Porter, 2008). Two of these initiatives involved the development of predominantly video-based learning resources, their placement in E-Learning sites for subjects and subsequent evaluation in terms of impact on learning outcomes (Aminifar, 2007).
The remaining focus on sustainability arises from the stakeholders who have provided funding for this project. The project is funded through the "Leadership for Excellence in Learning and Teaching Program" which supports "systematic, structured and sustainable models of academic leadership in higher education" (ALTC, 2007). In Australia at a national level there is recognition of the need to build leadership capacity in the tertiary sector.
At UOW we recognised that to address the mathematics learning support needs of our students we in fact were addressing a national issue and what was needed was a team of leaders, engaging others as leaders in the development, acquisition, reviewing and sharing of learning support resources across disciplines and universities. Within the mathematics discipline there is a need to develop leaders amongst mathematicians to produce generic mathematics learning support materials and to provide peer review. Across disciplines there is a need to develop leaders who can engage colleagues to define the contexts in which their students confront mathematics, so that learning materials may be contextualised and made relevant. The task is too big for a small group of individuals, hence our emphasis on leadership as engaging others to engage others in this process.
Specific aims included the development of: (1) a framework for building leadership capacity in the development of mathematics learning support resources across disciplines and universities; (2) enhanced leadership capacity to build coherent frameworks that map student need, and develop, acquire and align mathematics learning support resources so as to enhance student learning outcomes; (3) and, advances in the standard of Mathematics and Statistics learning support and resources, across disciplines and across universities.

Description of the model for solution

The project as conceptualised was to build upon cross-faculty networks to encourage the dissemination of knowledge and ideas (Lefoe, 2006). This project sought to extend the model from involving the sharing of creative ideas to the sharing of technology-based resources. It was planned to follow a cascade dissemination model based on the Effects Project in the UK (Fullerton & Bailey, 2001) whereby, at an appropriate point in time, new collaborative partners would be invited to participate in the project. The project was premised on the belief that there is a knowledge base for leaders on how to develop leadership (Marshall, 2006), different assumptions as to what is needed to create or be a leader (Anderson & Johnson, 2006). and different concepts as to the nature of leadership (Marshall, 2006) and these have implications for how leadership is transferred/created/developed. The developmental work at UOW leading to this project has involved a progression of different leadership approaches to complete certain tasks. Recent UOW leadership programs that have been developed have provided for their participants: express outcomes in terms of increased engagement with the university; networking with peers; improved strategic focus; improved work/life balance; improved delegation; specific skill development; and increased levels of confidence (Morgan & Denny, 2007). There was, however, a need to examine how leadership could be developed when it arises in the context of staff coming together to undertake a task rather than undertaking a leadership program.

Planned Approach

This project has an action-based methodological approach (Creswell, 2003), collecting evidence from staff and students in order to define and redefine directions. Using a reflective-practice approach (Schon, 1991), the participants in this study will reflect upon their perceptions of leadership. The planned execution of the project involves a systematic, multilevel building of leadership across two institutions, the University of Wollongong and its collaborative partner, Central Queensland University. The process involves four stages, preparation, assessment of need or review, implementation, and evaluation and dissemination.

(1) Preparation involved: identification of teams and potential leaders within the two participating institutions; identification of a Leadership Advisor in each institution; and leadership building activities that supports participants' reflection on leadership.

(2) Assessment of need and review involved: a self-assessment by team members/potential leaders at all institutions determining the nature of support, needed for developing more effective leadership; mapping student needs in relation to mathematics learning support; constucting a joint action plan for developing, acquiring and sharing resources; identifying an independent evaluator; preparing evaluation surveys to collect baseline and follow-up data; designing a website for, accumulation and housing of mathematics resources; and, investigation of a means to provide academic staff with ready access to video resources which can be directly downloaded into the institutions learning content system."

(3) Implementation included: engaging wider participation and the elicitation of leadership issues through a symposium held early in the project; drawing on materials from other leadership programs to construct a program for building leadership in the development, acquisition and sharing of learning resources; holding meetings and workshops through the Access Grid to address both the leadership issues arising and review developments associated with the learning resources task; making decisions regarding the allocation of resources; determining how to share the resources made during this project and ensuring they are available within and between institutions; developing accountability reporting mechanisms; developing a list of standards for the resources which allows for granularity and sharing among institutions to fit within the institutions own resource needs; and, provision of peer review opportunities to further build a range of resources suitable for the sharing between institutions.

(4) Evaluation and Dissemination were to involve: using the accountability reporting processes within institutions to engage and inform others; iwebfolio to document progress on the action plan ensuring accountability; analysis of all data collected regarding process, content, successes and failures in the leadership program and the impact on student learning; and, a symposium toward the end of the project, inviting the participation of other universities, creating a vision for the future and initiating the mentoring of other universities.

The extent to which the model was successful

As anticipated it was expected that issues and opportunities would present and that both the leadership roles and the approach taken to develop leaders would be further defined and expanded. With the project having only just met its halfway mark there remains much to do and much to learn. However there are several aspects of the project in terms of its commencement that framed its early progress. These included how the project was developed and initiatives undertaken.

(1) The development of the proposal was highly centralized with the project leader having developed the cases for funding of seed projects, having established and worked with teams, having led the discussion with faculty Deans and personnel to identify areas of need (Porter, 2008). The initial phases of the project also involved the project leader in determining direction, chairing meetings and in general assuming the mantle of responsibility. As the project was about building the capacity for leadership distributed across disciplines and universities, this needed to change. Three significant steps have led to a greater distribution of responsibility and leadership. These were the appointment of a project manager who could share responsibility and then the establishment of small committees with areas of

responsibility and the rotation of chairing responsibilities through members of the team. Formal reporting back to the full committee provides an accountability mechanism.

(2) The team had come together with the primary intention of developing and sharing mathematics resources. Many participants did not initially see the need to develop leadership or found this somewhat onerous. There was a need to explore perceptions regarding the nature of leadership required in this project and to focus on how to build leadership capacity which in our instance means preparing and engaging others to take a leadership role in the project. The initial reframing of leadership was to recognise that leadership could be exercised in many ways. We coined the term *leadership spaces* and identified several ways in which members of the team could contribute as leaders. These spaces or areas of activity included: copyright and intellectual property issues to allow sharing of resources; mapping needs in different disciplines and universities; training others to create resources in new technologies such as Tablet PCs; developing effective designs for resources; sharing resources including expertise; providing an easily accessible effective online repository of resources; creating effective learning designs; peer review of resources; use and evaluation of resources; developing effective dissemination strategies for people in the office next door to colleagues in other nations and to present the project to other institutions to invite them to participate.

(3) Holding a symposium early in the schedule of activities has meant the activities dissemination and an invitation to participate are occurring simultaneously. This has meant the early engagement of other institutions with the project and an ongoing process of dissemination rather than the cascade occurring toward the end of the project

Possibilities for transfer of the model to different environments

The intended outcome for this project has been the building of capacity in leadership in the development of mathematics and statistics resources across disciplines and across universities. Early evidence suggests that it is an approach that is building capacity in different disciplines and in other institutions.

At this early stage the UOW teams are optimally operating in three faculties, Science, Informatics and Engineering and would appear to be viable in the longer term with the capability of producing mathematics and learning support resources and extending the work to support learning in other discipline contexts. In these faculties the development of skills training and resource development is underway. The staff and faculties are also purchasing equipment signalling intent to continue. In these faculties, staff members have also engaged in practices inviting their colleagues to become involved. The value of the project from a university perspective is reflected in the participation of staff from core support units, Personnel, Learning Development and the Centre for Educational Development and Interactive Resources. In other faculties, Commerce, Health, Education and Medicine there are participants but not yet momentum in the development of resources or commitment to purchasing suitable equipment that can suggest long-term viability, however the project is only mid-term. University policy in the area of IT specifications and support are also being impacted as new tablet PC technologies are requested by staff wishing to engage with the developments. For long-term viability IT policy changes need to be effected to allow tablet PC technology to be adopted by staff as an option when renewing equipment rather than staff needing to seek additional funding.

In terms of the model working in different environments, our partner institution Central Queensland University has a cross discipline team engaged in the development of resources across several campuses. Two months after our initial symposium, one symposium participant has assisted through negotiating a price reduction for the PC tablet technology and participants from two other universities in Queensland have requested ongoing participation. Participants seek to provide leadership in the area of agricultural mathematics and to join the team working on Maths in Science.

The Access Grid has enabled members from these other institutions to take part in meetings with the University of Wollongong and Central Queensland University and to firmly build what can best be described as a Community of Practice. It is expected that the participation from other institutions will continue to grow and as sub-groups become larger, specialist groups will emerge setting up hubs of activity, dispersing leadership and thus increasing capacity throughout the sector. These specialist groups we expect to link back to the core through a few participants from each group so that the sharing of expertise and resources may continue.

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The use of visualization for learning and teaching mathematics

Medhat H. Rahim
Lakehead University
Thunder Bay, Ontario
CANADA
mhrahim@lakeheadu.ca

Radcliffe Siddo
Lakehead University
Thunder Bay, Ontario
CANADA
rsiddo@lakeheadu.ca

Abstract

In this article, based on Dissection-Motion-Operations, DMO (decomposing a figure into several pieces and composing the resulting pieces into a new figure *of equal area*), a set of visual representations (models) of mathematical concepts will be introduced. The visual models are producible through manipulation and computer GSP/Cabri software. They are based on the van Hiele's Levels (van Hiele, 1989) of Thought Development; in particular, Level 2 (Informal Deductive Reasoning) and level 3 (Deductive Reasoning). The basic theme for these models has been *visual learning and understanding through manipulatives and computer representations of mathematical concepts vs. rote learning and memorization*. The three geometric transformations or motions: Translation, Rotation, Reflection and their possible combinations were used; they are illustrated in several texts. As well, a set of three commonly used dissections or decompositions (Eves, 1972) of objects was utilized.

Introduction and Background

Why Visualization?

In the literature, visualization has been described as the creation of a mental image of a given concept (Kosslyn, 1996). As such, and from the teaching point of view, visualization seems to be a powerful method to utilize for enhancing students' understanding of a variety of concepts in many disciplines such as computer science, chemistry, physics, biology, engineering, applied statistics and mathematics. Specifically, there are many reasons that substantiate the use of visualization for learning and teaching of mathematics at all levels of schooling, from elementary to university passing through the middle and high school levels. The literature also indicates that the activity of 'seeing' differently is not a self-evident, innate process, but something created and learned (Whiteley, 2000; Hoffmann, 1998). As cognitive science suggests, we learn to see; we create what we see; visual reasoning or 'seeing to think' is learned, it can also be taught and it is important to teach it (Whiteley, 2004, p. 3; Hoffmann, 1998). Thus, teachers who have learned and became skillful in the use of visualization and 'seeing to think' would be able to reinforce mathematical concepts and improve the learning process in the classroom. The literature further suggests that brain imaging, neuroscience, and anecdotal evidence confirm that visual and diagrammatic reasoning is cognitively distinct from verbal reasoning (Butterworth, 1999). Also, studies of cognition suggest that visuals are widely used, in a variety of ways, by math users and mathematicians (Brown, 1998). Moreover, Whiteley (2004) stated that "I work with future and in-service teachers of mathematics: elementary, secondary and post-secondary. They are surprised to learn that modern abstract and applied mathematics can be intensively visual, combining a very high level of reasoning with a solid grounding in the senses" (p. 1).

Visualization as Justification and Explanation

Visual justification in mathematics refers to the understanding and application of mathematical concepts using visually based representations and processes presented in diagrams, computer graphics programs and physical models. There are several distinct characteristics of visual justification (reasoning) in many disciplines:

- Visual justification in solving problems is central to numerous fields beside mathematics such as statistics, engineering, computer science, biology and chemistry.
- Visual reasoning is not restricted to geometry or spatially represented mathematics. All fields of math contain processes and properties that provide visual patterns and visually structured reasoning. Combinatorics is very rich in visual patterns. Algebra and symbolic logic rely on visual form and appearance to evoke appropriate steps and comparisons.
- Visually based pedagogy opens mathematics to students who are otherwise excluded. Studies suggest that students (and adults) with autism and dyslexia may rely more on visual reasoning than verbal reasoning (Grandin, 1996; West, 1998; Whiteley, 2004; Gooding, 2009).

Types of Visual Representations

This article covers the following visual representations, (1) Diagrams, (2) Computer graphics programs and (3) Physical models.

(1) Diagrams: Visually based Representations and Processes

Visually based representations and processes are utilized in a variety of math subjects. This article will focus on the following subjects:

a) Geometry
b) Functions and Trigonometry
c) Number Patterns, and
d) Algebra.

a) Geometry: Due to space limits, the focus will be on visual representations for the *derivation of area formulas* of all commonly used polygons in school mathematics. Thus, a number of examples will be presented. Throughout these derivations, *Dissection-Motion-Operations, DMO, were utilized.* The DMO process consists of *two* components:

(1) Decomposition of a shape into parts by *Dissection* operations (vertical, horizontal, oblique),

(2) Composition of the parts into new shapes of *equal area* through *Motion* operations (translation, rotation, reflection). DMO were primarily introduced for 2-D shape transforms among polygons (Rahim, 1986; Rahim, Bopp, & Bopp, 2005; Rahim, Sawada, & Strasser, 1996; Rahim & Sawada, 1986, 1990); they were extended for 3-D prisms (Rahim, 2009). In 2-D, the area of the rectangle is taken as given, $A = base \times height$; it can be verified by the graphs in Figure 1.

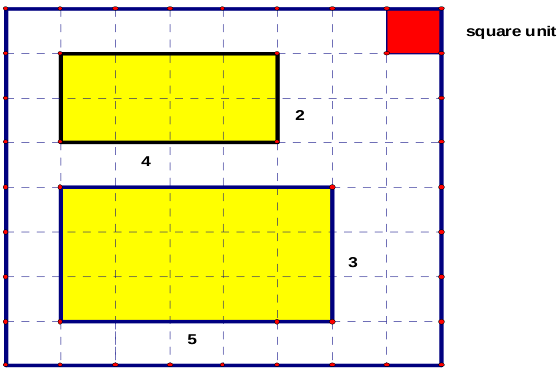


Figure 1: Based on the square unit, the Area of the rectangle = b×h

Examples

Below are a number of examples for visual representations in 2-D.

Example 1: Figure 2 below shows where the area formula of *any* triangle, $Area\ of\ \Delta = \frac{1}{2}\ b \times h$, did come from through DMO applied on *each* type of the triangle.

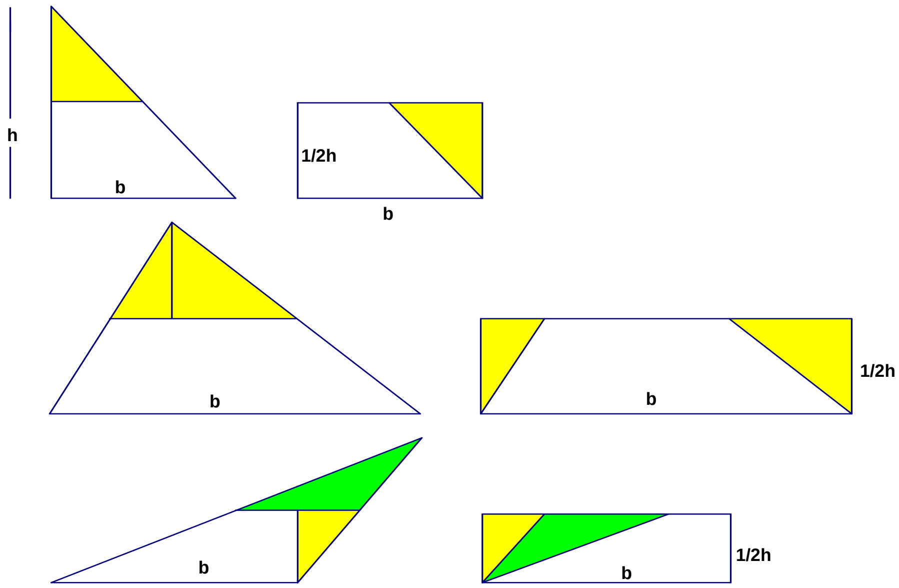


Figure 2: Through DMO, Area of Δ = area of rectangle = ½ b×h

Example 2: Figure 3 below shows where the area formula of a trapezoid did come from. The visual representations below show that the Area of the trapezoid = area of the resulting Δ. That is, Area of the trapezoid with height h and bases b_1 & b_2 = area of Δ with height h and base $(b_1 + b_2)$. Thus, area of the trapezoid = area of Δ = ½ b×h = ½ $(b_1 + b_2) \times h$.

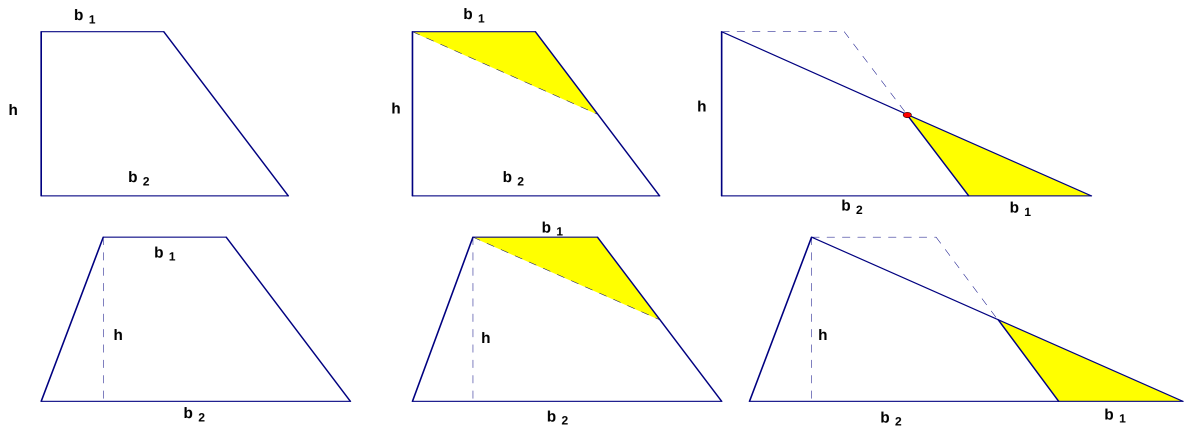
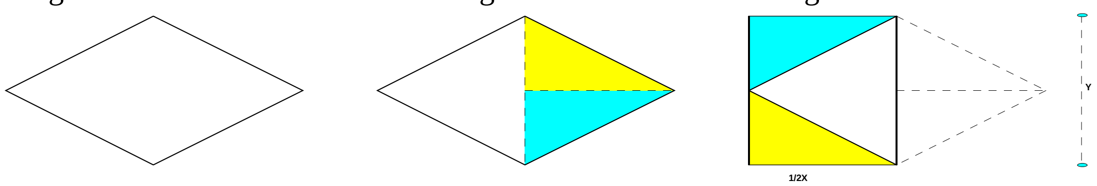


Figure 3: Area of the trapezoid = area of Δ = ½ $(b_1 + b_2) \times h$

Example 3: Figure 4 below shows the rhombus visual representations of its area derivation through DMO where X = horizontal diagonal and Y = vertical diagonal of the rhombus.



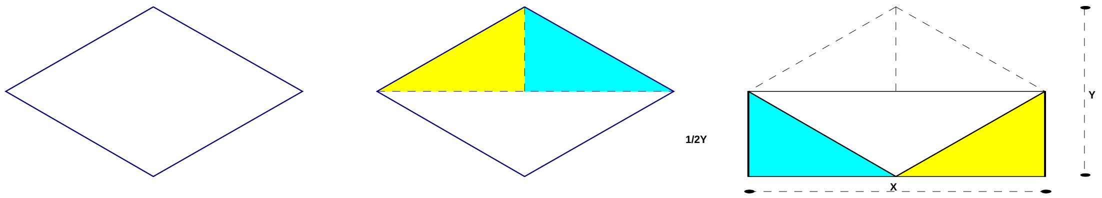


Figure 4: Area of the rhombus = area of rectangle = base×height = ½ X×Y

b) Functions and Trigonometry: Example 4. In this example, f(x), g(x) and h(x) were given below using GSP. By animation through GSP, as point C travels along the circumference of the circle B, the measures for the distance AC and angle ABC vary and the graphs of the three functions will correspondingly get in motion simultaneously. Vital observable information about the characteristics of each of the functions f, g and h with the relationships among them will be available. For example, students would observe what would happen when C coincides with A?

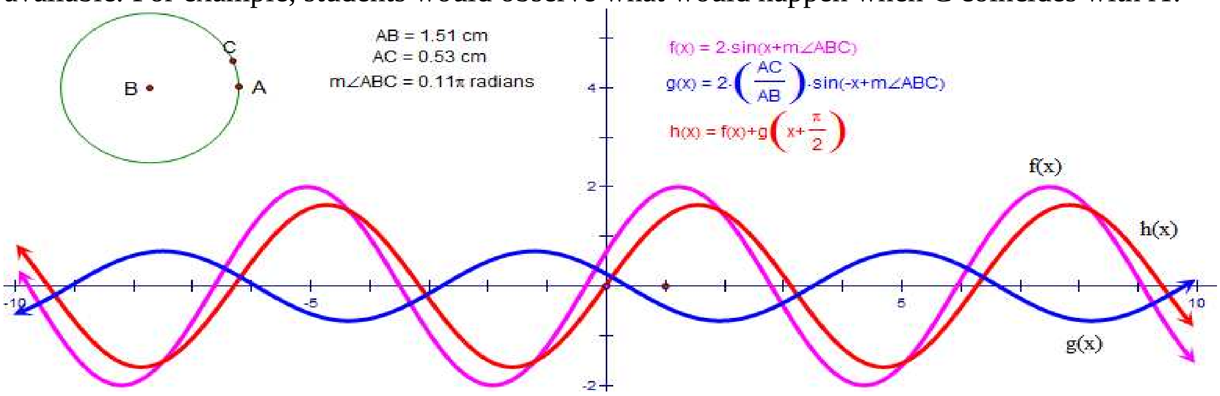


Figure 5: As C moves, the graphs of f, g and h get in motion revealing crucial properties

c) Number Patterns: Among many visual numbers' patterns, the *Symmetric Multiplication* is particularly attractive (Posamentier, Smith, & Stepelman, 2009). Consider the three symmetric multiplication patters shown in (A), (B) and (C) below. Students can multiply (A), (B) and (C) by conventional means (calculators for checking). After they attempted to use conventional method, they may welcome a more elegant solution by considering the *rhombic* method in (A) and (B) and explore the *pyramid* method in (C). The intention of these patterns is to introduce interesting properties of numbers multiplication.

```
    7777
x   7777
------------------
      49
    4949
  494949
49494949
  494949
    4949
      49
------------------
60481729

(A)
```

```
   555
 x 333
--------------
    15
  1515
151515
  1515
    15
---------------------
184815

(B)
```

```
1x1 = 1
11x11 = 121
111x111 = 12321
1111x1111 = 1234321
11111x11111 = 123454321
111111x111111 = 12345654321
1111111x1111111 = 1234567654321
11111111x11111111 = 123456787654321
111111111x111111111 = 12345678987654321

(C)
```

d) Algebra: Example 5. Visual representations: Signs' multiplications.

3x3 = 9	The pattern: a
3x2 = 6	reduction by 3
3x1 = 3	each new line.
3x0 = 0	
3x(-1) = ?	Must be -3
3x(-2) = ?	.... -6
3x(-3) = ?	.... -9
....	
(+)x(-) = (-)	.... (A)

3x3 = 9	
2x3 = 6	The pattern:
1x3 = 3	Identical to
0x3 = 0	case (A).
(-1)x3 = ?	.... -3
(-2)x3 = ?	.... -6
(-3)x3 = ?	.... -9
....	
(-)x(+) = (-) ... (B)	

3x(-3) = -9	By Case (A)
2x(-3) = -6	The pattern:
1x(-3) = -3	an increase
0x(-3) = 0	by 3.
(-1)x(-3) = ?	Must be +3
(-2)x(-3) = ?	.... +6
(-3)x(-3) = ?	.... +9
....	
(-)x(-) = (+)	.... (C)

2. Computer Graphics Programs: E.g., GSP and Cabri. For Figure 5 content, GSP was used.
3. Physical Models: A physical model to justify $\mathbf{(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3}$ will be displayed whenever presenting this article.

Finally, over the centuries, mathematicians, philosophers and some artists have recognized and highlighted the artistic aspects of mathematics. G. H. Hardy (1877-1947), a well-known mathematician at Cambridge University once stated: "a mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas" (O'Daffer & Clemens, 1992, p. 12).

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CLEARNESS AS A PRINCIPLE OF THE TEACHING OF MATHEMATICS

Ildar S. Safuanov, Prof. Dr., Moscow State City Pedagogical University,
Irina G. Shamsutdinova, Prof. Dr., Moscow Pedagogical State University shams53@mail.ru

Abstract In this paper, the psychological aspects of clearness in teaching mathematics are considered and some suggestions for the achieve the clearness are given

Speaking about the basic principle of teaching of mathematics in universities, B.V.Gnedenko (1981, p.57) indicated that it is necessary “to teach in such way that the students could clearly imagine the origin of basic concepts...” He emphasized he importance of “the transparency, new vision”, «consideration from new points of view” (ibid., p.120) in the teaching of mathematics. Used by scientists - mathematicians, teachers and psychologists verbs: “to find out”, “to explain”, “to clear up”, “to understand” - express the characteristic for the teachers methodical actions aimed at the achievement of clearness in the teaching of mathematics. Considering the importance of lectures, B.V.Gnedenko noted that “most important is to understand the essence of the subject matter, to find out its nature”, “to elucidate” the central idea of the discourse (bid, p. 134). A.A. Stolyar (1974, p.73) indicated the necessity of clearness of symbolic representations of the language of graphs. Explanation is one of the basic stages of the process of teaching. Clearness is one of the principles of the teaching of mathematics.

The clearness of mathematics for students is a characteristic of their perception of mathematics, and also a characteristic of expression of a mathematical thought of a teacher, of his explanation, of his speech. The clearness in the teaching of mathematics is even more difficultly achievable in comparison with other educational disciplines because of its abstractness, of the presence of various forms, ways and languages of representation of information.

How to provide the clearness of mathematics for the students? It is possible to deduce from the statements of A.Ya. Hinchin (1963, p.3) that we teachers:

little care about the clearness about the purpose mathematical elegance of expressions and statements;

too frequently... do not see the necessity... to find out the connection of a theorem or a concept with other, earlier acquired concepts, propositions, problems

should provide the presence “of clear view of the role and place of various parts of the studied theory “in the consciousness of students”;

should not allow “the mixing and jumps, resulting in mess and mistakes in reasoning” (ibid., p. 143).

Note that mixing, jumps and mess, absence or lack of precise branching in reasonings of the teacher create the ambiguity of mathematics in minds of students. This phenomenon is characterized by following complaints of the students: “a complete fog”; “I look at my notes and can not understand anything”; “it is impossible to understand”; “I do not understand what is connected with what”; “a complete mess”, ”the confusion in my head”; “everything has mixed up and confused”; “ everything is fallen down and does not make sense”; “ a mixture of formulas”; “a heap of theorems in my poor head” etc.

In Russian vocabulary the clearness acts as a synonym of definiteness, sharpness - alongside with such characteristics as: exact expression, intelligibility, articulateness, transparency etc. Among meanings of the word “clear” are: “light, not shaded”; “well seen, heard or perceived, understood”; “good organized, precise”.

The clearness of students” perception of the contents of mathematics is determined by the transparency of its representation (e.g. on a blackboard); by the absence of “noise”, i.e. insignificant symbols, of excessive variety of notations, of the interference of symbolical systems (vector, coordinate etc.).

The requirement of clearness of speech is always urgent. The great attention in rhetoric was given to this from ancient times.
The speech is called clear, if it is perceived without difficulties.
The clearness of speech of the teacher of mathematics usually is connected with its accuracy (inambiguity of mathematical terms, adequacy of used words to their meanings, taking into account the character and volume of the speaking competence (vocabulary) of students (the strict selection of the necessary minimum of terms), explanation of new terms (with the help of comparison, visual analogy, translation to the simple language with the help of synonyms, organization of a context etc.), exception of verbosity, superfluous words and long periods of silence.
How is the character and volume of the vocabulary of the students taken into account in the teaching of mathematics?
Many teachers with wide experience of work can formulate the answer to this question. The more various concepts (not contained in the vocabulary previously), and concepts of abstract character are encountered by the students, the less is the clearness of such speech. A.Ya. Hinchin (1963) indicated also to the necessity of literary elaborateness, accuracy of expressions. '
“In order to become clear, a new concept should be precise expressed, distinct from known concepts, correlated to a wider concept, precisely structured, considered in the logical relations with other concepts (equivalence, contrariness, equivalence, subordination, contradiction etc.).
One of the basic principles of teaching is its scientific character, and it is not reduced only to the requirements of strictness of the language of a discourse, clearness of concepts, consistency, completeness and provability of statements. It assumes that the arrangement of a material should correspond to the psychological features of the cognitive activity of the students. The known classification of cognitive actions includes:
actions of perception (to notice, to distinguish, to identify, to compare, to determine the degree of explicitness of an attribute etc.);
actions of imagination (to mentally transform an object etc.);
- Logical actions (reasoning, conclusions, generalization etc.);
What can do a teaching of mathematics in order to make the subject matter, its rules clearly perceivable, easily noticeable, precisely distinguishable, recognizable in other contexts etc. What efforts can a teacher undertake for to achieve clearness of a material to the students? She/he “explains”, “finds out”, “clears up”. Each of these methodical actions has the semantical peculiarity. To explain means to represent, to transfer clearly, completely, fully, to make clear as a whole. To find out means to establish together with students the essence and the nature of subject matter, to reveal it, to outline, to represent sharply etc. To clear up means to make unclear clear.
It is known that a person perceives those objects on which her/his attention is attracted. Therefore, the clearness of perception in many respects depends on the organization of attention during teaching. In psychology of attention the metaphor of “searchlight of attention” is used as a device that consequently lights up different parts of some area. This metaphor can be rather fruitfully used can be enough fruitfully used for the design of the methodical toolkit for the teaching of mathematics at universities.
Neuropsychological studies (General psychology, 2006) show that the shift of the “searchlight of attention” consists of three operations:
1) distraction, release,
2) actually movement,
3) attraction, “catching” of attention.
Quite often during the learning, unfortunately, occurs the distraction just from the object of cognition, , and the “searchlight of attention” is directed not on the notes of a lecture or on he textbook but on other things - a window, a fiction book, a newspaper, a crossword puzzle etc. If the

attention is released from the object of cognition, the clearness of perception is impossible. However this initial moment in teaching quite often is neglected.

It is known from the psychology of attention that the beam of the “searchlight of attention” is indivisible. Depending on a task the “illuminated” part of a visual field either is narrowed, or is expanded, but never split. It means that in the teaching of mathematics, the teacher should to think not only about the mathematical content and tasks, but also about the statement of cognitive tasks to students on each moment of teaching. For example, the statement of a cognitive task can be expressed in the formulation of the following cognitive purposes:

determine distinctive attributes of one concept in comparison with another;

establish the basic parameters and basic dependence between concepts used in the formulation of a theorem;

determine conditions of a theorem and its statement etc.

The teaching should have epistemological support adapted to the cognitive of activities of students, their attention, perception, thinking. Just the cognitive task in many respects determines the narrowing or expansion of a the “illuminated” (by the “searchlight of attention”) area of a visual field, degree of the distinctness of objects in that area, i.e. clearness.

In the recent history of psychology of attention there is a phenomenon of “blindness by the inattention”, i.e. functional blindness, which consists in inability of the observer to apprehend clearly distinguishable stimulus, if her/his attention is engaged in the analysis of other stimulus presented simultaneously with the given or prior to it. What the student will notice and what will she/he not notice if several target objects will be presented in the visual field simultaneously or with an insignificant interval of time? On what the attention will be focused, if “searchlight of attention” “is not split”? What interval of time of presentation of objects will be better for the clearness of their vision? Based on psychology of attention mathematics to develop, it is necessary to develop appropriate methods of teaching of mathematics for more effective organization of attention of the students at for the maintenance of clearness.

The second operation in moving of the “searchlight of attention” is actual movement. Research of B.G.Ananyev (1977) has shown a role of the relation “horizontal – vertical” in the cognitive movement, in the shift of attention. It is the necessary to investigate how the “movement" of attention in a visual field takes place.

The third operation is the attraction or “the catching” of attention. Quite often teacher in the process of teaching draws the attention of the students not to the central moments, but to minor, collateral things, withdrawing thus the “searchlight of attention aside, fixing it not on the essence, which remains not clear. It is necessary to find out how to determine and keep the central moment in a topic, not replacing it by superfluous illustrations, how long it is possible to keep attention of the students on one question, without loss of concentration.

Clearness of perception, the size of the “illuminated areas” perceived by the student, the characteristics of her/his “searchlight of attention” depend on the organization of her/his cognitive activity by the teacher, from the efficiency of the management of students’ attention. The order of presentation of cognitive objects, their size, grouping, arrangement in the visual field, sharpness, distinctness (visual and semantic) can influence the clearness of perception. The order of the arrangement can be determined by the cognitive (or motivational) importance of objects for students, the size of objects, their semantic connections, possible degree of their familiarity to students.

The features of visual perception(recognition) of mathematical objects (sizes, parameters, dependencies, formulas etc.). Students and the teacher are main subjects of the process of study and from their mutual understanding each other in many respects the productivity of the study depends. It is necessary to distinguish the clearness of the explanations of the teacher and the clearness of the representation in students’ minds.

Four relations are possible: the teacher explains clearly and the student understands clearly, the teacher explains clearly, but the student perceives unclear representation, the teacher explains not clearly, but the student perceives a clear representation, the teacher explains not clearly and the student represents the subject matter not clearly.
In the first case the scheme of explanations is strongly close to the scheme of the perception by the student.
In the second case the scheme of the explanation does not fit the student's scheme of the perception. The orientation of students' attention is not developed enough for making unclear clear independently.
In the third case the student clears up the information communicated by the teacher with the help of her/his own hidden orientation of attention that is superiority to the scheme of a discourse of the teacher). Such students manage their attention, they can also sometimes better than the teacher reconstruct the missing organization of visual attention in a visual field. At last, the fourth case, most adverse for the teaching - the student can not or does not try to make unclear clear. The reasons of that case may be the following:
loss of educational motivation, rejection of a perceived material because of its ambiguity;
absence of the students' own strict orientation of attention (anyway it developed enough for making unclear clear), though the educational motivation is not lost;
insufficient development of student's skills of the independent analysis and search (i.e. skills of mental activities);
absence of student's of methodological knowledge, which would allow her/him to have some general invariant, some kind of the scheme of perception.
What are the criteria of clearness? From above-stated, one can deduce the following features:
Distinctness;
Dissolution into parts (for example, of the course of reasoning);
Demarcation (for example, of the different symbolical systems: algebraic, vector, geometrical, trigonometrical);
Adequate orientation of attention, completeness of its scope for the basic objects;
Proper order of concepts and designations,
Sharpness, structure etc.
Important in the consideration of clearness is the question of its levels. According to the theory of learning activity, the mastering of knowledge occurs through the interiorization of knowledge externally developed in material or materialized form. On this basis it is appropriate to consider four levels of clearness of a mathematical contents for students:
The external clearness (possibly even only local), that means that externally everything is clear (precise, distinct), but the interiorization is complicated, the opportunities of translation of knowledge into the internal side are limited;
Clearness of the own scheme of mastering of that material, which is stated,
Internal clearness of the acquired material;
Clearness of expression (reproduction, interpretation and use) of the acquired knowledge.

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Creative mathematical activity of the students in the model of differentiated teaching in Russian Federation

Ildar S. Safuanov, Prof. Dr., Moscow State City Pedagogical University,
Valery A. Gusev, Prof. Dr., Moscow Pedagogical State University gusevvalmat@yandex.ru

Abstract. In this paper, creative mathematical activities of school pupils in conditions of the differentiated teaching in Russian Federation are described. Various forms of differentiated teaching (internal – level, external – profile) are characterized. Ways of using entertaining problems for detecting and fostering mathematical abilities are revealed. New course of geometry for differentiated teaching is introduced.

Creativity and differentiated teaching.

Creativity is one of the most important concepts in mathematics education. The creative approach is understood as certain abilities and readiness of a person for creating something new. The purpose of educational process at school is the education of a person who would use creatively approach for solving scientific or practical problems, would think independently and critically, a person able to develop and protect her/his opinions and beliefs, regularly acquire new knowledge by self-education, increases her/his qualifications, creatively use them for changing real world.
Among various school subjects, the mathematics is the best one for promoting the development of creative thinking, for encouraging the creative approach to the life in pupils.
In this paper we will consider the ways of promoting creativity in conditions of differentiated teaching.
There are various kinds of the differentiated teaching. Researchers distinguish the following kinds of the differentiation of teaching: internal (by level), external (profile differentiation), wide differentiation, diagnostical differentiation and continuous differentiation.
The internal, or level, differentiation of teaching is very important, as its means and methods penetrate entire teaching process at school.
G.V.Dorofeyev et al. (1990) interpret the level differentiation as such system of teaching at which each pupil, having acquired some minimum of the general educational preparation (being universal and providing opportunity of adaptation in constantly changing living conditions), gets the right and the guaranteed opportunity to pay primary attention to those directions which to the greatest degree answer her/his inclinations. Thus the level differentiation is expressed as follows: being taught in one classroom, under the same syllabus and the textbook, pupils can acquire the knowledge at various levels. Note that such internal differentiation is present also in all forms of external differentiation because at the level of the "selected" pupils their individuality also plays the important, and it is simply impossible to neglect it.
The question about level differentiation is closely connected with a problem of planning of obligatory results of teaching and with educational standards.
In our view, the elaboration of precise and practically usable standards for all school subjects still requires a lot of efforts.
It is possible to study and introduce various means of internal differentiation of teaching. However, the main complication here is connected with the coordination of mass forms of teaching and individual character of developments of pupils, of processes of mastering and application of knowledge.
Thus, the "internal differentiation" comes to the foreground. This kind of the differentiation of teaching is carried out in conditions of usual daily learning in a classroom. It is focused on all pupils and based leaning on their individual possibilities, needs and abilities.
Textbooks on the mathematics used in XX-th century, did not take into account mechanisms of level differentiation in educational process. They textbooks taking into account such mechanisms start to appear now. For example, authors include texts for individual reading (Gusev, 1999 and 2002, Sharygin and Erganjieva, 1992), much effort is applied for the differentiation of system of problems, etc.
The external, or profile differentiation of teaching in secondary school assumes promoting pupil's opportunity to receive education in various directions, under different curricula and syllabuses. A version of profile differentiation is profound studying of subjects. Note that this direction of profile

differentiation in our country has the wide experience, described in a lot of interesting books. The prominent mathematician, academician M.L.Lavrentyev wrote about the profile differentiation: “In fact the teacher teaches not an abstract pupil, but a quite concrete person with certain inclinations and abilities. I think, that it is useful as early as from the seventh grade to begin the differentiation of teaching on interests and propensities, that is to offer to pupils several optional courses... (Kobzev and Gorbachev, 1981).

Recognizing and fostering the creativity by solving entertaining problems.

An efficient way for recognizing and fostering the pupils’ mathematical abilities is solving entertaining problems, as their contents is usually clear for pupils at initial stages of studying mathematics. In the structure of these problems the exhibition of such parameters of mathematical abilities as sharpness, ingenuity, curiosity and inquisitiveness is incorporated. The mankind accumulated the huge quantity of problems useful as a material for revealing mathematical abilities, for the satisfaction of demands of the pupils possessing these abilities, and in general for the exhibition of the fascination of mathematics As a rule, these are not problems usually solved at school at the base level of mathematical education. It is a pity that these interesting, fascinating problems are insufficiently included in textbooks of this base level.

When there appeared the necessity to teach and learn mathematics, the mankind first of all addressed to entertaining problems and to mysterious stories: “To learn playing” was the first methodical instruction. This approach very much fits to the modern elements of the theory of motivation of teaching.

The popular author of books about mathematics Y.I.Perelman (1973) considered one of features of the entertaining science which, in its opinion, consists in that “its means do not exclude work of mind, rather, on the contrary, stimulate the work of thought”. Really, “brainwork is closely connected with acquisition of knowledge, and the entertaining science does not struggle to release from it at all. It helps only to make this work interesting, and, consequently, also pleasant, trying to deny a thousand-year old motto about “a bitter root of learning” (Ibid.).

Unfortunately, in practice of school it is not stipulated, to solve problems of entertaining character directly at the lesson (there is no direct indication in the curriculum, there are no recommendations in the educational literature, there is almost no appropriate material in textbooks). The teacher can independently decide to use or not to use such tasks, but “in fact for the majority of the people interested in mathematics, the first impressions of this science are connected with problems or books of entertaining character” (Ibid.).

In modern works of psychologists and researchers in mathematics education devoted to the study of cogitative mental activities in the process of acquisition of mathematical knowledge, not only the certain positive attitude to the entertaining mathematical tasks is expressed, but also the attempt to give the psychological and-pedagogical characteristics of various kinds of non-standard problems is made.

The psychological characteristic of an entertaining mathematical material (problems-puzzles) can be found in S.L.Rubinshteyn's works about the process of thinking. Describing the role of the processes of the analysis and synthesis in the solving of entertaining problems, S.L.Rubinshteyn (1958) argues that “so-called problems-puzzles are not the special funny thing, standing separately from the general laws of thinking... They are connected with the general laws of thinking”. Defining thus the nature of these problems, S.L.Rubinshteyn emphasizes the similarity of puzzles to creative tasks because both are composed on the basis of knowledge of laws of thinking, and he also draws attention to the fact that the essential conditions leading to the solution in puzzles are usually disguised by the non-essential circumstances: «... The puzzle arises when its statement specially emphasizes circumstances for insignificant its solution so that essential conditions of a problem become disguised, covered by insignificant, external circumstances» (Ibid.).

Characterizing the psychological side of the process of solving puzzles, S.L.Rubinshteyn emphasizes the role of the analysis in their solution, the role of a guess as the organic part of the process of thinking. On the basis of his experiments S.L.Rubinshteyn's revealed the “secret” of the occurrence of a guess during the solving. The guess is usually preceded by the careful analysis, distinguishing the essential conditions in a problem: «... In essence, a guess is a product of the analysis staying behind it» (Ibid.). Thus, in the opinion of S.L.Rubinshteyn, the solution of problems-puzzles occurs as a result of the precise analysis of their conditions during which search of the ways of solving is carried out.

Among few works devoted to entertaining and non-standard problems, worth of attention is a series of experimental studies directed by A.N.Leontyev (1954).

A.N.Leontyev put a problem of finding of a specific part of mental activity consisting of the emergence of a guess, of the idea of the solution. Experimental works performed under his direction were aimed at finding out the conditions under which "the experience of the pupil leads him to the correct solution based on the guess " (Ibid.).

In the research devoted to the analysis of problem solving, G.Polya (1957) distinguishes "problems on finding" and problems on proof [177]. The purpose of problems of the first kind is to determine an unknown element of a problem. G.Polya considers puzzles as problems on finding. For solving them he recommends to use the general rule for all problems on finding: to understand a problem, to choose, recollect auxiliary problem, to solve a part of a problem, to keep only a part of conditions, omitting the rest.

B.L.Kordemsky (1958) emphasizes special significance of problems-puzzles in developing essential elements of mathematical thinking in pupils, promoting the aspiration to search independently ways and means of solving a problem; fostering pupils' ingenuity, flexibility and criticality of mind.

Thus, problems of entertaining character can serve as the tool for revealing parameters of mathematical abilities of pupils and a good mean for stimulating pupils' interest to studying mathematics.

Taking into account the wide variety of various kinds of entertaining, joyful problems, for the maintenance of their purposeful and effective use some classification of entertaining problems would be useful. Consider some classifications available in the methodical and mathematical literature.

G. Lenhauer (1940), telling about the hall of mathematical entertainments in Saint Petersburg, shows sets of various entertaining mathematical problems. They are divided into following groups:

Problems not demanding very little mathematical knowledge and based on the ingenuity and a guess.

1. Problems demanding elementary mathematical knowledge or forcing to recollect knowledge once received at school.

3. Questions and problems which aimed at checking mathematical knowledge of a pupil. These are mainly unexpected comparisons and conclusions, sometimes paradoxes etc. This group is divided into three series according to the age of pupils.

4. A series for enthusiasts of difficult and witty mathematical problems. These problems demand for their solving decent mathematical preparation, however not exceeding the volume of a school course.

5. Problems for children of 8-12 years.

6. Problems-jokes, mathematical focuses and entertainments.

M. Gardner (1978) has divided problems into six categories: combinatorial, geometric, number-theoretical, logical, procedural and verbal.

He notes that these categories of problems are not disjoint, they inevitably intersect with each other.

B. Kordemsky (1958) distinguishes two categories of problems for out-of-class activities.

1. Problems close to the school course of mathematics, but of the increased difficulty – this is the kind of problems usually offered at mathematical Olympiads.

2. Problems of the type of mathematical entertainments. B.L.Kordemsky wrote about the second category: "The second category of problems for out-of-class activities (very mixed in their contents) does not have the direct connection with the school program and does not assume the strong mathematical preparation" (Ibid.)

Logical and geometrical problems can be successfully used for revealing and developing the mathematical abilities of pupils. From our experiments, we deduced following conclusions:

1. It is important to reveal pupils' skills to make conclusions from the problem's conditions.

2. It is important to track and develop the activities of the type "synthesis through the analysis". Pupils, making conclusions, are guided by an ultimate goal this process goes. However, this process goes chaotically, it is not subordinated by the general idea, but only uses rather standard analysis.

3. It is important to find out, whether the pupil is able to plan from the very beginning the strategy of the solution - to put forward an idea and to apply non-standard methods and means to the solution.

4. Often enough, planning the strategy of the solution, pupils choose already known, often used ideas. It is interesting to track a birth of a non-standard idea (during the process of the analysis through synthesis).

5. It is important to develop the pupil's skill to use analogies.

New course of geometry.

In our new geometrical course for the secondary school (Gusev, 1999), the process of development of means of thinking activity is carried out through a specially selected system of exercises. Thus the directedness of this system of exercises on shaping of analytic and synthetic activity is determined not only by the contents of these exercises, but also by structuredness of questions, tasks and pieces of knowledge distributed in six groups, each of which has precisely determined purposes. The system of exercises begins with two groups of problems: [↑] - exercises on development of skills to make correct conclusions of a condition of a task (to get corollaries) – this is a kind of synthetic activity; [↓] - exercise on finding out the reasons of a property of an object, i.e. the detection of characteristics of an object: this is one of the foundations of analytical activities. These two groups of exercises constitute the foundation of all thinking processes, all reasonings and proofs.

In each section of a text-book, there are other two groups of tasks, designated by characters [T] and [I]. [T] is a group of tasks, which have somewhat higher level of difficulty, namely tasks aiming in the development of the creativity in pupils. The sign [T] includes the character T which is the first character in a word «creativity» (tvorchestvo), and one may name these tasks "creative".

[I] designates the last group of problems (or tasks), the number of which will be not so much; they require more effort for the solution and assume doing some small research (sometimes this research can connect various themes of a course and even of various courses). Such problem tasks can not be completely solved in a classroom, and assume serious homework (sometimes, of several pupils). The character I is the first character in a word «research» (Issledovanie), and the meaning of these problems is "research" tasks.

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Innovations in Podcasting and Screencasting Course Materials To Bring Mathematics to Life

Paula Savich, MS PhD candidate
Instructor of Mathematics
Mayland Community College
Spruce Pine, North Carolina
USA
psavich@mayland.edu

Sandra Pierce, MS
Math Programs Coordinator
Mayland Community College
Spruce Pine, North Carolina
USA
spierce@mayland.edu

Abstract

Online and other forms of distance learning are a permanent fixture in the educational landscape. Mathematics taught in distance formats pose an even greater challenge to students and teachers alike. As mathematics is a skill subject, demonstration of concepts and processes is crucial, if not critical, to learning, particularly to visual and kinesthetic learners. Video podcasts and screencasts are the answer to distance students' need for demonstration and explanation of mathematical topics. In the current economic climate, however, expensive audio/video capture software and hardware, as well as a lack of technical media support, make it virtually impossible to create such course materials. Also, there is the question of ownership of intellectual property if created with institutional funds and/or resources. Free capture software and internet video hosting sites make it possible for an individual to create his or her own podcasts and screencasts for student use, retaining ownership of the created materials.Materials developed for online students can be made available to students in seated class. This benefits students who are unable to attend class, but can be made available to the entire class.

Workshop Summary

The primary purposes of this workshop are to demonstrate the creation of podcasts and screencasts for classroom use and to present data demonstrating student success rates where podcasts and screencasts are used in traditional seated classrooms as well as the online teaching environment. Concentration will be on the use of freeware or other inexpensive software for the creation of podcasts and screencasts, as well as free video sharing websites to make them available to students. While in the workshop, participants will be invited to create a short podcast or screencast and post each to the internet.

The workshop involves the use of technology for innovative teaching practices and utilization of new paradigms in teaching and learning. Podcasts and screencasts can be used in both seated and online learning environments. Both instructors have extensive experience in teaching online with a variety of platforms as well as nearly 50 years of combined classroom teaching experience. It was found that the materials originally prepared for distance learning classes were beneficial for students in a seated class for the review of concepts, or for students who were unable to attend the original lecture.

History of Podcasts and Screencasts

Podcasts are short digital media videos originally created for the small screen of the Apple iPod and available by download via web syndication. Despite the source of the name, it is not necessary to view podcasts from an iPod or other portable device. Podcasts can be viewed on any desktop or laptop computer that can display media files. As a result, the term "podcast" has been backronymed to mean, "**P**ersonal **O**n **D**emand broad**CAST.**" Podcasting came to the technological forefront somewhere around 2004, but the ability to distribute audio and video files has been around since the birth of the internet. In 2006, the Apple Trademark Department went on record stating that Apple does not object to third party usage of the term "podcast" and that Apple does not license the term "podcast."

The following is a logo that indicates a podcast:

Screencasts are digital media recording that capture the output of a computer screen, with optional voice-over. Another name for a screencast is a "video screen capture." Just as a screenshot captures a still image of a computer screen, a screencast captures the screen and all the changes over time, much like a video or a movie. Screencasts were originally developed to teach users the features of new software. Since 2004, the screencast has gained widespread use. Screencasts have a wide range of possibilities. For example, one could screencast a Power Point presentation, adding the voice over. This is particularly useful to individuals who do not desire to appear on camera.

How to Produce a Podcast/Screencast

This presentation is merely an introduction to podcasting to encourage you to step out and do it yourself. You will also be given some free resources. Teaching mathematics online basically means that the students are essentially on their own. Even with excellent resources, including a readable text, written lecture notes, summaries, email after email, students still need more. But it is easy to feel like the online students do not receive the same quality of instruction that the students in seated classes receive. Online students needed face to face assistance, particularly for visual and auditory learners, just like a traditional seated class. This is where podcasts and screencasts have their **greatest value.**

A few things you need to remember when recording a video are:

- Keep each podcast 20 to 30 minutes, if possible. If it's a particularly long lecture, break it up into two parts. Just assume all your students have Attention Deficit Disorder (ADD).
- Remember that an ipod screen is small. So, if an individual is actually viewing the lecture on an ipod, the text must be as large and as readable as possible.
- Have the examples prepared ahead of time on overheads to use with a document camera, if available (show examples) to cut down on time. You can use multiple colors to emphasize points, but avoid red and green (there can be disability issues over color blindness).
- Rehearse the lecture or have a script prepared in order to avoid stammering and stuttering, using to many uh's, etc. It also helps to use the technology prior to recording so you are familiar with how everything works. If it makes recording any easier, recruit a few students to have in front of you and teach to them instead of the camera, particularly if you are a bit camera-shy.
- Don't wear white or red, checks or stripes. It makes the camera crazy.
- Be careful when including video clips or music on your podcast to avoid copyright infringement. Get permission prior to recording or use public domain materials.

A professionally produced podcast requires a studio, technicians, digital capturing devices, and expensive editing software.

There are several options for capturing the raw video.

Digital video camera. This requires a technician or additional person to operate the camera and make sure it is focused and positioned correctly. Later, the images must be edited on editing software.

Debut (http://www.nchsoftware.com/capture/index.html) open source freeware for Windows. It is a simple, easy to use video recorder program that lets you capture video files directly using a webcam (video camera) or capture device (from video). It can also record almost anything that can be played or displayed on the computer screen. The program automatically saves the video on your hard drive as avi, wmv or other file types.

CamStudio (http://camstudio.org/) open source freeware currently only available for Windows computers. CamStudio is able to record all screen and audio activity on your computer and create industry-standard AVI video files and using its built-in SWF Producer can turn those AVIs into lean, mean, bandwidth-friendly Streaming Flash videos (SWFs).

Video Monkey (http://videomonkey.org) is open source software for Macintosh computers that can convert video to other formats. For example, if you have produced a screencast in AVI format and you want to distribute it using iTunes, you can use Video Monkey to convert the screencast. The developers of this software plan to support several different formats, but at this time only Macintosh formats are supported.

Next, if editing is necessary, there are several things to remember:

Once the podcasts are recorded, if you want them to be polished and maybe a bit flashy, then they can be edited. This can be as simple or as elaborate as you can handle. You can add music to your podcast, however, be careful of copyright infringement. You must use public domain music or pay royalties.

One good source for free music is Freeplay Music (www.freeplaymusic.com). It's a good idea to write Freeplay and let them know what you want to do with the music. Make sure they know it won't be used for general public broadcast viewing, that it will be on a website for only your students to use. Their website says that if you are using Freeplay Music for educational, non-commerical use limited to student use on school grounds for in classroom projects - nonbroadcast, then you can use their recordings for free. For further information contact them at 212-974-0548 or email Julie@freeplaymusic.com or Rox@freeplaymusic.com.When editing, you need a fairly powerful computer, one with at least a 500 gb hard drive. Most video is recorded in SD (Standard Definition). However, if you record in HD (high definition), remember, that HD video is 4x the size of the SD video.

Again, to do any fancy editing, you need to have video editing software in order to do any cutting, pasting or adding to your raw feed. It helps to be able to edit in order to take out any bloopers you recorded, thus eliminating the need to start over recording a particular lecture if you make a mistake. You also may need to compress the video before putting it out there for viewing. These skills are way beyond my expertise.

There is editing software for both Windows PC and Mac. Here are a few and what they cost.

PC:

1. Windows Movie Maker - Cost Free - Windows Movie Maker 2 is an XP-only download that gives you the tools to create, edit, and share home movies. Compile and edit a movie from video clips , add special effects, music, and narration.
2. Adobe Premier Pro CS4 - Cost $349 USD - Edit files from the latest tapeless formats. Find sections of content quickly for editing. As an educator, you qualify for educational pricing.
3. Adobe Premier Elements - Cost $139.99 USD - Adobe® Premiere® Elements makes it easy to create incredible movies with automated moviemaking options, add visuals and sound, and share your movies everywhere. With the additional Photoshop.com Plus membership, you can turn video clips into polished movies with special effects, back up on 20 GB of memory storage and access your video from virtually anywhere.

Macintosh:

1. iMovie - Cost $79.00 USD - Comes with Apple's iLife when you purchase a new Macintosh computer - iMovie makes it quick and easy to browse your library and create new movies. And iMovie is built for sharing. You can easily publish videos them on a website, and create versions for iPod, iPhone, and Apple TV.
2. Final Cut Studio 2 - Cost $1,299 USD - This is the industry standard, as many full length feature films currently in the theater were done completely on this software. This software is probably overkill.
3. Final Cut Express 4 - Cost $199 USD - Import video and edit like a pro, add special effects, titles and music. This is a scaled down version of Final Cut Studio 2.

Once you finish any editing and enhancing, then there are several different video formats you can use.

Flash - this is a smaller file size, good for a slow computer connection.

Quicktime - This is a larger file format with higher quality resolution than flash.

iPod download - You can subscribe to a podcast feed that will allow you to download the videos to your iPod or other video player for viewing.

Once the podcast is finished, you must store it somewhere that gives students access, yet protects your intellectual property from those who would just take what you have done, perhaps selling it to others.

YouTube will only allow you to save 10 minutes of video. Also, on a site like YouTube, you have no control over who views it. Anyone can access it, and anyone can steal it. Google video is basically YouTube.

iTunes now has iTunes U for educational purposes. It is free (at the moment). You must sign a contract for storage space and they handle distribution. It takes the load off the home server and band width. Students must subscribe and have a user name and password to access your podcasts to protect the content.

Vimeo (http://www.vimeo.com/) is a site that allots you 500 mb per week of video. Podcasts are generally about 50 – 60 mb per video, so this site is perfect for this. It also allows you to mark your videos private and require a password to access them. This way your intellectual property and hard work is better protected from being stolen and used without your permission.

Creating your Podcast

We will create and upload a podcast at the conference with a laptop, webcam, CamStudio and Vimeo.

Results

When we did this we found that at least some of our students made very good use of the material. Some of them reported viewing the podcasts multiple times to help them with material that they were having difficulty with. Based on this experience, we have decided to expand this idea and develop podcasts and screencasts for more of our classes. Even though this is still at a beginning stage, we felt it was worth sharing with others.

Concept Literacy in Mathematics and Science: experiences with the development and use of a multilingual resource book in Xhosa, Zulu, English and Afrikaans in South Africa

Marc Schäfer, BSc, H.Dip.Ed., BEd(h), MEd, PhD (Curtin)
Associate Professor in Mathematics Education, Education Department
Rhodes University, Grahamstown, South Africa M.Schafer@ru.ac.za

Introduction

It goes without saying that the understanding of key concepts in mathematics and science is fundamental to the teaching and learning of these disciplines. Research confirms that one of the key dimensions to understanding concepts is language. The intimate relationship between language and the understanding of concepts is well documented. For example, the poor performance of South African learners in the 1995 and 1999 TIMSS is largely ascribed to the problem that learners and teachers have in studying and teaching through English as a second or even third language. To address this problem a multilingual learning and teaching resource and support book (Grade 9 – 10 levels) was developed at the Centre for Applied Language and Literacy Studies and Services in Africa (CALLSSA) at the University of Cape Town in collaboration with Rhodes University and the University of Kwa-Zulu Natal. The book provides detailed meanings and explanations for key mathematics and science concepts in Zulu, Xhosa, Afrikaans and English. It is argued that when learners and teachers have access to these concepts in their own languages, they can transfer such understanding to their dealing with English as the language of learning and teaching (LoLT). The book was validated through a collaborative process involving the three universities. The validation process was enhanced by a research process of trialing and evaluating the book in classroom practice. This *inter alia* included an investigation of:

- the accuracy of the concept explanations in the four languages used;
- the appropriateness of the translations;
- the general effectiveness of the book as a learning and teaching resource.

The research involved the participation of Grade 10 teachers in the Western Cape, Eastern Cape and Kwa Zulu Natal of South Africa. This paper aims to share some of the experiences encountered in the development of this book by briefly describing the development process and the content of the resource book, and also highlighting some of the research issues that were encountered with special reference to code-switching practices as a central pedagogical strategy in many South African classrooms.

Copies of the book will be distributed at the presentation for discussion.

A concept literacy resource book

The problem of language proficiency as an obstacle to learning mathematics and science is well documented (Adler 2001, Howie 2002, Setati 2005). Young, van der Vlugt and Qanya (2004) suggest that this problem can be addressed at two inter-related levels:

a) concept understanding and use, and
b) language/discourse contexts and forms in which these concepts are embedded.

The notion of **concept literacy** that framed the development of the multilingual resource book can be described as “understanding, through reading, writing and appropriate use, basic learning-area specific terms and concepts in their language contexts” (Young at al, 2004). Kilpatrick, Swafford and Findell (2001), describe conceptual understanding as a critical component of mathematical proficiency that is necessary for anyone to learn mathematics successfully. Conceptual understanding implies an understanding of knowledge that not only revolves around isolated facts but includes an understanding of the different contexts that frame and inform these facts. Kilpatrick et al (2001) suggest that “students with conceptual understanding …have organized their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know”.

The idea of a concept is controversial and difficult to define. It ranges from a personal idea or construct to a statement that is universal and generic. The definition that underpins the resource book suggests that a concept is a "mental picture which has a standard and universally accepted meaning" (Young et al., 2004). Similarly the notion of literacy is difficult to pin down as it no longer simply refers to the ability to read and write. Young et al. (2004) argue that literacy implies a capacity to recognize, reproduce and manipulate the conventions of text, spoken and written, shared by a given community. Concept literacy therefore emphasises the interaction between context and content. It is a process that is dynamic and that changes over time as the concept is internalized and understood. From a Vygotskian and constructivist perspective this implies that concept literacy involves the modification of prior conceptions and experience. Fundamental to this process is of course language proficiency. Young at al. (2004) correctly argue that it is therefore likely that modifying one's prior knowledge if one is learning in an additional language (a language other than the first language) can be problematic, particularly if one is not proficient in that additional language.

For most South African teachers and students of mathematics and science the LoLT in these disciplines is English – an additional language that for many is difficult to understand and use. To address these difficulties CALLSSA embarked on writing a learning and teaching resource and support book for Mathematics and Science. This book provides detailed meanings and explanations for key concepts in Xhosa, Zulu, Afrikaans and English within the framework of the Revised National Curriculum Statement (RNCS) at a Grade 9 – 10 level. It lists about 60 key RNCS mathematics and science concepts that are grouped under the themes time, space and number for mathematics and energy, matter, earth and life for science. Very importantly, attention is also given to the everyday English meanings of these specific concepts. As Young et al. (2005) note, the words and language forms of mathematics and science often differ markedly form those of everyday use of the same words. For example, concepts like power, force, revolution, work and pressure have very different everyday meanings to their specialized meanings in mathematics and sciences.

The development of the book was a collaborative process with teachers of mathematics and science in the Western Cape, Eastern Cape and Kwa-Zulu Natal, and validated by expertise in mathematics and science education, Xhosa, Zulu and Afrikaans from across South Africa. In its introduction the book acknowledges some of the dilemmas that were faced in translation and explanations of concepts:

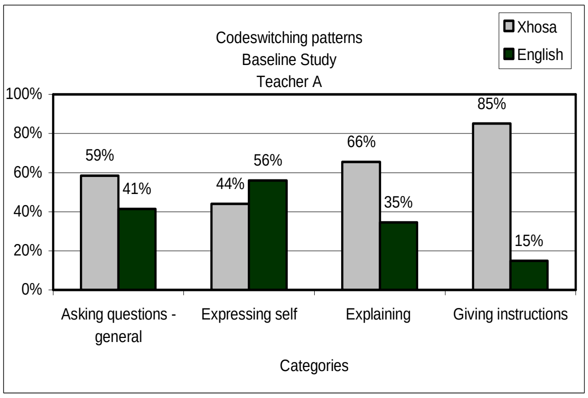
> We are very aware of questions about which Xhosa or Zulu words or terms for these concepts are 'correct', standardized forms. We have, wherever possible, tried to ensure that our uses of both Xhosa and Zulu are correct. Until these two languages, and other African languages, are fully standardized, our text must serve as an interim attempt to offer translation equivalents in Xhosa and Zulu for English concepts dealt with in this book. We think it is better to present work close to the ideal as a starting point rather than to have nothing available!

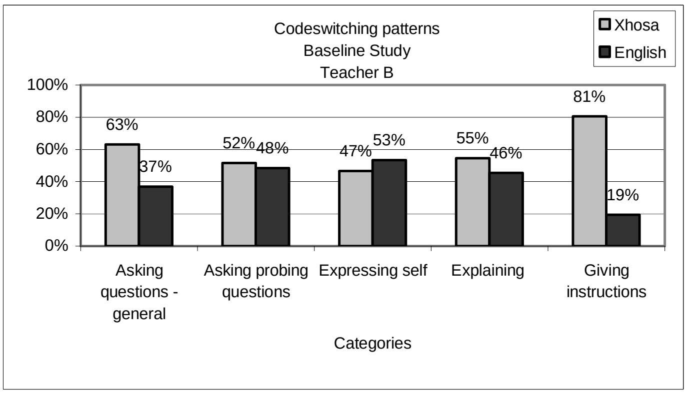
Teachers from across the three provinces participated in the development phases of the book by trialing sections of the initial manuscript and providing feedback on their experiences. Lessons were videotaped and deconstructed with the participating teachers. Issues such as inaccuracies in translations and the use of inappropriate and inaccessible diagrams were identified and noted. These were then incorporated in the final version of the book. The book was marketed in all the provincial education departments of South Africa and those provinces where Xhosa and Zulu are particularly prevalent, have ordered copies for their teachers.

Code switching and some tentative research results

South Africa is a multicultural and multilingual country with a diversity of 11 official languages. Although the Language-in-Education Policy insists that the LoLT in the first four years of schooling is mother tongue, the use of code-switching is common practice in most schools where

the home language of teachers and learners is not English. In South Africa these are mostly schools that, in the apartheid years, were classified as black- or township schools. Code-switching is the practice where "an individual (more or less deliberately) alternates between two or more languages" (Baker, 1993:76 – 77). As Setati (2005:91) notices, code-switching "can be between languages, registers and discourses". In the South African classroom, code-switching would typically involve an indigenous language and English. Despite the fact that the medium of instruction changes to English after Grade 4, the practice of code-switching is often sustained for the entire duration of schooling. It is argued that code-switching can be a powerful and effective pedagogical tool to overcome language barriers to teaching and learning. As Setati and Adler (2001) noticed, many teachers in South Africa have a dual task of teaching both mathematics and English at the same time. It goes without saying that by the same token, learners also have to cope with the language of mathematics and the language in which it is taught – and this in many instances is a second or even third language.
Recent preliminary pilot research by Tokwe and Schafer (2009) explored how the Concept Literacy book in question impacted particularly on the code switching practices of selected Xhosa speaking Grade 10 mathematics teachers. Four teachers (A, B, Y and Z) were involved in the pilot study. Their code switching practices were observed and documented over a number of lessons before the Concept Literacy book was introduced to two of the teachers, Y and Z. After using the book over a period of a two terms their code switching practices were once again observed and documented.
The figures below illustrate the code switching practices of teachers A and B.





It is interesting to note that when giving instructions, both teachers preferred to use the vernacular. This trend however changed when the teachers started to explain and illustrate mathematical concepts and terms. The use of English became more prominent and the practice of code-switching increased. This is illustrated in the following conversation:

Consider the situation whereby *siza kuthatha ii* **triangles** *zethu ezimbini sizibeke* **on top of one another**. **What I'm trying to say is this** [drawing 2 triangles adjacent to each other]. Translation: **Consider the situation whereby** *we will be taking our two* **triangles** *and put them* **on top of one another**. **What I'm trying to say is this** *[drawing 2 triangles adjacent to each other]*

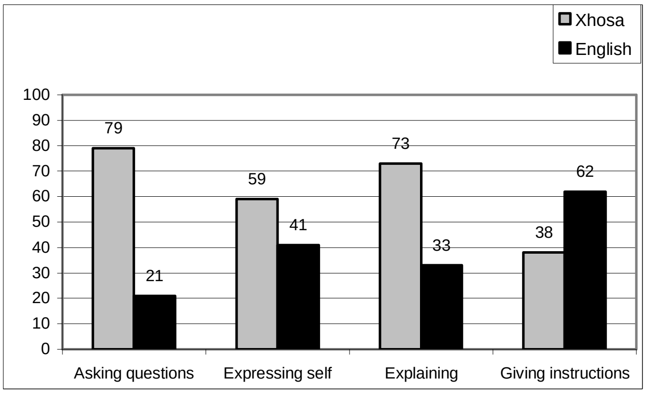
If you say now all angles of a triangle are equal, *ingaba i* **angle** *inye kuzo* errr *ndicinga ukuba…………. Ingaba inye iza kuba* **how many**? (Teacher and learners respond simultaneously.) "*Ngu* 60 **degrees**." Translation: **If you say now all angles of a triangle are equal**, *is it that one of the* **angles** *err.. I think that…………. How many will one of them? (Teacher and learners respond simultaneously.) "It is 60* **degrees.***"*

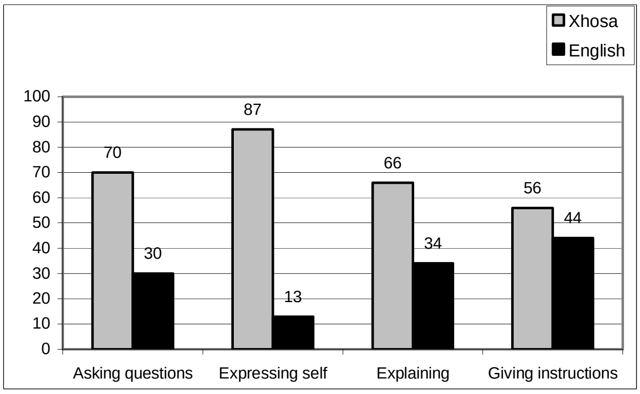
"In other words, *ukuba siza kuthi le yi* **parallelogram**, **so that means** *eli cala lingapha liza kuba* **parallel** *kwela cala lingaphaya* **and** *eli lona libe* **parallel** *kweli lingaphaya."* Translation: "**In other words**, *if we say that this a* **parallelogram**, **so that means** *this side here will be* **parallel** *to that side on the other side* **and** *this one will be* **parallel** *to the one on the other side.*

The above scenario is however not surprising if one considers the lack of a mathematical register in Xhosa and the dearth of mathematical resources and texts in that language. There are many constraints in mother-tongue education, as Probyn (2002:10) states "…there are [numerous] linguistic and economic constraints on mother-tongue education: the fact that indigenous languages have not been used for academic purposes means that the necessary terminology and textbook resources have not been developed".

Teacher Y and teacher Z displayed very similar practices. When going to press the data was not yet analysed and processed sufficiently to illustrate these practices graphically.

The figures below illustrate the code switching practices of teachers Y and Z after the Concept Literacy Resource book had been used over a period of time.





It is interesting to observe that after the Concept Literacy Resource book intervention, the use of Xhosa increased markedly for the following categories of communication: asking questions, expressing self and explaining. Notwithstanding the small sample and the tentative nature of the pilot, this suggests that the Concept Literacy Resource book had an impact on the code switching practices of the participating teachers. Their use of their first language increased and they appeared more confident in using Xhosa in mediating mathematical concepts.

In general, the Concept Literacy Resource Book was well received and initial classroom visitations revealed the following:

Deep Xhosa vs everyday Xhosa. A number of the teachers felt that the Xhosa that was used in the resource book was at times difficult to understand. They felt that the translations were dominated by 'deep' Xhosa – sometimes also referred to as rural, old, traditional, formal Xhosa as opposed to 'township' Xhosa – also referred to as everyday, modern Xhosa. According to the teachers, many of the learners expressed similar sentiments.

Inconsistent use of Xhosa. In some instances it was felt that the translation used in the text was not consistent with some of the dictionaries that the teachers had access to (Schäfer, 2005).

Assistance in conceptualization. A number of teachers said that the Xhosa text assisted in their own conceptualization of a particular concept. This also applied to many of the learners who were provided with photocopies of the text in various lessons (Schäfer, 2005).

More comprehensive translation. There was widespread consensus that the entire book needed to be translated into Xhosa and not just the key concepts (Schäfer, 2005).

Texts in mother tongue. Many of the teachers felt that they themselves were not aware of the existence of some of the Xhosa terminology and were surprised when they encountered some of the terms in their own language. There was consensus that a standardized Xhosa mathematical register needed to be developed as soon as possible. There was a strong commitment from the teachers to the preservation of Xhosa and many felt that it was important to teach through the medium of Xhosa. It was however also recognized that in an era of globalisation and market driven economies, the dominance and power of English cannot be ignored (Schäfer, 2005).

Resistance to Xhosa. There was resistance to the use of Xhosa by some learners. They felt that English was the international and dominant language and hence they needed to learn mathematics and science in that language. Incidentally, numerous teachers commented that a similar sentiment existed amongst some parents, who felt that teaching should be done through the medium of English and not through the mother tongue (Schäfer, 2005).

Support of textbooks and other learning areas. The resource book was used to support the textbook in lesson preparation and implementation. Some teachers photocopied pages out of the book to hand to the class (Schäfer, 2005).

Conclusion

Our research into the use of the Concept Literacy Resource book shows that a multilingual text of this nature is long overdue and could play an important role in enhancing the role of indigenous language in the teaching and learning of mathematics in South Africa. The development of a mathematics and science register in all indigenous languages is fundamental to the realization of the vision that asserts that each child should have the choice of his/her language of instruction.

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Single Sex Mathematics Classes: A Critical Analysis of the Impact at a Secondary School

Angelique Seifert, Ph.D., Charlotte-Mecklenburg Schools, Charlotte, NC, USA
David K. Pugalee, Ph.D., University of North Carolina at Charlotte, Charlotte, NC, USA
Angelique.seifert@cms.k12.nc.us; David.Pugalee@uncc.edu

Abstract
Single sex classes have recently been emphasized as an effective way to promote mathematics learning. Despite their popularity, the research on the effectiveness of such programs is mixed underscoring the need for additional research and discussion. This research is set in one of the twenty-five largest public school systems in the United States, where schools have recently been allowed to begin instructional initiatives with same sex classes in mathematics. Preliminary data on the effectiveness of one program will be highlighted. Achievement data, compared to traditional classes, will be considered to demonstrate the academic effectiveness of the project. Qualitative data analysis will provide a rich description of the affective issues relative to this innovation. The current project will be framed in critical analysis of the research literature and will discuss the potential benefits and disadvantages both from this current project and from the related literature.

Introduction

Single sex classes have been gaining the attention of educators across the United States. Such efforts fly in the face of coeducational proponents who argue that single sex classrooms reflect the real-world interactions required of students and are more likely to prepare students for cross-gender interactions and eventual integration into society (Mael, 1998). Schools are attracted by the purported claims that such classrooms hold the answer to poor academic results in mathematics as well as other subjects.

Research Perspectives

Research on single-sex classes is mixed and is marred by a large number of poorly conceptualized and executed studies. Salomone (2006) posits that the relationship between program planning, implementation, and assessment should guide the exploration of questions and methods related to such programs. Hubbard and Datnow (2005) conducted a systematic review of the literature where outcomes in the majority of cases were related to short-term academic achievement and short-term socio-emotinal development. The review found mixed results with 53% showing no difference and 10% showing mixed results. Likewise, Lee (1998), focusing on the school as the unit, found no consistent patterns of effects for promoting either single-sex or coeducational schools for boys or girls. Riordan (2002) posited that the effects of single-sex classrooms are relatively small when compared to the effects of socio-economic status and curriculum variables.

Hubbard and Datnow (2005) found in their anthropological study of single sex classes in California involving low-income and minority students schools' successes were more likely due to interrelated contributions of the organizational characteristics of the school, positive student-teacher relationships, and sufficient resources.

Little is known about the practices that comprise instruction in single-sex classrooms. Martino, Mills, & Lingard (2005) found that teachers in single-sex environments frequently modified their pedagogical practices and the curriculum to respond to stereotypical constructions about boys and girls perceived oppositional orientations to learning. Teachers' knowledge and assumptions about gender influence how they execute pedagogy in single-sex classrooms. Many of the instructional practices supported in single-sex classrooms could be found in any effective school (Bracey, 2006). One of the criticisms leveled about the research of same sex classroom initiatives is that they are poorly controlled research designs (Salomone, 2006) The lack of quality research designs makes it difficult to isolate variables that may impact the implementation of single sex classroom practices. .

Research Design

Research Questions

This study investigated preliminary findings related to the effectiveness single sex classes in algebra. More specifically, the study sought to respond to three broad questions:

1. Do single sex algebra classes positively affect the behavior of students?
2. Do single sex algebra classes positively affect the academic performance in algebra of at-risk students?
3. How do students in the single-sex algebra classes describe their experiences?

School Setting

The current study is set in an urban school district in the southeastern part of the United States. The district serves over 134,000 students in 172 schools. Students represent 161 different countries and 140 native languages. The secondary school which is the focus of this study serves approximately 2,500 students. The school population is diverse with 51% white and 37% black students with 27% eligibility for free or reduced-price lunch which is a federal program to assist households in low socioeconomic levels. The school has initiated freshmen academies to address challenges unique to ninth graders. In addition, single sex classes are being piloted in algebra and English. This instructional initiative has one male and one female algebra class with enrollment limited to students with lowest levels of achievement on eighth grade state-mandated mathematics scores (level 1 and level 2). Level 1 is defined as performance illustrative of two plus years below grade level and level 2 describes the achievement level of those students performing one year below grade level.

Procedures

First, student behavior data consisting of discipline referrals to the school administration and out-of-school suspension records for all level 1 and level 2 students (lowest levels of academic performance) for their eighth grade and first semester of ninth grade. Data for students in the single sex classrooms was disaggregated for comparison purposes.

Second, first semester exam data for Algebra 1A will be used to compare academic performance. The first semester exam is a district mandated assessment created at the school site as a common assessment administered to all Algebra 1A students. Scores for all students in the specific high school who were at level1 or level 2 upon entrance to ninth grade were obtained. Scores were disaggregated to compare the performance of the single-sex Algebra 1A students to the scores for the level 1 and level 2 students in coed Algebra 1A classes.

Third, open-ended surveys will be used to elicit responses from the students relative to their experiences within the single-sex algebra classes. The questions were formulated and field-tested by the researchers. Data from the surveys will be coded based on searching for themes and difference in the responses.

Results

School site administrators decided to initiate two single sex algebra classes during the 2008-2009 school year. It was believed by some administrators that single sex classrooms could reduce discipline problems and improve academic performance.

The rationale supporting the creation of the program was formed by a review of results from a similar initiative in some South Carolina schools, a desire to better serve at-risk students, and an interest in implementing instructional strategies which recognize gender differences in how students learn. The following table provides the means and standard deviations for the final numerical grades for Algebra IA. Algebra IA is the first part of the Algebra IA and Algebra IB sequence which earns the student a credit for Algebra I.

Table 1. Final Algebra IA Averages for Classes

	Male Algebra IA	Female Algebra IA	COED Algebra IA
Mean	73.38	77.58	64.6
SD	10.3	8.89	14.5

The same teacher taught a coeducational section and either the male or female same sex class. The class means are based on the final course averages which includes a school constructed (common assessment) final exam which is administered to all students at the completion of Algebra IA instruction. Therefore, all Algebra IA students at the school take the same final exam.

The coed scores are based on two sections. In one of those sections, three students out of 18 received a zero on their final exam. If we omit those scores from the data, the average for the coeducational classes is 72.75. This would raise questions about the efficacy of the reported results and the effectiveness of the instructional initiative. The 72.75 average is comparable to the all male class average of 73.35 and approximately five points lower than the all female class average.

Behavioral data in the form of number of referrals (in-school suspension and out-of-school suspension) is being collected for the entire academic year. Comparisons of behavior among the various class types will be made. Initial analysis of first semester data reveals little difference in overall number of referrals but shows a significant difference in the number of referrals when looking at male versus female with twice as many suspensions in each category.

Qualitative surveys will be administered at the end of the academic year. The qualitative surveys will provide data about the perceptions of the students and their teachers regarding their experiences in the single sex class.

Conclusions

Advocates of single-sex classrooms believe that responding to gender differences is a positive step in addressing the diverse needs of students and empower students to succeed. The researchers do not disagree with this basic premise; however, differentiation of instruction might be best conceived and implemented in all classrooms to reach all students. Hubbard and Datnow (2005) reported that the majority of single-sex classroom experiments served primarily nonwhite and high poverty students in urban areas. From a critical perspective, one might question the underlying motivation for such studies as perpetuating a bias toward low income students and more specifically towards minority students.

The data do not provide a substantive basis on which to base programmatic decisions relative to single sex classes. Despite the limited data and the incompleteness of the analysis process, the school is moving forward to expand the instructional initiative to additional sections of algebra and English, and include social studies and science classes for ninth graders. The researchers recommend caution in basing decisions about programs on limited data, particularly when potential variables such as teacher efficacy for particular approaches and the potential differential pedagogical approaches are not controlled for in designing studies on which to inform decisions about structuring of classes.

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Visual Modeling of Integrated Constructs in Mathematics As the Base of Future Teacher Creativity

Smirnov Eugeny, PhD Habilitation, Professor of Mathematics and Mathematics Education,
Department of Mathematics, Yaroslavl State Pedagogical University,
Burukhin Sergei, PhD Habilitation, Professor of Physical Training, Department of Physical Training,
Yaroslavl State Pedagogical University, Respublikanskaya, 108, Yaroslavl, Russia,
Smirnova Irina, Master Degree, Yaroslavl Theatre Institute, Pervomaiskaya, 45, Yaroslavl, Russia
smirn@yspu.yar.ru

Abstract

Visual modeling concept of integrated constructs (essence) of mathematical objects in teacher training of humanistic area is presented as technology of education in problem solving. The main goal of innovative approach is student's activity in mathematics on generating of concrete essence manifestations on concepts, methods, theorems, algorithms, procedures and so on. Such student's activity should be:

- Success in an area of actual interests and person's experience and reached by perception;
- Have high level of variability in visual modeling;
- Success in domain of reflection process stimulation.

Similar creative behavior of persons is typical for actors, dancing, and figure skating and so on.
Now we show that such technology will be fruitful for teacher training in mathematics for humanistic specialties.

Introduction

Actualization of productive links between mathematics and social sciences as subjects taught at high school, being the basis for vocational education of students are responsible for powerful humanitarian potential, determining processes of socialization and adaptation to the changing phenomena of world around as well as stimulating development of intellectual forces and personal qualities of students. Social sciences (foreign languages, for example) always aims at solving its problems with the help of wide base of verbal information, social interactions and practical , experimental realization of theoretical constructions by using personal experience; mathematics wants to achieve logic completeness, create mathematical knowledge based on models and integrity, which serve social processes and phenomena. Contents of mathematical education of students on humanitarian faculties () consists on set theory, discrete mathematics, mathematical logics and probability theory, calculus and statistics. Students have some difficulties in learning of mathematics, do not see the real links between mathematics and special subjects, feel himself weak ability to understanding of mathematical ideas. Indeed we remark the whole cohort of problems of teaching mathematics for students on humanitarian faculties: -Weak motivation to learning of mathematics; - Some break between person's lines of experience to self-realization and aims of mathematical education; - Thinking, concerning with verbal and visual components, and weak level of mathematical generalizations; - Tendency to concrete activity with mathematical objects but weak perception of theoretical constructs.

Influence of social sciences (humanities) and mathematics on formation of substructures of a person will be especially powerful, if the process of their teaching and learning (as well as selection of the appropriate contents) is as much interconnected as possible. But it's worth mentioning that impact of social sciences on mathematics and mathematics on social science is not proportional and has certain particularities in essence and forms of presentation. Mathematics, being objectively a highly formalized science demanding a high level of abstraction and derivation from realities of the actual world, requires activization of concretizational, motivational and activity-modeling processes during its learning. It defines the following basic components of influence on the humanitarian contents on learning of mathematics with developing effect: - *Motivational* (determining personal meaning of activity in the direction of the "purpose – result" vector). For example, occurrence of motivations, stimulated by the humanitarian contents, can be manifested in the following criteria: integrity (presence of anticipation for manifestation of cognitive experience; anticipation can take shape both in reproductive as well as in productive learning activity; thus, in the first case, they can take shape of humanitarian problems, phenomena, processes resulting in motivated introduction of mathematical concepts, procedures and theorems; in the second case it can take shape, for example, of quasi-research activity of students in small groups aimed at solving of humanitarian problems by means of mathematical instruments and modeling. Another criteria are: achievements (creation of the social situations problem, which stimulate coming to life of new mathematical information); background (creation of conditions for directed perception by activization of mental activity; - *Self-determination* (creation of the situational dominant of a social position choice of the pupils while solving social- humanitarian problems with maximal use of mathematical resources and modeling).

In this situation we should define innovative methods, forms, resources and technologies of teaching mathematics then research activity of students are fixed on the background of humanities and mathematics integration. Students must show in research activity with mathematical knowledge such attributes of scientific thinking as insight and nonlinear thinking, visual modeling and anticipation, founding, reflection and mental efforts in context of social interactions.

Background

The basic conception of our paper is an understanding of mathematics for students on humanitarian faculties by using the idea of creation and decision of mathematical tasks by students themselves on the base of teacher's pattern (visual modeling) and student's creativity and personal experience. Exactly, research activity of students is personal activity including in process of new creation (also, personally modern) and having in-system and out-system shift of knowledge and skills in new situation, changing of conditions and methods of actions (as internal or external) during the problem solving. Such approach will be promote the person's motivation and creativity grow, activity and thinking of visual modeling, founding of person's experience as sequence of crossing from concrete forms of experience to model ideal of process or phenomenon.

Motivation. So we base on detailed structure of student's interests components, which consists from three area of characteristics: A – motivation of results achievement, R – motivation of self-realization, E – motivation of thinking efforts. Based on this position we define the *interests of students (I)* as vector (oriented) psychological category:

$$\vec{\mathbf{I}} = \vec{\mathbf{A}} + \vec{\mathbf{R}} + \vec{\mathbf{E}}$$

All of these characteristics should be actualized by special pedagogical instruments, actions, resources according to educational aims using student's creativity and experience developing.

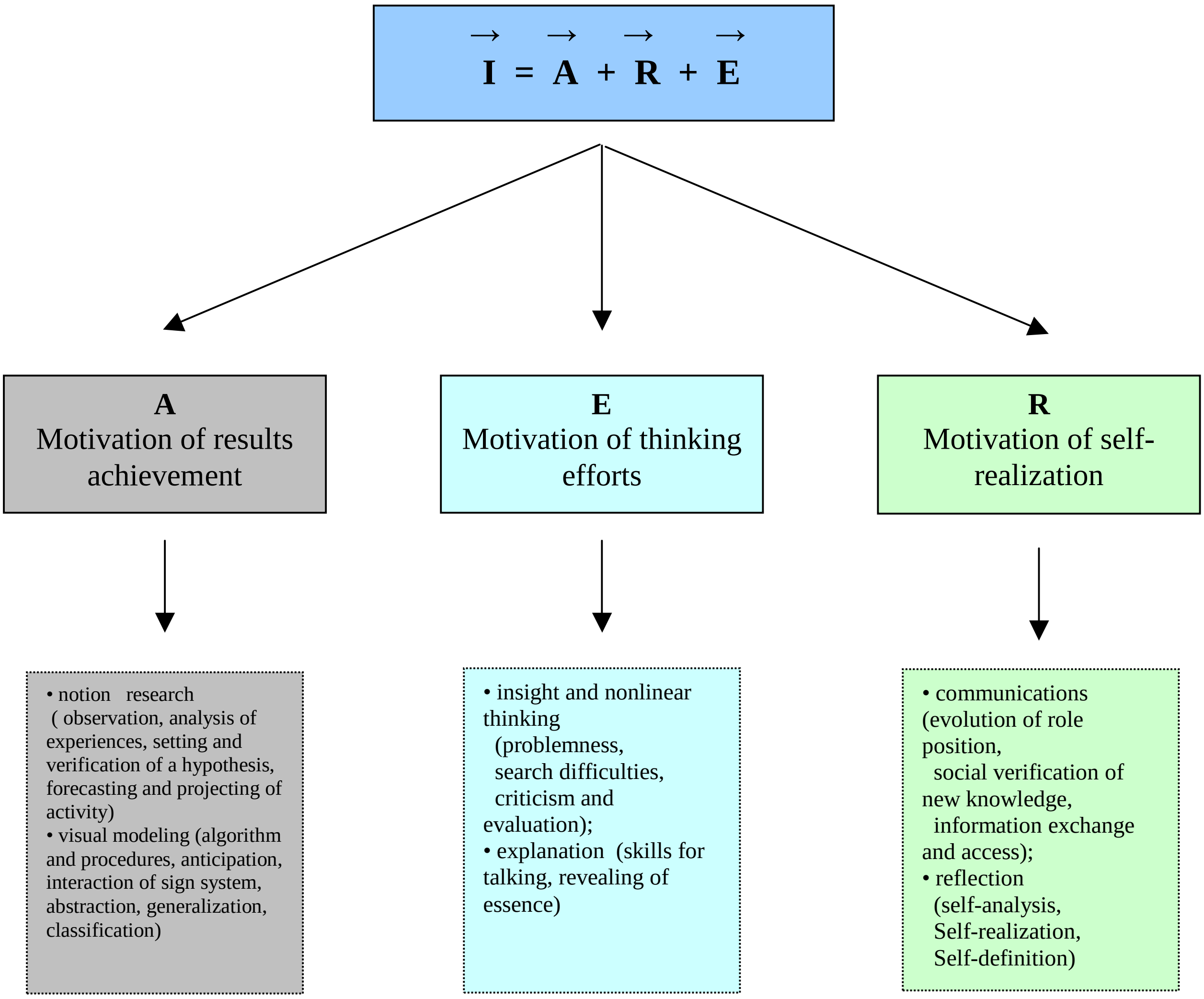
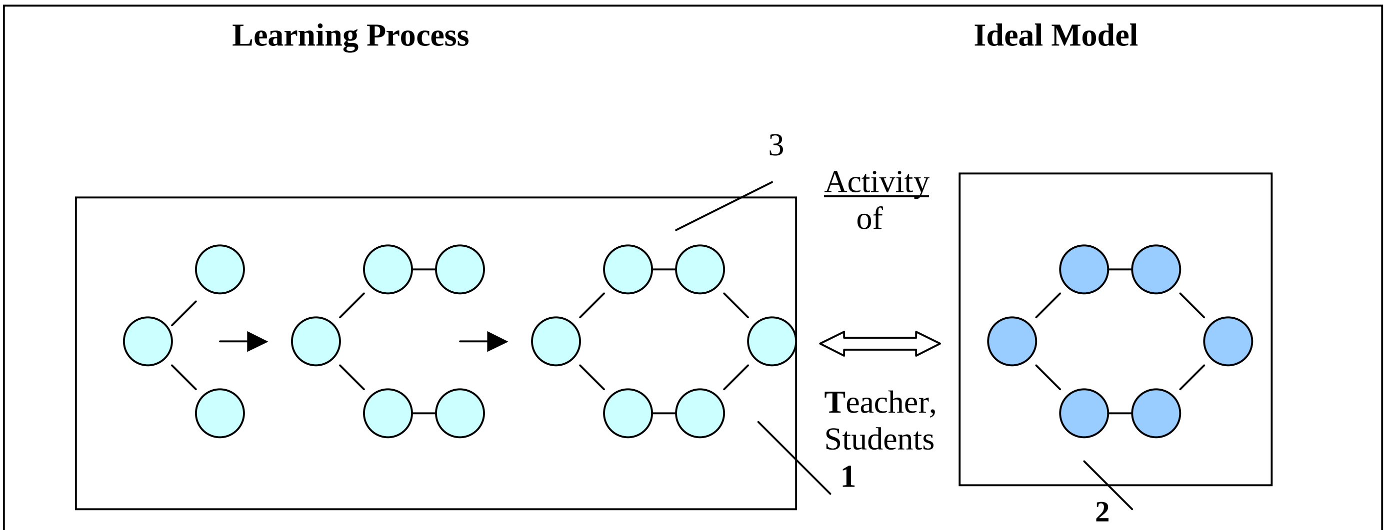


Fig.1. Characteristics of Components of Student's Interests

Visual Modeling. The pedagogical *technology of visual-modeling learning* of humanities and mathematics plays a fundamental role in the proposed didactic system of humanities and mathematics integration of knowledge and actions. This technology makes it possible to achieve stochastically guaranteed result of teaching of various qualitative levels of learned material as well as integrity of representation of the basic humanities, information and mathematical structures.

Visual modeling methods of learning present: -“a priori” modeling the essential links of the object of perception; - a process of forming an adequate category of ultimate purpose of the learners’ internal actions during the process of immediate perception; - all teachers’ managing actions, modeling of separate pieces of knowledge or an arranged set of knowledge for stabilizing the learners’ immediate perception.



1 – The Procedure of Arranged Set of Knowledge (The Object of Perception); 2 – An Ideal Model of the Object of Perception According to the Didactical Aim; 3 – The Result of the Student Internal Actions Connected With Immediate Perception .

Fig.2. Visual Modeling of Mathematical Object (Procedure)

The process of perception of the given visual model presupposes all key qualities of the science, information or mathematical object. It is especially important when information is of great volume (or contains a mix of mathematical (physical) and informatics knowledge or actions). It is necessary to keep in mind such actions when separate pieces of knowledge or an arranged set of knowledge are given. We can deal with proving theorems, solving problems, constructing the algorithm, modeling the real phenomena, learning some parts of scientific and mathematics analysis in its various logical correlations, with a single lesson presentation, a lecture etc.

As has already been mentioned, according to A.N. Leontiev (Russian psychologist), when visual methods of learning and teaching are used, it is necessary to proceed from the psychological role, which they (methods of learning or teaching) play in the perception of new material. He chooses two functions of visual methods of learning or teaching:

- the first is aimed at extending the sensible experience;
- the second is aimed at developing the essence of the processes or phenomena under study.

Founding. We based our assumptions on the fact that concordance or optimisation of interaction of fundamental and professional components in the general structure of pedagogical education is a key moment in training of a student at a teachers’ training college. It is obvious, that fundamentalisation of mathematical knowledge without considering it as a pedagogical aim will hinder professional training of a teacher. At the same time it is beyond doubt that a teacher is not in able to fulfil his professional functions successfully without a certain volume, structure and quality of fundamental knowledge. Thus, in a nutshell, the problem is to find means, forms and ways to bring to concordance fundamental and professional lines in the process of pedagogical education.

In order to realise the principle of founding it is necessary to define the basis for helical diagram of the basic knowledge, skills and experience of mathematical training of students at teachers’ training colleges modelling. If founding of various school subjects is to be carried out layer by layer, then volume, content and structure of mathematical training must undergo considerable changes in respect of practical realisation of theoretical generalisation of school knowledge based on the principle of a “boomerang”. If the knowledge is being founded in such a way, the teacher, who possesses knowledge of the subject, together with the student will master methodological side of teaching. The school knowledge will act as a structure-formation factor, making it possible to select theoretical knowledge from mathematics of a higher level, via which school knowledge has been founded. The layer of founding provides perfection and extension of practical skills, projected by approximate basis of learning activity. In the activity aspect of pedagogical process realisation of foundation principle acquires a helical character, which corresponds to dialectical understanding of a system of knowledge development.

Development of spirals of personal experience via integral constructs of ancestral generalisation and technological comprehension of its specific manifestation render integrity and orientation to the projected didactic system. At the same time construction of such a model absorb in its unique and particular manifestation all main features of theoretical knowledge about foundation process of basic educational elements of mathematics. Creation of system-genetic block of spiral's founding makes it possible to define a stable nucleus of educational information content, which projects elements of approximate basis for educational activity of students. On the other hand, projecting of theoretical generalisation (ancestral concept) onto specific diversity of special case in forms of updated practical applications creates a stable motivational effect in acquiring mathematical knowledge. But even comprehensive, well-timed and modelled spiral of founding will not carry a cognitive and professional component of prospective activity, if methods and elements of educational activity are not shaped in the process of education, which develop its component composition, if structure, particularities of perception and comprehension, stimulation of motivational and emotional sphere of students and definition of mechanisms for checking and correction of spirals of founding are not taken into consideration.

Approaches and methods of educational activity

We try to use methodological ideas of problem solving, visual modeling, founding, activity in small groups, humanizing mathematics education:

- *setting of the productive humanitarian problems with mathematical decision* (actualization of humanities and mathematics knowledge on the basis of integration; participation in discussion and statement of educational tasks; construction of humanities and mathematical model of process or the phenomenon; ability to consolidation (in thinking of the pupil and activity) the initial data for the decision of the problem);
- *educational activity of students by using of model pattern* (quasi-research activity of students aimed on creation and search of new patterns ; search experiment using numerical methods and computing procedures, diagnostics of information dynamics of parameters; monitoring and correctional interaction of obtained results, search of integrative knowledge and prospect of development; skills of visual modeling and estimation of real processes);
- *efficiency of using resources (material, materialized, ideal) for activization of cognitive processes and social interaction* (presence of adequate results in practical activities; joint analysis, information interchange, presentation of results; visual modeling in educational activity; reflection and internal plan of students action).

Conclusions and suggestions

The analysis of these results made us feel confident that the hypothesis concerning the opportunity to increase motivation in learning of mathematics by incorporating into research activity with personal experience and mathematical knowledge is consistent and logical. It can be achieved by means of development of resource lessons and activization of cognitive and creative activity of students.

The conducted research has shown the importance of the chosen topic and has partially confirmed the put forward hypothesis about the significance of the integrated approach to interaction of humanities and mathematics. Research of the innovative approach in visual modeling of humanitarian and mathematical processes, activation of motivational and cognitive processes have promoted positive changes in personal development and successful mastering (learning) of teaching material. Founding of personal experience of students, as basic form of realization of interaction of humanities and mathematics has shown its efficiency and opportunity for further research. It is recommended to develop the *cycles of pattern of developing mathematical models* (integrated constructs) in teaching of mathematics for students on humanitarian faculties of universities and to carry out a detailed analysis and feasibility of the technological innovations.

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Using Technology to Discover and Explore Linear Functions and Encourage Linear Modeling

Tanja Soucie MSEd
Matka Laginje Elementary/Middle School, Zagreb, Croatia
tanja.soucie@zg.t-com.hr
Nikol Radović M.S.
Faculty of Geodesy, University of Zagreb, Zagreb, Croatia
nradovic@geof.hr
Renata Svedrec
Otok Elementary/Middle School, Zagreb, Croatia
renata.svedrec@zg.t-com.hr
Helena Car
Otok Elementary/Middle School, helenacar52@gmail.com
Zagreb, Croatia

Abstract
In our presentation we will show how technology enables us to improve the teaching and learning of linear functions at the middle school level. Through various classroom activities that involve technology such as dynamic geometry software, graphing calculators and Excel, students explore functions and discover basic facts about them on their own. Students then work with real life data and on real life problems to draw graphs and form linear models that correspond to given situations as well as draw inferences based on their models. Participants will receive complete classroom materials for the unit on linear functions.

Introduction

One of the *National Council of Teachers of Mathematics* (NCTM) principles states: "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning." In order to meet the new demands placed upon teachers, thoughtful changes should be made in regards to curriculum and the way we approach teaching and learning. Some of the critical problems we have to address are students' inability to see the application of mathematics in real life, the belief they will never use mathematics outside the classroom, and the belief that mathematics is difficult and boring. It is crucial to convince students otherwise through the use of various classroom activities. We believe the use of technology can help bridge some of those gaps. The use of technology makes mathematics accessible to more students and enables the use of real life data which in turn puts mathematics in a context for students. When students see the application of mathematics in real life they begin to understand its importance and beauty as well as gain more appreciation for the subject. In addition, the use of technology allows students to work at their own pace, make mistakes and correct them on their own, as well as feel free to take risks.

Model of instruction

After a brief introduction to linear functions, students explore linear functions on their own using dynamic geometry software. They discover how parameters a and b relate to the equation $f(x) = ax + b$, how to find the x- coordinate of a zero of a linear function, how to determine whether the graphs of the functions are parallel or perpendicular from the equation and how to solve systems of linear equations graphically. Next, students engage in a mini-project about medicine. Students determine the linear equation that describes a dosage based on body mass and draw graphs (using Excel) as well as interpret them to answer questions such as: Why shouldn't we skip dosages?, Should we take a double dosage if we did not take our medicine on time?, etc. Next, students engage in a study of their choice. While working on their project, students can use a graphing calculator or Excel.

1. Vitruvian man – students take various measurements and discover the relationships between e.g. height and shoulder length, ear length and face length, etc. as well as describe these relationships using functions. Students compare their equations with human proportions recorded in Book III of the treatise De Architectura by the ancient Roman architect Vitruvius and used in Leonardo da Vinci's drawing of a man.
2. Cost of living – students describe the costs of a gas bill, phone bill, internet bill, etc. using functions and answer various questions using these equations as well as determine the best phone and internet provider for their consumer needs.
3. Exploration in physics – students take measurements and explore relationships between mass and force recorded on the spring scale, mass and volume, the radius and height of a cylinder, the volume and height of a cylinder to determine whether these relationships are linear. If a relationship is linear, students determine the equations of the function.
4. Math in sports – students shoot baskets from various distances and describe the number of baskets made based at each distance from the basket using linear functions and answer various questions based on this model. Students also graph data about long jump world records and determine linear equations that predict the length of the jump based on the year. Students will try to answer questions such as: How far can a man jump? Will women ever jump farther than men?
5. CAT/ RAT simulation – students use a computer simulated experiment to investigate the actions of standard drugs on the cardiovascular system. They inject drugs into an animal's circulation and explore relationships between drug dosage and the values of heart rate and blood pressure.
6. Food and water in the animal world – students explore relationships between body mass and the time of digestion and body mass and the velocity of losing water for different kinds of animals. Using graph data students will try to answer questions such as: How long does a rabbit take to digest its food? How fast does a rabbit lose water from its organism?

Rubrics for assessments are provided.

Transfer of the model to different environments

Sometimes the lack of technology is an issue in schools. In classrooms where computers are not easily available, teachers can show presentations using dynamic geometry software and students can explore linear functions as a class through discussions. All of the graphs and equations can be found using Excel if no graphing calculators are available. In case there is neither a computer nor a graphing calculator available, students can draw graphs by hand and use scientific calculators to arrive at linear models (by using two variable statistics).

Conclusion

The strategies used in our model create a motivating and engaging environment where technology allows students to discover mathematics on their own as well as model real world data. The use of technology allows students to work at their own pace, make mistakes and correct them on their own, and feel free to take risks. The choice of topics and the choice of technology used empowers students to make their own decisions based on their own interests and personal preferences. We believe this model is superior to traditional lecture based instruction because it requires active engagement of students and encourages higher level thinking.

Virtual Manipulatives: Design-based Countermeasures to Selected Potential Hazards

William R. Speer, Ph.D.
Associate Dean for Academic Affairs, College of Education, Director, UNLV Center for Mathematics and Science Education, University of Nevada Las Vegas, william.speer@unlv.edu

Abstract
Virtual manipulatives are employed by both preservice and inservice teachers to enhance the instructional effectiveness of physical manipulatives and related tools by addressing limitations of access, cost, and adaptability. While research into the use of emerging technologies continues, there are several variables to consider when measuring the effects of virtual manipulative use. Research design, sampling characteristics, and the type of manipulative used may influence achievement. Variables that may influence the effectiveness of virtual manipulatives include: previous experience with computers, grade level, mathematical topic, treatment length, student attitudes toward mathematics, and computer-to-student ratio.

Introduction

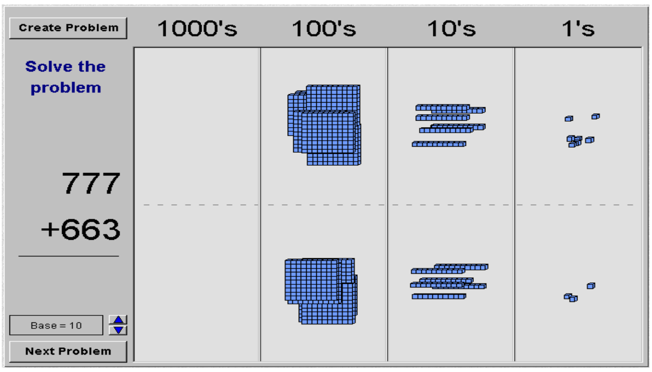
One pedagogical technique commonly used in mathematics education is to provide students opportunities to actively manipulate certain aspects of the phenomenological world (Heddens, Speer & Brahier, 2009; Izydorczak, 2003; Moreno, 2005; NCTM, 2000). This technique relies on careful construction of those phenomena that exemplify the mathematical concept being conveyed. In essence, these phenomena serve as concrete analogies of mathematical concepts and, in the language of mathematics education, are said to *model*[1] those concepts. Pedagogical tools specifically designed for this type of active manipulation are called "manipulatives." With the advent of digital technology, this basic idea of manipulatives has been extended to the computer-based manipulatives or "virtual manipulatives" (Schackow, 2007; Tversky & Morrison, 2002).

This paper identifies and discusses 1) some potentially detrimental effects that the use of virtual manipulatives may have on mathematical learning, and 2) possible ways to address these effects. The issues identified in this paper are based on observations of students interacting with a number of virtual manipulatives found on the Internet, namely, at the National Library of Virtual Manipulatives for Interactive Mathematics site. This paper reports on test cases and anecdotal evidence that lack the necessary basis for definitive conclusions and, consequently, is presented as a preliminary study upon which further, more rigorous, investigation may be formulated.

Methodology

This paper is based on two different types of observations. One type was based on a series of one-hour sessions with 4th grade students. Sessions consisted of a series of addition exercises each first attempted using the virtual manipulative and then using the physical manipulative (Mousavi, Low, & Sweller, 1995; Schnotz, 2005).

This screen shot shows an exercise with the virtual Base-10 Blocks manipulative:

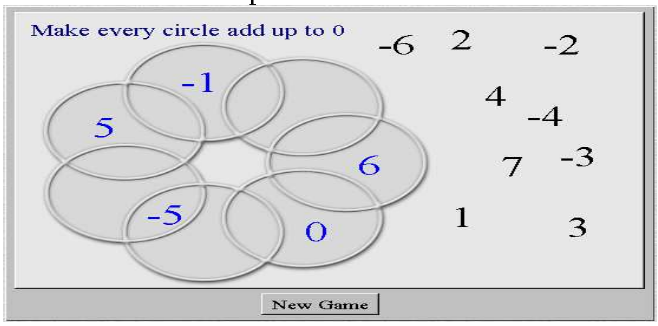


Here, the object of the manipulative is to 1) aggregate 10 pieces in one unit into a single piece in the next higher unit, and 2) place each piece in the columns to which they belong.

Additional observations were based on three "computer-lab" sessions of three different middle school mathematics classes. Most of the students worked individually, each with a dedicated computer. However, due to a limited number of computers, a small minority of students worked in groups of two. Also, on occasion, especially when students appeared to be "stuck" in a particular scenario, the observer interjected with questions and suggestions to the students. Each lab session began with the students exploring the

[1] It is curious that the notion of *modeling* in mathematics education is a mirror image of that found in science. In physics, for example, mathematical concepts are used to *model* physical phenomenon. Here, the object of study is the phenomenological world and mathematics is a language used to describe that world. In mathematics education, on the other hand, the object of study is mathematics itself and the phenomenological world is used to *model* mathematical concepts.

Circle-0 virtual manipulative.



The object of Circle-0 is to place all the numbers (using the drag-and-drop technique with the mouse) within the circles so as to make each circle to "add up to 0".

Potential Issues

This section of the paper identifies 4 different ways virtual manipulatives may be counter-productive to learning. Each is discussed in terms of: characterization of the potential issue; observations that support this characterization; ramifications of this issue for effective and efficient learning; potential design solutions to address these issues; and, procedures devoid of concepts.

Under certain circumstances, manipulations may not be accompanied by their intended conceptual counterpart. Students may acquire procedural expertise needed to successfully complete the manipulatives without internalizing the concepts that the manipulatives were designed to model (Atkinson, 2002; Izydorczak, 2003).

An observation that prompted this issue was with a student's interaction with Base-10 Blocks. With virtual Base-10 Blocks, the procedure for combining 10 pieces in one unit required surrounding 10 *or more* pieces with a bounding rectangle with the mouse. Once those pieces were successfully bound, the computer automatically transformed 10 of those pieces into a single piece of the next higher unit. It became clear that the student became focused on the operation of surrounding all of the pieces in each unit with the bounding rectangle. From the student's perspective, this was a reasonable strategy for the mastery of that skill - irrespective of any concepts that may be associated with it - was what was necessary to complete the exercise.

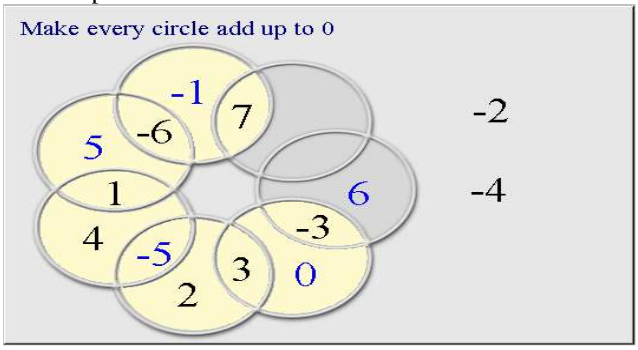
The fact that the student had difficulty replicating his solution with the physical Base-10 Blocks lent additional credence that there was a chasm between his understanding of the procedural requirements of the manipulative and their corresponding meaning in the number system. With a clear understanding of the relationship between the two, one would expect that changing the medium of the manipulatives - from virtual to physical - would have had less impact than was observed (Drickey, 2001).

Clearly, when students "play the game" devoid of the concepts the manipulative is designed to demonstrate, the effectiveness of the exercise in meeting the intended pedagogical goals is likely limited (Atkinson, 2002). However, this potential for procedural expertise devoid of conceptual understanding seems to be an inherent vulnerability of manipulatives in general (Thompson, 1992). Manipulatives are, in essence, phenomenological analogies for concepts and, as such, always carry the possibility of being misconstrued. Of course, the task of educators is to limit the likelihood of those misconceptions and to resolve them when they do occur.

What, then, are some potential strategies to 1) limit the likelihood of conceptually empty procedural manipulations, and 2) resolve them when they occur? With respect to the virtual Base-10 Block example cited, the virtual manipulative may be designed so that: the student must explicitly collect exactly 10 pieces of one unit rather than simply surround 10 or more pieces within a bounding rectangle; the student must explicitly request (say, using a button) to convert the 10 pieces to a single piece of the higher unit; and/or the student must replicate the manipulation with symbolic operations. In general, by requiring of the student greater responsibility of the manipulations (as opposed to automated manipulation by the system) and corresponding symbolic operations, the likelihood that the student will make the conceptual connection with procedural operations may be increased (Pass, Renkl, & Sweller, 2003).

Local Minimum

Some students seem to get "stuck" in a local minimum of the search space. As a result, those unwilling or unable to backtrack (i.e., give up some of the gains seemingly achieved) were unable to complete the problem (Schnotz & Bannert, 2003). The level of challenge represented by these local minimums may be counter-productive to some students. This issue was observed particularly with the Circle-0 virtual manipulative.



In this example, 5 out of 7 circles have met the criteria, i.e. they "add up to 0". There are two numbers left (-2 and -4) which must be placed in the two remaining spots (between 7 and 6) so that the remaining two circles also add up to 0. It is easy to see that neither of the two combinations of placing the remaining numbers produces the solution. Therefore, in order to reach the solution where all the circles add up to 0, the partial solution generated so far must be sacrificed.

The question remains as to whether such perplexity is conducive or counter-productive to learning. It seems that, in general, the answer depends on the student (Schnotz & Rasch, 2005). The detrimental effect of too much challenge seems particularly relevant in the area of self-efficacy. Specifically within the context of Circle-0, the level of challenge posed by a solution that seems so close and yet so far is almost always inappropriate for the students for which the manipulative is target, i.e. students learning to add single digit integers.

One way to control the potential level of local minimum is through additional constraints. In the case of Circle-0, additional starting numbers may be added so as to remove the possibility of deep local minimums. Given the size of the search space and the computational speed of even modest computers, it should be feasible to check all possible points on the search space so as to ensure maximum level of difficulty in terms of local minimums.

Disengagement

Some students seemed to be disengaged with the problem at hand and repeatedly hit either the "Hint" or the "Reset" button without demonstrating any attempts to actually solve the problem presented. To the observer, it was as if, once they had developed the pattern, they were stuck in a mental mode of simply pressing those buttons. Needless to say, such a lack of engagement with the problem is counter-productive to learning.

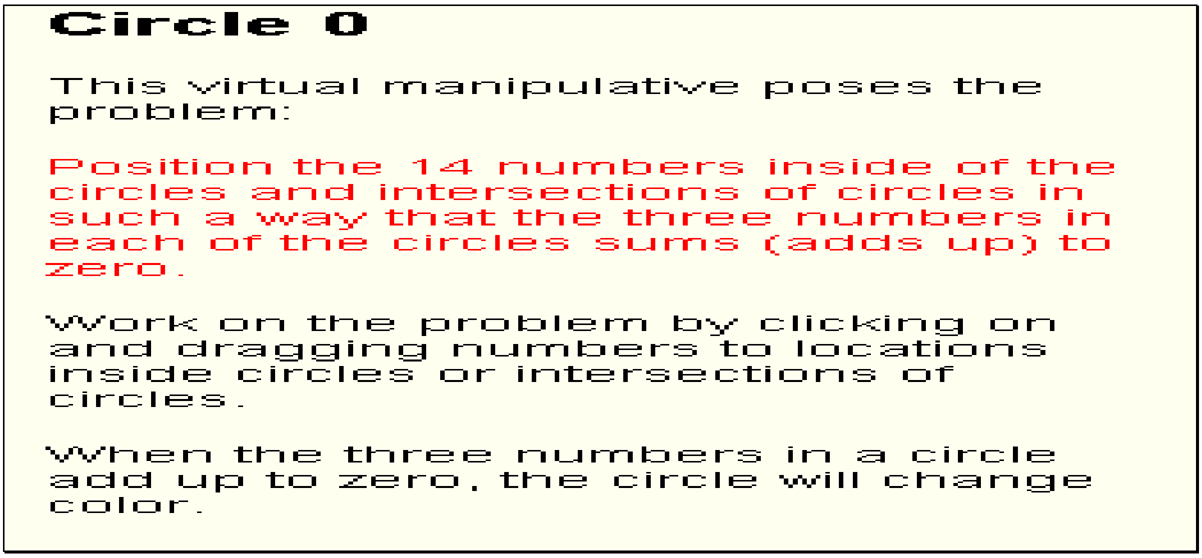
The problem of disengagement seems not so much a problem with virtual manipulatives, but a manifestation of a more general issue in learning (Mayer & Chandler, 2001). A student's inability or unwillingness to effectively engage a problem likely points to more fundamental issues in learning and may require higher levels of intervention. Therefore, it seems unlikely that this type of disengagement would be eliminated by simple design changes in the manipulatives. At least in principle, however, it may be possible to enable the virtual manipulative system to automatically 1) detect certain patterns of use (or rather misuse) of the program and 2) to provide some type of interjection (or notification to human instructor).

From a technological perspective, the level of computational sophistication needed for such functionality is qualitatively different than what is found in implementations of virtual manipulatives at the National Library of Virtual Manipulatives for Interactive Mathematics site. This type of functionality - automated diagnosis of student performance and feedback based on that diagnosis - has yet to be effectively demonstrated in much of educational technology. It is the author's view that the development of this class of functionality will become a focus of research.

Rule Confusion

Some students seemed to not understand what was being asked of them even after they spent some time reading the accompanying instructions. One group of students seemed in a state of bewilderment with the Circle-0 manipulative. However, once they were shown how to fill in one or two circles, they immediately and quickly progressed with the remaining circles. The amount of cognitive and emotional support seemingly required during these periods of confusion is likely counter-productive. Certainly, from the learner's point of view, this can be both unpleasant as well as counterproductive.

One way to increase the understandability of the instruction may be to simplify the language of the instruction. The online instruction for the Circle-0 manipulative, for example, is shown:



It is sometimes unclear for whom the instructions are intended. Using a language more suitable to younger students may increase the likelihood of being understood. An even more intuitive and effective way (albeit, more difficult and costly) to convey the instructions for using the manipulatives may be through animated demonstration of its use. Animation may be effective in communicating not only basic instruction but also different strategies for tackling a problem (Atkinson, 2002).

Conclusion

Perhaps, it should be axiomatic that every technology has its limitations and, therefore, can be misused. This paper discussed some the potential inherent and design limitations of virtual manipulatives and how these limitation may be addressed. One of the conclusions based on the observations cited in this paper may be that appropriate supervision is needed to maximize/minimize the potential benefit/detriment of virtual manipulatives. It is the view of this author that one of the major areas of research in educational technology is the development of assistive mechanisms to effectively and efficiently support this type of interaction.

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Teaching for the objectification of the Pythagorean Theorem

Panagiotis Spyrou, Andreas Moutsios-Rentzos & Dimos Triantafyllou
Department of Mathematics, University of Athens, Greece pspirou@math.uoa.gr

Abstract

This study concerns a teaching design with the purpose to facilitate the students' objectification of the Pythagorean Theorem. Twelve 14-year old students (N=12) participated in the study before the theorem was introduced to them at school. The design incorporated ideas from the 'embodied mind' framework, history and realistic mathematics, linking 'embodied verticality' with 'perpendicularity'. The qualitative analyses suggested that the participants were led to the conquest of the 'first level of objectification' (through numbers) of the Pythagorean Theorem, showing also evidence of appropriate 'fore-conceptions' of the 'second level of objectification' (through proof) of the theorem. The triangle the sides of which are associated with the Basic Triple (3,4,5) served as a primary instrument for the students' objectification, mainly, by facilitating their 'generic abstraction' of the Pythagorean Triples.

Introduction

The *embodied mind* framework (Varela, Thompson & Rosch, 1991) seems to be compatible with *realistic mathematics* (Gravemeijer & Doorman, 1999), especially in Geometry and its elementary theorems, which are immediately connected with the perception of environmental stimuli. Building on ideas extensively discussed in Lappas and Spyrou (2006), we propose a realistic teaching design that links gravity and the embodied verticality with the Pythagorean Triples. Through similarity, the Pythagorean Triples can help the students to link kinaesthetic actions with mathematics and to formulate the Pythagorean Theorem (*ibid*). The proposed teaching takes place before the introduction of the Pythagorean Theorem to the participants at school. In this way, the students have the opportunity to experience the 'transformation' of a subjective conception into an objective mathematical idea: its *objectification* (Derrida, 1989, Radford, 2003).

Theoretical framework

According to the 'embodied mind' framework, mathematics can be viewed as structures deriving from within the human bodily functions (Varela, Thompson & Rosch, 1991). Evidence from neuroanatomy shows that gravity plays an essential part in the function of the *equilibrial triad* (visual, proprioceptive and vestibular system; Noback, Strominger, Demarest & Ruggiero, 2005), the input of which is evaluated by the brain for optimum equilibrium, motor planning and spatial orientation. Hence, gravity can be linked with the humans' ability to identify verticality, which roughly is the ability to identify the perpendicular to the ground, thus supporting the claim that gravity and embodied verticality can be linked with perpendicularity (Lappas & Spyrou, 2006).

Lappas and Spyrou (2006) identified *two levels of objectification* in mathematics: the first level is realised through *numbers*, whereas the second through *proof*. Moreover, they argued that, historically, the Pythagorean triples is one of the first results in Geometry that derived from the act of 'objectively' ascribing the perception of the 'shape' of perpendicularity via numeric relationships (thus, first level objectification).

We argue that ideas put forward from the embodied mind research area could be compatible with 'realistic mathematics' (Gravemeijer & Doorman, 1999). Bearing in mind that the way the human body experiences gravity is invariant through history, we attempt a teaching of the Pythagorean Theorem based on the sensory experience of gravity with the intention to *reactivate the primordial act* of *objectification* (Derrida, 1989; Radford, 2003) in mathematics "through an *adaptive didactic work*" (Radford, 1997, p. 32), which was "redesigned and made compatible with modern curricula in the context of the elaboration of teaching sequence" (*ibid*).

In this study, we investigated a realistic teaching designed to facilitate the students' objectification of the Pythagorean Theorem, incorporating ideas from the 'embodied mind' framework, history and adaptive pedagogy. Note that this study focused only on the first level of objectification due to curriculum constrains related with our sample, affecting the level of mathematisation that we expected the students to reach.

Sample and procedures

The study was conducted with 12 students (7 males and 5 females), who were in the second grade of the Greek Gymnasium (14 years old). The participants were grouped in six pairs based on their friendship (as suggested by their teacher), so that they would cooperate better in the various activities of the study.

A structured teaching of around an hour took place in the school lab. All the activities were videotaped and qualitative analysis was conducted on the video data.

The teaching design

The teaching design consisted of *seven phases*. The *first* phase is labelled as '*Falling ball*'. Each student was given a small ball and was asked to let the ball fall from his/her hands. This triggered a discussion about the vertical trajectory of the falling objects and its relationship with gravity.

The *second* phase is labelled as '*Plumb-bobs*'. Once the students realised the relationship between gravity and verticality, we asked them to suggest ways of marking the trajectory of a downfall. The use of the plumb-bob in the construction of a vertical wall was presented to them.

The *third* phase is labelled as '*Bottle containing coloured liquid*'. The purpose of this phase was for the students to realise the necessity of constructing perpendicularity and the horizontal plane. Thereby, a discussion was initiated about the construction of a perpendicular line to a vertical wall. The students were presented with a bottle containing a coloured liquid (see Figure 1). The researcher held the bottle against the vertical wall in a variety of angles, making evident that the surface of the contained liquid remained horizontal. The surface of the coloured liquid, embodying the horizontal plane, and the plumb-bop, embodying the vertical line, formed a *natural* example of perpendicularity for the students to see and act upon.

The *fourth* phase is labelled as '*Wooden sticks*'. In this phase, the focus was on the Basic (Pythagorean) Triple (3,4,5). A series of activities was designed with the purpose to link the right angle with the right-angled triangle and the Basic Triple. Each pair of students was given three wooden sticks (90 cm, 120 cm, 150 cm) coloured with a different colour every 30 cm, embodying the Basic Triple. We asked the students to place the 120 cm long stick against the wall and to try to construct a right-angled triangle. Subsequently, we asked them to construct with these sticks a right-angled triangle on the floor. Thus, the students were led to the Basic Triple:

Verticality →Right angle →Right-angled triangle →Basic Triple

Subsequently, the students, starting from the Basic Triple (3,4,5), they were led to the construction of a right angle, thus realising that the converse is also true:

Basic Triple →Right-angled triangle →Right angle

The *fifth* phase is labelled as '*Basic Triple on the millimetre'*. We asked the students to draw a right-angled triangle with the perpendicular sides being 3 cm and 4 cm. The students were asked to find the length of the third side. Thus, the students were led to the Basic Triple (starting from 'right angle'; see 'wooden sticks' above).

The *sixth* phase is labelled as '*Basic Triple and angles*'. First, we explored, the students' prior knowledge of the various types of angles (acute, obtuse or right). Subsequently, we asked them to draw a triangle with two of its sides being 3 cm and 4 cm and the angle between them being different from 90°. The students were asked to measure the third side of that triangle and to note any rule that they might have found.

The *seventh* phase is labelled as '*Figurative numbers*'. The students were presented with the first four *figurative numbers* (1^2, 2^2, 3^2 and 4^2; in the Pythagorean sense, with dots, see Figure 1) and were explained the process of constructing such numbers. We asked the students which number would be the next figurative number and whether they could draw it. Subsequently, the students constructed three squares with their sides being 3 cm, 4 cm and 5 cm respectively. We asked them to amount the dots and to do possible operations with the numbers. The students found more Pythagorean Triples, constructed the respective triangles and confirmed the truth of the Pythagorean Theorem in those cases. We initiated a discussion about the case in which the sides of the triangles were not natural numbers. Finally, we disclosed to the interviewees that an informal ‘proof’ using areas would be presented to them in class.

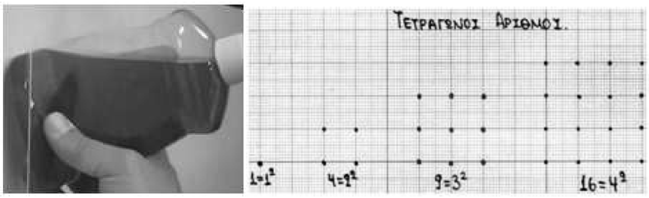

Figure 1: ‘Bottle containing coloured liquid’ (left) and ‘Figurative numbers’ (right).

Results

During the ‘Falling ball’ phase the students were asked about the trajectory of the falling ball. Nine students answered “a straight line”, one implied the same answer, while the rest of the answers were “a curve” and “it will follow the law of gravity”. We asked them whether they had heard of another word for ‘perpendicular’, expecting ‘vertical’ (*cf.* Manno, 2006). All the participants answered negatively, although most of them were aware of the word “vertical”, when we asked them. Subsequently, the students were asked: “How can we find a way to materialise the vertical line that gravity creates?”. Three of the children thought of builders constructing houses, while one girl, Daphne, commented: “A! We will use a piece of rope or string or something like that and we will draw it [the vertical line] while we hold the piece of rope”.

In the ‘Plumb-bobs’ phase, once we explained the use of plumb-bobs to the students, we asked them: *“Since we have established verticality through gravity, how can we establish that a straight line is perpendicular to this line* [the vertical]*? That is, how can we construct a right angle?”* The students were given time to realise the significance of the questioning and to try to discover something new for them. Some students considered using a triangle ruler, but we clarified that this was not consistent with the fact that “the [right-angled] triangle has not been constructed yet”. Therefore, they resorted to an empirical answer:

Researcher: And how do you know it would be vertical?

Nikos: ‘By the eye’ [meaning a rough, visual estimation]. We will see it.

In the ‘*Bottle containing coloured liquid*’ phase, the students appeared to realise the relationship between the surface of the liquid and the vertical line:

Researcher: Observe. What’s this [referring to the bottle]?

George M.: It’s a kind of liquid.

Re: Look carefully. I move this bottle and I have a surface [referring to the surface of the liquid]. What is the relationship of this surface with this piece of thread?

George K.: If we put it perpendicularly…it will be a right angle.

In the ‘Wooden sticks’ phase, the students were given to inspect the three wooden sticks. One pair of students produced an expression using proportion: “This is ... 1, 2, 3, 4, 5 ...this is three fifths ...”. The students were then asked to use the sticks in order to construct a right-angled triangle on the wall and subsequently on the floor, with the purpose to facilitate the students’ relating the Basic Triple (3,4,5) with perpendicularity independently from gravity.

During the ‘Basic triple on the millimetre’ phase, the students faced various difficulties concerning the angles, including the identification of the various types of angles, the differentiation among these types and the appropriate naming of specific angles using capital letters. Nevertheless, most of them (8 students) successfully coped with the activity itself (measuring the third side of the triangle).

In the ‘Basic triple and angles’ phase, the students were asked to draw a triangle with two sides being 3cm and 4cm and the angle between these sides being different from 90°. Most of the children did not face any difficulties with this activity. They were then asked to measure the third side of that triangle and note any rule that they could find concerning the lengths of the sides. The majority of the students observed that in a triangle with two sides being of a fixed length the third side would be longer than the respective side in a right-angled triangle when the angle was obtuse and shorter when the angle was acute:

Researcher: Can you give us a rule by comparing these cases?
Aggeliki: That in an acute triangle…the third side is…
Daphne: Yes, it doesn't have the same number…the angle is not 90˚ like in the right-angled triangle.
Re: Aggeliki would you like to complete your thought, as well?
Agg: Emm…this is essentially what I wanted to say.
Re: But your phrasing was different.
Agg: That in an acute triangle…
Re: Yes…
Agg: The third side will be smaller than it will be in the right-angled triangle.
Re: And in the obtuse triangle?
Agg: Bigger.

In the 'Figurative numbers' phase, once the students were familiarised with the figurative numbers, we asked them to construct three squares with sides respectively 3cm, 4cm and 5cm. Most of the students pointed out the relationship between the squares of numbers 3, 4 and 5 ($3^2+4^2=5^2$) indicating with their hands the areas of the square: "The sum of the two small sides equals to the big one" (Aggeliki). The students were asked to think about the relationship of these numbers and to try to find more triples satisfying this property. Some interviewees noticed that the multiples of the Basic Triple were cases of such triples:

Researcher: What other combinations of numbers could lead us to the formation of a right-angled triangle?
Nantia: Their doubles.
Re: Meaning?
Eleni: 6, 8 and 10.
Re: Why [did you choose] these numbers? What were you thinking of?
N: That their doubles will also do [form a right-angled triangle]. It will just be a bit bigger…that the double [referring to the triangle with sides of double the size] will be exactly the same shape.

In the above excerpt, Nantia noted that although the size of the sides in the new triangle is double, the shape "will be exactly the same". In a similar vein, Kostas explicitly uses the word 'analogy' to describe the reason why the multiples of the basic triple are also suitable.

Researcher: We have $3^2+4^2=5^2$. Observe that these numbers are in a certain relationship with their squares. Are there any more numbers like that? I wonder.
Kostas: Let's try the numbers (6,8,10).
Re: Why [did you try] these numbers?
K: Because they are the doubles of (3,4,5).
Re: And what do you make of this? What will happen if we double the numbers?
K: That there is an analogy ... we will have the same result.

Nevertheless, other students noticed that the basic triple consists of consecutive numbers and, therefore, they hypothesised that triples of consecutive numbers might also be suitable. The following excerpt is an example of such a case.

Researcher: The triad (3,4,5) ensures the formation of a right-angled triangle. Is it possible for us to find another triad that also forms a right-angled triangle?
Aggeliki: 6, 7 and 8.
Re: Why are you saying this?
Agg: They are just like 3, 4, 5 were. They are consecutive [numbers].

Note that Aggeliki argues that (6,7,8) is "just like" (3,4,5). It can be argued that this is also a case of 'similarity', which however is qualitatively different from the previous examples. Kostas, Nantia and Eleni expressed the *geometrical* similarity through numbers, while Aggeliki found similarities, meaning *numerical patterns*, between two numerical triples, *unrelated* with the *geometrical meaning* of the Basic Triple. Hence, (geometrical) 'similarity' seems to be crucial for the students' identification of suitable triples (*cf.* Lappas & Spyrou, 2006).

At the end of the final phase of the teaching, three students (Kostas, Daphne and Nantia) wrote down the arithmetical equations and formulated the Pythagorean theorem by saying that "*in a right-angled triangle, the square of one vertical side plus the square of the other vertical side equals to the square of the hypotenuse*".

Discussion

During the teaching, the students appeared to appropriately link gravity with perpendicularity. They drew upon the Basic Triple, in order to identify a sufficient condition for determining and identifying perpendicularity. Initially, through an investigation into the links between the length of the sides of the triangle and its angles (acute, obtuse or right angle), the students identified a *fore-conception* (Sierpinska, 1992) of the Pythagorean Theorem by linking perpendicularity with the lengths of the sides. Furthermore, the idea of (geometrical) similarity appeared to be crucial for the students' generation of new suitable triples. The students discovered new triples, appearing to reach *generic abstraction* (Harel & Tall, 1991) of the Pythagorean Triples. The activities with the figurative numbers helped the students to identify and state special cases of the basic equation of the Pythagorean Theorem. The students realised that in the case in which the lengths of the sides were not integers the theorem could not be immediately generalised. The figurative numbers embody both the numerical and geometrical representation of the square numbers and can be used as the intermediate link between the geometrical squares and the arithmetical ones. The students appeared to realise that the combination of two representational systems (numerical and geometrical) suggests the independence of mathematical objects from the individual's perception, thus facilitating the students' objectification (first level) of the Pythagorean Theorem (Derrida, 1989; Radford, 2003). Overall, most of the students appeared to follow the desired cognitive path. First, they linked gravity with verticality and verticality with the Basic Triple. Subsequently, through the Basic Triple they managed to differentiate between verticality and perpendicularity. The 'generic abstraction' of the Pythagorean Triples and the figurative numbers allowed them to partially mathematise the situation, reaching the first level of objectification of the Pythagorean Theorem. Note that these claims seem to be supported by the post-test data analysis (not presented in this paper).

In conclusion, in this study, we created a comprehensive learning environment that settles the Pythagorean Theorem within the students' experiences, thus making it meaningful to them. Moreover, the students are allowed to construct important proto-mathematical ideas based on authentic experiences and to be aware of the role of gravity in the construction of a crucial geometrical concept. Hence, the students can realise the constructive, non-arbitrary and *constitutional* function of the mathematical concepts. Finally, this teaching is the result of 'adaptive didactic work' and, therefore, we argue that it can be introductory to the lesson traditionally taught at school.

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Mathematics education reform:The role of coherence within the complexity of change

Christine Suurtamm, Associate Professor
Barbara Graves, Associate Professor
Faculty of Education, University of Ottawa, Canada
suurtamm@uottawa.ca bgraves@uottawa.ca

Abstract
This paper draws on data gathered from a large-scale, multi-year research project, Curriculum Implementation in Intermediate Mathematics (CIIM), that examines the implementation of a reform (inquiry-oriented) mathematics curriculum in Grades 7 – 10 in Ontario, Canada. To describe classroom practices and ways that teachers have been challenged and supported in implementing an inquiry-oriented approach, the data included teacher questionnaires (n =1096), focus group interviews with mathematics educators across the province, and nine case studies. While some of our data align with the research of others who show that teacher change is complex and inquiry-oriented pedagogies are slow to emerge (Frykholm, 1999; Jacobs, Hiebert, Givven, Hollingsworth, Garnier, & Wearne, 2006), we also have evidence of teachers engaged in a variety of classroom practices that involve students in inquiry-oriented mathematics learning.

Introduction
The NCTM *Standards* (1989) served as a catalyst to prompt reform in mathematics education. Current thinking in mathematics education and, in many jurisdictions, current mathematics curricula reflect these reform views and recognize that knowing mathematics means more than simply knowing procedures but includes being able to reason and communicate mathematically and to engage in solving mathematical problems (Artelt, Baumert, Julius-McElvany, & Peschar, 2003; Ball 2003; Boaler, 2002; Hiebert, 1997; NCTM, 1989, 2000). However, research suggests that although the catalyst document is now 20 years old, evidence of reform teaching practices is not as prevalent as one might expect (Jacobs et. al., 2006). Facilitating mathematical inquiry is a complex process that involves the posing of problems, the generation of thought-provoking questions, and most importantly listening and responding to student thinking. These are not practices that are easily prescribed and they require a substantive re-orientation not only of teachers' practices but also of their beliefs about mathematical ideas and mathematics teaching and learning (Borasi, Fonzi, Smith, & Rose, 1999; Frykholm, 1999). Where reform mathematics curricula exist, "teachers often transform such new materials in light of their own knowledge, beliefs, and familiar practices; as a result, the 'enacted curriculum' can be quite different from the 'written curriculum'" (Sherin, Mendez & Louis, 2004, p. 210). In this paper we shed insights on the complex process of enacting a reform curriculum by presenting the results of a large-scale study that examines the implementation of a reform mathematics curriculum.

Design of the study
The Curriculum Implementation in Intermediate Math (CIIM) research project is a 3-year study designed to provide information about how the Grade 7 – 10 Ontario mathematics curriculum is understood, taught, and supported. Data were gathered through an analysis of the Ontario mathematics curriculum, focus group interviews with leaders in mathematics education (i.e. mathematics consultants) and mathematics teachers, an extensive questionnaire that was distributed across the province to teachers of Grades 7 – 10 mathematics (n =1096), and nine 1-week case studies of mathematics classrooms where there was evidence of inquiry-oriented classroom practices, or in other words, the written and enacted curriculum were fairly well-aligned. In this paper we will focus on describing our evidence of the emergence of reform-oriented practices and report on ways that such emergence has been facilitated and supported.

Context of the study
Our analysis of the Ontario mathematics curriculum suggests that it indeed reflects a reform curriculum. There is a pronounced emphasis on problem solving and investigation as part of classroom practice. The curriculum states that problem solving "forms the basis of effective mathematics programs and should be the mainstay of mathematical instruction" (OME, 2005a, p. 11 & 2005b, p. 12). It also indicates that using a variety of tools, including concrete materials and technology, is an essential part of classroom practice to help students learn concepts and develop

flexible thinking. Communication is a key element of instructional and assessment practices and classroom strategies that promote student-to-student dialogue about mathematical ideas are encouraged to enhance students' understandings of mathematics. The curriculum also takes the view that assessment is on-going, embedded in instruction and should support student learning.

While a curriculum describes the learning expectations for students, teachers often need resources and professional development (PD) to get a better sense of what a curriculum might look like in a classroom. In Ontario, the production of resources and PD occurred through the collaboration of the Ministry of Education with the provincial mathematics education organization and organization of mathematics coordinators. Thus, leaders in mathematics education teamed up with policy makers and practicing teachers to work together to support new teacher learning. The curriculum was also supported through provincial funding of resource materials such as manipulative kits, software and graphing calculators.

Classroom Practices

One of the items on the teacher questionnaire asked teachers "In this class, how often do the following occur?" and then listed a variety of classroom practices. Table 1 provides a summary of the responses.

Table 1: Summary of teacher classroom practices

Classroom practices	Never	Some lessons	Most lessons	Every lesson
Students work on practice questions	0%	9%	39%	52%
The teacher explains, demonstrates or provides examples	0%	8%	43%	49%
The teacher provides solutions to problems	1%	32%	42%	25%
Students provide solutions to problems	1%	25%	52%	22%
Students justify their answers and explain their reasoning	0%	34%	47%	19%
Students work on problems with multiple solutions	4%	62%	25%	9%
Students work on investigations to determine relationships or mathematical ideas	2%	62%	29%	7%
Students work with concrete materials or manipulatives	7%	69%	20%	3%
Students use computer software or graphing calculators	14%	73%	11%	2%

In this table we see that practices such as the teacher explaining, demonstrating or providing examples, students working on practice questions and providing solutions are part of most or every classroom lesson. However, we also see that reform oriented practices such as students justifying their answers or explaining their reasoning, working on problems with multiple solutions, working on investigations, or using technology occur in some or most classroom lessons.

Our case study data also provide evidence of these reform practices and help us to describe what these practices look like. During each case study, we interviewed the teacher(s) and school principal and observed and video-recorded the teachers' mathematics lessons over a period of 5 days. For the discussion in this paper, we focus on the classroom practices of providing opportunities for problem solving, encouraging the use of mathematical thinking tools, and facilitating mathematical communication. These categories reflect reform-oriented practices and also align with the practices that appeared as less frequent practices in Table 1.

Opportunities for problem solving. In all of the nine case studies, our video data show teachers posing problems, students moving in and out of groups to work on the problems, the encouragement of multiple solutions and the use of a variety of representations to model problems. Students shared their solutions with the class and through discussion the teacher consolidated understanding. In one case study, the teacher might begin a lesson with a problem to introduce the topic such as in a lesson that introduced partial and direct variation with a problem about selling programs at a baseball game. At other times, small problems were interjected in the lesson for students to think about, discuss with a partner and then share their ideas in a whole class discussion. In another case study, the class moved very easily in and out of group problem solving activities, as though a culture of problem-solving had been established. Students worked in pairs and were encouraged to investigate, discuss, and seek assistance from each other. The students had access to a variety of manipulatives and the students used these to construct mathematical models, examine their properties, and conjecture connections to other (pictorial, symbolic, and language) representations. All teachers spoke about the importance of problem solving in the current math curriculum. One teacher discussed an opportunity he had to observe a school in Hong Kong that uses the Ontario curriculum and he had been impressed to see the students' willingness to struggle with problems and thus he allows his students time to struggle.

Mathematical thinking tools. We use the expression "mathematical thinking tools" for materials that students use in class that help them create, think about, and discuss mathematical ideas. In the case study classrooms we saw students using a variety of mathematical manipulatives such as linking cubes, algebra tiles, and two-colour counters. In one case, the entire secondary mathematics department focused on including mathematics manipulatives in all their math courses. We observed the teacher assigning each group of students a second difference and asking them to build a quadratic function with that second difference using linking cubes. The students then shared their models and the teacher asked students to use their models to create a table of values and to examine their relationships using a graphing calculator. In later lessons she worked with students on multiplying binomials and factoring trinomials using algebra tiles.

We also saw a range of technologies being used that included graphing calculators, interactive white boards, clickers, motion sensors, and computers, particularly in the Grade 9 and 10 classrooms. In one Grade 9 classroom, the teacher makes extensive use of technology that includes all of the above as well as virtual algebra tiles and even the students' own ipods. In interviews, she suggested that not only does technology help students represent mathematical ideas but it also increases students' motivation. In another Grade 9 classroom several lessons included investigations where graphing calculators and other devices were used to collect data for which students would then create mathematical models.

Mathematical communication. In all case study classrooms, students were observed discussing mathematical ideas with one another in both small group and whole class discussions. In two classes in particular, the teachers were modeling the creation of a math talk community or math congress based on Cathy Fosnot's work (e.g. Fosnot & Dolk, 2001) and in all classes we saw students presenting solutions while other students observed, paraphrased, and asked questions. All of the case study teachers emphasized the importance of students learning from one another's solutions. As one of the Grade 8 teachers stated:

> I believe that children learn constantly from the world around them. They hear each others' voices much louder than adults' voices. Often peers can influence and teach each other quite effectively. In our classroom, you'll usually see students working together to discover new things and explain why to each other. (Angela, interview)

Another Grade 8 teacher revealed that she uses language to create an environment of respect and comfort in her classroom as well as a way of sharing ideas. She talked about using the term 'mathematician' to describe her students, telling them that they are all mathematicians and they need to share their ideas.

Facing dilemmas and uncertainties

While our case study data show teachers who are successfully engaging in reform practices, we also saw them face uncertainties and dilemmas. The practices that are being asked of teachers are often difficult to define, feel unfamiliar, and require a certain level of risk-taking. We observed teachers in the case studies reflecting on their lessons and often questioning whether they did the right thing at particular moments. They discussed the degree to which they needed to adapt moment to moment as the lesson changed direction based on what the students were doing and saying. Time also became an issue and teachers commented that lessons often took longer than expected.

The dilemmas that teachers faced when changing their practice did not always come from within the classroom. Teachers were not only influenced by their own beliefs, knowledge, and practices but were also influenced by student responses, colleague and administrator impressions, and parent concerns. In some cases, teachers were challenged by colleagues or parents who questioned what they were doing in their class as it appeared different from others teaching the same course. Teachers were also worried whether students would be adequately prepared for moving on to the next grade where the teacher expectations may be very different. Grade 9 teachers worried about student performance on the large-scale assessment that is administered to Grade 9 students in June. At times, these teachers felt isolated in their schools. In many cases, they were set apart as they were viewed as leaders in their schools and thus, did not necessarily have someone at the school level with whom they could discuss their concerns and share ideas and resources.

Support for Reform-Oriented Practice

We draw on a variety of data sources to discuss the ways that teachers feel supported in their implementation of new teaching practices. Two items in our questionnaire ask teachers how they learn about new ways of teaching and what resources or learning opportunities have supported their implementation of the curriculum. While teachers report that the top resource to support their teaching is the textbook, their second most valuable resource or learning opportunity comes from dialogue with colleagues. An open-ended item on the questionnaire asked teachers to describe a professional experience that positively influenced the way they teach mathematics. Interestingly, approximately one quarter of the 757 responses to this item mentioned an experience that involved collaboration, such as a professional learning group, a lesson study or dialogue at workshops.

Data from focus group interviews with both teachers and leaders in mathematics education support this view. As one teacher stated:

> The best thing that I got out of the workshop was just having the time to talk about how things work in my classroom in comparison to how things are going in other people's classrooms and having the same concerns. (Teacher Focus Group 2)

The value of collegial support was also seen in case studies. While many of the case study teachers did not necessarily have colleagues in their school with whom they could discuss their ideas, most of them had made strong networks outside of the school through their involvement in district or provincial initiatives. For instance, some had written some of the provincial resource materials that supported the curriculum and others had been involved in district lesson studies or professional learning communities. These initiatives gave them the opportunity to meet and dialogue with colleagues who had ideas similar to theirs and to try out and discuss new ideas in their classrooms. Such opportunities helped to support their changes in practice and provided them with confidence to continue with their work.

Several of the case study teachers were also supported at the school level by the principal and/or the department head. Administrative support was seen as crucial to the confidence and comfort of the teacher in trying out new ideas. This support was realized in a number of ways including scheduling adequate blocks of time for math to allow for problem-solving activities, while also providing time for teachers to meet together to talk about their work. In one case study, the entire department worked together to integrate manipulatives in all of their secondary math classes through the direction of the department head. This department head was, in turn, supported by the principal who provided release time so that the department head could work with new teachers on the integration of manipulatives in their courses. In another case study, teachers had been supported to take part in a lesson study initiative in a family of schools setting. This initiative was led by the mathematics coordinator, a secondary Vice-Principal and two elementary school Principals who not only supplied release time but who also attended and participated in the preparation sessions with the teachers. These administrators remarked on how much they themselves learned through participation.

> As an administrator you get time, while they're having their discussions, you sort of sit back a little bit and listen more than anything else and watch them and listen and you learn . . . (Elementary Vice-principal B, Case study 3, interview)

Conclusion

Fullan (2001) points out the value of coherence and shared meaning in implementing new ideas. Since the development of the curriculum and resources was a collaborative effort, the messages in the curriculum, resource materials and PD were common and reflected current thinking in mathematics education. Teachers appeared to be able to build new meaning and enact the new practices called for in the curriculum as they engaged in a dynamic cycle of discussion with their colleagues and testing out new ideas in their classroom. As Rosenholtz (1989) suggests:

> It is assumed that improvement in teaching is a collective rather than individual enterprise, and that analysis, evaluation, and experimentation in concert with colleagues are conditions under which teachers improve (p. 73).

Our findings support this. We also see that teachers report that they need greater opportunities to work with their colleagues and we saw the important role that administrators and policy makers can play in helping to facilitate such collaboration. The challenge we face at this juncture is to keep the momentum strong so that networks continue to develop.

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Mathematics games: Time wasters or time well spent?

Paul Swan and Linda Marshall
Faculty of Education and Arts, Edith Cowan University, Joondalup, Western Australia, 6027
p.swan@ecu.edu.au l.marshall@ecu.edu.au

Abstract:
Globally education authorities are placing increasing emphasis on the development of literacy and numeracy in primary schools. This paper reports on research designed to assist teachers to improve the numeracy of their students by making the use of mathematics games a more focused aspect of the teaching and learning experience in mathematics. Classroom experience and anecdotal evidence suggest that games are often used without really focussing on the mathematics involved in playing the game, and are justified simply on the basis of children having 'fun'. In this paper we report on the use of one game, Numero and how teachers made use of the game and the impact on the children's learning when using the game.

Introduction

The use of mathematics games is often cited as an effective strategy for teaching mathematics. However, the researchers were unsure whether games are being used as supplementary activities for children who finish their 'real' work early, as busy work, or used with a real purpose. Although children may be having 'fun', the belief is that this is not sufficient reason for their inclusion in a mathematics program.

The ECU team, in collaboration with Industry Partners The Association of Independent Schools of WA (AISWA) and R.I.C. Publications, were awarded a grant to investigate the use of mathematics games in primary schools in Western Australia. This investigation involved offering extensive Professional Development to a group of teachers from a variety of independent schools, with follow up in-school support. As part of this process, the teachers were expected to conduct some aspect of action research on the topic, and report back their findings to the group.

Literature Review

While it may be assumed that the use of games in the teaching of mathematics has been researched, a brief literature review revealed that surprisingly little empirical research into the use of games has been carried out. Often teachers assume that the use of games is an effective teaching tool. This may not be the case (Bragg, 2006).

Many authors have presented the use of games as a beneficial tool in the mathematics classroom (Bragg, 2006; Booker, 2000; Gough, 1999; Ainly, 1990). Also numerous authors assert that games should not just be restricted to practice and that they can be an effective vehicle for teaching new concepts to children (Bright, Harvey & Wheeler, 1985, Kamii & De Clark, 1985; Thomas & Grows, 1984; Burnett, 1992; Booker, 2000)

Clearly, what is important is the structure of the games used (Ainley, 1990; Badham, 1997) and the literature does highlight that if this structure is not provided learning does not always take place (Onslow, 1990; Burnett, 1992).

Connecting School and the Home with Games

Games are used worldwide as a means of developing mathematics concepts. Often parents purchase educational games as a means of supporting learning in the home. There is a large educational games industry referred to as "Edutainment" that is designed to tap into the parent market. The games used in this research tended to be more 'classroom games', although games such as Battleships that cross over the school/home market were explored. Numero is a game that crosses the school/home divide. There are many games of educational value played in the classroom that help make some topics less onerous; however, children would not choose to play at home. If a game has proven educational value and children choose to play them out of school, then the children would be spending more time on task and it could be assumed become better at the task or in this case the embedded mathematics. The fact that Numero is played both at school and home as of interest to the researchers.

The researchers consider Numero to be more than a simple practice game as it includes elements of problem solving when children make use of strategies to maximise their point score. For detailed explanation of the card game Numero see http://www.ricgroup.com.au/bookshtml2/Numero.pdf.

Asplin (2003) noted in her study that simply playing Numero as a time-filler or casually was not sufficient to improve mental computation skills. The authors argue that if a game is deemed worthy of playing then it should be elevated from the status of a time filler or activity for early finishers but rather be an integral part of a teaching sequence. Asplin went on to note that in the class where the teacher encouraged students to describe orally their moves, and where the children were grouped according to ability, gains in mental computation ability were much higher. However, Bragg (2006) who studied the use of two games in the teaching of mathematics, raised questions about grouping children according to ability when playing games. In this research the children within classes were not grouped according to ability but, in at least one case, the class had been formed because the children were less able.

The larger research project explored, in some depth, the use of games in the teaching of mathematics, in particular number, and aimed to develop guidelines for maximising the effectiveness of games in the teaching of mathematics. These guidelines are noted using Numero as an example.

Research Framework

This research employed a mixed-method design.

- Children were given a pre-test prior to the study and a post test after participating in playing the games over a period of time. As the selected game, Numero, is a ‘skills’ practice game, a pre-existing and validated basic number skills test was used. A test that reports on basic facts in the four operations was deemed the most appropriate tool to determine if any learning had taken place (Westwood, 2000). According to Westwood (2000), “the test-retest reliability of the One Minute Basic Number Facts Tests ranges from 0.88 to 0.94 according to age level” (p. 107).
- Classroom observations were made of the children playing the game.
- The teachers participating in the study were interviewed to determine their beliefs about the use of games as a pedagogical tool.

Action research (Kemmis & McTaggart, 1988) was undertaken by individual teachers and groups of teachers involved in this research. These multiple techniques were employed to ensure triangulation of data.

As the research methodology involved a mix of qualitative and quantitative methods, Miles and Huberman’s (1994) processes of data reduction, data display and conclusion drawing provided the qualitative data analysis base for this project.

Sample

A sample of 16 – 20 teachers were drawn from the Association of Independent Schools of Western Australia and asked to participate in a four-day professional learning program. This program highlighted the findings from the literature and introduced the teachers to a variety of different games. Participants were introduced to the action learning process and given the opportunity to explore an issue associated with the use of games, for example, the development of number skills through playing Numero. Not all of the participants elected to use the game Numero as their action research project, however those who chose to look at this particular game had a further meeting to examine preliminary data. Later, this sub-group met with the entire research team to share their findings.

Findings

The card game Numero is a practice game involving mental computation that, at its simplest level, deals with addition of numbers from 1 to 15. This can be extended to include subtraction, multiplication, division, fractions, square and cube roots, square numbers and cubic numbers. In this study the teachers did not go beyond the four operations when playing the game.

After playing Numero on the second day of the Professional Development component of this project, teachers were asked to fill in a Review sheet on the game. The questions in the Review sheet were:

- Does this game support/reinforce a concept that I expect the children to understand? Why?
- Could this game be used to introduce a ‘new’ maths concept?
- Comment on the rules of the game. Is it appropriate that all children in the class need to know this game? If so, why? If not, why?
- Some people suggest that children learn by playing games. What might children learn from playing this game?

- Additional comments (for example, if you had the money, would you purchase this game for your classroom? Why?)

Four teachers believed that playing Numero was beneficial in *reinforcing previously taught concepts* such as the four operations and BODMAS (an acronym for remembering the order of operations), rather than using it to introduce new concepts. However, another teacher reported that he used it to introduce the concept of order of operations as well as to practise their basic facts. He believed that a big advantage of the game was that the rules are constant, but the level of difficulty could be adjusted according to the ability of the groups of children playing the game. One teacher stated that, "The rules are appropriate to all levels and the children are able to conceive numbers in abstract ways"; another said, "One child can teach another as their ability develops".

One Case

One teacher reported his results in detail. He taught students who were in Year 7 of a Middle School, so they turned 12 that year. The range of pre-game results was higher for the 'critically low' range than for the 'average range', according to the tests. He reported that addition and subtraction were generally the best operations, with division by far the worst. Table 1 below shows the comparative data.

Table 1: results from Westwood One Minute Tests of Basic Facts

Operation	***Initial Results***		***Post –Game Results***	
	Average scores	***Critically low***	***Average scores***	***Critically low***
Addition	12	11	19	4
Subtraction	14	9	18	5
Multiplication	8	15	15	8
Division	5	18	14	9

After playing Numero regularly, the critically low scores for addition dropped from 47.8% to 17.4%; subtraction from 39.1% to 21.7%; multiplication from 65.2% to 34.8% and division from 78.2% to 39.1%. These data are reported because the teacher chose to involve parents in the research. He took the unusual step of phoning each of the parents to share the results from the pre-test and urged them to come along to learn to play Numero. The researchers supplied a pack of Numero to parents who attended the training session.

The children played the game three times a week for around 15 minutes per session. Parents, with the exception of one who did not play the game at home, reported playing the game from at least once a week to playing most nights of the week. The researchers suspect that the children of parents who played the game regularly at home made the most gains but the data were such that direct matches to children could not be made.

A teacher from a different school reported that her students did not experience gains. This was possibly due to the conditions not being suitable to foster improvement. However, the teacher commented on the improvements in attitude towards the learning of mathematics when games were employed. In this case parents were not involved in playing the game. The teacher commented that she had not been able to motivate the children to play beyond set times during mathematics. Clearly, if children are motivated and engaged in mathematics they are more likely to benefit from participating in the game, task or activity.

Criteria for Assessing Games

At the completion of the first day of Professional Development on games, all teachers involved in the project were interviewed. One of the questions they were asked involved describing the types of games they preferred. Table 2, below, shows the responses according to criteria of their choosing. These comments were provided before the teachers were shown the card game Numero.

Four main themes emerge from Table 2. These were illuminated by the comments made by teachers during the semi-structured interviews and in conversations during the workshops.

Classroom management issues

The most common criteria for selecting games as reported by teachers in the study were concerned with classroom management issues. Some of the comments related to trivial issues about pieces going missing or taking too long to set up and pack away. However, the authors do not wish to trivialise the impact on the smooth running of a lesson and the associated loss of teaching time.

Table 2: Teachers' criteria for selecting games

Description	*No. of responses*
Classroom management issues	
Simple, easy to explain	3
* Played in a small group	3
Not too noisy	1
Quick and easy to set up	1
Not too many pieces	1
Games that are quick to play	1
Games to make rather than buy	1
* Everyone participates at same time (not taking turns) ie whole class	2
Games that the teacher can observe	1
Types of games: Skills	
Relies on skill rather than luck	2
Practise a skill	2
Mental practice	2
Number games	1
Types of games: Concept	
Clear links to concepts	1
Place Value	1
Strategy games	2
Games that are relevant to children's stage of learning	1
Motivation and engagement	
Interactive	2
Challenging	1
Other	
A variety – chance for children to experience them all	2
Card games	1
Board games	2
Visual, spatial awareness games	1
Children use manipulatives alongside game	2
Where children move around physically	2

There were conflicting comments about number of players. In two cases the teachers preferred to play whole class games, while three teachers preferred playing games in small groups because it maximised the time children were thinking about the concepts embedded in the game.

Short games appeared to be favoured over longer ones. This allowed for more flexibility as to when and how games were used in the classroom. Simple rules were favoured as less time was spent introducing the game and sorting out conflicts based on misinterpretation of rules.

Clear links to skills and concepts

While the teachers liked the idea of using games from a motivational point of view, they preferred the game to be linked to a specific skill or concept. Several factors appear to affect this concern.

Teachers feel pressured to 'cover' a great deal of content and felt that devoting too much time to games without there being a direct link to specific content would erode their teaching time. The introduction of a National testing program during the conduct of this research weighed heavily on teachers' minds. The need to justify the use of games in terms of concept or skill learning was apparent. The teachers reported pressure from parents for their children to be seen to be completing some rigorous mathematics. In some cases, this translated to completing sets of algorithms on a page as evidence of having ‘worked hard’ in the lesson. The extreme case is the reported pressure to complete all the pages in a textbook before the end of the year.

Motivation and Engagement

Most teachers would choose to use games as motivators for engaging in mathematics. Games are often employed to make practice more pleasant. Difficult concepts such as fractions may be embedded in a game format to encourage deeper thinking about the concept. Bragg (2006) noted in her research that once the challenge of a game is lost, motivation wanes. Children are less inclined to engage with the game. For example, once you know how not to lose in Noughts and Crosses (Tic Tac Toe), it seems pointless to even begin a game.

Conclusion

Preliminary data collected from recorded interviews with teachers in the sample suggested that the research evidence presented during the Professional Learning sessions was valuable in changing teacher attitudes toward the use of games. Prior to attending the Professional learning sessions and carrying out their own action research, some teachers did not use games while others had only used them as a reward or with children who finished their class work early. Those teachers now report that they would use games as an integrated part of their mathematics program.

This research has helped to show that, given the right conditions, games can achieve an increase in basic fact skills in a stimulating and enjoyable environment.

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On Evaluation Problem of the Quality of Educational Models

Vladimir A. Testov,
Ph.D., Professor on Mathematical Education, Head of Department , Faculty of Physics and Mathematics, Vologda State Pedagogical University, Russia, 160035, Yologda, Orlova, 6
e-mail: vladafan@inbox.ru

Abstract The current approach to assessing the educational quality applicable to assessing objects and processes formed and realized in producing spheres is widely spread. However, as education is a much more complicated anthropological, social and cultural object in comparison to that of production, the above mentioned approach is least effective. In education both "strong" and "weak" models are used. There do not exist measurement instruments for accurate assessing mild results. Self control, expert assessing method and portfolio are being put forward.

In the past years, both in Russia and other European countries much attention has been paid to the study of the problem of the quality of both secondary and higher education. Different models and techniques of assessing the quality of teaching specialists are being elaborated; numerous bodies of its monitoring are being set up. Nonetheless, there are no evident results to judge the efficiency of the existing system due to, first and foremost, to the fact that a number of fundamental methodological questions have not been settled. Although the notion of “educational quality” is wildly used in cotemporary society neither its essence nor meaning has been described by scientists, practitioners, educationalists and educational authorities. Researchers of the problem do not demonstrate unanimous understanding of the fundamental terminology and basic concepts. Nevertheless, educational quality is under study and attempts to measure it are made by both educational authorities and researchers.

From the philosophical point of view the quality of objects or phenomena reveals itself as a combination of their characteristic features. Quality is part and parcel of the object and is connected with it as a whole. It cannot lose its quality without commencing to exist. As for education it means that from the philosophical point of view its quality is part and parcel of education, its essence, i.e., if there is education there its quality, otherwise there is no education whatsoever.

Most of contemporary researchers have rejected the philosophical definition of quality. Measure of educational quality is claimed first and foremost instead on the ground that the philosophical category of quality is not of estimating character which makes the question of measuring quality and differentiation between good and bad quality absurd. Taking this ground into consideration a number authors took another approach which is applicable to objects and processes realized in production spheres. This approach is most vividly presented in the concept of the Total Quality Management (TQM), International and Russian Standards of Quality (ISO 9000:2000, ISO 9000), according to which education is to meet the demands of the consumer that is the state, employers, students and their families, the society as a whole.

In the models and technologies worked out on this basis a well-known technocratic approach reveals itself. It is educational quality which is of primary importance in education rather than the quality estimation, which is of secondary importance. However, we sometimes observe a topsy-turvy picture, like the situation in Russia after the introduction of the USE when the importance of the examination for the quality assessment overcame its harm on the educational quality in the eyes of the authorities. We witness lack of understanding or will to understand that educational results can be of two types: those measured through controlling in quantities and those connected with the functions of education to bring up and develop, which is very difficult to undergo any analysis and measurement.

Educational results, especially those of the fundamental one, do not reveal themselves at once. This is why it is difficult enough to assess the students’ results objectively right after their graduation or a year later, because educational system is organized so that it follows the traditions of the society cultural development and tendencies rather than due to the necessity to meet the market demands. It is apparent that there exists a certain very important connection between the person’s professional success and his education, but the person’s individual career is influence by a number of other factor, which have nothing to do with education, (like person’s personality, connections and bonds, even his/her outward appearance). Thus neither the results fixed by the USE or through the Internet Test nor the fact that university graduates are given jobs after graduation can be reliable criteria of the quality of education. Ideally it is the level of applicants’ and graduates’ breeding and scholastic abilities which are to be measured and the goodness of the educational institution can be judged only if the students' cultural, moral and intellectual potential is growing.

Education as an object has complicated enough and quite sophisticated social and cultural connections and relationships, which makes the educational quality be defined from a principally different approach with the consideration of both the needs of various outer and inner educational processes, nature and essence of education.

The nature of the pedagogical system, its organization is reflected in the notion of educational model, which has been used by educationalists for a long time and this notion appearance is logical as the system of education has become far more complicated. In science, including pedagogy, it has been assumed that there can be various models and schemes of the same system in accordance with different research concepts and paradigms.

All pedagogical models can be divided into two types: "strong" and "weak" models which were first described by prominent Russian mathematician V.I. Arnold in his articles and reports, where he convincingly showed the usefulness of weak (mild) economic, ecological and sociological models characterized by somewhat non-determined and variable ways of development and the danger of strong (tough) models for which the one and only way of development exists [1]. The importance of educational strong and weak models was spoken of in our work [3]. The model reflects the nature of the way educational system is organized which is done first and foremost due to its goals. In the strong model the goals are concrete and are to be gained by the definite way, while in the weak model the goals are of a more general character and one can gain the goals by different ways, which are not determined and never reach them. That is why unlike in the weak models educational progress in the strong model is easily checked when compared with the targeted ones.

From the times of Renatus Cartesius and Isaac Newton strict predetermination of construction has been dominated in science. First it started in natural sciences and mathematics, then, this outlook penetrated into humanitarian spheres. Predetermination penetrated into pedagogies in with Y.A. Komensky, the result of which was an attempt to make education an ideally functioning tool. According to the then dominating theory in order to educate a person (child) it is enough to learn how to run such a tool, i.e., to turn education (learning-teaching process) into kind of an industrial and technical process. Thus, technological approach was started to be applied to the learning-teaching process and reproductive activity of students started to be predominant. The most important contemporary achievement of the technological approach in teaching is said to be setting up concrete diagnostic aims to be reached in a definite school period. So the greater part of technologies, made in the years, can be considered as examples of tough educational model.

A question if this toughness is useful arises. The system of teaching goals elaborated by B. Blum has been considered most popular in recent years. But his teaching targets have been transformed into learning activity, which determines the levels of the learning progress and his system parameters are mainly oriented to knowledge and not to the students' development.

Strong technology supposes concord of the result and the aim. On the contrary, creative activity presupposes discord of the aims and the results. If the teaching goals are set for a long period of time (say a school year or a number of years), then definite strong targets can appear either impossible to achieve or even harmful; they will have to be changeable or they are to be of more general character. Tough model is a way to erroneous prognostication. Moreover, under definite circumstances striving to plan and optimize a few years ahead can lead to a catastrophe. The strong educational model supposes that students and the teacher are forced to achieve the defined goals. But compulsion is always non-efficient and ruinous.

When constructing pedagogical models including those of education quality, it is necessary to take into account the social and economic state of affairs, which are changing very fast, principal uncertainty, variability of challenging life situations, demanding from the students to learn to live and study in the conditions of choice.

In the past decades changes in the whole system of mentality have emerged on the basis of discoveries in natural sciences (I. Prigozhin, G. Khaken and others): a transfer from the images of order to the images of chaos has occurred, science is no longer associated with determination, there developed ideas of non-determination, non-predictability of the evolution ways of complicated systems. There appeared new sections (theory of catastrophe, fractal geometry, theory of uncertain sets, polysemantic logics etc.) in mathematics. These are the basis of mathematical theory of weak models, the usefulness of which was found quite recently, that is why few researchers take the new ideas as a clear-cut theory, and striving for determined constructions and making strong models still predominate.

Far too few educationalists understand the usefulness and necessity of educational we weak models, although as far back as the 1980s Russian researcher and educationalist E.N. Gusinsky formulated the principle of uncertainty for humanitarian educational systems according to which the results of joint actions and development can not be predicted in all details [2]. Thus in the learning-teaching process there are always small changes, fluctuations of different pedagogical systems (both individuals, groups of students and knowledge system), which are not planned. This is why at the basis of the present day educational models there is to be a principle of uncertainty of a number of teaching and governing parameters.

Due to this factor a feedback, estimation of the state of affairs is necessary for making decisions depending on the real state of affairs and not only plans. Thus educational goals are to be changing continuously or they have to be of non-rigid character, so that there could be various ways to achieve these goals. The point of ramification of the ways in science is called the point of bifurcation, the existence of which is a characteristic feature of systems capable of self-organizing. In the past decades science of self-organizing systems – synergetics – has been developing. Like many other modern theories synergetics caused a great number of both scientific and pseudo-scientific publications. Nonetheless, thanks to profound answers to simple questions a chain of remarkable impressions compelled to take synergetic approach seriously. It is synergetics that weak models emerged.

The development of synergetic ideas could not but tell on the development on pedagogies. In the past 15 – 20 years the interest to the theory self-organization in pedagogical spheres has been growing. Unfortunately, only a few enthusiasts are making attempts to put theory into practice.

Educationalist who stick to strong models do not understand, that in school and university there is to occur some chaos, that the micro-level fluctuations play a very important part in revealing tendencies and teaching aims for the nearest future.

The conclusion synergetic arrives is the following: governing and management of the self-organizing system can only exist in case that it has stepped on its own way of development, rather than through rigid plans and schedules, which are part and parcel of strong models. This is what makes the essence of the educational weak model approach as well as in checking educational quality, based on the search for inner tendencies of educational systems their self-development, self-organization without being imposed from aside by alien ways of development.

Weak models are the wisdom of flexible conduct of the learning-teaching process through advice and recommendations, which in fact makes this way of conducting the process self-governing. The best way to monitor the quality is self-monitoring and the best control is self-control. It is the ways students learn the necessary knowledge, search for the necessary information, ways of self-education rather than the ways the teacher and lecturer teach, that makes the essence of education. One of the forms of weak models in learning-teaching process is heuristic. Heuristic, enlightening way of thinking is an example of non-linear mentality, the result of which is that it is impossible to plan and measure precisely.

Both tough and mild educational models have long existed in education history. E.g., Socrates' system of teaching is an example of a mild model. Among our contemporaries there are bright examples of systems: that of M. Montasorry and Russian educationalist and innovator M.P.Stchetinin. During hundreds of years rivaled and added to one another. Contemporary educationalists with linear way of thinking strive to present the learning-teaching system as a whole quality control system to constructing tough models with measurable results, which is supposed to be ruinous .

The unified approach to define the educational quality imposed on our schools and higher educational institutions can be very harmful. In order to provide systematic, full fledged character of the educational quality one has to use a complex of methods. For weak models beside self-control such methods as expert assessment, method of portfolio etc. are used. Special attention is to be paid to collective expert assessment of final educational quality (kind of pedagogical council). Only this approach can help to assess the real results of education and are held in real time.

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HOW CAN A SYSTEM WITH NO PUBLIC EXAMS BE FAIR?

Kerry J Thomas, Head of Mathematics, The Southport School, Queensland, Australia.

Abstract:

For 25 years, I have worked in a high school education system, where for the final 2 years of schooling, teachers at each school write their own programs of work and write their own assessment items. They then mark and report on this assessment. There are no final public statewide exams, and as an outcome students right throughout the State are ranked for University entry.

What follows is an exploration into the procedures that are put in place to ensure that each and every student is treated fairly and equitably. I will discuss the various levels of moderation that take place between schools, the processes that aid in keeping a level playing field for all concerned.

Introduction:

Imagine teaching in a system, where there are no final exams, teachers organize their own coursework, then their own assessment items. All in a student's last 2 years of high school before entering university.

How can such a system be fair? How can there be a guarantee that students at school XYZ are not advantaged or disadvantaged when compared to school ABC in the next suburb, the next town?

First however I will need to give you a little history and geography of my country, Australia.

Australia, a spacious country for its 20 million inhabitants, is made up of 6 states and 2 territories. Each state has its own Governmental Education Body, and as such, each of the States and Territories tends to set their own educational agenda. As a result there is a different curriculum, assessment and reporting framework in each state.

My State of Queensland, which has 20% of Australia's population, and is the fastest growing state, adopted this new school-based assessment in 1972, when public exams where abolished. The Queensland Studies Authority (QSA) is the state body responsible for the overseeing of all learning and reporting in years 11 and 12. Bear in mind that what follows applies to every subject offered but I will concentrate on Mathematics as it is what I am most familiar with.

Overview:

The QSA develops, reviews and approves syllabuses for subjects for the Senior Certificate. Senior syllabuses form the basis for the preparation of work programs and study plans by schools.

Teachers assess student work and determine levels of achievement according to criteria-and-standards descriptors outlined in the subject syllabus.

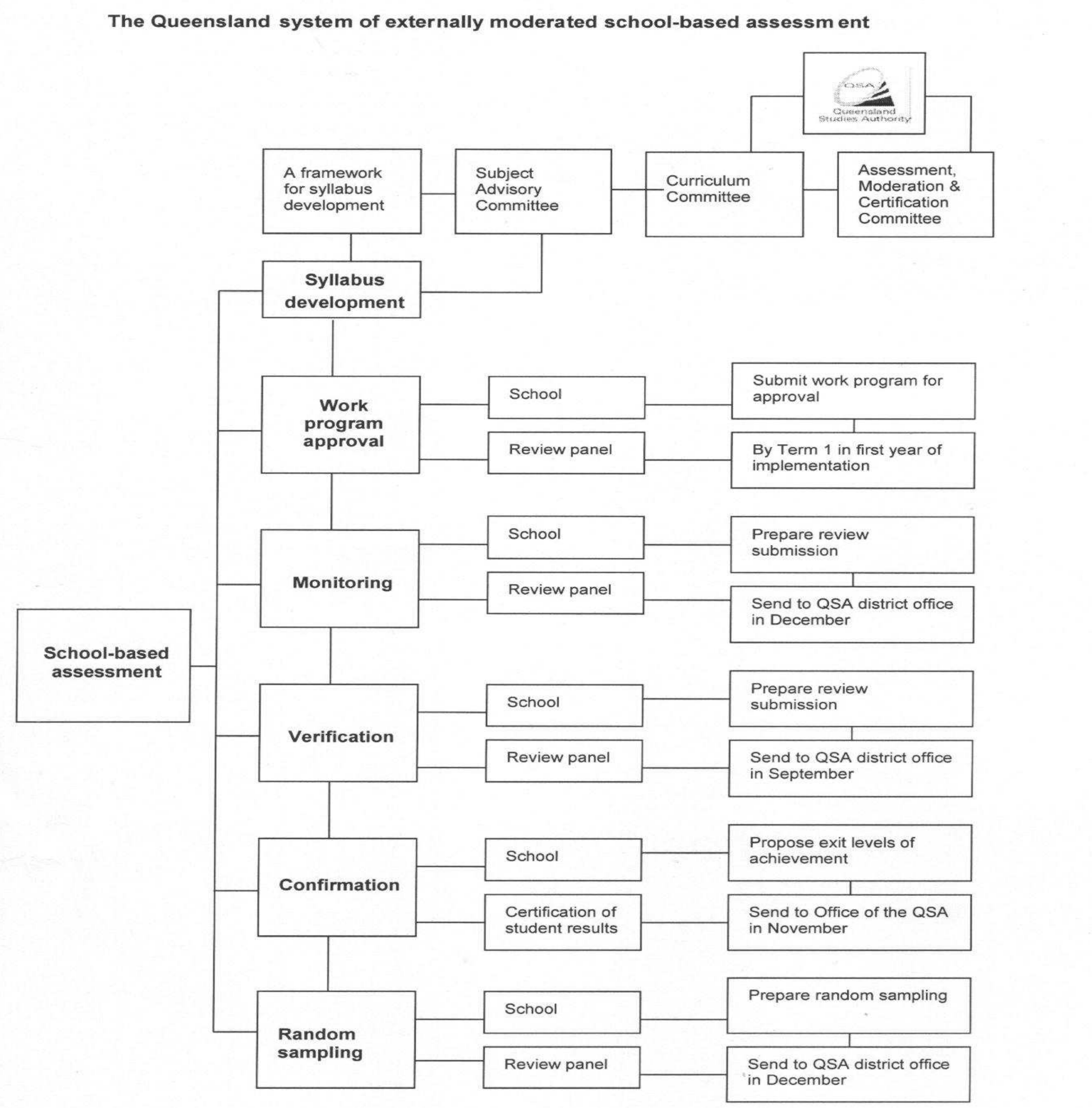
Schools provide learning experiences and assessment opportunities for their students based on work programs approved by the QSA. Schools are responsible for setting up appropriate accountable processes and procedures for assessing student achievement and communicating these processes and procedures to students.

The school principal (or nominee), acting as the school moderator, ensures that implementation of assessment and judgments of standards within the school are consistent with the procedures outlined by the QSA. This person is responsible for the total assessment program in the school and for moderation processes within the school.

Within each school, subject moderators (subject teachers, subject coordinators or heads of department) are directly responsible for preparing and implementing work programs, setting assessment standards consistent with syllabus descriptors, internal moderation processes with the subject where these are required, and preparing external moderation submissions.

Assessment in Authority subjects is externally moderated. The QSA, through state and district review panels, operates quality-assurance procedures, approval of work programs, monitoring of standards of assessment, reviewing (verification and confirmation) of proposed levels of achievement before certification of results, and random sampling of student folios after certification.

The following diagram summarises the Queensland system of externally moderated school-based assessment. (taken from “Moderation Processes for Senior Certification”, QSA, 2005)



Quality Assurance Framework – Stage 1

Developing a Work program from a Syllabus:

The QSA develops a syllabus for each of the three Mathematics subjects on offer to Year 11 and 12 students. The subjects are called Mathematics A, Mathematics B, and Mathematics C.

The Contents page of the Mathematics B Syllabus gives a good idea of the standards set. (taken from "Mathematics B Senior Syllabus", QSA, 2008)

From this document, teachers at each school develop a Work Program. This is submitted to a QSA panel to ensure the Syllabus requirements are met. This usually takes place in the 1st term of the school year. A panel will consist of a group of a dozen or so teachers in the district.

This process will be repeated about once ever 7 years when each syllabus is revised or re-written.

The detail given to each of the many topic in the syllabus is outlined in the example which follows. Note that the Suggested Learning Experiences (SLE) have been truncated. (taken from "Mathematics B Senior Syllabus", QSA, 2008)

Exponential and logarithmic functions and applications (notional time 35 hours)

Focus
Students should be encouraged to develop an understanding and appreciation of exponential and logarithmic functions and the relationships between them. They should be conversant with the three methods of representation (algebraic, graphical, numerical). Emphasis should be placed on the application of these functions to solve problems in a range of life-related situations (e.g. finance and investment, growth and decay). The use of technology should help students in these processes.

Subject matter

- *index laws and definitions (SLEs 1, 3, 5, 9)*
- *definitions of a^x and $\log_a x$, for $a > 1$ (SLE 1)*
- *logarithmic laws and definitions (SLEs 1, 2, 9)*
- *definition of the exponential function ex (SLEs 4, 6)*
- *graphs of, and the relationships between, $y = a^x$, $y = \log_a x$ for $a = e$ and other values of a (SLEs 3, 6, 12, 14, 16, 17, 18)*
- *graphs of $y = e^{kx}$ for $k \neq 0$ (SLEs 3, 6, 8, 14, 17, 18)*
- *solution of equations involving indices (SLEs 5, 8, 9, 11)*
- *use of logarithms to solve equations involving indices (SLEs 8, 9, 11, 13)*
- *development of algebraic models from appropriate datasets using logarithms and/or exponents (SLEs 2, 5, 7. 10, 17, 18)*
- *derivatives of exponential and logarithmic functions for base e (SLEs 3, 6, 7)*
- *applications of exponential and logarithmic functions, and the derivative of exponential functions (SLEs 2, 5, 7, 8, 10, 11, 13, 14)*
- *applications of geometric progressions to compound interest including past, present and future values (SLEs 19, 20, 21)*
- *applications of geometric progressions to annuities and amortising a loan (SLEs 22–32).*

Assessment:
Underlying principles of exit assessment:
The policy on exit assessment requires consideration to be given to the following principles when devising an assessment program for the two-year course of study.

1. Information is gathered through a process of continuous assessment.
2. Balance of assessments is a balance over the course of study and not necessarily a balance over a semester or between semesters.
3. Exit achievement levels are devised from student achievement in all areas identified in the syllabus as being mandatory.
4. Assessment of a student’s achievement is in the significant aspects of the course of study identified in the syllabus and the school’s work program.
5. Selective updating of a student’s profile of achievement is undertaken over the course of study.
6. Exit assessment is devised to provide the fullest and latest information on a student’s achievement in the course of study.

Assessment techniques in this syllabus are grouped under categories. The following categories of assessment techniques may be considered:

- extended modelling and problem-solving tasks
- reports
- supervised tests.

Assessment of student achievement should not be seen as a separate activity, but as an integral part of the developmental learning process which reflects the learning experiences of students. There should be variety and balance in the types of assessment instruments used, thereby enabling students with different learning styles to demonstrate their understanding.
Assessment techniques other than supervised tests must be included at least twice each year and should contribute significantly to the decision making-process in each criterion.

<u>Quality Assurance Framework – Stage 2</u>

MODERATION via Monitoring, Verification and Comparability.

So now we have a Work Program that has been endorsed by the QSA panel and we can commence teaching and assessing over the 2 years of the course.

- At the end of the first year (Year 11) half a dozen students scripts together with a ladder of student achievement are submitted to the QSA panel for **Monitoring**.
- Three quarters of the way through the second year of the course (Year 12), 10 students scripts and a more detailed ladder of student achievement are submitted to the QSA panel for **Verification**.
- And finally there is the process Random Sampling that ensures **Comparability** between the districts in the state.

These three processes are now explained in more detailed.

MONITORING

Monitoring is the process by which review panels consider the schools' implementation of the course and the standards of assessment in Authority subjects after Year11.

The focus of the monitoring meeting is on the quality of implementation of the course and organisation of the submission. That is, monitoring is about answering the question:

"How well is the school implementing the course?"

Monitoring review panels are required to give schools advice on:

- Implementation of the course of study, as indicated by the assessment instruments and possibly the course organisation in the work program
- Standards of assessment at this stage of the course
- Quality of decision-making about student **Levels** of Achievement at this stage of the course
- Organisation of the submission
- Assessment design that will inform future design and practice

Monitoring review panels are NOT required to give schools advice on:

- **Relative Achievement** of each sample student
- The overall distribution of relative achievements of students nor should any inference or implication be drawn
- An alternative rung placement on the R3 ladder form for any sample students
- An individual assessment item without consideration of the entire package
- The shortcomings of an approved work program

The bottom lines:

The Monitoring review aims to provide feedback on the school's decisions about the implementation of **the course**

If a syllabus does not demand it, then a panel CANNOT demand it

VERIFICATION

Verification is the process by which review panels advise schools on standards of student work and the relative achievement of students in Year 12.

The focus of the verification meeting is the quality of each schools' decision making, informed by comparing standards of student work with the level of achievement descriptors outlined in the criteria and standards matrix of the syllabus. That is, verification is about answering the question:

"How good are the schools' judgements about student achievements?"

Verification review panels are required to give schools advice on:

- Overall standards of STUDENT work compared with exit standards descriptors
- Quality of schools' decision-making with respect to SAMPLE students
- RELATIVE ACHIEVEMENT of SAMPLE students
- HOW WELL SUPPORTED AND ACCURATE are the schools' judgements

Verification review panels are NOT required to give schools advice on:

- The modes of assessment undertaken
- Work program issues
- Placement of other students on the R6 ladder other than the Sample students.

The bottom lines:

The Verification review aims to substantiate the school's decisions wherever possible regarding relative achievement of sample students

If a syllabus does not demand it, then a panel CANNOT demand it

COMPARABILITY

Comparability is the process by which state review panels ensure understanding of and advice about standards and levels of achievement by district review panels are consistent across the state.

The focus of the comparability meeting is matching the judgments concerning interim levels of achievement of agreed-to submissions of districts to the syllabus standards. That is, comparability is about answering the question:

"Do judgments made in schools across the state match the syllabus descriptors of standards?"

Comparability review panels are required to give the Queensland Studies Authority feedback related to:

- the appropriateness of levels of achievement that have been awarded
- Quality of advice provided by district panels to schools.

Comparability review panels are NOT required to:

- Conduct Verification reviews of the sample submissions
- Necessarily review every folio provided

The bottom lines: The comparability review aims to ensure that standards and levels of achievement are maintained across Queensland and judgments made match syllabus standards.

Conclusion:

So that explains the multi-level Moderation System used to ensure there is equity for all. So what are **the benefits of a moderated school-based assessment** program?

Studies done by the QSA show that Queensland's system produces reliable and comparable assessment of student achievement - at significantly higher levels than typically found in the marking of standardised public exams. These are some of the added benefits:

- Teachers can write work programs that reflect the school's clientele in terms of interests and issues, and that make best use of school and local facilities.
- Teachers can use a range of assessment techniques - including group work, oral presentations and supervised exams - to cater for the varied learning styles found in any group of students.
- Continuous assessment provides more opportunities for teachers to give feedback to students about how they might improve their performance.
- Students are judged on their performance over 2 years rather than in a once-only exam.
- The external moderation process helps teachers improve their understanding of assessment and provides valuable professional development.

The next question is that as a teacher, working in this system for 25 years, as well as being a QSA panel member, **"Does it work?"**

In a nutshell, yes I believe it does. I'm not saying that the system is not open to abuse, but on the main, there appears to be equity for all. Teachers certainly have an increased workload, particularly setting and marking assessment items that they know are going to be openly scrutinised. Many are also members of the various QSA panels that look at other schools work, but even though there is time commitment (approx 3 days per year) and responsibility aspects to these positions, there is also the positive outcome of enrichment and professional growth.

Teaching Mathematics in Eniaio Lykeio (Unified Upper-Secondary Education) with the use of New Technologies

Dr Eleni Tsami
Department of Statistics and Insurance Sciences, University of Piraeus, Greece
Tel. +306974-813711. Email: etsami@yahoo.gr

Abstract

In the teaching of the subject of Mathematics and in particular, in the teaching of the linear function f(x) = ax + b, the use of Microsoft Office Excel programme (1st grade of Eniaio Lykeio/ Unified upper-secondary school) equally facilitates both participants of the learning process, as the particular programme is incorporated in the context of the learner-centered educational procedure. Within the framework of this point of view and with the aim of effectively compiling the syllabus, the application of twelve (12) basic principles hinging on the active participation of learners in mutual cooperation, is considered necessary. Self-evaluation and the need to establish specific incentives and set concrete aims and objectives constitute indicative examples of basic principles.

Within the frame of the afore-mentioned educational principles, it is suggested that the class is divided in groups of 2-3 students and new technologies are implemented, with the ultimate goal to clarify and comprehend concepts and applications relevant to the subject. Criteria for the design of such an activity are the exploitation of learners' background knowledge and experience as well as the experimental involvement in new teaching practices. As prerequisites, we pose the formulation of conjectures and conclusions and the 'de-penalization' of errors in the mind of learners.

Introduction

In a traditional – teacher centered lesson, learners are taught definitions of concepts, theorems with their proofs, and finally conclusions, with the aim to solve exercises and mathematical problems. The tools employed to assist learners are: paper, pencil, ruler and a pair of compasses. The main disadvantage of the above tools is that, often, it is not possible to achieve accuracy in drawing or designing and as a result, fundamental geometrical principles are not disclosed. For the lesson we have chosen to deliver within the context of this assignment, we will make use of Microsoft Office Excel sheet.

A model lesson of the 1st grade of the Hellenic Eniaio Lykeio (Unified upper-secondary school) is included; more specifically, the paragraph *"the linear function f(x) = ax + b"*. The learner's worksheet, the teacher's worksheet as well as the Excel sheet in which learners can experiment under the guidance of the teacher, follow.

Psychological- pedagogical principles

This excel sheet resolves the lack-of-accuracy problem and facilitates learners' trials and conjectures. It also facilitates the disclosure, with ease and clarity, of the fundamental relations hidden in the figures. Excel helps the teacher to plan and implement certain educational activities of high standard which are oriented towards rich and fruitful teaching/ learning objectives hinging on the constructive model, that is, activities that focus on thinking and understanding rather than on mere drilling and barren memorization.

The psychological principles described further down relate to education and recapitulate some of the significant findings of current research on learning. They make an attempt to incorporate researches from various fields of psychology such as educational, developmental, cognitive, social and clinical psychology. These researches have provided new ideas concerning the learning process and the evolution of knowledge in several fields of study. As a result, today, school curricula and the teaching process undergo transformations in an effort to become more learner- than teacher-centered, link school to real life conditions and focus on understanding and thinking rather than on memorization and merely drilling.

The 12 principles can be more easily comprehended if perceived as an organised unified whole where each principle supports the others. The principles are put forward as a unified framework for the designing of school curricula and teaching methods. Indeed, they are behind a number of innovative programmes in schools all over the world.

We begin with the discussion of three principles which are widely acknowledged as the ground on which teachers should design the learning environments of modern school. That is, learning environments which encourage learners to actively participate in the learning process, cooperate with other learners and be involved in meaningful activities. Then, we proceed with seven principles which focus on cognitive factors that are basically internal, but also interact in significant ways with environmental factors. Teachers should take into consideration these principles so as to design more effective syllabi and teaching procedures. In the end, we discuss developmental and individual differences as well as the impact of motives on learning. These two last areas are very important for teaching and learning and they are worth being the focus of separate sheets so that they are adequately elaborated upon. The 12 principles are the following:

1. Learning requires the active and constructive participation of the learner.
2. Learning is primarily a social activity. Learner's participation in school social life is a prerequisite for learning to take place.
3. Learners learn more easily when they get involved in activities which they consider useful in real life and are related to their culture.
4. Learners' new knowledge is structured upon the basis of their beliefs and understandings.
5. Learning is enhanced and accelerated when students learn to employ effective and flexible strategies to resolve problems.
6. Learners should know how to plan and monitor their learning process, set their own objectives and rectify their errors.
7. Sometimes background knowledge can block the way towards new learning. Students should learn how to resolve internal conflicts and restructure the already existing concepts, whenever it is necessary.
8. Achieving learning is facilitated when the material used, hinges on general principles and explanations and does not rely on memorizing isolated facts and procedures.
9. Educators can enhance their students' ability to implement the knowledge gained in order to resolve real life problems.
10. Learning is a complex cognitive activity that demands no precipitation. A lot of time and practice are required for a skill to start being formed.
11. Research has shown that there are significant developmental learning differences among learners.
12. Learning is affected by the existence of incentives/motives on the part of the learner. Psychologists have distinguished between two types of motivation: a) extrinsic motivation and b) intrinsic motivation. Extrinsic motivation t is realized when positive methods of reinforcement are employed, such as rewards, high grades etc., while intrinsic motivation emerges when students fervently participate in meaningful activities without the need for any rewards. Teachers, thus, through their attitude and the tasks they bring into the lesson, can draw their students' attention and stimulate their interest so as to establish motivation which is an essential precondition for the realization of learning.

Learners in the P/C laboratories

Given the fact that we rely on the afore mentioned psychological and pedagogical principles as well as on the potentials of Excel, we propose the following educational activity within the context of a two-hour teaching session in 1st grade, Eniaio Lykeio. It is recommended that the teacher divide the class in groups of two (2) to three (3) students and distribute the Learner Work Sheet that follows. Each group has their own P/C. We propose that the activity is realized in four (4) stages:

Stage I: The teacher gives instructions to the learners concerning Excel so that they experiment on the figure of the educational activity.

Stage II: Learners start working individually.

Stage III: In each group, learners exchange opinions. Whether the correct solution is reached or not, by any of the groups, is the last thing that should preoccupy the teacher.

Stage IV: An open discussion on the solutions of each matter takes place.

Teaching/ learning aims:

At Cognitive level

Within the framework of the traditional teaching practice, the attempt to approach concepts, especially abstract concepts such as the 'straight line', which are not part of learners' direct and natural experience, is problematic. Thus, the development of incentives and the active character of learning are restricted. As a result, it is not possible to resolve cognitive conflict and structure new cognitive schemata. Nevertheless, the simplicity, the speed and the accuracy of the Excel program, in combination with its dynamic operation and its ability to demonstrate concepts, contribute to learners' familiarization with and comprehension of difficult-to-perceive concepts which constitute the primary aim of the activity mentioned.

Also, our aim is that learners:

1. Activate and make use of their prior knowledge regarding: a) the Pythagorean Theorem, b) the distance between two points, c) the concept of the tangent angle, d) the concept of the parallel lines and the concept of the perpendicular lines, e) the solution of an equation and f) the solution of a system of linear equations.
2. Understand the concept and function of a variable. Focal points are: a) the manner in which we select a variable, the reason for using it and what it represents, b) how a figure is transformed when an element of its structure changes due to the variable we have selected.
3. Feel the need to come up with one or more strategies in order to resolve a number of problems.

Implementation of New Technologies

A parallel teaching objective of the activity is for learners to become familiar with the computer and practice basic calculating skills of the software. Thus, our goal is that learners, after understanding the basic commands of the main menu of the software, are able to:

1. work with the "mouse" as a tool of dynamic manipulation.
2. depict, in a system of rectangular axis, the graphic representation of the function f(x)=ax+b and interpret it.
3. save and retrieve their assignment.

The Learning process

The criteria for the planning of this educational activity include:

1. The activation and exploitation of learners' prior knowledge and experience.
2. Learners' direct manipulation of and experimentation with the figure as a result of the exploitation of their intuitive thought.
3. The notion according to which, the existence of a problem waiting to be solved, creates the need to refer to Mathematics – without a problem, there is no need to be engaged in Mathematics.
4. The resolution of a problem creates the learning framework within which the learner structures Mathematical knowledge.
5. Learners contact with the methodology of conducting experiments, that is, with:
   a) the formulation of conjectures related to the intersection, the parallelism and the perpendicularity of two straight lines
   b) the checking of the pertinence of these conjectures
   c) their verification
   d) the formulation of conclusions.
6. Learners' need to invent a certain strategy as a methodology towards actively acquiring knowledge.
7. The studying of the 'straight line' concept
8. The cultivation of learners' analytical and synthetical thinking.
9. The creation of new pedagogical roles both for the teacher and the learners
10. Learners' engagement in procedures that promote cooperative learning and communication.
11. Learners' practice of skills concerning the expression of oral and written word.
12. The 'de-penalization' of the concept of error in the mind of the learners

v) Learner-Teacher Work Sheets, Excel Sheet

LEARNER WORK SHEET

The linear function f(x) = ax+b (§ 7.3)

Open the **line1.xls** file (activate the macro instructions).

a) Complete the following table (Use boldly the scroll bars to change the values of the parameters a, b).

x	-0,37	1,26	a
y = 2,5x +1,4			
y = 2,5x -1,4			

➢ Conclusion:

b) Examine whether the points A(-1, -1.1), B(-2, 3.6) and C(0.4, 2.4) belong to the straight line y = 2,5x +1,4. Justify your answers.

c) Find the equation of the straight line that passes from the points A(2,4) και B(-2,16).

a) Complete the following table:

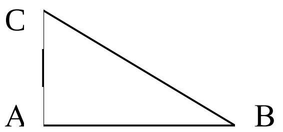
y = 2x+1	
x = 0 y =	Geometric interpretation:
x = y = 0	Geometric interpretation:
y = -2x+3	
x = 0 y =	Geometric interpretation:
x = y = 0	Geometric interpretation:

➢ Conclusion: b) Do the lines l_1 and $l_{2,}$ in figure 1, cross?

Make a conjecture.

For the following you can use the Windows Calculator.[Start > Accessories > Calculator (View: Scientific)]

Also, remember that in a right triangle *ΑΒΓ (Α=90°) apply:*



$$\boxed{tanB = \frac{AC}{AB}}$$

If **y = 2,5x + 3,4** (line $\mathbf{l_1}$) find the points of its intersection with the axes and then with the angle $\mathbf{w_1}$ which is formed by l_1 and the axis x´x.

a_1	intersection l_1 with xx'	intersection l_1 with yy'	tan w_1	w_1

If **y = 2,5x + 1,8** (line $\mathbf{l_2}$) find the points of its intersection with the axes and then with the angle $\mathbf{w_2}$ which is formed by l_2 and the axis x´x.

a_2	intersection l_2 with xx'	intersection l_2 with yy'	tan w_2	w_2

If **y = - 0,4x + 1,2** (straight line $\mathbf{l_3}$) find the points of its intersection with the axes and then with the angle **φ** which is formed by l_3 and the axis x´x.

a_3	intersection l_3 with xx´	intersection l_3 with yy´	tan φ	φ

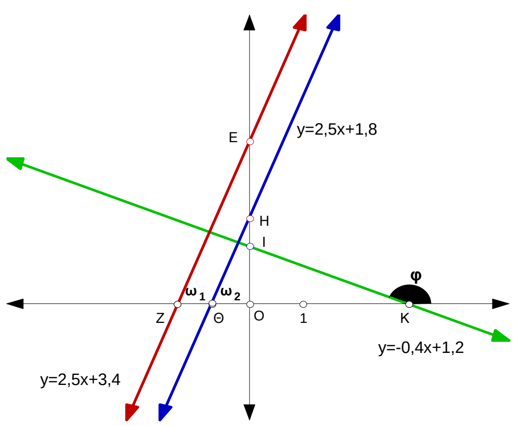


Figure 1

If **w** is the angle which is formed by the axis x´x and the line **l,** with equation y**=ax+b**, then complete the table based on the above:

lines	w	tan w	a
l_1			
l_2			

- Conclusion: What is the relation between the *'coefficient of direction'* **a** and angle **w**?

7. If two lines are parallels, then the coefficients of direction of the lines are equal. Does the inverse apply; (figure 2)
Make conjecture

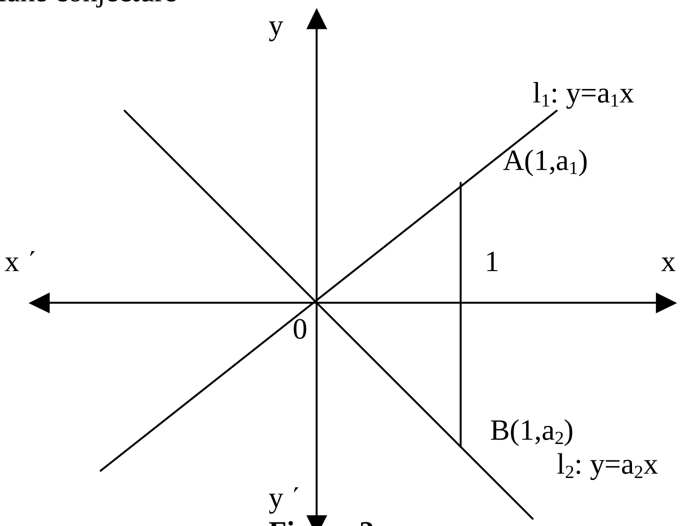


Figure 2

Find the angle which is formed by the straight line δ and the straight line l_1 of figure 1.
If two lines are perpendicular then the product of the coefficients of direction equals –1 and inversely (figure 2). Make a conjecture.

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Math lessons for the thinking classrooms

Ariana-Stanca Văcărețu
Mathematics teacher
Romanian Reading and Writing for Critical Thinking Association, Cluj-Napoca, Romania
ariana.vacaretu@vimore.com

Abstract

Teaching mathematics means teaching learners to think – wrote Polya in How to Solve It? 1957.

This paper intends to offer mathematics teachers suggestions for incorporating reading, writing, and speaking practices in the teaching of mathematics.

Through explicit examples and explanations we intend to share ways of engaging students in deep learning of mathematics, especially using and producing written and oral texts. More specifically, we plan to broaden and deepen teachers' understanding of strategies for guiding students' thinking so that they grasp mathematical concepts and processes, and also bridge the divide between mathematical processes, and written and oral communication.

This paper presents a core math lessons which provides numerous opportunities for the students to get actively engaged in the lesson and think about the new concepts, algorithms and ways of solving problems/ exercises.

The lesson was designed for the 7^{th} graders (13 year-olds). It was chosen to illustrate teaching by using reading and writing for understanding math processes. The teacher's reflections after the lesson and some samples of the students' work and feedback are included in the paper.

The material in this paper is based on the author's own extensive teaching experience; and her work in the Reading and Writing for Critical Thinking project in Romania.

Introduction

Unfortunately, lately, in Romania, math has been less and less about teaching learners to think, and more about teaching learners enough math to pass a national examination. So, teachers "show" students algorithms and recipes for solving exercises and problems in the final exams. Many students learn the drill given by the teacher, and solve problems never asking "why am I doing this or that?"; although they understand almost nothing, they can get a good grade in the final examination. In most cases, *thinking* is not a process involved in this scenario.

According to G. Polya (How to Solve It?, 1957), teaching students to think does not necessarily mean sharing a lot of information with the students, but rather finding ways to develop the students' abilities to use information.

Through this lesson we will demonstrate concrete ways in which using a variety of written and oral texts can increase student participation and support thinking and learning in the mathematics classroom.

Informed by a cognitivist perspective, the lessons is organized around a three-part structure – A-B-C - of planning and instruction that consists of: **Anticipation**, in which the teacher uses reading, writing, and discussion strategies to activates students' background knowledge; **Building Knowledge**, in which the teacher guides the students' inquiry and helps them construct an understanding of the new content; and **Consolidation**, where students are led to summarize, apply, interpret, critique, and innovate relying upon the understandings they have constructed, assess their own learning, and ask new questions.(Crawford *et al*, 2005)

The lesson: The Square of a binomial

In this lesson we will demonstrate several methods of using writing to facilitate the understanding of mathematical processes. In mathematics classes, students are often required to prove theorems and properties and to solve problems and exercises. A large part of the content learned by students in mathematics classes consists of methods of proving and solving.

Students must know how to approach a problem or proof, how to reason, and what to do in order to come to the proof's conclusion or the problem's solution.
Comprehending the text of the problem or theorem is essential if students are to be capable of constructing the solution or proof. Throughout my years in the classroom, my students have often told me that they did not understand a problem or theorem when, in fact, it was its text that they did not understand, because they had not read it carefully. Students are very rarely asked to read the proof of a theorem or the solution to a problem. Reading mathematical texts – problems, theorems, proofs of theorems, and solutions to problems – and understanding them provide a basis for mathematical learning (Mower, P., 2003). On the other hand, the cognitive processes involved in the writing process require students to think before they use algorithms or solve problems. Often, students solve mathematics problems mechanically, working through the steps of the solution without understanding their purpose. If, however, they explain these steps in writing, they are encouraged to think beyond the algorithm and to see the logic behind the succession of steps of the solution. The writing activities exemplified in the following lesson require students to understand why a specific algorithm leads to the solution of the exercise or problem, encourage them to reflect upon the process, solicit the analysis of mathematical processes, and allow students to explore the logic of mathematics.
This is the fourth lesson of the *Algebraic Computation* unit, seventh grade and the first lesson of the *Strategies for Shortening Computation* sub-unit. In this unit, in previous lessons, students learned and practiced calculations with real numbers represented by letters (addition, subtraction, multiplication, division, and integer exponentiation).

Preliminary concerns	
What is the topic or question? What question and information should students investigate during this lesson?	Strategies for shortening computation – square of a binomial Where does it come from? How do we use it?
Why does it matter? Why is this knowledge worth having? What opportunities for thinking and communicating does this lesson afford?	The square of a binomial is often used in mathematics – in order to shorten the algebraic computation – when calculating algebraic expressions, solving quadratic equations. The lesson, through both oral and written communication activities, will allow students to analyse mathematical processes, to prove that there is more than one way to demonstrate a certain formula, to think about ways mathematicians work.
What are the objectives? What knowledge should the students gain? What should they be able to do with that knowledge? What strategies for thinking, investigating and communicating will they learn?	Content objectives: Students will be able to: ▪ Explain the meaning of *the square of a binomial*; ▪ Demonstrate the formula of *the square of a binomial*; ▪ Use the *square of a binomial* formula in two types of simple exercises; ▪ Understand the algorithm of using the formula of *the square of a binomial* in simple exercises. Process objectives: Students will be able to: ▪ Reflect on and learn the process of using the formula of *the square of a binomial* in simple exercises; ▪ Communicate using the vocabulary of algebra; ▪ Transfer mathematical content into their own words; ▪ Practice with basic proof reading and proof writing; ▪ Consider various approaches of a proof writing.

The lesson we will present uses the A-B-C structure. In this lesson we will use the following strategies: directed listening thinking activity (DL-TA), concept definition, proof reading, proof writing, and writing solution steps.

Anticipation: In this stage of learning, the students are prepared for what they will acquire from the lesson. Here, anticipation is accomplished through a **directed listening thinking activity** (**DL-TA**).

Activity 1 - Task: please answer the question: *What do we know about how mathematicians work?* – by thinking about the way you work with math. Please draw a line down the center of a sheet of paper so that you have two columns. In the first column write down some of the ways you work when you are learning math, and in the second write down how mathematicians work by thinking whether your ways of working would be used by a mathematician.

Students work in pairs and a few pairs are invited to present their answers in front of the class.

Activity 2: The teacher tells the students a joke about mathematicians. She starts the joke, but at one point she stops and asks students to tell her what the punch line is.

> ***Mathematician Joke***
>
> *A mathematician and a physicist are talking over a cup of coffee in the hallway of their university. Suddenly, for no apparent reason, the coffee machine bursts into flames. The physicist grabs the fire extinguisher from the wall and puts out the fire.*
>
> *The following week, the two scientists are once again having a cup of coffee in the hallway, next to their new coffee maker. Suddenly, this coffee maker also bursts into flames.*

The teacher stops and asks students to take a guess at what the end of the joke could be, by thinking of what they've discussed about the way mathematicians work.

Some students tell the end of the joke and the teacher asks each of them to argue their answer.

The teacher appreciates the students' endings for the joke and presents in front of the class the ending she knows:

> *The mathematician takes the fire extinguisher and gives it to the physicist to put out the fire, thereby reducing the problem to one solved previously.*

Building knowledge: The teacher will introduce the notion of the square of a binomial through a **concept definition** activity and the derivation of the shortened computation formula for the square of a binomial through a **proof reading** activity followed by a **proof writing** activity. In the proof reading activity, the teacher will require students to examine an algebraic proof using a variety of representations: numerical, geometric, and algebraic.

Activity 3: Task: Please fill in, individually, the second column of your concept table after thinking what *the square of a binomial* means.

Concept	*What I think it means*	*Definition*
Square of a Binomial		

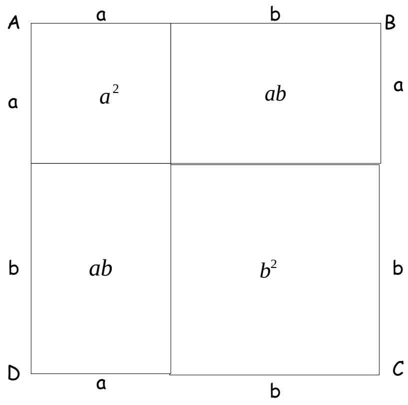
After filling in your concept table, discuss with your partner what you've written in the second column.

The students work individually and then discuss in pairs while the teacher oversees their activity. Some students are invited to present their answers in front of the class. The teacher then leads a brief discussion related to the concept definition and finally introduces the definition of a square of a binomial.

Activity 4: Task: Read the text on the worksheet carefully, write down the proofs, justifying every step in writing, and give another numerical example.

> **Worksheet 1 – The Square of a Binomial**
>
> We will show that $(a + b)^2 = a^2 + 2ab + b^2$
>
> 1. Algebraic Proof
>
> $(a + b)^2 = (a + b)(a + b) = a(a + b) + b(a + b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$

2. Geometric Proof
Look at the figure:



The square ABCD has the length of a side equal to a + b, and thus will have area: (a + b) (a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2
3. Numerical Example
If a = 2 and b = 5, we have $(2 + 5)^2 = 2^2 + 2(2)(5) + 5^2 = 4 + 20 + 25 = 49$
and $(2 + 5)^2 = 7^2 = 49$.
The students read the proofs on the worksheet, write them down while justifying their steps, and, finally, give a numerical example.
In this interval, the teacher moves around the classroom and monitors the students, offering help to those who need it.
Activity 5: Task: You will derive the formula for the square of a binomial containing the minus sign on your own. You will work in pairs and after ten minutes you will present the proof you have come up with. The pairs that finish early will also give a numerical example for the formula they have derived.
The students work in pairs. The teacher moves around the classroom and monitors the students' activity. Some students are invited to present their answers in front of the class.
Consolidation: In this lesson, consolidation will be achieved with the help of a **writing of solution steps (**it is called **method of operation** in Mower, P., 2003) activity. Students will use the formula for the square of a binomial in problems, writing down every step of their solutions.
Activity 6: Homework task: Solve the exercises on worksheet 2, answering the questions on the sheet.

Worksheet 2 – Applications of the Formula for the Square of a Binomial

A) Calculate $(2x – 3y)^2$ using the formula for the square of a binomial. Below, write each step of your solution:

1. ____

… Add as many lines as necessary for the solution.

B) Compute $(x + 1)^2 – (x + 2)^2 + (x – 3)^2 – 6$ using the formula for the square of a binomial. Below, write each step of your solution:

1. ____
2. ____
3. ____

… Add as many lines as necessary for the solution.

C) Write the following expression as the square of a binomial:
$4x^2 + 4x + 1$. Below, write each step of your solution:

1. ____

2. ____

… Add as many lines as necessary for the solution.

D) One thing we discovered today about formulas for shortening computation:

____

____

Homework is checked and discussed in the next lesson.

Samples of the students' work

Activity 1: *What do we know about how mathematicians work?*

How do I work in mathematics?	*What do we know about how mathematicians work?*
I write for solving exercises and problems. I use writing when calculating. I write detailed solutions.	They write proofs. They do mental calculations. They choose the shortest/ most ingenious solutions and look for formulas already known.
I read the exercises/ problems and analyse them.	They read and analyse the context.
When I don't know how to solve a problem I get support or I give up.	When they need support they work with other colleagues. They never give up.

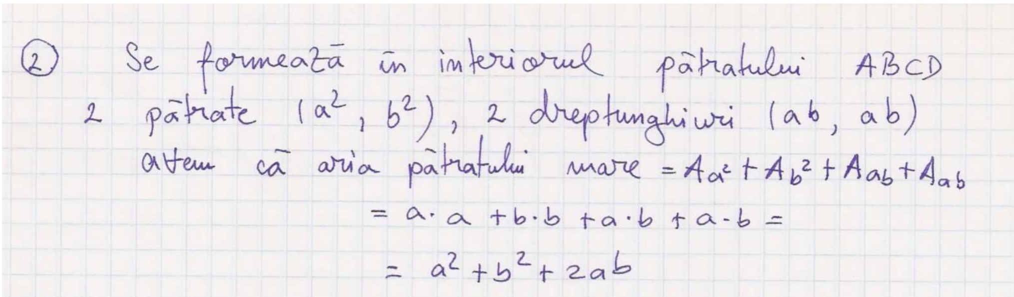
Activity 3: What I think it means the square of a binomial?

Student A: the square of a two figures number, the square of a two numbers product;

Students B: the square of "something with two";

Student C: the square of a two terms sum.

Activity 4: Geometrical proof



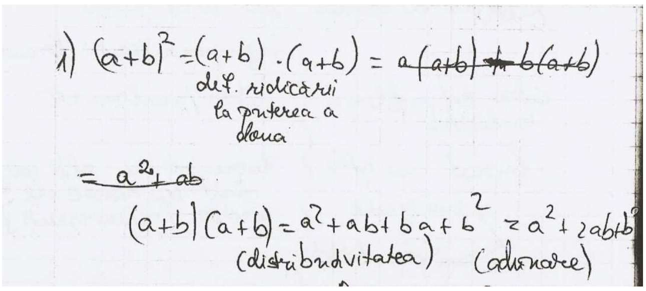
② Se formează în interiorul pătratului ABCD
2 pătrate (a^2, b^2), 2 dreptunghiuri (ab, ab)
avem că aria pătratului mare $= A_{a^2} + A_{b^2} + A_{ab} + A_{ab}$
$= a \cdot a + b \cdot b + a \cdot b + a \cdot b =$
$= a^2 + b^2 + 2ab$

The square ABCD has side length a + b and thus area $(a + b)^2$ – because the area of a square is the length of its side squared.

The area of square ABCD is equal to the sum of the areas of the figures that make it up: the square of side length a, which has area a^2, the square of side length b, which has area b^2, and the two rectangles with dimensions a by b, which have area ab.

If we equate the two ways of writing the area of square ABCD, we have:

$(a + b)^2 = a \cdot a + b \cdot b + a \cdot b + a \cdot b = a^2 + b^2 + 2ab$.



1) $(a+b)^2 = (a+b)\cdot(a+b) = \cancel{a(a+b) + b(a+b)}$
def. ridicării la puterea a doua
$\cancel{= a^2 + ab}$
$(a+b)(a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$
(distributivitatea) (adunare)

Teacher's and students' reflections

During the lesson, each student actively participated. Activities 1 and 2 offered students good reasons for learning Strategies for Shortening Computation, as they didn't ask, as other students did in the previous years, "why are we studying these formulas?".

By being engaged in activity 4, the algebraic proof, many students had problems in identifying the propriety/law which allows writing the second and the third equal signs in the proof. They mixed up the distributive law with its reverse, factoring out the common factor, even if they are able to state the distributive property and to spontaneously use it in arithmetic calculations. We'll have to facilitate students' awareness of the distributive property.

For activity 5, all pairs, except one, gave an algebraic proof – geometry is still difficult for students or it is hard for them to use geometry during algebra lessons.

Checking the homework I observed that only 60% of the students used the formula for exercise A); the others solved the exercise by calculating, without using the formula. For exercise B), worksheet 2, 80% of the students used the formula. When discussing this issue with the class, some students who didn't use the formula for exercise A) and used it for the exercise B) told me that they used the formula for exercise B) because there was too much to calculate for exercise B).

Students' answers for D) – worksheet 2 – showed that they discovered about formulas for shortening computation that they help finish the calculations more quickly. Some students said that they discovered that "the formula for shortening computation is an equation with two elements. In some exercises we are given the right-hand element and asked to find the left-hand one, and in other we are given the left-hand element and asked to find the right-hand one." Three students said that they discovered that the square of a binomial can be used without proving it each time it is used.

I continued, for four lessons, to ask students to use the **writing of solution steps (method of operation)** when using the formula in exercises. After these four lessons, they could decide if they wanted or needed to continue using it. After two months, 15% of students were still writing the solution steps.

Approaching the lesson: *Difference of squares* by using almost the same writing strategies showed that students were quick and very good, this time, at defining the concept (difference of squares) and writing the proof. Finally, it is a learning process. Again, they were amazed by the geometrical proof but they considered it difficult.

I enjoyed the lesson; my main goal, as a math teacher, is to make students enjoy mathematics and to think, question and analyze mathematical issues as mathematicians do. During this lesson, we were on the right track to reach my main goal.

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A Stochastic Model for the Process of Learning

Michael Gr. Voskoglou, MSc., PhD
Professor of Mathematical Sciences, Faculty of Technological Applications , Graduate Technological Educational Institute (T.E.I.) 263 34 Patras – Greece voskoglou@teipat.gr

Abstract
A Markov chain is introduced to the major steps of the process of learning a subject matter by a group of students in the classroom, in order to obtain a mathematical representation of the above process. A classroom experiment for learning mathematics is also presented illustrating the applicability of our results in practice.

Introduction

There are very many theories and models developed from psychologists and education researchers for the description of the mechanisms of learning. Nowadays it is widely accepted that any instance of learning involves the use of already existing knowledge. Voss (1987) developed an argument that learning consists of successive problem – solving activities, in which the input information is represented of existing knowledge, with the solution occurring when the input is appropriately interpreted.

The whole process involves the following steps: ***Representation*** of the stimulus input, which is relied upon the individual's ability to use contents of his (her) memory to find information, which will facilitate a solution development; ***interpretation*** of the input data, through which the new knowledge is obtained; ***generalization*** of the new knowledge to a variety of situations, and ***categorization*** of the generalized knowledge, so that the individual becomes able to relate the new information to his (her) knowledge structures known as schemata, or scripts, or frames.

The process of learning a subject matter in the classroom

In order to describe the process of learning a subject matter in the classroom one must keep in mind that, as it frequently happens, a learner may not be able to pass successfully through all the steps of the learning process in the time available into the classroom. Therefore it is convenient, for purely technical reasons, to include in this case one more step in the sketch of the process described in the previous section, the step of ***failure to reach categorization.***

We are going to construct a 'flow-diagram' representing the whole process. For this, let us denote by S_i, i=1,2,….,5, the steps of representation, interpretation, generalization, categorization, and failure to reach categorization respectively. The starting state is always S_1. From S_1 the learner proceeds to S_2. Facing difficulties there he (she) may return to S_1 to search for more information that will facilitate the interpretation procedure. Then he (she) must go back to S_2 to continue the process. From S_2 the learner is expected to proceed to S_3, unless if he (she) is unable to interpret the input data during the learning process in the classroom. In this case he (she) proceeds directly to S_5, and the process finishes there for him (her). From S_3 the learner, if he (she) has difficulties during the generalization procedure, may return to S_2 for a better understanding of the subject. Then he (she) comes back to S_3, wherefrom he (she) proceeds either to S_4 or to S_5 and in both cases the process finishes there.

According to the above description the flow-diagram of the process of learning a subject matter in the classroom by a group of students is that shown in Figure 1.

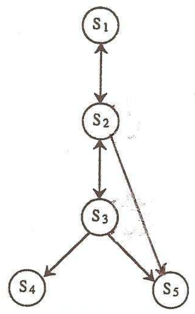

Figure 1: Flow-diagram of the learning process in the classroom

The stochastic (Markov) model

Roughly speaking a ***Markov chain*** is a stochastic process that moves in a sequence of phases through a set of states and has "no memory". This means that the probability of entering a certain state in a certain phase, although it is not necessarily independent of previous phases, depends at most on the state occupied in the previous phase. This is known as the ***Markov property.***

When its set of states is a finite set, then we speak about a ***finite Markov chain.*** For special facts on such type of chains we refer freely to Kemeny & Snell, (1976).

Here we are going to build a Markov chain model for the mathematical description of the process of learning a subject matter in the classroom. For this, assuming that the learning process has the Markov property, we introduce a finite Markov chain having as states the five steps of the learning process described in the previous section. The above assumption is a simplification (not far away from the truth) made to the real system in order to transfer from it to the ***"assumed real system"***. This is a standard technique applied during the mathematical modeling process of a real world problem, which enables the formulation of the problem in a form ready for mathematical treatment (Voskoglou, 2007; section 1).

Denote by p_{ij} the transition probability from state S_i to S_j, for i,j=1,2,3,4,5, then the matrix A=[p_{ij}] is said to be the ***transition matrix*** of the chain.

According to the flow-diagram of the learning process shown in Figure 1 we find that

$$A = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ p_{21} & 0 & p_{23} & 0 & p_{25} \\ 0 & p_{32} & 0 & p_{34} & p_{35} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array},$$

where we obviously have that $p_{21}+p_{23}+p_{25}=p_{32}+p_{34}+p_{35}=1$

Further let us denote by $\varphi_0,\varphi_1,\varphi_2,\ldots$ .. the successive phases of the above chain , and also denote by

$$P_i=[p_1^{(i)}p_2^{(i)}p_3^{(i)}p_4^{(i)}p_5^{(i)}]$$

the row - matrix giving the probabilities $p_j^{(i)}$ for the chain to be in each of the states S_j, j=1,2,3,4,5 in the phase φi, i=1,2,.... , where we obviously have that

$$\sum_{j=1}^{5} p_j^{(i)} = 1.$$

The above row-matrix is called the ***probability vector*** of the chain at phase φ_i. From the transition matrix A and the flow-diagram of Figure 1 we obtain the tree of correspondence among the several phases of the chain and its states shown in Figure 2.

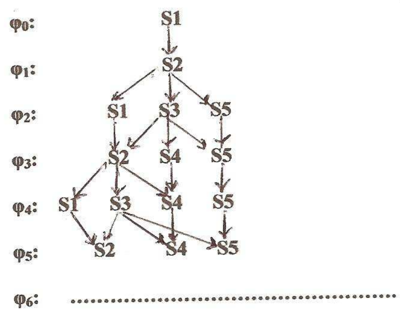

Figure 2: Tree of correspondence among states and phases of the Markov chain

From the above tree becomes evident that P_0 = [1 0 0 0 0], P_1 = [0 1 0 0 0], and P_2=[p_{21} 0 p_{23} 0 p_{25}]. Further it is well known that

$$P_{i+1} = P_iA, \quad i=0,1,2,\ldots .$$

Therefore we find that

$$P_3 = P_2A = [0 \;\; p_{21}+p_{23}p_{32} \;\; 0 \;\; p_{23}p_{34} \;\; p_{23}p_{35}+p_{25}] \quad (1),$$

$P_4=P_3A=\ldots\ldots$, and so on.

Observe now that, when the chain reaches either state S_4, or S_5, it is impossible to leave them, because the learning process finishes there. In other words S_4 and S_5 are absorbing states of the chain. Further, from Figure 1 it becomes evident that from every state it is possible to go to an absorbing state (not necessarily in one step). Thus we have an ***absorbing Markov chain***. Applying standard techniques from theory of absorbing chains we bring the transition matrix A to its ***canonical (or standard) form*** A* by listing the absorbing states first and then we make a partition of A* as follows:

$$A^* = \begin{array}{c} \\ S_4 \\ S_5 \\ \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{cccccc} S_4 & S_5 & & S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} 1 & 0 & | & 0 & 0 & 0 \\ 0 & 1 & | & 0 & 0 & 0 \\ - & - & | & - & - & - \\ 0 & 0 & | & 0 & 1 & 0 \\ 0 & p_{25} & | & p_{21} & 0 & p_{23} \\ p_{34} & p_{35} & | & 0 & p_{32} & 0 \end{bmatrix} \end{array}.$$

Symbolically we can write

$$A^* = \begin{bmatrix} I & | & 0 \\ - & | & - \\ R & | & Q \end{bmatrix},$$

where Q is the transition matrix of the non absorbing states and R the transition matrix from the non absorbing to the absorbing states.

Next we consider the ***fundamental matrix*** N of the chain, which is given by

$$N = (I_3-Q)^{-1} = \frac{adj(I_3-Q)}{D(I_3-Q)} \;,$$

where I_3 denotes the 3X3 unitary matrix, adj(I_3-Q) denotes the adjoin matrix of I_3-Q, and D(I_3-Q) denotes the determinant of I_3-Q. A straightforward calculation gives that

$$N=\frac{1}{1-p_{23}p_{32}-p_{21}}=\begin{bmatrix} 1-p_{32}p_{23} & 1 & p_{23} \\ p_{21} & 1 & p_{23} \\ p_{21}p_{32} & p_{32} & 1-p_{21} \end{bmatrix}.$$

Finally we consider the 3X2 matrix

$$B=NR=\frac{1}{1-p_{23}p_{32}-p_{21}}\begin{bmatrix} p_{23}p_{34} & p_{25}+p_{23}p_{35} \\ p_{23}p_{34} & p_{25}+p_{23}p_{35} \\ (1-p_{21})p_{34} & p_{32}p_{25}+p_{35}(1-p_{21}) \end{bmatrix}.$$

We write symbolically $B=[b_{ij}]$, with i=1,2,3 and j=4,5. It is well known then that b_{ij} gives the probability that, starting at state S_i, the process is absorbed at state S_j. Thus the probability for a learner to pass successfully through all the states of the learning process in the classroom is given by

$$b_{14}=\frac{p_{23}p_{34}}{1-p_{23}p_{32}-p_{21}} \quad (2).$$

The calculation of b_{14} enables the teacher to check the efficiency of his (her) lectures. It also could be used either as a measure of comparison of the efficiencies of the lectures of different teachers, or as a measure of the learning abilities of different groups of students. The following classroom experiment for learning mathematics illustrates the applicability of our model in practice.

A classroom experiment for learning mathematics

The present experiment took place recently at the Graduate Technological Educational Institute of Patras (Greece), when I was teaching to a group of 30 students of the School of Technological Applications (i.e. to future engineers) the use of the derivative for the maximization and minimization of a function. During my 2 hours lecture I used the method of rediscovery (Voskoglou, 1997). Thus, after a short introduction to the subject, I left my students to work alone on their papers. I was inspecting their works, and from time to time I was giving them some instructions, or hints. After the basic theoretical conclusions I gave them some exercises to solve first, and at the final step some problems including applications to constructions and economics.

During the experiment I found that 4 students were completely unable to understand the subject. Also 10 students faced difficulties before understanding the basic ideas (they looked back to their notes of my previous lectures and/or asked for help). Furthermore 5 students, although it seemed that they understood the basic theoretical ideas, were unable to apply them in order to solve the given exercises and problems. The other 21 students solved the exercises, but 8 of them faced difficulties before they came through. At the last step 10 students solved the problems and 11 they didn't (or solved a small part of them). Interpreting these data with respect to the flow-diagram of Figure 1 I was led to the following conclusions, which are represented in Figure 3.

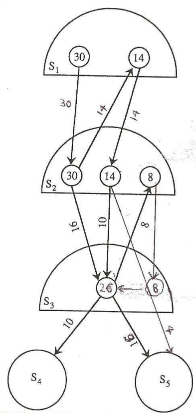

Figure 3: Representation of the experiment's data

- Initially the 30 students proceeded from S_1 to S_2, but 14 of them faced difficulties to interpret the input data. Therefore they returned to S_1 to search for more information that will facilitate the interpretation procedure, wherefrom they came back to S_2. Finally 4 of them reached directly the absorbing state S_5, because they didn't manage to interpret the new knowledge.
- The remaining 26 students proceeded to S_3, but 8 of them faced difficulties to generalize the new knowledge to a variety of situations, and they returned to S_2 for a better understanding of the new information. Then they came back to S_3.
- At the last step 10 students, who solved the exercises and problems, completed successfully the learning process in the classroom and therefore they reached the absorbing state S_4. The other 16 students, i.e. 5 students who didn't manage to solve the exercises and problems, and 11 who solved the exercises, but not the problems, reached the absorbing state S_5.

Therefore, since we had a total of 52 'arrivals' to S_2, 14 'departures' from S_2 to S_1, 34 'departures' from S_2 to S_3, and 4 'departures' from S_2 to S_5, it follows that $p_{21}=\frac{14}{52}$, $p_{23}=\frac{34}{52}$, and $p_{25}=\frac{4}{52}$. In the same way one finds that $p_{32}=\frac{8}{34}$, $p_{34}=\frac{10}{34}$, and $p_{35}=\frac{16}{34}$.

Replacing the values of the p_{ij}'s in equalities (1) and (2) of the previous section we get that $P_3=[0 \ \frac{22}{52} \ 0 \ \frac{10}{52} \ \frac{20}{52}]$ and $b_{14}=\frac{1}{3}$. Interpreting these data with respect to our model we find that the probabilities for a student to be in phase φ_3 of the process of learning in the classroom (i.e. 3 phases after its start) at the steps of representation, interpretation, generalization, categorization, or failure to reach categorization are approximately 0, 42,31%, 0, 19,23%

and 38,46% respectively, while the probability to pass successfully through all the steps of the process is approximately 33,33%.

Remarks and further examples

Most real world problems concerning applications of finite Markov chains can be solved by distinguishing between two types of such chains, the absorbing (e.g. the case of our model in the present paper) and the ergodic ones (Voskoglou, 2006; section 3). We recall that a Markov chain is said to be an ***ergodic chain***, if it is possible to go between any two states, not necessarily in one step.

In Voskoglou (1996) an ergodic chain is introduced for the study of the analogical problem-solving process in the classroom, while in Voskoglou and Perdikaris (1991) the problem-solving process (in general) is described through the introduction of an absorbing Markov chain to the main steps of the process.

In Voskoglou (1994) an absorbing Markov chain is introduced to the major steps through which one would proceed in order to effect the study of a real system (modelling process). An alternative form of the above model is introduced in Voskoglou (2007) for the description of the mathematical modelling process in the classroom. In this case it is assumed that after the completion of the solution process of each problem a new problem is given from the teacher to the class and therefore the process is repeated again. Thus the resulting Markov chain is an ergodic one.

In Voskoglou (2000) an absorbing Markov chain is introduced to the main steps of the decision making process performed in order to choose the best among the existing solutions of a given problem, and examples are presented to illustrate the applicability of the model to "real" decision making situations.

We could mention very many other known applications of Markov chains for the solution of real world problems in almost every sector of the human activity, but this is rather out of the scope of the present paper.

Final conclusions

The theory of Markov chains is a successful combination of Linear Algebra and Probability, which enables one to make forecasts for the evolution of various phenomena of the real world.

In the present paper we built a Markov model for the description of the process of learning a subject matter by a group of students in the classroom. In this way we succeeded to calculate the probabilities for a student to be at any of the major steps of the learning process in each of its phases in the classroom, as well as the probability to pass successfully through all the steps of the learning process in the classroom. Our results are illustrated by a classroom experiment for learning mathematics.

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Developing explanatory compentencies in teacher education

Anke WAGNER, Ludwigsburg
Claudia WÖRN, Ludwigsburg
Sebastian KUNTZE, Ludwigsburg

Abstract
When interviewing school students for what constitutes a good mathematics teacher, the first characteristic usually listed is the ability to explain well. Besides well-founded content knowledge most important for classroom episodes of teacher explanations is knowledge about how to present mathematical concepts in a comprehensible way to students. This encompasses competencies in the area of verbal communication as well as the conscious use of means for illustrating and visualising mathematical ideas.
We report about an analysis of explanatory processes in math lessons and about an analysis of prospective teachers' explanatory competencies. As a result we identify improvements in teacher education at university.

1. Introduction

Concerning negotiation processes, teachers everyday have to come up with an appropriate way to make mathematical content understandable for pupils. Permanently they wonder about questions like: How can I explain certain mathematical content to pupils? During the preparation of class teachers have to think of well suited exercises, possibilities on how to make certain mathematical content clear and intelligible to all pupils, and ways of engrossing, diversifying and transferring mathematical thoughts.
These tasks must be accomplished everyday anew, concerning every mathematical content, every course and every pupil, not to forget pupils' different cognitive levels that have to be considered.
It is astonished that despite this knowledge in teachers education - and we are speaking out of our own experience - the competency of "adequate explaining" which teachers need in everyday class preparation, is not being taught.
This dilemma is caused by various reasons, which shall not be discussed in this paper.
Apart from that we emphatically point out, that the deficits of prospective teachers in decision-making and responsibility can not be compensated by in-service teacher training after university due to the complexity of the current everyday life at school.
In our opinion these competencies have to be developed in academic courses, what makes a change in educational content become mandatory.
In the following, theoretical basics of explanation are mentioned. Engaging in these basics, the lack of literature dealing with arrangements of explanatory sequences in class becomes obvious in the form of missing national empirical studies as well as in missing empirical models. Even though some theoretical models do exist, those models can not be adapted to use in class from our point of view. This will be explicated in chapter 2.1.1. The same conclusion can be made regarding the usage of representations within explanatory sequences in math class (see chapter 2.1.2). Although in didactical literature it is always advised to use representations due to their significance in gaining insights, there are only very few statements on whether and how to use these representations in explanatory sequences effectively.

In this paper theoretical basics to explanatory sequences in math class are presented at first. Based on actual literature as well as on our own empirical studies we afterwards present necessary changes in teacher education and concrete approaches towards the arrangement of academic courses.

2. Background

2.1. Theoretical basics

2.1.1. Theories of explaining

In literature different theoretical models for the concept of "Explaining" are mentioned. For instance Hempel and Oppenheim's deductive-nomological model is well-known in the philosophy of science. It characterizes scientific explanations primarily as deductive arguments with at least one natural law statement among its premises (Hempel/Oppenheim, 1988).
In the field of education Kiel's model has to be mentioned (Kiel, 1999). Kiel in contrast to Hempel/Oppenheim does not focus on the explanation as a product but on the process of explaining which is sub-divided in eight different steps. Each of these steps is allocated to three categories of cognitive operation: analysis (recognizing the constituents of an explanation), synthesis (recognizing the function of an explanation) and syncrisis (comparing with other explanatory objects and if necessary integrating in one's cognitive structure) (Kiel, 1999, S. 267 ff.).
As we formerly have adumbrated these models are not or only partially adoptable to teaching math, due to the following points:

- Class room explaining always refers to consignees. In Hempel/Oppenheim's model consignees are irrelevant which leads to our denegation of this model.
  Although Kiel takes the consignees into account, he does not go into the variety of explanatory approaches which teachers have to have available concerning to such a multiple set of consignees.
- Whether an explanation is applicable or not depends on such multifaceted factors as the type of school, the headcount in class, the cognitive level of pupils, etc.
  When deciding for a special way of explanation those factors have been considered. As an exemplification the introduction of negative numbers can be accomplished by using different approaches such as the model of step, the law of permanence, real world model, etc.
- The intention behind an explanation (producing occasions of communication, reflection of one's own or others problem-solving-strategies, defining terms, …) is crucial to the way of explaining.

Although the field of explanation is complex mentioned above, when looking on the current state of mathematical didactical research, there are no national empirical findings in the field of explanatory processes in math class.

2.1.2. Using representations within explanatory sequences
To illustrate mathematical terms and operations, in math class often representations are used in explanatory sequences.
Bruner (1966) distinguishes three modes of representation: enactive representation, iconic representation, and symbolic representation. By using the example of tying a knot he explicates these representation modes.
In the enactive mode the knot is represented action-based. "With respect to a particular knot, we learn the act of tying it and, when we "know" the knot, we know it by the habitual pattern of action we have mastered." (Bruner 1966, p. 6)
The iconic representation is comprehended by Bruner in different aspects: the picture of the knot in question, its final phase or some intermediate phase, or, indeed, even a motion picture of the knot being formed (Bruner 1966, p. 6).
On the third mode of representation, the symbolic one, Bruner comments (1966, 7):
„For symbolic representation, whether in natural or mathematical "language", or whatever the medium uses to combine the discrete elements by rule. Note, too, that whatever symbolic code one uses it is also necessary to specify whether one is describing a process of tying a knot or the knot itself (at some stage of being tied). There is, moreover, a choice in the linguistic description of a knot whether to be highly concrete or describe this knot as one of a general class of knots. However one settles these choices, what remains is that a symbolic representation has built-in features that are specialized and distinctive." (Bruner 1966, p. 7)
Zech (1998, S. 106) references Bruner's statements but states certain "levels" of representation. He subdivides Bruner's symbolic representation into a symbol-based representation (according to what Bruner calls the "mathematical language") on the one hand and a language-based (according to Bruner's "natural language") representation on the other hand (Bönig 1995, S. 60).

Lompscher (1972) eminently refers to the different modes of language (besides the modes of visualisation), depending on the different levels of cognition which are similar to Bruner's modes of representation: first, language is the medium of mental action, second, it is a supporting element e.g. to coordinate action or to document results.

Recapitulatory there are only few empirical hints that recommend certain representations in explanatory sequences and state how these could be supported by language. Also, there are no recommendations on the use of certain representations concerning one special explanatory object: What kind of enactive representation is suited e.g. for the introduction of reducing fractions?

Which explanatory model is appropriate for introducing cylinder volume to low-achieved pupils and which one is appropriate for high-achieved pupils?

2.2. Empirical studies

2.2.1. Data

In our empirical study we focus on explanatory sequences comprising pupils working on open ended as well as closed tasks. Our data comes from 45 videographed math lessons from primary school, secondary school with low-, middle and high-achieved pupils. In primary school grade 4 and in secondary school grade 7 was participating. Prior to the video recordings the pupils have slowly been accustomed to the video camera. To avoid disturbing effects triggered by videographing, the cameramen have been instructed to act in an inconspicuous way and not to take part of interaction with pupils. Thereby pupils' interest in the cameramen was reduced to a minimum. The video recordings took place in the pupils' familiar class room. During the recordings, observations made by the researchers and were logged in detail. Furthermore three out of the seven teachers were interviewed promptly after the lessons.

Beside these data additional data were taken out of academic courses: Students at the end of their academic studies were asked to answer explanatory tasks dealing with different mathematical content.

The corpus of our data finally consists of (a) video recordings, (b) pupils' documents, (c) researchers' notices, (d) recall-interviews and (e) academic students' documents.

2.2.2. Results

As a result of the transcripts analysis as well as of the students' documents it appears that explanatory sequences are built up in a certain, mostly unchanged structure.

Furthermore some special phenomena could be realized in explanatory sequences.

a) Structure of explanatory sequences

The structure of explanatory sequences in math class is explicated below:

Cause of explanatory sequences

The cause of explanations is the beginning and therefore the activator of explanatory sequences. Depending on the teacher's know how, complex explanandi are determined already during class preparation. In this case, explanations are well-planned and prepared. Beside this unplanned explanatory causes are the result of unexpected or spontaneous situations of cognitive disequilibrium (Kiel 1991, S.74), followed up by adhoc-explanations (Schmidt-Thieme & Wagner, 2007).

Initiation of explanatory sequences

When a cause of an explanatory sequence is identified, it is of interest to see, who explicitly calls for an explanation and in which way this calling is uttered. This incident shall be denoted as the "initiation of the explanatory sequence".

Process of explanatory sequences

The Explanandum comes to the fore in the process of explanatory sequences. It should constitute the core in this part. However, often the explanatory process does not proceed in a straight-forward and linear way, but there are interposed explanations broaching the issue of something else. (Schmidt-Thieme & Wagner, 2007). In both cases explanatory processes can fulfill different didactical functions.

Beside the elimination of cognitive disequilibrium metacognitive skills can be initiated (Schütte, 2002) as well as opportunities for communication can be created (Bauersfeld, 2002). Here you can consider that different mode of representation of Bruner (1966) were used to adequately support verbal processes.

Coda of explanatory sequences

Irrespective of the success of one explanatory process, this process comes to an end, which is called "Coda".

b) Realized phenomenon's within explanatory sequences

In our data, explanatory sequences are mostly initialized by teachers. In contrast to this, the explanatory processes, themselves part of explanatory sequences, are mainly accomplished by students (Wagner & Wörn, 2009). In this context, teachers' role has changed - they act rather in a supporting than in a teaching way. Altogether it is noticeable that explanatory sequences mostly take place only by verbal interaction and without any further usage of representation. Just in a few cases language is supported by another representation (in the kind of an intermodal transfer), whereas the linking-up of three modes of representation has hardly taken place in the analysis of our data. The few cases that have been recognized all refer to the teachers' acts. In addition to that, it is conspicuous, that in comparison to pupils, teachers use a higher diversity of combining language and other modes of representation.

c) Students' explanatory competencies

In grade 7 math class of middle school the circumcircle of triangles is part of the curriculum. Along with this topic typical pupil questions arise. To figure out whether prospective teachers do have explanatory competencies in this field, the following explanatory task has been given to them:

Think about a pupil who has realized that the circumcenter can be constructed by the perpendicular bisectors of the sides. His question now is, why you have to use these perpendicular bisectors of the sides and not other construction lines such as for example bisecting lines of angles.

By analyzing students' documents you can notice three categories of an explanation. They deliver insight into the current status of the students' explanatory competency, which will be needed just one year after university at school.

Category 1: No explanatory concept.

Student A's explanation can be used as a representative of this category. He does not seem to have an idea about the task at first. His first note - a question mark - can be used as an evidence for this interpretation. Beyond this, it is conspicuous that the explanation is very short and formulated in key words or rather fragmentary sentences. He for example only figures out that a perpendicular bisectors is a line from whom all points are equally far away for example from point B and C. The student apparently is not able to verbalize an explanation which is adequate and target-aimed to the consignees. In lieu thereof he refers to a definition. Maybe he tries to reduce the complexity of the problem by merely focussing the segment $\overline{BC}$

Category 2: Minimal support (proposal of strategy) devoid of explanation.

Student B's explanation - as a representative of this category - does not answer the task. He gives a new advice to the pupil instead: "Try to construct the circumcircle by other construction lines like bisecting lines of angles." If we think of a pupil following this recommendation and acting this way (what means constructing the bisectrices which also subtend each other in a single point), he may realize his failure. However his primal question is still not being answered.

Category 3: Detailed explanation.

Student C's explanation stands for this category. In this kind of an explanation it is evident, that the student first specifies the task by identifying the important components of the task (in this example the "circumcircle") and its characteristics (the distance from each point A, B, C to the centre is equal). After that he divides the complex problem into certain sub-aspects. Therefore he first focusses the segment $\overline{AB}$ and explicates the concept of the perpendicular bisector of the beside. This sub-aspect explanation is repeated concerning the segments $\overline{BC}$ and $\overline{AC}$. Finally, by combining these sub-aspects the original task is re-focussed and solved.

3. About possible improvements of teachers education at university

Based on the lack of theoretical basics concerning class room explanation as well as on our own research findings, we advocate the intensely analysis and reflection of explanatory sequences already at university.
Already at university academic students shall acquire knowledge and competencies about

- adequate representations in dependence on the explanatory object
- advantages and disadvantages of respective representation
- intermodal transfer and its importance due to gaining insight
- the importance of internal representations
- the structure of concrete explanatory sequences on the basis of videos or rather transcripts
- specific ways of teachers' behaviour: asking technique or rather impulse technique, interventions including modes of assistance, feedback, usage of representation
- possible "good" explanations by means of adequate tasks
- possible explanatory scenarios in math class
- proper usage of material (by self-experiment)
- development of creative and multifaceted tasks

A lack of knowledge concerning these competencies leads to limitation in structuredness, clearness, comprehensibility and achieving the focussed aims in math class. Thereby essential characteristics of quality of teaching (Helmke, 2008) do not get fulfilled. To avoid this, the explanatory competencies mentioned above have to be acquired necessarily during the teacher education. Only hereby teachers can cope with the complexity of class, at least to a certain extend.

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Improving Student Interest, Mathematical Skills, and Future Success through Implementation of Novel Mathematics Bridge Course for High School Seniors and Post-secondary Students

Derek Webb, PhD, Professor of Statistics and Mathematics, Bemidji State University, Bemidji, MN, USA dwebb@bemidjistate.edu
Glen Richgels, PhD, Marty J. Wolf, PhD, Todd Frauenholtz, PhD, Ann Hougen, MS
Bemidji State University

Abstract

We present a new course titled "Introduction to the Mathematical Sciences." The course content is 1/3 algebra, 1/3 statistics, and 1/3 computer science and is taught in a laboratory environment on computers. The course pedagogy departs radically from traditional mathematics courses taught in the U.S. and makes extensive use of spreadsheet software to teach algebraic and statistical concepts. The course is currently offered in area high schools and two-year post-secondary institutions with financial support from a Blandin Foundation grant (referenced under BFG). We will present empirical evidence that indicates students in this course learn more algebra than students in a traditional semester-long algebra course. Additionally, we present empirical evidence that students learn statistical and computer science topics in addition to algebra. We will also present the model of developing this course which depended on increasing future student success in a variety of disciplines at the post-secondary level of study.

Introduction

Schools in the United States have been unsuccessfully trying to address two major problems in mathematics education: a lack of interest in the mathematical sciences by students who become worse as they progress through the grade levels *and* students choosing to not take any mathematics classes in 11^{th} or 12^{th} grade or only taking mathematics classes that satisfy requirements but do not help them to be successful in post-secondary education. These two problems are exacerbated by the fact that most high school mathematics curricula in the U.S. are based on what we call the *calculus model* which is four years of curriculum culminating in an advanced placement (AP) calculus course. The measure of success for this model is the AP calculus success rate and the number of students in the high school taking AP calculus. The problem is that only approximately 10% of a given high school has students ready and capable of taking AP calculus their senior year. The rest of the students (90%) may suffer from the *calculus model* by virtue of their exclusion. Problems with the post-secondary model of mathematics education also created a strong need for developing this course. Over a five year period (2001-2006), Glen Richgels examined Bemidji State University (note that BSU is a medium size liberal arts university typical of many post-secondary institutions across the U.S.) data and discovered that approximately 78% of all graduates across all programs need one or more statistics courses to graduate, just 12% need one or more calculus courses. The *calculus model* in high school is not benefiting most students at college.

With these problems in mind, during the spring of 2007, faculty at BSU (four from mathematics: Todd Frauenholtz, Ann Hougen, Ryan Hutchinson, Glen Richgels, one from statistics: Derek Webb, and one from computer science: Marty Wolf) created and piloted a novel new course titled "Introduction to the Mathematical Sciences."

Creation and Content of the Course

The creation of our course occurred in the winter of 2007. The course was experimentally taught at BSU in the spring of 2007 and simultaneously at Lincoln High School in Thief River Falls, Minnesota. The initial experimental offering of the course at BSU in the spring of 2007 was supported by a Minnesota State Colleges and Universities grant (referenced under IPESL).

Choosing course content was novel compared to how most mathematical sciences courses are created. Course content focused on three areas of mathematical science: algebra, basic statistics, and basic computer science. Topics for the course were chosen based on their usefulness and applicability in various fields of study outside the discipline of mathematics that students may pursue at the post-secondary level. Our overarching goal was to populate our course with topics that would contribute to student success at the post-secondary level in a wide range of programs of study.

Topic choices were based on over 20 interviews of faculty at BSU in many different programs. Examples of the diverse programs from which faculty were interviewed include: political science, psychology, theatre, geology, physics, applied engineering and business administration. The following table contains topics included in the course.

Algebra Topics	Statistics Topics	Computer Science Topics

Functions
• Represented by formula, table, graph, words
Graphical and Tabular Analysis
• Tables and trends
• Graphs
• Solving linear equations
• Solving nonlinear equations
• Optimization
Linear Functions
• The geometry of lines
    • Linear Functions
    • Modeling data with linear functions
    • Linear regression
    • System of equations
Rates of Change
    • Velocity
    • Rates of change of other functions
	Collecting and displaying data
	• Types of data
	• Creating data files in spread sheets
	• Displaying data in tabular format
	• Bar charts, histograms, pie charts, box plots, scatter plots
	Populations and samples
	Measures of central tendency
	• Sample mean, median, and mode
	Measures of dispersion
	• Sample range, standard deviation, and inter quartile range
	Shapes of distributions
	• Skewness, symmetry, and modality
	Correlation and association
	Introduction to linear regression
		Syntax and Semantics
		Understanding Processes
		• Describing processes used to solve specific problem
		• Generalizing processes to solve general problem
		• Converting processes into computer solutions
		The notion of a "variable" in computing
		• Variable names, references, and values
		Formulas and expressions
		• Operations, evaluation order, results, and errors
		Making decisions
		• Logical and rational operators and their values
		• Conditional syntax
		• Conditional semantics
		Using functions
		• Function syntax and semantics.

Pedagogy and Learning Environment of the Course

The pedagogy and learning environment of the mathematics and statistics portions of the course are in alignment with the National Council of Teachers of Mathematics (referenced under NCTM) recommendations. The pedagogy and learning environment consist of the following components:

- Algebra, statistics and computer science topics are all presented in the context of real-world problems taken from many disciplines. This is especially critical in the teaching of algebra topics. In our class, algebra is not taught as a set of rules and symbol manipulation skills, which is what students typically see in traditional algebra classes. Students readily see the applicability of the algebra topics they are studying.
- Algebra, statistics, and computer science topics are interwoven and not taught in isolation. Our course content is not three topics taught separately. Rather, it is three topics taught in concert making use of natural relationships. Students understand how the "mathematical sciences" is a cohesive discipline, not silos of information.
- Most algebra, statistics, and computer science topics are taught using spreadsheets. Students are much more engaged in the learning of algebra topics using spreadsheets and they also have a much better understanding of, and need for, proper order of operation and algebraic syntax.
- Students spend at least half their class time in a computer laboratory environment. This pedagogical aspect of the course depends on the available facilities the school. If possible, we prefer that the course be taught entirely in a computer laboratory environment. If not, students should spend at least 50% of classroom time in a computer laboratory.
- The classroom time commitment for this class is approximately double that of a typical three credit college algebra course. This is very important because it allows enough classroom time for students to work together on their own, in student groups, and with the instructor to complete the majority of their "homework." That way, they know they are being successful and do not struggle

in isolation at home. This ensures that the majority of work is completed and students remain engaged in learning.

Initial Implementation and Expansion

In the spring of 2007 the course was piloted at BSU and at Lincoln High School. Afterwards, adjustments and improvements were made. One adjustment was to add content to the course when offered at high schools because high schools usually have many more course contact hours than colleges or universities. The course was then offered at BSU in the fall of 2007 and at two high schools: Bemidji High School and Lincoln High School. Bemidji State University continues to offer the course once per year and has accepted the course into the university's permanent curricular offerings.

In the fall of 2008 Derek Webb and Glen Richgels were awarded a large grant from the Blandin Foundation to offer the course in multiple high schools and post-secondary institutions (referenced under BFG). The map below shows all partnering schools where the course is being offered. The institutions are color-coded in the following way:

Red: post-secondary institutions offering the course starting in the spring of 2009 – Northwest Technical College in Bemidji; Northland Community College in Thief River Falls, Red Lake Nation College in Red Lake and White Earth Tribal and Community College.

Blue: high schools offering the course starting in the spring of 2009 – Lincoln HS, Bemidji HS, Red Lake HS, and Win-E-Mac HS.

Purple: post-secondary institutions offering the course starting in the fall of 2009 – Leach Lake Tribal College and Fond du Lac Tribal and Community College.

Green: high schools offering the course starting in the fall of 2009 – Clearbrook-Gonvick H S, Grand Rapids HS, Floodwood HS, Cass Lake HS, and Walker-Hackensack-Akeley HS.

Empirical Evidence of Success

Assessment instruments were used to assess content knowledge gained during the course. A standard placement test was used to assess algebra knowledge. This placement test has been routinely used by various universities in Minnesota to place students into their initial collegiate mathematics course, including various algebra courses. The test was given at the beginning of the semester and again at the end. Statistics and computer science tests were also created and were given at the beginning and at the end of the semester. Appropriate and consistent pre-test and post-test assessment protocol was followed at all institutions every time the course was assessed.

During the spring semester of 2007, one traditional college algebra class and two liberal education mathematics classes were also studied as control groups. Students in each of these classes were given the same standard placement test (pre and post) to assess algebra knowledge gains throughout the semester. These students were not given the statistics or computer science tests because these topics were not covered in the algebra classes. In the liberal education mathematics class, statistics was only briefly discussed and computer science was not discussed at all. The pre-test and post-test results were analyzed using paired *t* tests. Significant increases in post-test scores vs. pre-test scores were found in the placement (n = 14, p-value = 0.010), statistics (n = 17, p-value = 0.000), and computer science (n = 17, 0.000) tests for our experimental course. Interestingly, no significant increase was found in the placement test for the students in the traditional college algebra class (n = 16, p-value = 0.308) or in the liberal education classes (n = 9, p-value = 0.087 and (n = 13, p-value = 0.151).

The course was again taught at BSU in the fall of 2007 and is being taught in the spring of 2009. The course was piloted at Lincoln High School in the spring of 2007 and again in the spring of 2008. The course was piloted at Bemidji High School in the fall of 2007 and again in the fall of 2008. The pre-test and post-test assessment data are presented below. Note that for some institutions, the statistics and computer science (CS) pre-tests and post-tests were combined so only one result is given. This was due to how the test results were compiled by the individual instructors at the partnering institutions.

Institution	Placement Test	Statistics Test	CS Test
BSU – fall 2007	n = 15, p = 0.218	n = 14, p = 0.013	n = 14, p = 0.000
Bemidji HS – fall 2007	n = 11, p = 0.008	n = 11, p = 0.000	
Bemidji HS – fall 2008	n = 22, p = 0.000	n = 22, p = 0.000	
Lincoln HS – spring 2007	n = 12, p = 0.263	n = 12, p = 0.000	
Lincoln HS – spring 2008	n = 22, p = 0.019	n = 22, p = 0.000	

Based on the placement exam pre-test and post-test results, the algebra knowledge gains of students in our class are not consistently statistically significant. There are two classes that had no statistically significant gain. The reason why most classes showed gains in algebra and two did not is not known and is a focus of ongoing research. There has been consistent statistical evidence of statistics and computer science knowledge gains in all the classes offered to date. This is a very positive result and a strong point of the course because, not only are the statistics and computer science topics engaging to most students, students are also increasing their content knowledge in these areas of mathematical science.

Summary

Most high school math curricula in the U.S. are based on what we call the *calculus model* which is four years of curriculum culminating in an advanced placement (AP) calculus course. We have developed a novel mathematical sciences class that we believe is a more appropriate and valuable alternative to the *calculus model* for many students in the U.S. in 11th and 12th grades. The course pedagogy was developed based on sound mathematics and statistics education research and the course content was created based on content needs in a variety of disciplines. The content is interwoven and consists of topics from algebra, statistics, and computer science. The content is largely delivered through projects and investigations that create and hold student interest and promote learning.

The course was piloted at Bemidji State University and two area high schools and by the end of 2009 will be offered in approximately 15 high schools and post-secondary institutions.

Through our research and the process of creating the course, we believe course content and pedagogy for mathematical sciences and, specifically, mathematics courses that have wide student exposure should be developed with input from many areas of study, not just from pure mathematicians. We believe this improves the pedagogy of courses and the content of courses focuses on relevant topics, not purely theoretical topics or topics for which students do not see any applicability. Too many mathematics courses in the U.S. remain unchanged through many years and are out of alignment with current mathematics education research and findings.

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IPESL – Initiative to Promote Excellence in Student Learning grant program through the Minnesota State Colleges and Universities awarded authors $63,374 grant. Title of project: Building Student Success on a Foundation of Preparedness. Grant awarded in November, 2006.

NCTM – Principles and Standards for School Mathematics, published by the National Council of Teachers of Mathematics, 2000.

Family Maths and Complexity Theory

Paul Webb and Pam Austin
Centre for Educational Research, Technology and Innovation, Faculty of Education, Nelson Mandela Metropolitan University, Port Elizabeth, South Africa
Paul.webb@nmmu.ac.za; Pamela.austin@nmmu.ac.za

Abstract

The importance of family involvement is highlighted by findings that parents' behaviours, beliefs and attitudes affect children's behaviour in a major way. The Family Maths programme, which is the focus of this study, provides support for the transformative education practices targeted by the South African Department of Education by offering an intervention which includes teachers, learners and their families in an affirming learning community. In this study participating parents were interviewed to investigate their perceptions of the Family Maths programme mainly in terms of their engagement, enjoyment and confidence levels. The major themes and ideas that were generated in this study include the development of positive attitudes, parents and children working and talking together, and the skills exhibited by Family Maths facilitators. These findings are analysed within the parameters of complexity science and the pre-requisite conditions for developing a complex learning community, viz. internal diversity, redundancy, decentralized control, organised randomness and neighbour interactions.

Introduction

Early-childhood studies reveal that, when it comes to children's cognitive development outcomes, parents' behaviours are more important than other highly publicised factors such as daycare arrangements (Belsky, Vandell, Burchinell, Clarke-Stewart, McCartney & Owen, 2007). In the light of these findings, Bouffard and Weiss (2008) stress the importance of reframing family involvement within a complementary learning framework (one that directly supports what is being taught in schools) across ages and settings through the co-constructed efforts and shared responsibilities of many stakeholders. However, historically investment in family involvement in schools has been limited and usually consists of parents assisting teachers in their classrooms, chaperoning, and parent-teacher conferences (Bouffard & Weiss, 2008). These approaches and roles still persist in schools despite research findings that demonstrate that family involvement should be broader and is most authentic and effective when it is 'linked to learning' (Henderson, Mapp, Johnson & Davies, 2007).

The Family Maths programme, which is the focus of this study, provides support for transformational education practices targeted by the South African Department of Education(2002; 2003), but goes further by extending these efforts beyond the school walls to the community at large through offering a creative education practice that impacts on teachers, learners and their families. The data generated have been analysed through the lens of complexity theory in an attempt to interrogate and understand what conditions influence parents' engagement, enjoyment and confidence levels and enable the development of the complex learning community that we attempt to create when bringing together parents, children and teachers in the Family Maths collective.

Complexity science

Complexity science first arose in the mid 20th century as a result of the confluence of cybernetics, systems theory, artificial intelligence and non-linear dynamics (Davis & Simmt, 2003). Events such as the collapse of stock markets, the sudden spread of ideas in society, the collapse of communism in the Soviet Union and Eastern Europe, the rise of life on Earth, shoals of fish which change direction at the same time, etc. are not only examples of what has been of interest to complexity scientists, but are phenomena that have provided the stimulus for the continued emergence of the study of complexity (Waldrop, 1992). A complex system is seen as something greater than the sum of parts, it is the product of the parts and their interactions – something that is self-organising and can adapt (Capra, 2002; Johnson 2001). Complexity scientists describe this type of self-organising phenomenon as a 'learning system'. As such, Davis and Simmt (2003) describe complexity science as the science of learning systems. These authors understand learning in terms of the "adaptive behaviours of phenomena that arise in the interaction of multiple agents" (Davis & Simmt, 2003: 137) and suggest that complexity science is defined more in terms of its objects of study than in its modes of investigation. In this study, the object of study is the beliefs that parents have of what influences their levels of engagement, enjoyment and confidence in mathematics while participating in the Family

Maths programme; a programme which is envisaged as a complex learning system with multiple agents, viz. parents, children, teachers and the ideas they create.

There are several necessary but not sufficient conditions that need to be met for complex systems to arise and maintain themselves (Davis & Simmt, 2003). These conditions, which have been adapted from Bloom (2000), Casti (1994), Kelly (1995) and Lewin and Regine (2000), are internal diversity, redundancy, decentralised control, organised randomness and neighbour interactions. Internal diversity reflects the different ways in which members of a community can respond and interact. Redundancy is the complement of diversity, i.e. redundancy is the 'sameness' of the individuals within a system. This 'sameness' in a learning community may be a factor of knowledge, purpose, background, etc. Redundancy in this sense may be recognised by the participants' degree of commonality of expectation and purpose and is essential to triggering a transition of *me's* to a collection of *us* (Davis & Simmt, 2003). Lewin and Regine (2000: 28) call the area of intersection between redundancy (commonality) and internal diversity as the "zone of creative adaptability", a notion somewhat similar to Vygotsky's (1978) 'zone of proximal development' in that both ideas refer to immediate possibilities for co-activity, but which are limited by certain criteria.

Another condition that is necessary but not sufficient for systems to evolve and maintain themselves is decentralised control; a situation where power and authority are distributable, the locus of learning is the individual, the system itself 'decides' what is and what is not acceptable, and understandings and insights are co-specified and shared. The condition of organised randomness is the delicate balance between enough organisational control to direct activities, and enough randomness to allow flexible and varied responses. Within the notion of organised randomness, Davis, Sumara and Luce-Kapler (2000) coined the term 'liberating constraints', i.e. those that are not too prescriptive (such as 'turn to page 17 and do the geometry examples number 1-7') or too open-ended ('write down everything you know about geometry), but something more enabling such as 'tell me what you consider to be the five most important things about geometry'. Finally, there are agents within a complex system that affect ideas and activities. These agents are termed neighbour interactions. In the sense of complexity theory and learning communities promoted by Davis and Simmt (2003: 156) these 'neighbours' are not "physical bodies or social groupings", but "ideas, hunches, queries and other manners of representation" which must "bump" against one another. It is the interaction of concepts and understandings that make possible a mathematics learning community.

In this paper we attempt to position parents' perceptions of what influences their levels of mathematics engagement, enjoyment and confidence within the frameworks of complexity science. We do this in order to better understand the influences of necessary, but not sufficient, conditions required for the development of the complex mathematical learning community we wish to promote. A clearer understanding of these influences should also better inform future practices and positions that aim at promoting family involvement in children's education and developing complex family-oriented learning communities.

The Family Maths programme

The 'Family Math Program', which was conceptualised and designed at the Lawrence Hall of Science in Berkeley, California as a subset of the EQUALS programme (which aims at promoting mathematics for all, but particularly amongst girls and minority groups) is designed to allow meaningful links to be made between school and home learning via cooperative learning strategies (Thompson & Mayfied-Ingram, 1998). The 'Family Maths' programme, which has operated in South Africa since 1996 is an adapted version of the American precursor, but similarly aims at dispelling negativity toward mathematics and encouraging learners, parents and other family members to translate new experiences and concepts into workable solutions through discussion and the use of hands-on, minds-on, process-oriented, inquiry-based activities (Kreinberg, 1989).

Negative sentiments about visiting their children's schools are often voiced by parents, particularly in the context of previously disadvantaged schools in poor South African communities (Austin & Webb, 1998), and therefore a less publicised, but nevertheless underpinning, aim of the programme is to bring to the school 'hard-to-reach' parents who mostly have only experienced the negative contexts of teacher and principal's complaints about their children's weak performance, poor attendance, bad behaviour, etc. At each Family Maths workshop there are usually four or five stations where mathematical activities are displayed. At each station a teacher, who has been trained in inquiry-based teaching and learning strategies, facilitates the activity. The facilitators are trained in techniques to encourage learners, their parents and other community members to engage, explore and discuss the problem at hand. This is to be done by asking questions, rephrasing the problem statement, giving clues when necessary, and by asking probing questions to direct participants' discussion and thinking. As there is a paucity of data on the effects of family orientated educational programmes in South Africa in general, and on maintaining parental involvement in particular, this study examines parental perceptions of what they believe encourages and promotes their level of engagement, enjoyment and confidence.

Methodology

The study focused on a 'convenience' sample (Grinnell & Unrau, 2005) of volunteer parents (n=12) out of a total of 140 parents/family members of intermediate phase (grade 4-6, age 10-12) learners who participated in a series of seven quarterly Family Maths workshops. The workshops were implemented in urban, peri-urban and rural primary schools in the Eastern and Southern Cape. All parents interviewed were English second-language speakers with either Xhosa or Afrikaans as their first language. Open-ended discussions with all parents prior to the first workshop revealed a high level of negativity towards, and fear of mathematics. At the conclusion of each Family Maths workshop, volunteer parents were given an opportunity to reflect on their experiences at the workshop, their attitude towards mathematics, and their perceptions of facilitators' inquiry-based teaching and learning skills. Data were collected by means of semi-structured interviews which enabled the generation of first-hand, in-depth, rich, unexpected and relevant information from the interviewee (Kvale, 1996). The researchers felt that individual, rather than focus group interviews, should be conducted as individual interviews would probably enable parents to express their perceptions and feelings honestly, while group interviews may be intimidating for some parents who might feel pressured to concur with others in the group.

Data analysis

The interview questions focused on the first three consecutive stages of the five stage model of the inquiry approach to instruction which is outlined by Layman (1996). These three steps for each problem activity include engaging participants, allowing participants to explore, and encouraging participants to explain mathematical concepts. The interview responses were recorded *verbatim* and then coded via an inductive process that involved breaking up and categorising text to form descriptions and broad themes (Creswell, 2005).

Ethical measures

In this study ethical measures included assurance of anonymity and obtaining the informed consent of interviewees. The aims of the study, research design and methodologies were communicated to all (Mouton, 2001). The participants were told that they could withdraw from the process at any stage and that their decision to participate or not to participate would in no way be viewed negatively by the programme facilitators or the researchers.

Results

Overall, the data suggest that the common thread running through the participants responses was *'a sense of achievement'*. The facilitators' approach was the most important factor in making the parents' feel comfortable during the Family Maths sessions. Every parent interviewed noted directly or indirectly that they were happy to ask questions regarding the problem-solving activities. The fact that the facilitators also encouraged and enabled the participants to work in teams was also noted, e.g. *"I enjoyed working as a team", "The teacher encouraged me and my daughter to try different ways of solving the problem" "The teacher gave us a hint and we solved it together", "I enjoyed working in a team, helping each other and asking questions. I felt comfortable."*, and that the *"How do you know that?"* questions by their peers really helped them reflect and develop their understanding of the problem.

Each parent interviewed said that the facilitator had assisted them to solve the problem by engaging them in one or more of the following ways; encouraging them, making them feel relaxed, and not rushing them (42%); explaining clearly what the problem was, asking questions and giving clues when they needed them (42%); not giving them the answer or telling them how to solve the problem, but encouraging them to keep trying (16%). These responses, amongst others, suggest that encouragement, clear explanations, and the fact that the facilitators were not willing to give the participants the answers, but allowed enough time for them to engage sufficiently with the problem, can be identified as themes running through the parents' perceptions of successful engagement in the Family Maths process.

All parents said that the facilitator had encouraged them to explore the concept in one way or another. Parents gave the following reasons for being encouraged to explore the concept; teachers asked questions which made them think and gave clues when they needed them (50%); teachers gave them time to think and keep trying and they did not feel as if they were being rushed (36%); and they had tried different ways of solving the problem (14%).

It appears evident from the data analysis that the majority of parents who participated in the Family Maths programme entered the programme with minimal knowledge or experience in inquiry learning. This finding was deduced from parental statements such as *"After a workshop I had learnt new ways of thinking"*, while another said, *"We are learning maths we did not do at school"*.

Discussion

The major themes and ideas that were generated in this study were those of developing a sense of achievement; being encouraged; feeling comfortable; working in a team; engaging in activities with others, skilful questioning by the facilitators; using concrete examples; developing new ways of thinking; and learning maths they did not do at school (novelty). The respondents' references to challenges and engaging with one another, signs of exploratory talk, identification of facilitator skills, and the fact that they noted that these activities influenced their self esteem, suggest that most of the pre-requisite conditions for developing a complex learning community were met in the workshops. Comments by the participants that they were empowered because they felt that their ideas counted suggest that there was sufficient redundancy within the group for believable judgements to be made by their peers, children and facilitators. The fact that they expressed ideas that others appreciated, indicates that there was sufficient internal diversity within the groups for this condition to be met while parents' comments on the facilitators' ability to question skilfully, use concrete examples effectively, introduce a sense of novelty and get them go about thinking about things differently suggest that the facilitation process did promote organised randomness and that the facilitators were able to provide sufficient liberating constraints for a community of learning to emerge.

While issues of internal diversity and redundancy are easily considered to be inherent characteristics of the group membership, it is tempting to believe that the condition of decentralised control is a product of the facilitation process. However, our observations suggest that in this study the collective emerged and sustained itself through shared projects (activities) and neighbour interactions, not through planning or other deliberate strategies. This possibility is supported by the research of Buchanan (2000), Johnson (2001) and Varela (1987) who argue that complexity theory allows different perspectives on what influences the development of a learning community, perspectives opposed to the tendency to suspect the existence of a coordinating agent, something which is probably rooted in habits of cause-effect thinking. These perspectives provide different pointers as to what can be done to promote the emergence of complex learning communities, and allow different interpretations of outcomes.

Understanding the dynamics of interactions, and an awareness of the necessary, but not sufficient, conditions which have influence in the emergence or non-emergence of a learning community are not merely useful tools for post-activity analysis of classroom events, they can also be used to structure engagements with learners (Davis & Simmt, 2003). When structuring engagement, the notion of internal diversity suggests the need to develop activities that can be adapted by learners to their particular knowledge, understandings and interpretations. Redundancy points to the need for shared experiences and clear terms of engagement. Decentralised control and organised randomness highlight the need for careful planning in terms of prescription and awareness of liberating constraints, while neighbour interactions focus attention on how ideas might be represented and juxtaposed.

It is accepted that even if all conditions for complexity are met, there is no assurance that complex possibilities will arise (Davis & Simmt, 2003). On the other hand, it is reasonable to expect that if they are not met, a complex learning community will, to a high level of probability, not emerge. For this reason, we believe that thinking in terms of complexity deepens our understanding of the dynamics within programmes such as Family Maths and raises the chances of influencing the emergence of a complex community of learning. In turn, we believe that there is a better chance of improving parents' understandings, beliefs and attitudes in terms of promoting their children's educational development and their own participation in educational family life if they are able to engage with their children and their teachers within a complex learning community, such as the collectives envisaged by the Family Maths programme

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Elementary Mathematics from an Advanced Standpoint and Elementary Views on Advanced Mathematics

Ysette Weiss-Pidstrygach

Mathematical Institute, Georg-August-University, Göttingen, Germany weiss@uni-math.gwdg.de

Abstract

What kind of and how much mathematics should a high school maths teacher know? The experience with a math camp, an innovative form of bringing together high school pupils, university math students and math teacher students as well as university professors in the common aim to teach mathematics sheds new light on this question. Different interests define different positions. The different actors have little common aims since they rarely form a joint community of practice.

Over the seven years of its existence the math camp has evolved from a classical lecture-centred activity for gifted pupils to a much more encompassing experience illustrating the importance of a two way communication between advanced mathematics and elementary mathematics in schools.

What are communities of practice?

The term was introduced by Lave and Wenger (1991). A community of practice is a group of people informally bound together by shared expertise and passion for a joint enterprise. It has an identity defined by a shared domain of interest. Membership therefore implies a commitment to the domain, and therefore a shared competence that distinguishes members from other people. A community of practice is not only a community of interests, it implies sharing information, learning from each other and developing a shared repertoire of resources: experiences, stories, tools, ways of addressing recurring problems ...in short a shared practice (see Etienne Wenger).

Many math departments undertake activities for gifted pupils, where university math professors take care of the next generation by introducing pupils to topics of higher maths. Often this forms a community of practise evolving around a joint enthusiasm for advanced mathematics and operating with modern mathematical notions. Rarely, however, are students involved especially not students wanting to become math teachers. The intention of the math camp project was to broaden the scope of participants, involving a broader set of pupils, as well as students both from mathematics as well as prospective math teachers.

The first math camp- organizational frame

The activity started in 2002 as a programme for mathematically gifted pupils aged 15 to 19. The camp was organized as a week long workshop where pupils were introduced to invariant and knot theory. The topics were covered by lecture series given by two math professors, and eight exercise classes held by prospective math teachers and maths students. Maths students and prospective math teachers were supposed to hold the exercise classes as tutors in pairs. All the materials, the problems for the exercise classes as well as the lecture notes were chosen by the professors and given to the tutors two month before the workshop. The problems were supposed to motivate and illustrate notions and concepts introduced in the lectures, some shaped for exploratory learning, some to repeat techniques explained during the lecture. During the exercise classes the 40 participants were split in groups of eight. Every group was supposed to work out a presentation of a solution to "their" favourite problem. The problem given on the poster advertising the camp showed the expected standards: Given two (concrete) positions of a position game called Schiebefix, is there a move to get from the first to the second position. The pupil could start with trial and error and then develop a theory for certain positions. To find a general solution for any two positions was quite a difficult task without the theory of invariants.

Tutor training

In the preparation of the workshop maths students and prospective teachers jointly started to solve the problems given to them and to prepare themselves for the variety of possible solutions by the participating pupils in the forthcoming workshops. Usually the math students developed solutions and explained them to the prospective teachers who often assumed the role of pupils. Quite a few of these solutions, however, were based on the theory presented in the lectures. Prospective teachers, who could not rely on a thorough understanding of the theory made clear that this implied conditions that seemed rather artificial and incomprehensible from a school mathematics context. Teaching methods for the exercise classes and the preparation of the presentations were rarely discussed in the preparatory meetings of the tutors. At the first math camp the prepared solutions thus used techniques of invariants given in the lecture notes.

The pupils

Pupils applying for the workshop had to attach their last school report including all marks and a short explanation why they would like to take part. 40 pupils between grade 10 and 13 were chosen: one third with prior experience in out school mathematical activities and two thirds knowing maths only from school. All participants underlined in their application apart from their joy in doing maths that they would like to meet other pupils sharing the same interest. We intentionally chose one third of the participants where the interest in math was not reflected in good marks in school math, hoping to get hold of gifted but underchallenged and therefore no longer motivated pupils.

Workshop implementation

During the workshop the prospective teachers hardly used the prepared solutions. Most pupils came up with their own solutions, some of them working in groups some on their own. During the tutorials another division of labour established itself: prospective teachers organized and supported group work, listening to different approaches, dealing with problems arising from competition, different age, different sex, adolescence. The activity of the math students were normally confined to checking solutions and talking with outstanding (usually single working) pupils about the lecture and related to maths. The final presentations differed considerably: those guided by the math students resembled mathematical lectures, using definitions and generally working in a deductive manner, whereas those accompanied by prospective teachers were much more individual also presenting the process of finding the solution.

Evaluation of the first workshop

At the end of the math camp pupils as well as tutors were asked to fill out a feedback sheet that showed that pupils as well as tutors enjoyed the work shop enormously. Amongst others teachers appreciated being backed by specialists and the math student valued the social capabilities and engagement shown by the prospective teachers. Pupils even suggested subjects for the next math camp.

The prospective math teacher were satisfied with their performance during the work shop, however they did not assume that the materials they had worked through during the camp could be reasonably used in schools.

The formation of a community of practice

The experiences of the first camp triggered the following adjustments:

- the theory taught in the lectures has to be developed based on the problems to be addressed in the exercise class; these problems should be formulated without notions from higher maths;
- the lecturing math professors take part in the preparation of the tutors and adjust the propaedeutic and introductive problems to a level the prospective teachers feel confident with;
- in the preparatory meetings the lecture drafts are discussed with the aim to find formulations that are informal, close to common speech or school math notions and descriptions;
- the “favourite problem presentation” should encourage to reflect the process of finding the solution and should preferably be derived in group work
- based on the feedback of the preparatory meetings and the progress during the work shop the lecturing math professors suggests suitable subjects for bachelor and master theses related to the problems

Supported by the existence of an enthusiastic kernel of students and staff members organising and running the camp repeatedly the activity became more and more a community of practice based on the shared interest to bridge the gap between conceptual mathematics and problem solving example oriented school mathematics.

Experiences intensifying the self conception of the prospective teacher as a mathematicians were:

- teachers’ solid mathematical background knowledge and interest in modern maths supports acceptance and respect from the pupils;
- conceptual approaches and knowledge of general methods and techniques in problem solving can often cover missing immediate creativity;
- complex mathematical concepts like classification are universal and if recognised and used intentionally prove extremely useful in everyday life;

- the prepared elementary approach to a subject in higher maths can be used for project work and activities for gifted children in school;
- the joy of approaching mathematics with the spirit of a researcher and problem solver.

As a result some of the prospective math teachers wrote master theses about motivations to notions or elementary introductions to areas of higher maths. The evaluation showed another important experience coming from the mentoring during the workshop: gifted but not motivated teenagers behave very differently when with like-minded people and can find in this environment a new starting point.

Changes in attitude and in the sense of responsibility for the next generation were also noticeable with the mathematicians:

- some math students continued teaching children outside school in math clubs;
- the elaborated elementary introductions to some notions in higher maths found their way into the lectures in algebra, analysis, topology and geometry;
- links to questions explored in school mathematics were used in introductory courses at the University;
- non formal discussions of well understood mathematics (not only related to the math camp) subjects became routine

There is perhaps no universal answer to the question: What kind of and how much mathematics should a high school maths teacher know? Indeed, decisions about which subjects of higher maths are becoming "elementary" now, how much conceptual background is needed to feel secure in the classroom, are individual. However the joint work in this community of practice helped the members to find their own position and a joint frame related to the subjects treated.

The idea to integrate the two approaches "Elementary Mathematics from an Advanced Standpoint" and "Elementary Views on Advanced Mathematics" into teacher training is not new. The new curriculum for the high school teacher training at the ETH Zürich includes such courses (see Urs Kirchgraber).

To have a wide domain of joint experiences the subjects of the math camps are changed annually. So far they were:
- Invariants and knot theory
- Symmetries of Rubik's Cube and different geometries
- Prime numbers and coding theory
- Symmetry and its use in differential equations
- Projective geometry and other geometries
- Invariants in topology
- The projective plane

The Göttinger model to link programs for mathematically gifted children and maths teacher training is easy to adjust to different educational and training systems. In most countries the national programmes for pupils gifted in mathematics are personally linked to, or even run by maths departments. Examples are summer schools, master classes, specialised maths classes and maths schools. These are environments where elementary introductions to modern mathematics and science are permanently developed and informal discussions create the new mathematical culture.

Göttingen counts with a long standing tradition with similar approaches; over a hundred years ago Felix Klein, himself a brilliant mathematician as well as extraordinarily gifted teacher, organized a community of practice of maths researchers and educators in Göttingen in the form of seminars. Felix Klein's books "Elementary Mathematics from an Advanced Standpoint" provides the conceptual background from higher mathematics to the school maths as it is taught until today. Changes in the curriculum (Meraner Reform), participation in DAMNU and IMUK and teacher training reforms were problems which the participants, like R. Schimmack and W. Lietzmann dealt with at the seminars. Today's rapid political, social and cultural changes in schools make it necessary to establish communities of practice, uniting not only researchers and educators but also prospective teachers and pupils.

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DeltaTick: Applying Calculus to the Real World through Behavioral Modeling

Michelle H. Wilkerson-Jerde, Uri Wilensky
Northwestern University, Evanston, IL, USA {m-wilkerson;uri}@northwestern.edu

Abstract Certainly one of the most powerful and important modeling languages of our time is the Calculus. But research consistently shows that students do not understand how the variables in calculus-based mathematical models relate to aspects of the systems that those models are supposed to represent. Because of this, students never access the true power of calculus: its suitability to model a wide variety of real-world systems across domains. In this paper, we describe the motivation and theoretical foundations for the DeltaTick and HotLink Replay applications, an effort to address these difficulties by a) enabling students to model a wide variety of systems in the world that change over time by defining the behaviors of that system, and b) making explicit how a system's behavior relates to the mathematical trends that behavior creates. These applications employ the visualization and codification of behavior rules within the NetLogo agent-based modeling environment (Wilensky, 1999), rather than mathematical symbols, as their primary building blocks. As such, they provide an alternative to traditional mathematical techniques for exploring and solving advanced modeling problems, as well as exploring the major underlying concepts of calculus.

Introduction

"Calculus has been so successful because of its extraordinary power to reduce complicated problems to simple rules and procedures – thereby losing sight of both the mathematics and of its practical value." (Hughes-Hallett et al 1994)

As Hughes-Hallet notes, many complicated problems can be solved easily using the tools of calculus – not only in science, technology, engineering and mathematics (STEM), and importantly also in the social, behavioral, and economic (SBE) sciences. But by focusing only on analytic procedures, calculus educators have largely failed to help students understand why or how those procedures work. While it is clear that recent reforms in calculus education have addressed some of these issues by helping students to better understand the concepts that underlie calculus rather than simply the procedures and methods involved, researchers have yet to show that students are learning how to apply these concepts generatively to describe, model, and explore real-world phenomena (Ganter, 2001). At the same time, there are an increasing number of calls from science and industry for students to understand calculus in an interdisciplinary, applied, and generative manner (NCTM, 1989; AAAS, 1990). Because calculus is an important tool for so many different domains and topics, students without a calculus background can only explore a limited, highly simplified set of problems and ideas, and students are rarely exposed to the true power of calculus: its suitability to model a wide variety of real-world systems across domains.

In this paper, we provide the motivational background and brief description of the DeltaTick and HotLink Replay applications, designed to address these issues by a) enabling students to model a wide variety of systems in the world that change over time by defining the behaviors of that system, and b) making explicit the links between that system's behavior and the mathematical trends that behavior creates. These applications employ the visualization and codification of behavior rules within the NetLogo agent-based modeling environment (Wilensky, 1999), rather than mathematical symbols, as their primary building blocks. As such, they provide an alternative to traditional mathematical techniques for exploring and solving advanced modeling problems, enabling even students without a calculus background to explore those topics, while at the same time introducing students to many of the major underlying concepts of calculus.

The DeltaTick project builds on work that complements typical algebra-based representations of calculus concepts such as rate of change, integration, and differentiation with alternative – and in some ways, more accessible – representations using computational media. It extends this work by representing these concepts in the context of *systems* – that is, scientific and social phenomena for which many components and behaviors in a system contribute to a single measurable quantity that changes over time. In the following sections, we will describe these representational shifts and the applications we are developing in more detail, and then illustrate how a number of important concepts in calculus can be explored – and how interesting systems can be modeled and studied – using this alternative media.

Background

The DeltaTick project leverages the power of computers for enabling new ways to represent the mathematics of change, and is rooted in design-based research on representational infrastructure shift (diSessa, 2000; Papert, 1996; Kaput, Noss & Hoyles, 2002), the "restructuration" of domains (Wilensky & Papert, 2006), and the computational and pedagogical affordances of agent-based modeling (Wilensky & Resnick, 1999; Wilensky & Reisman, 2006) to provide an alternative method for exploring and creating models of the real-world systems. In this section, we briefly review this literature, provide a description of what we mean by a *system* in the "calculus of systems", and conclude with an illustration of how the notion of *rate of change* in calculus and differential equations can be fruitfully "restructurated" as a collection of the behaviors executed by elements or agents in the case of a predator-prey system

Restructurations and Shifts in Representational Infrastructure

Computation enables dynamic encoding of information that was previously only encoded using static media (diSessa, 2000; Papert, 1996; Kaput, Noss & Hoyles, 2002; Wilensky, 2006). Over the past 20+ years, researchers have explored how the ability to codify and execute rules over time using computers can enable students to explore the important ideas underlying the mathematics of change. A number of computational learning environments have been developed that leverage students' intuitive thinking about the mathematics of change – primarily in the context of one- and two-dimensional motion (Nemirovsky, Tierney & Wright, 1998; Roschelle, Kaput & Stroup, 2000), but also in the context of other simple phenomena such as banking or other dynamical systems (Wilhelm & Confrey, 2003; Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Herbert & Pierce, 2008). These programs enable students to visualize, collect, or even produce their own meaningful quantitative patterns of change over time – all while relating these systems with graphs, number tables, and other more "mathematical" representations of these quantities.

Although these computer-based models look very different from their typical equation-based counterparts, both models encode what can be very similar information about how the system changes over time. This is an example of what Wilensky and Papert (2006) refer to as *restructurations:* changes in the encoding of disciplinary knowledge as the result of changes in the technology that is used to encode that knowledge. The language of calculus was developed using static media, so that if one is interested in how a certain rate of change affects the trend of a quantity over time, they must use symbolic "tricks" like integration to *mathematically* predict that trend. Now that computers enable the simple externalization and rapid execution of rules (that themselves define rates of change), we can replace those symbolic tricks with the writing and execution of rules that *computationally* predict the trend.

Wilensky and Papert note that a given knowledge "structuration" reflects the technologies and techniques available to a given community when that knowledge was created. As such, the classical or common encoding of knowledge for a given domain is not necessarily the *best*, *most complete*, or *most accessible* way of presenting it. It might be that as new technologies and techniques emerge, new knowledge encodings grant more people access and facility of use of that knowledge – even though at the same time, those different encodings may reduce other affordances provided by their alternatives, such as efficiency or precision.

Agent-Based Modeling and the NetLogo Modeling Environment

The DeltaTick project differs from other environments that leverage the representational affordances of computers, because we use *agent-based modeling* as a means to represent different kinds of systems that change over time. Agent-based modeling is a computational technique in which a system comprised of a number of homogenous elements or *agents* each execute relatively simple individual-based rules, producing often unexpected or interesting aggregate, system-level patterns. Whereas typical calculus-based treatments of many systems involve an equation representing the quantitative patterns of aggregate measures of phenomena (such as the pressure of a gas, ratio of chemicals in a solution, or trends in a population), agent-based models involve programmatic instructions representing the behavior of individual elements of the system that produce those measures (such as the Netwonian motion of individual gas particles, the reactions of individual and pairs of molecules in a chemical solution, or reproduction and death of individuals in a population).

Agent-based modeling has been increasing in popularity for not only educational, but also scientific purposes, and a number of environments for creating agent-based models exist. One such agent-based modeling environment, NetLogo (Wilensky, 1999) was developed with the explicit goal to be "low-threshold and high-ceiling" – making it very simple for beginners to build models, but also possessing enough power to allow for the development of highly sophisticated models. It has been shown that the restructuration of a number of domains in mathematics and science (such as biology, Wilensky & Resnick, 1999; chemistry, Levy, Kim & Wilensky, 2004;

materials science, Blikstein & Wilensky, 2006; and electromagnetism, Sengupta & Wilensky, 2008) within the NetLogo environment can help to make understanding and analyzing these systems accessible to many more students. We believe such models – especially if explicitly related to more typical calculus-based notions of concepts such as rate of change – may also provide an accessible means through which the mathematics those systems (namely, calculus-based equations) can be introduced to students. Both the DeltaTick and HotLink Replay applications have been developed to work with NetLogo, but to explicitly emphasize the relationships between this form of modeling and that mathematical models typically used to explore quantitative patterns of change over time.

Defining a Calculus of Systems

While a great deal of research has been done investigating student understanding of change over time, little is known about how they understand change over time in *systems* – that is, in phenomena for which calculus is used to model change over time of single values that are affected by a number of elements and behaviors. For example, a single population measure that changes over time is influenced by the reproductive and resource-based behavior of each member of the population, and different behaviors (such as reproduction, or the absence of reproduction due to a lack of resources, or death) can change the population measure in different wants. When we use the term “calculus of systems”, we are specifically referring to phenomena which always involves multiple components or agents, and which can also involve multiple behaviors for each of those agents (shaded in the table below) – but not systems that only involve one or a small number of agents.

		Number of Agents	
		One	*Many*
Number of Behaviors	*One*	Example: One-dimensional motion Agent: Moving object Behavior: motion in single direction	Example: Simple exponential (Malthusian) population growth Agents: members of population Behavior: reproduction
	Many	Example: Projectile Motion Agent: Projectile Behaviors: inertia, downward acceleration due to gravity	Example: Lotka-Volterra predator-prey system Agents: members of population Behaviors: reproduction, predation, death

Different classes of phenomena that can be modeled using Calculus. We are interested in classes that involve many agents (shaded).

These situations comprise a particularly interesting and important class of phenomena for students to explore and understand for a number of reasons. First, a large and increasing number of important concepts in science, technology, engineering, and mathematics; as well as in the social and behavioral sciences, involve a number of components (atoms, particles, people, markets, and so on) that all interact to produce some outcome of interest. Second, enabling students to connect ideas in calculus to these more complex systems increases the number of real-world and personally meaningful phenomena that they are able to explore and model. Finally, by better understanding how seemingly different scientific and social systems might produce similar or different quantitative trends over time, students might have more access and encouragement to draw interdisciplinary links between topics they learn in school and/or find personally interesting. If students do possess qualitative resources for reasoning about mathematical patterns and relationships in complex systems similar to those that they exhibit when reasoning about simple change-over-time situations, this points to exciting implications for calculus and science education at earlier levels of education.

Macro-Level Rates of Change as the Result of Micro-Level Behaviors

Finally, we would like to provide an example of how a relatively complex calculus-based representation of a system that involves multiple rates of change can be re-conceptualized in terms of the behaviors that produce those rates of change, and thus, the quantitative patterns of interest that we wish to model. Consider, for example, the mathematical models used to describe a typical predator-prey system (the Lotka-Volterra equations), here involving wolves (*W*) and sheep (*S*):

$$\frac{dS}{dt} = b_S N_S - k_1 N_S N_W \qquad \frac{dW}{dt} = k_W N_S N_W - d_2 N_W$$

A detailed description of the meaning of these equations is included in Wilensky & Reisman (2006), but for our purposes it suffices to state that these equations describe how wolf and sheep populations change over time as a result of the sheep birth rate (b_S), the predation rate at which wolves consume sheep (k_1), and the death rate of wolves (d_2), multiplied by the total number of sheep and wolves in the system (N_S and N_W).

Another way to conceptualize this system is not in terms of rates, but in terms of the relevant behaviors of a wolf or sheep, and the effects those behaviors can have on the population of each. For example, each wolf can reproduce or die, which leads to an increase or decrease in the wolf population. Since wolves prey on sheep, a wolf death also affects the likelihood that a sheep will be eaten, and so on. Each of these behaviors can be mapped onto the corresponding rates of change that are included in the traditional differential equations featured above, and serve to explain the mechanisms through which those rates of change manifest: clearly connecting the mathematics of the system with the system itself.

This alternative method for representing rate-based mathematical models of systems resembles systems dynamics models (such as STELLA; Peterson & Richmond, 1992), but includes a number of differences. First, it is defined at the *agent* level rather than the *aggregate* level: in other words, rather than describing a collection of elements of a system (such as wolf and sheep in this example, or a collection of atoms in a gas, and so on). Second, the basic building blocks of these systems are *behaviors* that each of those agents produce, rather than rates of change as is the case in systems dynamics models – providing students with more information to build an integrated understanding of both the mechanisms and mathematical trends of systems of interest.

DeltaTick and HotLink Replay: Facilitating Construction of Rate-Based Systems Modeling

DeltaTick: Building the Model. To use DeltaTick, users first define the actors or agents in a system. They are then are asked to define how the system changes over time as a result of the behaviors of those agents - specifically, how certain properties (such as the number, size, position, and so on) of agents change per discrete unit or ``tick" of time. Although users can create the behaviors they would like to assign to agents from "scratch", DeltaTick comes equipped with libraries of behaviors that can be simple dragged onto agents for common models that might be of interest to users: for example, the "population growth" library includes behaviors that can be added that fundamentally change the shape of the curve produced by the model: reproduce-clone, wander, die-randomly, enough-space?, partner-here?, and so forth. Once added, the user can also modify these included behaviors to their own preferred specifications.

DeltaTick allows this construction to occur within a freeform drag-and-drop interface, and makes explicit to users which values of interest (that is, which aggregate-level outcome measures, such as population in the example above) are affected by which behaviors. When a model has been created, it can be saved and loaded into the NetLogo modeling environment, or run and recorded directly by the HotLink Replay application.

HotLink Replay: Running the Model. Once the model is defined, users are able to load it into the HotLink Replay system and execute it over several ticks of time. The model will produce both a visual and graphical representation of the phenomena, so that the student is able to directly relate what goes on in terms of behavior to the graph that characterizes that behavior quantitatively. After the model stops running, the student will be able to move back and forth over the duration of the model, evaluating important features of the graph (such as inflection points, limits, maxima and minima that may occur in different models) and comparing them to the visualized behavior of the model that produced those features.

Though conceptually very simple, this approach to constructing models of, and exploring, systems that change over time affords a great deal of flexibility in terms of what can be modeled, and how complex a model can be. In fact, the two examples used above -- of continually compounding interest and population growth -- are not typically presented to students until well after the first year of calculus. The basic underlying principle of this system is that ideas of derivative, integral, change, and so on can be presented within, and understood in the context of, real-world systems that change over time.

Using DeltaTick, a number of important underlying concepts of calculus are re-conceptualized in the context of an agent-based model that emphasizes the behaviors, rather than the quantitative trends, of systems. A model developed can be envisioned as an equation, for which different parameters can be inputted and which can be executed for a given period of time in order to compute results. An individual "tick", or iteration, of the model can be conceptualized as the differential: embodying a single collection of behaviors that contribute to the changes to quantities of interest for the smallest available unit of time. Finally, the integral or accumulation is obtained by running a model for several ticks; allowing for the repeated execution of those behaviors than affect outcomes of interest and build upon quantities derived from the previous execution.

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Algebraic Thinking- More to Do with Why, Than X and Y

Windsor W.J.J., M.Ed.
Lecturer Mathematics Education, Faculty of Education, Griffith University,
Mt. Gravatt Campus, Brisbane, Queensland, Australia w.windsor@griffith.edu.au

Abstract
Algebraic thinking is a crucial and fundamental element of mathematical thinking and reasoning. It initially involves recognising patterns and general mathematical relationships among numbers, objects and geometric shapes. Using historical evidence, this paper will highlight how the ability to think algebraically might support a deeper and more useful knowledge, not only of algebra, but the thinking required to successfully use mathematics. It will also provide a framework for educators of primary and middle years' students to develop the necessary thinking strategies required to understand algebra.

Introduction

Mathematics is often seen as the gate-keeper of the mathematically intensive vocations. For nearly thirty years this metaphorical gatekeeper has worked very effectively with governments' world wide identifying a steady decline in the participation rates of students undertaking advanced mathematics courses at a secondary school level. For example, only 12% of Australian students enrol in advanced mathematics courses with only one-third of these students being young women. The declining participation rates and limited engagement with mathematics is slowly impinging on the availability of competent individuals pursuing careers in the mathematical rich vocations offered at a tertiary level (Norton & Windsor, 2008). Critically, this non-participation is negatively impacting on the employment opportunities available to people. More alarmingly is the fact that a limited understanding of mathematics may directly hinder a person's effectiveness to participate in our modern society where information, discussions and rhetoric are immersed and in some cases shrouded by mathematics. As Booker, Bond, Sparrow & Swan (2009, p7) state, individuals who lack an ability to think mathematically will be disadvantaged and at the mercy of other peoples interpretation and manipulation of numbers.
Algebra is the crucial link between the predominantly arithmetical approach of the primary school curriculum and secondary mathematics subjects such as calculus, quadratics and trigonometry. However, in the recently published Foundations for Success (US Department of Education, 2008, p 18) it was noted that the sharp falloff in mathematics achievement begins as students reach middle school where, for many students, they are introduced to algebra for the first time. Arcavi (2008) states that algebra, in many ways, intimidates students and affects their attitudes towards mathematics. These conceptual and attitudinal impediments have long been seen as reasons why student struggle with some advanced mathematical concepts at a school secondary level. The question that needs to be addressed is how do educators ensure that all students have the opportunity to successfully participate in algebra? If this issue is addressed in the primary and middle school context then it may influence students to participate in the mathematically rich subjects undertaken at secondary school.
Simply bringing the subject of algebra to the earlier grades does little to address the underlying problems of student misunderstandings (Kriegler, 2006). Importantly, educators need to consider the thinking required for understanding algebra. It is widely acknowledged that to understand number, students initially use additive thinking structures before transitioning to multiplicative structures. Surely, to understand algebra students need to develop the thinking required to identify, understand and communicate generality which is the essence of algebra. To develop this thinking- often referred to as algebraic thinking- Kaput (2008) suggests an increasingly longitudinal view of algebra; that is, a view of algebra not as an isolated course or two, but rather as a strand of thinking and problem solving, beginning in primary school and extending through students' mathematical education. By connecting and seeking out the generalities inherent in number, geometry and measurement, algebraic thinking and algebra can become the unifying strand of primary and middle school curriculums.

Algebraic Thinking

Algebraic thinking promotes a particular way of interpreting the world. It employs and develops a variety of cognitive strategies necessary to understand increasingly complex mathematical concepts and builds upon students' formal and informal mathematical knowledge. Essentially students are using, communicating and making sense of the generalities and relationships inherent in mathematics, rather than just the identification of a single numeric answer or objective fact. Chazan (1996) implores that educators appreciate the algebraic thinking already done by students, their parents, and other members of the community, even though it is not necessarily expressed in x's and y's. Developing students' ability to think algebraically is a precursor not only for participation in the subject of algebra, but also importantly to be able to think broadly about problem situations. Algebraic thinking provides an extra dimension to an individual's understanding and use of mathematics because they seek out and understand the generalities, as well as the specifics of a problem.

Algebraic thinking can emerge from the number, geometry and measurement activities primary school students engage with daily at school. By illustrating ideas and using concrete materials, models, diagrams, tables and patterns of objects students can 'see' the relationships between the concepts. Students who think algebraically are aware of the inherent links and interconnectedness of mathematics and this thinking can be developed in all students. They understand that mathematics is a system of interpretation where the concrete and the abstract are interwoven. This would suggest that using concrete materials is fundamental because so many of the ideas of algebra are not intrinsically obvious. As Booker et al (2004, p.14) suggest students need to be assisted to develop algebraic thinking using structured materials, materials through which the underlying ideas are understood and appreciated. Furthermore, Lins & Kaput (2004) suggest that students who are engaged in algebraic thinking attempt acts of generalisation and seek to communicate those of generalities. Their thinking involves, usually as a separate endeavour, reasoning based on the forms of syntactically structured generalisations, informed by syntactically and semantically guided actions. By linking the concrete and the abstract, educators parallel the historical development of algebra.

History and Algebraic Thinking

Reflecting on and analysing the historical development of algebra can provide an awareness of the ways mathematical thinking and understanding has developed. A rich tapestry of information is available to link the epistemological and the historical. According to Ernest (2006), analysing history from a deep epistemological perspective for psychological purposes moves mathematics away from the traditionalist view of mathematics to a more humanistic position. Devising a pedagogical approach that is in sympathy with its historical development takes into consideration all the elements of knowledge creation and appreciates and values all human activities associated with mathematics. Using the history of mathematics can benefit and influence the way educators teach and importantly develop a greater sensitivity concerning how students learn algebra.

The history of mathematics informs us that the development of algebra and consequently the ability for individuals to interpret, think, and communicate algebraically, progresses through three distinct yet overlapping stages of development. Researchers (Katz 2007; Bashmakov & Smirnova 2000) define these three stages as the rhetorical stage, the syncopated stage, and the symbolic stage. This chain of development, first identified by G.H.F. Nesselmann in *Die Algebrader Griechen (The Algebra of Griechen, 1842 cited Puig & Rojano, 2004),* attempts to summarise how algebraic thinking strategies developed over a 4000 year period. From his work the commonly adhered to definitions for each of the three stages are; the rhetorical stage, where the calculations are expressed completely and in detail utilising everyday written and spoken vernacular; the syncopated stage, where frequently occurring concepts and operations are replaced by consistent abbreviations instead of the complete words; and finally the symbolic stage whereby all possible forms and operations are represented in a symbol based system.

History would suggest that to understand and solve problems of an algebraic nature, individuals operate and manoeuvre their thinking continually between the rhetorical, syncopated or symbolic stages. For example, the Lucus' Tower of Hanoi puzzle, whereby disks are moved from one rod to another in the least number of moves without a larger disk being placed on a smaller disk, can be examined algebraically. At the rhetorical and syncopated stages, students describe and identify the relationship between the minimum number of moves and the number of disks. The description may be summarised using a table, diagrams or simply a model to develop the generalities and identify the least number of moves, for any disk configuration. At the symbolic stage to fully understand the relationship between the disks and the number of moves, students will make links with the relationships identified at rhetorical and syncopated stages of thinking.

An Example of Algebraic Thinking within a Primary School Context

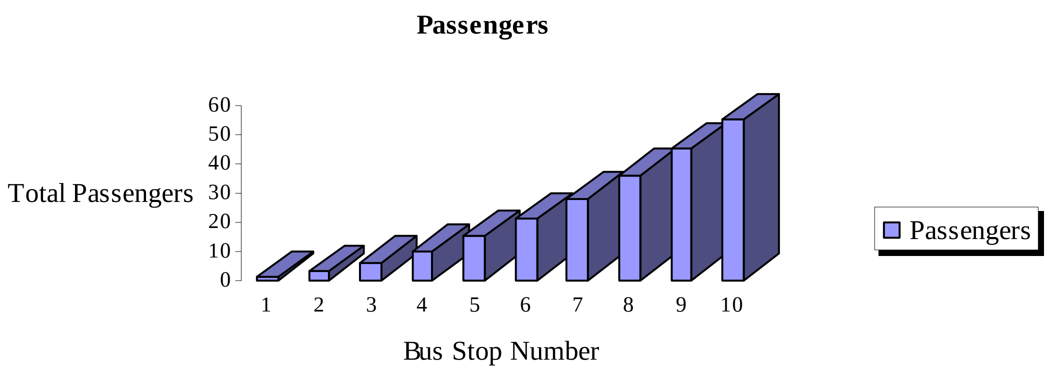
To build-on and extend a class of year seven students' numeration and computation understanding and to develop their algebraic thinking skills, a variety of different problems were presented to them. The class were required to work in small co-operative groups, whereby they would verbally present to their peers and teacher their understanding of the patterns and generalisations they identified within the problems. All of the groups were able to write a short explanation, however some went beyond these explanations and explored alternative representations of their thinking. For example, Dale a 12 year old boy, who had an excellent understanding of numeration and computation concepts, observed that the number of passengers boarding the bus was the same as the bus stop number. Because no passengers were 'hopping off' the bus, Dale identified the total number of passenger as the sum of all the bus stops. He represented the 'Bus Stop Problem' firstly, by identifying and summarising his thinking using a table, he then proceeded to graph the information. When asked why he constructed the graph he simply stated that the graph made it easier for him to see the pattern. In conjunction with his peers he formulated a description of the pattern.

Problem- The Bus Stop

One day the bus conductor noticed that passengers were boarding the bus in the following way. At the first bus stop, 1 passenger got on, 2 got on at the second stop, 3 at the third stop and so on. The capacity of the bus was 72. What was the number pattern that the conductor noticed?

Dale's Responses

Bus Stop	1	2	3	4	5	6	7	8	9	10
Passengers Getting On	1	2	3	4	5	6	7	8	9	10
Total Passengers	1	3	6	10	15	21	28	36	45	55



Rhetorical Description: The number of passengers getting on is the same as the bus stop number. The total number of people on the bus is the sum of the passengers already on the bus and the next bus stop. To work out the total passengers on the bus take the bus stop number people are getting on at, add one to this then multiply it by the bus stop number and half this total

Conclusion

Like many fundamental mathematical concepts, algebraic thinking is best learnt by communicating and linking real objects and materials with the symbols of mathematics. Importantly, by extending algebraic thinking beyond algebra's purely symbolic realms, educators may give all students the opportunity to learn how to generalise, justify and reason using algebraic methods. Crucially students must use materials to "see", describe and reason about generality. Secondly develop the necessary understandings to summarise those generalities by using graphs, tables or diagrams. Finally, using the representational systems of mathematics and algebra they communicate those generalities succinctly and with understanding. Educators can ensure that the catch cry of "algebra for all" is a legitimate goal.

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A New Pedagogical Model for Teaching Arithmetic.

David Womack
M.Sc.,B.Sc.,B.Ed. Senior Lecturer in Education, University of the South Pacific
davidwomack@uk2.net

Abstract

Young children's 'alternative' notions of science are well documented but their unorthodox ideas about arithmetic are less well known. For example, studies have shown that young children initially treat numbers as position markers rather than size symbols. Also, children often hold a *transformational* view of operations; that is, they are reluctant to accept the commutativity of addition and multiplication. This 'alternative' view of operations is often overlooked by teachers, keen to demonstrate the so called 'laws' of arithmetic. However, this paper argues that we should not be in any haste to replace these primitive intuitions; instead, we should show that *transformational* operations actually reflect how objects behave when acted on in the physical world. The paper draws on earlier research of the writer in which young children used signs for transformational arithmetic in game scenarios. In particular, it examines the feasibility of 'sums' in which the operator is distinguished from the operand. In short, this paper presents the theory behind an entirely new way of teaching arithmetic, based on children's 'alternative' intuitions about numbers and operations.

The problem which motivated the proposed innovation.

As a teacher of primary and slow-learning children for some years, I was constantly frustrated by the inadequacy and inconsistency of arithmetical notation. For instance, there seemed to be no coherent mathematical model (Womack, 1995) to explain why we have no 'linear' signs to show the exact relation between numbers A and B in such cases as: 'A adds to B' (to distinguish it from 'B adds to A') and 'A multiplies B' (to distinguish it from 'B multiplies A'). The problem also extends to the power operation: how do we write 'A powers B' linearly (rather than B is powered by A), using the same order in which we speak. In everyday language we can reverse 'actor' and 'acted-on' with the simple device of active/ passive mode (e.g. 'A pushes B' can be restated as 'B is pushed by A'.) The problem also extends to questions such as: why can't we write:- A subtracts from B, A divides B etc. (Womack, 1992).

Further inconsistency arises in trying to write *in a linear notation* 'the Ath root of B', 'the log of B to the base A'. This is a particular difficulty for continued roots and logs (cf continued fractions). All these problems can be separated into two issues which I have dealt with under the headings of Part I and Part II.

Part I: Children's belief in the non-commutativity of operations

Background research: In this necessarily short space I can only summarize the findings of Hughes (1986) and others. Hughes concluded that children do not immediately regard the signs of arithmetic as having any connection with the real world but rather see them as inhabiting a self-contained world having no significance, other than a stimulus to *do something* to the numbers. Rather than expressing a symmetrical relation between two numbers, the addition sign tells children that something should be done to one number with the other number. For most children, '3 + 2' means quite simply, 'we had 3 and added 2 more' (Gifford, 1990),

In my own research with 5-year old children, in an 'adding' game scenario, I used an arrow notation to distinguish between the position number and the number 'added' (Womack, 1997, 1998). However, if the notation was to be used more widely, I needed a notation which could be used much more generally. This problem was dealt with as follows.

Description of the proposed innovation ('dot notation).

The problem with the '+' sign (and 'x' sign) is that we cannot express the Active and Passive forms of the operation verb - and so we have no means to indicate which is Operator and which is Operand number [Note (1)]. Therefore, working with teacher trainee students, I sometimes used *a dot* to the right of the + sign to indicate the active mode, whilst a dot to the left indicated the passive mode. Hence the expression '8 .+3' will mean '3 is added to 8', whilst '3 .+8' will mean '8 is added to 3'. Effectively, this notation indicates the *active* or *passive* meaning of the operation (sign) as we see more clearly with the multiplication sign ('x.' means *multiplies*, '.x' means *is multiplied by*). To see how this notation is used, we need to consider the second issue.

Part II: Children's belief in two 'kinds' of inverse operations.

Background research: For those unfamiliar with a primary school perspective, I will summarize two well known investigations, designed to see whether children would use the familiar (to them) 'minus' sign in a practical situation. In the first study, children were encouraged to invent their own signs for 'subtraction' situations. For example, to show that a researcher had added six bricks, one seven-year-old English boy (Scott) didn't use the 'plus' sign which he already knew but instead drew six British soldiers marching from left to right. To show the *taking away* of five bricks he ignored the 'minus' (-) sign and drew five Japanese soldiers marching from *right to left* (Hughes, 1986).

In another series of school-based studies (Atkinson, 1992), young children were required to show how they found the numerical *difference* between pairs of numbers, thrown with two dice or represented on a number line. One girl (Koyser) invented a sort of 'skipping rope' sign linking the two numbers, the longer the string indicating the greater the difference (Gifford, 1990).

We can't read children's minds but perhaps Scott saw the numbers as *symbols for size* - appropriate to the situation, whereas Koyser saw the number symbols on the dice more as *objects which had positions* along the mental number sequence. My own research has also shown that young children can use different signs for the 'take-away' and 'comparison' aspects in game scenarios (Womack, 2000, 2001). Also adults can appreciate and understand different signs for these different 'aspects' of inverse operations (Womack, 1998a and 1998b).

Description of the proposed innovation to show inverse operations.

Therefore, this paper proposes that 'take-away' (to find the operand) and 'compare' (to find the operator) should be expressed more formally in signs. The problem is that again, there is no coherent mathematical model to explain these 'mental operations' to children. There is no attempt to link them to the parallel inverse operations of 'sharing' and 'repeated subtraction' and more importantly, to the mathematical operations of 'finding the root' and 'taking the logarithm'. Since in 'take-away', the operator is subtracted, whilst in 'comparison', the operand is subtracted, we can show this in 'dot' notation with the following example:- 8 .+3 = 11 implies the following:-

11 take-away 3 (the operator) = 8 (the operand) and

11 compare 8 (the operand) = 3 (the operator)

Intuitive subtraction: To show explicitly the difference between *take-away* and *compare,* we can annotate the '+' sign appropriately. For example, '11 – 3' can mean *'11 take-away 3' or '11 compared with 3'.*

'11 take-away 3' can be shown as 11 .-+3

'11 compared with 3' can be shown as 11 ./+3

Therefore, the appended 'hyphen' before (or above) the + sign, indicates the operator is to be subtracted, whilst the 'slash' before (or above) the + sign, indicates the operand is being subtracted.

Theory: The connection children see between numbers and operations.

Numbers as positions: The roots of many of children's beliefs about operations are closely tied to their understanding of drawing and writing. For instance, children consider all words (including number words) to be names, and believe that a name should refer to only one item. We see the application of this to numbers in one study where a group of 2 to 3 year-old children were asked to count five stars placed on a table before them. After they had finished counting, the researcher indicated all five stars by waving her hand over them and said to the children, 'Are these the five stars?' Instead of agreeing with the researcher, it seems the children thought the researcher was referring *the last counted object - the fifth star* (Fuson, 1988; cited in Durkin, 1993, p.152). In short, the meaning of 'five' was thought to be the unique object pointed to in the fifth position. Of course, children eventually realise that a number can (and usually does) represent size, but initially, this seems to be a 'sequence-size' tied to the learnt number sequence - as demonstrated in the study finding below.

Numbers as a sequence/ span: Another influence of writing on children's understanding of numbers is the belief that to represent (e.g.) three pigs requires three words. Children hold a similar view about numbers. That is, they believe that a single number word or symbol such as '3' cannot be sufficient to describe a collection of three objects – they expect that this should require three numbers (Sinclair, et al, 1983). For example, in another classic study, researchers placed five bricks on a tray and asked young children to count them. When the children had finished counting, they were asked, *How many bricks are there on the tray?* Instead of answering 'five', the children immediately began to count-out the numbers to the researcher, '1, 2, 3, 4, 5'. It seems, children thought the answer to the question 'how many', required not a single number but *a recitation of the numbers up to 5* (Fuson and Hall, 1983). According to Fuson, it was as if, for young children, the numeral '5' *meant* the *sequence* '1, 2, 3, 4, 5'. Children believe *the same thing* when they are asked to invent *written* symbols for numbers (cf. Sinclair, et al, 1983). Only after some time do children accept that the same number can represent both position *and* sequence (size) – what I have elsewhere, called *meta-symbolic thinking*.

In summary, children's initial intuitive understanding is that numbers are indicators of *position* (Fuson, 1988) or represent a *sequence* of numbers (Fuson and Hall, 1983).

A belief that numbers can indicate either positions or sequence-size, leads children to regard operations as non-commutative. For example, in my investigation of this using a classroom game, in order to 'reach' the position number 13, children can start from any position but each 'route' will be different. That is, from position 5 they can advance a sequence of 8 positions OR (equally), they can start from position 8 and advance 5 positions. These two routes are not the same so '8+5' (8, advance 5) does not represent the same situation as '5+8' (5, advance 8) [Note (2)]. Intuitively, it seems, children regard the position number as a sort of operand which is increased or acted-on by the number 'added' in order to reach a higher position number. That is, they consider the position number as an *intuitive operand* and the sequence-span number as an *intuitive operator* [Note (3)].

Possibilities for transfer of the model to other situations.

1. Intuitive division: This 'dot' and inverse notation can be extended to multiplication. For example, 4 .x3 will mean '4 multiplied by 3' (3 is the operator), and 4x. 3 will mean '4 multiplies 3' (3 is the operand) [Note (4)]. Since multiplication also has two inverse 'aspects', this inverse notation can again be used [Note (5)]. For example:

8 .x3 = 24 (where 8 is operand and 3 is operator) implies:-

24 shared by 3, which can be written as 24 .-x3 = 8

24 repeatedly subtracted by 8, which can be written as 24 ./x8 = 3

2. The power operation: However, the real pedagogical benefit of the notation is seen in its application to the non-commutative operation of powering. For example, let ^ be the operation sign. Then 3 .^4 = 81 implies:-

81 .-^4 = 3 (81 'rooted' by 4 = 3) and

81 ./^3 = 4 (81 'logged' by 3 = 4) [Note (6)]

3. Terminology: Because these inverse operations are so general I have found it convenient to refer to them as follows:-

The *INVER-operation* is the means to find the Operand.

The *ANTI-operation* is the means to find the Operator.

So we have consistent terminology and notation for:-

inver-addition (take-away), inver-multiplication (sharing), inver-powering (finding the root), etc and

anti-addition (comparison), anti-multiplication (repeated subtraction), anti-powering (finding the log), etc

4. Continued operations: The notation can also be used to express, 'root' and 'log' operations, in 'active' mode, such as, *3 'roots' 2* (the cube root of 2) and *3 'logs' 81* (log of 81 to the base 3) Both these are represented in linear (left to right) fashion.

Therefore, continued fractions can be expressed linearly. Eg 'phi' can be expressed as: 1 +1) –x.1) +1) /x.1) –x.1) +1) /x.1) –x.1) +1) /x.1) ………..

We can also express continued roots or continued logarithms in the same way.

Eg. 81 -^.81) -^.81) -^.81) -^.81) ……. Or 81 /^.81) /^.81) /^.81) /^.81) …….

This leads to interesting speculation as to the nature of the result of such repeated roots or logs. Also expressions such as 7/ (4th root of (6/ (log to base 5 of 30 could be represented linearly as 30 ./^5) -x.6) .-^4) -x.7 .

[Note that 'divides' can be inver or anti multiplication; also '-x.' means 'divides', not subtract, since the typography does not allow the hyphen to be written above the operation sign.]

Discussion:

The paper was based on the assumption that young children:-

-Initially regard numbers as representing either *positions* in the number sequence or as a *sequence-span* of numbers;

- Hold an 'action' (non-commutative) view of addition and multiplication;
- Recognise intuitively, two 'aspects of subtraction (take-away/ comparison) and two aspects of division (sharing/ repeated subtraction);
- Can use signs for these different 'intuitive operations' with the same facility they use the conventional signs of + (plus) and – (minus).

Some advantages of the notation are:-

- Children's intuitions would be built on rather than replaced.
- It provides a consistent interpretation of *all* operations.
- It provides a notational system to deal with higher operations, if and when necessary.
- If the linear notation were to be adopted by manufacturers of calculators, it would allow far more flexibility and efficiency when dealing with the calculations involving 'active' operations, such as continued fractions and similar expressions.

Conclusion:

It seems that if arithmetic is to reflect numerical aspects of the physical world, then perhaps its operations should mimic more closely, the action on objects in that world (Womack, 1995). Now actions on objects are not commutative, because an object (cf. operand) is not the same sort of thing as an action (cf. operator). It is characteristic of children to try to understand the new in terms of what they already understand. It may be therefore that children expect numbers and the operations of arithmetic to behave in the same way as objects in the real world. In short, their transformational view may arise because they see *operations on numbers* as akin to *physical actions on objects in the real world*.

Notes:

(1) Working with these two different concepts of number changes the orthodox meaning of addition and so we could call this 'addation'. However, more generally I have referred to non- commutative addition and non-commutative multiplication as 'operations' – though I have used the neologism 'operactions' in other papers.

(2) Note also that although we can add a sequence number to any position number, (to get a 'higher' position number), we cannot add a position number to a sequence number.

(3) I have introduced the term 'intuitive' since orthodox mathematics defines operator and operand (and operation) in a very specific way.

(4) Here and elsewhere, '4 multiplied by 3' is taken to mean 4+4+4, though an alternative interpretation will not affect the reasoning.

(5) Note that conventional inverse operations can also be written in either active or passive mode. For example, '8 subtract 5' is '8 .-5' and '5 subtracts from 8' is '5 -.8'. Also '12 divided by 4' is '12 .÷4' and '4 divides 12' is '4 ÷.12' .

(6) Another advantage of the notation, though not one with which we are concerned here, is the facility for extending operations indefinitely as repeated applications of lower operations. Note however, that this implies two such systems according to whether we bracket to the left or to the right.

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Solids of Revolution – from the Integration of a given Function to the Modelling of a Problem with the help of CAS and GeoGebra

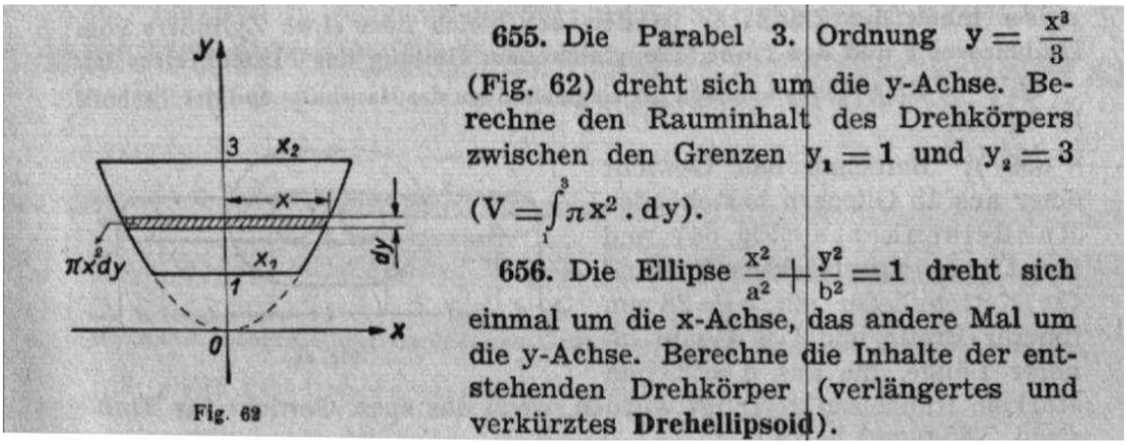
Dr. Otto Wurnig
Institute of Mathematics and Scientific Computing
University of Graz, Austria
adyotto.wurnig@aon.at

Abstract

After the students in high school have learned to integrate a function, the calculation of the volume of a solid of revolution, like a rotated parabola, is taken as a good applied example. The next step is to calculate the volume of an object of reality which is interpreted as a solid of revolution of a given function f(x). The students do all these calculations in the same way and get the same result. Consequently the teachers can easily decide if a result is right or wrong. If the students have learned to work with a graphical or CAS calculator, they can calculate the volume of solids of revolution in reality by modelling a possible fitted function f(x). Every student has to decide which points of the curve that generates the solid of revolution can be taken and which function will suitably fit the curve. In Austrian high schools teachers use GeoGebra as a software which allows you to insert photographs or scanned material in the geometric window as a background picture. In this case the student and the teacher can control if the graph of the calculated function will fit the generating curve in a useful way.

Introduction

After the 2nd world war the syllabus in Mathematics only recommended an introduction of calculus in form 11 and 12. This meant that only the power function had to be integrated. The teachers had to find applied problems. The students had to calculate areas and volumes of solids of revolution. The following problem 665 with Fig. 62 is typical for the next thirty years.(Rosenberg et al., 1974)



655. Die Parabel 3. Ordnung $y = \frac{x^3}{3}$ (Fig. 62) dreht sich um die y-Achse. Berechne den Rauminhalt des Drehkörpers zwischen den Grenzen $y_1 = 1$ und $y_2 = 3$ ($V = \int_1^3 \pi x^2 . dy$).

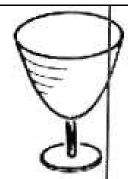
656. Die Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ dreht sich einmal um die x-Achse, das andere Mal um die y-Achse. Berechne die Inhalte der entstehenden Drehkörper (verlängertes und verkürztes **Drehellipsoid**).

$y = \frac{x^3}{3} \to x^2 = \sqrt[3]{9y^2}$ $\quad V = \int_1^3 \left(\pi.\sqrt[3]{9}.y^{\frac{2}{3}} \right).dx$ Result in the book: $\frac{3\pi}{5}\left(9.\sqrt[3]{3} - \sqrt[3]{9}\right)$

This was a typical result for that time as it was done without a calculator. At the end of the seventies the students were allowed to use calculators and the result was 20.546.

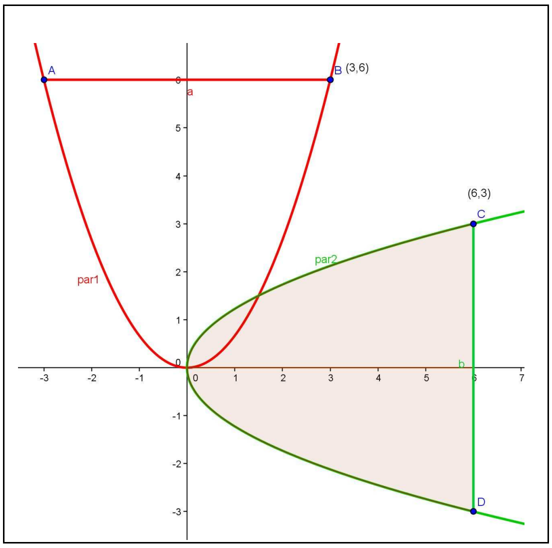
In course of time the students asked for problems which were more realistic. Problem 1162, taken from Szirucsek (1999), is an example for the move in the right direction.

1162 Der Hohlraum eines Weinglases ist im Wesentlichen ein Rotationsparaboloid. Das Glas ist 6 cm tief und hat 6 cm Öffnungsdurchmesser. Wie viel Wein ist enthalten, wenn das Glas gestrichen voll ist bzw. wenn es bis zur halben Höhe gefüllt ist? In welcher Höhe müsste die Markierung für $\frac{1}{16}$ Liter angebracht werden?



If students solve this problem, they have to choose between two formulas they have learned. They can take $x^2 = 2py$ (symmetrical to the y-axe) or $y^2 = 2px$ (symmetrical to the x-axe). If they substitute x=3 and y=6 in $x^2 = 2py$ and x=6 and y=3 in $y^2 = 2px$ they get the equations for the two parabolas par1 and par2:

par1: $x^2 = (3/2)*y \rightarrow y = \frac{2}{3} * x^2$ par2 : $y^2 = (3/2)*x$



Rotation round the y − axe :

$$V = \pi \cdot \int_0^6 x^2 \cdot dy = \pi \cdot \int_0^6 \frac{3}{2} \cdot y \cdot dy$$

$$V = \frac{3\pi}{2} \cdot \frac{y^2}{2} = 27\pi = 84.8$$

Rotation round the x − axe :

$$V = \pi \cdot \int_0^6 y^2 \cdot dx = \pi \cdot \int_0^6 \frac{3}{2} \cdot x \cdot dy$$

$$V = \frac{3\pi}{2} \cdot \frac{x^2}{2} = 27\pi = 84.8$$

To get these results, the students have to choose between two ways. In my tests I often asked the students for two ways of solution. It was a great advantage for the students that they could hope their solution was right if both ways led to the same result. But ex. 1162 is not really an applied problem. The numbers are chosen to allow easy calculating and the position of the coordinate system is fixed with the two formulas they have learned.

New possibilities with the introduction of a CAS, e.g. DERIVE

In Austria a great step in the right direction was taken with the introduction of a CAS in Mathematical Education. In the early 1990s Austria was the first country to buy a general licence for the use of DERIVE in all its grammar schools. One of the advantages was that the students could very quickly create tables and graphic representations of functions. Vice versa the students were enabled to find a function to a given table with the command *FIT(v, A)*. “The elements of the label vector v are the data variables followed by the parameterized expression.” A is the data matrix with the set of points. “When the number of data matrix rows equals the number of parametric variables, *FIT* returns an expression that exactly fits the data, to within roundoff error.”

“When the number of data matrix rows exceeds the number of parametric variables, *FIT* returns an expression that is a least squares fit to the data.” (Help of DERIVE 6)

FIT([x, a.x² + b.x + c], [-1.5, 0; 0.5, -2; 1.5, -1.5]) simplifies to $\frac{x^2}{2}-\frac{x}{2}-\frac{15}{8}$

Later on with DERIVE 6 it was possible to insert a photo, e.g. of a wine glass, in the graphic window with the commands „***Option Display Color Picture***“. After you have suitably positioned the co-ordinate system, you can mark, with the cursor, some points lying on the rim of the wine glass and enter the co-ordinates as a data matrix. Now you have to choose a suitable function f(x). If you interpret the rim of the wine glass as a polynomial function, you can get f(x) with *FIT(x, f(x), A)*. With the command *PLOT*, the graph of the function is quickly plotted in the graphic window and you can control how well the graph of f(x) fits the curve. (Böhm, 2006)

Fig.1: The green curve in the figure is the graph of the polynomial function
$y = 0.002716x^7 - 0.04097x^6 + 0.2281x^5 - 0.5596x^4 + 0.5593x^3 - 0.3814x^2 + 0.8343x + 3.816$

Now you can quickly calculate the volume of the solid of revolution round the x-axe:

$$V = \pi g \int_0^{4.727} (y(x))^2 g dx = 255.159$$

New possibilities with the introduction of GeoGebra (version 2)

The first Research Projects in Austria (1992-2004) organized by the **ACDCA** (**A**ustrian **C**entre for **D**idactics of **C**omputer **A**lgebra) had the goal to reform the teaching of Mathematics with the use of CAS. In the last projects (after January 2005) ACDCA and **GEOGEBRA** have been cooperating. The name of this software is taken from **GEO**METRY and AL**GEBRA,** because the great advantage of GeoGebra is that an expression in the algebra window corresponds to an object in the geometry window and vice versa. You can simply insert photographs or scanned material in GeoGebra in the geometry window as a **background picture** (Hohenwarter, 2008). Then you can take the coordinates of important points of a curve.

With the command ***Conic through five points*** you can find the curve of the conic and the equation of the conic through the chosen five points simultaneously.

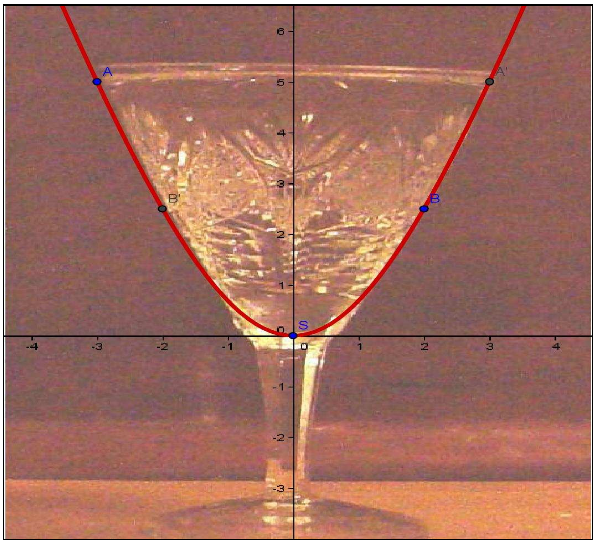


Fig.2: My own photo of a wine glass
hyp : $1875x^2 - 150y^2 - 2625y = 0$

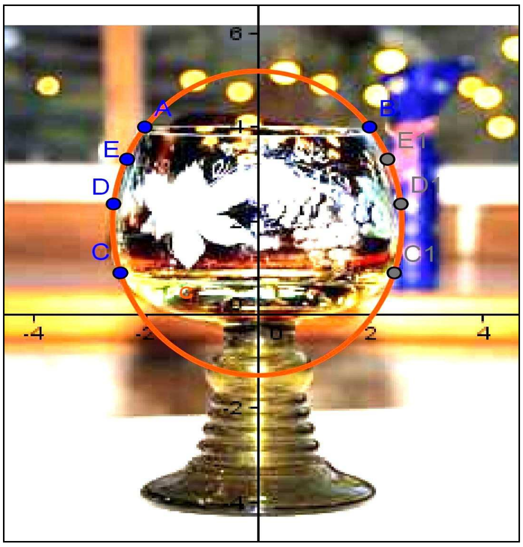


Fig.3: Photo of a wine glass (J. Böhm)
ell : $453.25x^2+287.07y^2-1119.7y = 1927.36$

If you have the goal to find a function to a curve, you have two possibilities:

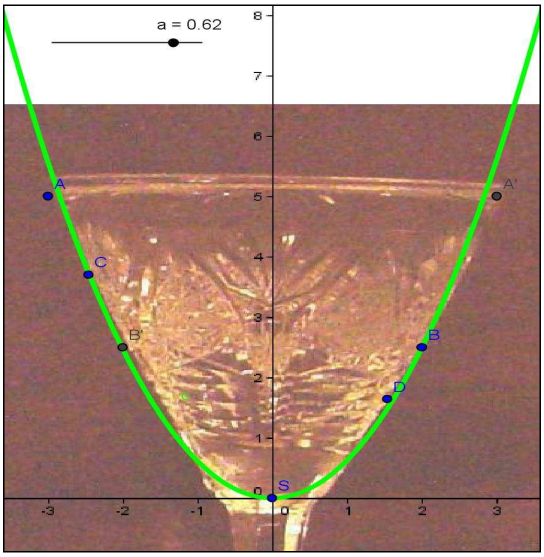


Fig.4: parabola y=a.x² with a=0.62

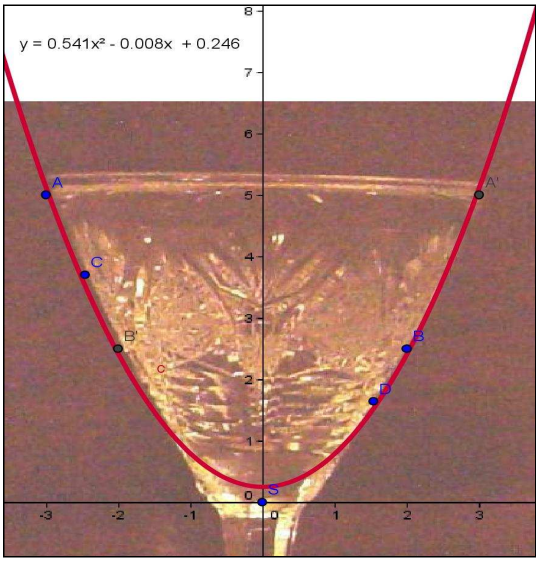


Fig.5: parabola with quadratic regression

In **Fig.2** the curve seems to be a parabola **$y = a.x^2$**, a **parameterized expression** of a function. You take the parameter **a** as slider and move it until the parabola will sufficiently fit the curve. The result for $a = 0.62$ is shown in Fig.4.
In **Fig.5** some points on the rim are marked. If you want to find a function through these points, you have to take a CAS. With TI-Nspire CAS you calculate with the command ***quadratic regression*** the parabola: **$y = 0.541x^2 + 0.008x + 0.246$**

The volume of the solid of revolution round the y-axe with GeoGebra:

In **Fig.4** it is easy to calculate: $V = \pi \cdot \int_{0}^{y(A)} x^2 \cdot dy = \pi \cdot \int_{0}^{5} \frac{y}{0.62} . dy = 63.34$

In **Fig.5** you need a CAS. First you have to solve **y = 0.541x² + 0.008x + 0.246** to x = f(y), then to compute $x^2 = (f(y))^2$ and to integrate. I did it with DERIVE 6 and TI-Nspire CAS and got the following result:

$$V = \pi \cdot \int_{0.246}^{y(A)} x^2 \cdot dy = \pi \cdot \int_{0.246}^{5} \left(\frac{2.(\sqrt{5}.\sqrt{54100y - 13307} - 2\sqrt{2})^2}{292681}\right).dy = 65.186$$

GeoGebra can integrate, too. "*Integral[Function, Number a, Number b]:* Returns the definite integral of function in the interval *[a , b]*. Note: This command also draws the area between the function graph of *f* and the *x*-axis." (Geogebra Help 3.2)

In **Fig.2** Geogebra calculated the hyperbola: $1875x^2 - 150y^2 - 2625y = 0$. If you transform to x^2, you get $g(y) = \frac{\pi \cdot (150 \cdot y^2 + 2625y)}{1875}$. In Geogebra you have to enter x instead of y!

$$g(x) = \frac{\pi \cdot (150 \cdot x^2 + 2625x)}{1875} \rightarrow V = Integral[g(x), 0, 5] = 65.45$$

New possibilities with the introduction of GeoGebra Pre-Release

If you look into the homepage **www.geogebra.at** , you will read that "GeoGebra is a free and multi-platform dynamic mathematics software for learning and teaching. It has received several educational software awards in Europe and the USA". If you use the order "*Start GeoGebra*" you load the last official version of GeoGebra. If you click "*Future*", you will get to **GeoGebra Pre-Release** and you can try out the currently developed version of GeoGebra. If you click "*help*" and "*about*" you will find the date of the latest version, often the date of the actual day.
I will now treat the wine glass of Fig.1 with **GeoGebra Pre-Release** from April 7, 2009.

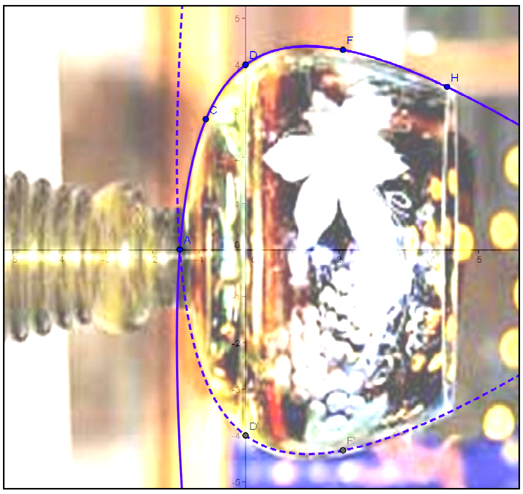


Fig.6: *Conic[H,F,D,C,A]* → hyperbola

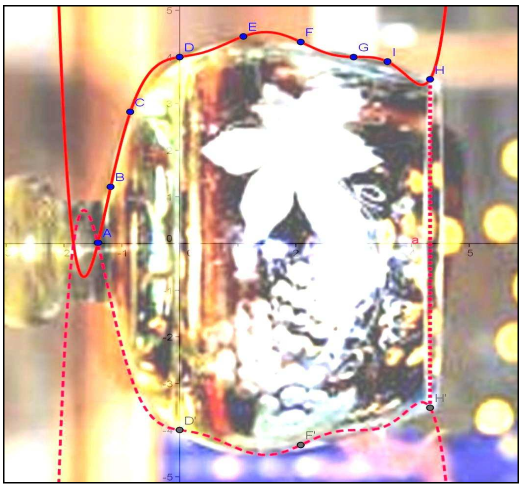


Fig.7: *Polynomial[A,B,C,D,E,F,G,H,I]*

The volume of the solid of revolution round the x-axe with GeoGebra:

In **Fig. 7** the new command "*Polynomial[A, B, C]* creates an interpolation polynomial of degree n-1 through n given points" (GeoGebra 3.0 release notes), with nine points:

$$y = 0.004x^8 - 0.042x^7 + 0.153x^6 - 0.063x^5 - 0.656x^4 + 0.996x^3 - 0.191x^2 + 0.195x + 4$$

V = π*Integral[f(x)², 0.5, 4.321]= **254.042**

In **Fig.6** the command *Conic through five points* computes the hyperbola:
$109.6x^2 + 148.79x.y + 21.4y^2 - 945.66x + 303.85y = 1557.83$
with TI-Nspire CAS: $y = 2.63892g(\sqrt{x^2 + 13.4336x + 17.6907} - 1.31736g(x + 2.04214))$.
with GeoGebra: V = π*Integral[f(x)², 0.5, 4.321] = **255.498**

In **Fig.1** the volume has been worked out with DERIVE: V = **255.159**

Summary of my experiences

Many students in grammar school often ask in Mathematics what the use of learning functions and conics really is and where they exist in reality. Whenever the students take their own photos with their digital cameras of objects which can be interpreted as solids of revolution and then insert them in the geometric window as background pictures, they learn to find conics and graphs of functions. Afterwards they can calculate the volume with integration and are content to have solved an applied problem. The solutions of the students are often very surprising for the teachers and sometimes the numerical results are a little bit different.

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A way of computer use in mathematics teaching -The effectiveness that visualization brings-

Shuichi Yamamoto and Naonori Ishii
College of Science and Technology, Nihon University, Funabashi, Chiba, 274-8501, Japan
syama@penta.ge.cst.nihon-u.ac.jp

Abstract

We report a class of the mathematics in which an animation technology (calculating and plotting capabilities) of the software *Mathematica* is utilized. This class is taught for university students in a computer laboratory during a second semester.

It is our purpose to make a student realize the usefulness and the importance of mathematics easily through visualization. In addition, we hope that students will acquire a new power of mathematics needed in the 21st century. For several years, we have continued this kind of class, and have continued to investigate the effectiveness that our teaching method (especially visualization) brings in the understanding of the mathematics.

In this paper, we present some of this teaching method, which is performed in our class. From the questionnaire survey, it is found that our teaching method not only convinces students that the mathematics is useful or important but also deepens the mathematic understanding of students more.

Introduction

Mathematics plays evidently an important role in the 21st century. However, it is very difficult for students to realize the significance of learning mathematics in a traditional class. We think that computer use gives students the motivation for learning mathematics. In particular, the remarkable evolution of animation technology enables students to visualize a lot of formulas and mathematical concepts. Through this visualization, we can lead students to realize the significance of learning mathematics. In addition, it is expected that students will acquire new mathematical abilities needed in the 21st century.

We report a class taught for university students in a computer laboratory during a second semester, in which animation technology (calculating and plotting capabilities) of the software *Mathematica* is used.

The characteristic of our class is as follows:

(1) We first teach the contents of the program that students will execute on *Mathematica*. Our class is designed so that students can learn the program step by step. In addition to watching a picture on the monitor of a computer, the learning of this program helps students understand the mathematical concept.

(2) We present the mathematical properties which we cannot explain on a blackboard, but which we are able to explain more easily if using a computer. This makes students realize the utility of mathematics better.

(3) We present the problems of mathematics which students can solve on the monitor of a computer by using mathematical concepts without a pencil and a paper. It is important for students to solve these kinds of problems in order to promote their conceptual understanding of mathematics.

In this paper, we introduce a teaching method of integration and give examples of the above (2) and (3). We also report the student evaluation for our teaching.

A teaching method of integration

In our university, most students can calculate the integral of a function such as x^2, but they do not know what the integration represents. Our aim is to teach what the following formulas mean.

$$\int x^2 dx = \frac{x^3}{3} + C, \quad \int_0^1 x^2 dx = \frac{1}{3}.$$

For the purpose, the students are provided the following two examples.

Example 1: Compute the Riemann sum which gets close to the definite integral $\int_0^1 x^2 dx$.

Example 2: Observe how a primitive function $x^3/3$ is obtained from a function x^2 on the closed interval $[0,1]$.

In Example 1, let $f(x) = x^2$ be a function defined on the closed interval $[0,1]$ and for the sake of simplicity, we consider the Riemann sum

$$S(n) = \sum_{i=0}^{n-1} f\left(\frac{i}{n}\right) \times \frac{1}{n}.$$

According to the program, the students define the Riemann sum $S[n]$ on *Mathematica*. First, let students evaluate $S[10], S[20], S[50], S[100]$ using calculating capability of *Mathematica*. Next, for a larger n, let students evaluate $S[n]$. As n gets larger and larger, the students observe that the value of $S[n]$ gets closer and closer to the definite integral of x^2 over $[0,1]$

$$\int_0^1 x^2 dx = \frac{1}{3}.$$

In order to make students understand the concept of the definite integral, it seems to be important for students to compute the Riemann sum directly.

In Example 2, let $f(x)$ be the same function as in Example 1. We consider the accumulation function defined by $f(x)$ to be a primitive function of $f(x)$. By definition, the accumulation function $F(x)$ is given by

$$F(x) = \int_0^x t^2 dt.$$

In order to observe how $F(x)$ is derived from $f(x)$, we put $x_k = k/n$ and approximate the value $F(x_k)$, the Riemann sum

$$y_k = \sum_{i=0}^{k-1} f\left(\frac{i}{n}\right) \times \frac{1}{n}.$$

Using *Mathematica*, we produce an animation of line graphs, which are created by connecting the points (x_k, y_k) ($k = 0,1,\ldots,n-1$) in numerical order, and run. Then students find a curve which the line graph gets close to as n $\to \infty$, and visualize this curve as the graph of the primitive function of $f(x)$.

At the last of the class which we did on 19th December in 2007, our teaching on the integral calculus was inspected by the questionnaire surveys to 30 students. The questions and results are as follows.

Question 1: Do you think our method which makes use of a computer is a better way than the traditional approach which utilizes a textbook?

The result is Table 1 which shows the indicated scales and the number of students who agreed to the scale.

Table1

Strongly think so.	Think so.	Neutral.	Do not think so.	Do not think so at all.
6	18	6	0	0

Question 2: Why do you think so? Choose a suitable reason (multiple answers are possible).

The result is Table 2 which shows the chosen reasons and the number of students who agreed to the items

Table 2

I can imagine the contents by watching a graph and an animation.	22
It is easy to understand the contents because studying step by step through a program.	9
The typing operations make me concentrate in learning.	3

Trigonometric functions and connections with waves

In the class that trigonometric functions are taught, only many formulas are emphasized and the role of these formulas is not explained so much. It is effective from an educational view point that students learn the connections between trigonometric functions and waves and discover the role of formulas for trigonometric functions. For example, what kind of wave motion phenomena does the function $\sin x \cos t$ give by changing the value of t ? What kind of behavior does the addition formula $\sin(x+t) = \sin x \cos t + \cos x \sin t$ give by changing t ? By helping students visualize the changes, which can hardly explain with a blackboard and chalk, they realize the significance of using mathematics.

The students learn the following.

Example 3: Observe how the graph of the sum of $\sin x$ and $\cos x$ is drawn.

Example 4: Observe how the graph of $\sin(x+t)$ shifts when t moves.

Example 5: Observe how the graph of $\sin x \cos t$ shifts when t moves.

Example 6: Observe how the graph of $\sin x \cos t + \cos x \sin t$ shifts when t moves.

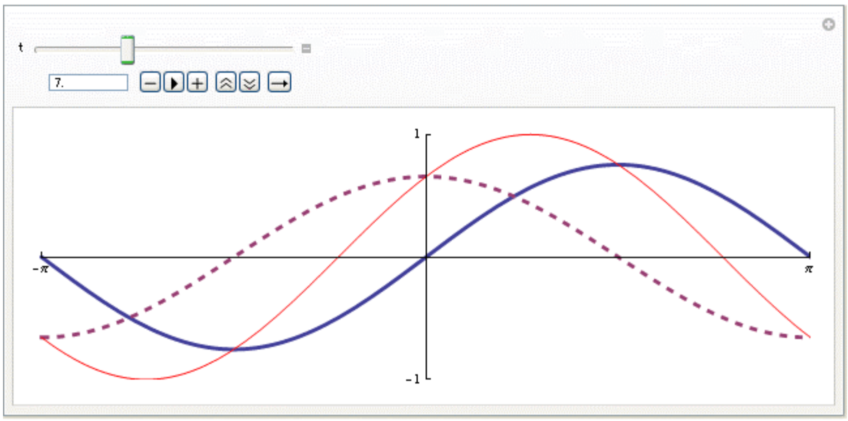


Figure1: An animation of graphs of $\sin x \cos t$, $\cos x \sin t$ and $\sin x \cos t + \cos x \sin t$ when t moves.

The graph in Figure1 shows the case in which $t = 7$. The students can observe the movement by clicking a triangle key.

The animation in Figure 1 shows graphically two waves shifting up and down combined to become the wave shifting to the left. From these observations, students realize that the trigonometric function plays an important role in wave theory. Moreover, students can understand the formula visually in connection with the behavior of waves.

Conceptual understanding of derivatives

The students learn the relationship between the shape of the graph of a function and the sign of its derivative. The graph of the derivative $f'(x)$ represents the instantaneous growth rate of $f(x)$ at a point x . When the value of $f'(x)$ is negative, the graph of $f(x)$ is going down. When the value of $f'(x)$ is positive, the graph of $f(x)$ is going up. Here we consider the following problem which is found in [3].

Problem 1: Figure 2 shows the graphs of two functions. One is the graph of the derivative of the other. Determine which is which and explain your reasoning.

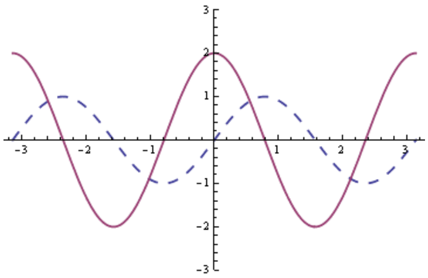


Figure 2: Graphs of a function and its derivative.

The solid line represents $f(x)$ and the dotted line represents $f'(x)$, but no equations are given, so students have to rely on their graphical knowledge of functions and their derivatives.

We requested the students who took the traditional class to solve this problem in 2007. Most students at the time were not going to use the relationship between the shape of the graph of a function and the sign of its derivative for solving the problem. The answers of students who chose these functions correctly were based on their knowledge that the derivative of $\sin x$ is $\cos x$.

We believe it is important in the new century to raise student's ability to solve problems through the graphical knowledge of functions and their derivatives. We attempt to raise this ability by using animation technology. In our class, the following examples and problem are given for students.

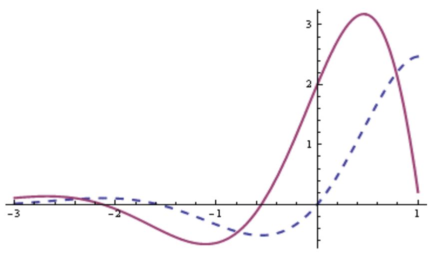
Example 7: Let $f(x) = x(x-t)(x-2)$. When the vale of t moves from -1 to 2 , observe the relationship between the shape of the graphs of $f(x)$ and the sign of $f'(x)$ by watching the graphs of $f(x)$ and $f'(x)$.

Example 8: Let $g(x) = \sin tx$. When the value of t moves from 0 to 3, observe the relationship between the shape of the graphs of $g(x)$ and the sign of $g'(x)$ by watching the graphs of $g(x)$ and $g'(x)$.

The students who studied the above examples come to solve the following problem by the graphical knowledge of functions and their derivatives.

Problem 2: Figure 3 shows the graphs of two functions. One is the graph of the derivative of the other. Determine which is which and explain your reasoning.

Figure 3: Graphs of a function and its derivative.



The performance of the students and summary

We conducted a student survey at the end of the semester. The investigation items are as follows.

1: The feeling that mathematics is more useful increased.

2: I brought myself to study mathematics further.

3: I realized the importance of mathematics more strongly.

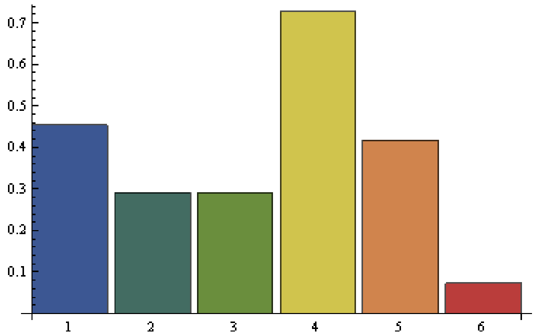
4: Being able to visualize mathematical concepts and properties, I have understood mathematics deeper.

5: Getting free from memorizing formulas, I could enjoy learning mathematics.

6: Others

The following Table 3 shows the performance of the 55 students who took our class in 2008.

Table 3: The performance of the students



In Table 3, the numbers of the horizontal axis correspond to those of the investigation items above and the vertical axis shows the ratio of the student who agreed to the item. Students could choose the multiple items.

From Table 3, we found that about 74% students felt that the mathematics was useful or important and about 73% students understood mathematics more deeply by the assistance of visualization. Most students believe that to learn mathematics is the same as memorizing ways of solving given problems. We want to pay attention to the fact that about 40% students could learn mathematics enjoyably. Our teaching method appears to produce improvement in the attitudes of students learning mathematics.

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Using physical experiments in mathematics lessons to introduce mathematical concepts

Simon Zell
University of Education Schwaebisch Gmuend simon.zell@ph-gmuend.de

Abstract
Physical experiments have a great potential in mathematics lessons. Students can actively discover how mathematical concepts are used. This paper shows results of research done how students got to know the different aspects of the concept of variable by doing simple physical experiments. Further it will be shown what other concepts could be touched by the same treatment.

Background

The project ScienceMath is an interdisciplinary European co-operation project for the promotion of mathematical and scientific literacy. The basic idea is to encourage mathematic learning in scientific contexts and activities of the pupils. The pupils shall experience Mathematics in an appropriate interesting and important way by the means of extra-mathematical references. With the aid of scientific contexts and methods the gap between formal mathematics and authentic experience shall be closed and on the other hand the variety of mathematic items shall be experienced. While doing experiments different mathematical concepts and methods are used, especially the concept of function and the concept of variable.

Experiments to introduce the concept of variable

Malle (1986) differentiates three aspects of variable. If a variable stands for an unknown item or unknown object it is considered as variable as an object. The placeholder aspect means variables can be substituted through a number. If variables stand for meaningless symbols, with which certain rules can be applied, he speaks of the calculation aspect. He further differentiates variable as an object into single number aspect and interval aspect. Single number aspect means an arbitrary but fixed number within a given domain. Variables which match to the interval aspect represent the whole domain. Within that interval aspect it can be differentiated between simultaneous aspect (representation at the same time) or changing aspect (representation in succession). Comparing the decomposition of the concept of variable according to Trigueros et. al (1996) into generalized number (representing a general entity, which can assume any value), specific unknown (representing a constant value, which might change in another situation) and variable in a functional relationship, generalized number can be attributed to variable as an object, which can be represented at the same time or in succession. Specific unknown is equivalent to the static component. To conceptualize variables in a functional relationship, knowledge of dynamic and static components is needed.

Regarding quantitative experiments a lot of the aspects mentioned above can be touched. Measuring values are unknown objects representing an interval of numbers. They may change causing changes of other related measuring values. The relationship between measuring values is given by specific unknowns. They remain constant in the same situation and might change if settings or the environment change. Since “students’ conceptions of a mathematical concept is determined by the set of specific domains in which that concept has been introduced for the student” (Michelsen, 2006), experiments have a great potential to introduce the concept of variable.

Using physical experiments to introduce mathematical concepts means putting emphasis on the mathematical aspects. That means some physical aspects should play a minor role. For example the way measuring instruments work, why an experiment is set up this way or another are not major concerns of mathematics. Major concern of mathematics is the reliability of the measuring values involved to do mathematics. Even then measurement errors occur, i.e. functional relationships between measuring values are ideal models and reflect reality only if one takes these errors in consideration. To minimize physical aspects which might trouble students, experiments should be easy to handle. Students should be acquainted to measuring devices. If they are not, they should be easy to understand then. Especially if examining inversely proportional relationships the measuring interval should be well chosen since there are intervals, in which slight changes of one measuring value cause a big change of the other. Therefore the variance of the products increases, making it hard to see that these products are constant.

Study design

Which aspects of the concept of variable and how these were touched was examined in three different schools with a total number of 60 students. 28 of them (S1-S18 and S51-S60) were

6^{th} graders of a German Gymnasium and 32 were students of a German Realschule of grade seven. They were invited to come to school an extra afternoon and were doing a physical experiment in pairs and filling out a worksheet and were interviewed afterwards by students of the University of Education and the author. Problem oriented interviews were chosen, since they allow an open atmosphere, wherein the students could speak freely within a given framework (Lamnek, 2005). Interviewers gave guiding open questions only. Even if students came voluntarily, both stronger and less strong students were part of the study. Three different experiments were used: buoyancy, thermal expansion and Boyle's law. In the buoyancy experiment students measure gravitational forces of a mass in air and in water and have to find a proportional relationship between theses forces. Because of thermal expansion the height of a pillar of liquid will increase if the temperature increases. In that case difference of height is proportional to the difference of temperature. Boyle's law describes an inversely proportional relationship between pressure and volume at a given temperature. In the experiment students measure pressure and volume of a given piston. The worksheets asked all the aspects of the concept of variable in a descriptive and a formal way using open tasks if possible. They started with an impulse from everyday life to encourage students and to make connections to the experiment. Then they shall experience the functional relationship qualitatively, i.e. change of one measuring value causes change of other measuring value and how they change. After measuring at least six pairs they were asked to come up with a formula describing the phenomenon. Then questions concerning possible values are asked followed by a question how changes done at the setting or the environment might change the functional relationship. To touch the specific unknown explicitly they were asked to set up a more general formula which would be valid in different situations, too . At the end they should write down what they have learned to a student who hasn't seen such a formula. The design allows students to work by themselves. Instructions were only given, when students were at a loss and if tasks were essential for the following tasks.

Aspects of variable touched by the experiments

Students are able to identify the measurands with their chosen variables

buoyancy experiment

S37 [1:35]: uhm, what was our formula?
S36 [1:38]: L divided by W equals 1.1.
S35 [1:40]: and translated: air (German: Luft) divided by water (German: Wasser) equals 1.1.
[1:42]: (S35, S36, S37 laugh)
S35 [1:46]: correct.

Boyle's law

I1 [7:52]: What are these cm stand for?
S1 [7:55]: Mm, if you look at that scale, there is 6 cm.
I1 [7:57]: Mmhmm.
S1 [7:58]: So for the respective number.
I1 [8:00]: And the x?
S1 [8:02]: Stands for the respective pressure.

Variables make sense to them. Most of the time they used words or the first letter of the measuring values involved. The 7^{th} graders who had had a formal introduction to the concept of variable in class a few weeks before the study often chose x and y as their variables.

All possible values of the measurand depend on the setting. When students were asked which values were represented by their variables, most of them wrote the interval they have measured. A few students extrapolated to all positive numbers. Considering functional relationships students on the one side experienced it statically like the students above, on the other side students touched the functional relationship dynamically. This aspect was very dominant.

Thermal expansion experiment

S60 [10:03]: Well, if the liquid raises, then the heat, that means the temperature has had to increase as well, otherwise the liquid doesn't change. This we have seen, once we added warm water, the pillar of liquid went higher immediately.

Boyle's Law experiment

S56 [2:42]: So, if you turn the piston, and for example the number of cm increases, then the pressures decreases.

Students are also able to explain, how the functional relationship can change due to a different setting. Both qualitative and quantitative explanations are given.

Thermal expansion

S10 [3:21]: that, if the glass tube is thicker, then it raises slower and if the glass tube is thinner, the liquid raises quicker.

Boyle's Law experiment

S42 [4:45]: Yes, it could be possible, that at position 5cm, the pressure now would be 2 or 1, depending how much air is in it.

Buoyancy experiment

I2 [10:52]: What changes at your given formula, if there is sea water?

S5 [11:06]: Well, then uhm it will be air divided by water with salt, too and that is equal to an unknown number.

Thermal expansion experiment

S17 [5:53]: The x is the constant, in our case it was 6 and that x can also change, if the pillar of liquid is smaller or thicker.

Therefore it is a good basis for a sophisticated discussion in class in which less strong students can join the discussion, too.

Possible obstacles

There are two different kinds of obstacles, inner mathematical obstacles and such caused by physics. Considering inner mathematical ones, especially weaker students had big problems finding a formula describing the phenomenon and got demotivated. Since most of them touched the different aspects of variable on a descriptive level, a worksheet asking on that level only, would be more appropriate. The introduction of variables would be afterwards then. Quite a few students from the second group, who got introduced to the concept of variable in advance, thought finding a formula had to be something with x. Students from the two other schools though, who hadn't had an introduction, chose more natural names as their variables. Two main obstacles caused by physics troubled the students. First some students had problems using the physical material. They didn't dare touching it or didn't know what to do with it. This was only some students though; other students were very curious trying out new material. Talking with the science teacher of a class about such possible problems would probably help to minimize or get rid of this obstacle. The second obstacle troubled most of the students, measurement errors. Due to measurement errors students' quotients weren't the same and long decimal numbers, but close to each other. Tough they got an introduction about measurement errors and how to handle those, they still had problems. So they were told to consider measurement errors a second time, when they were trying to find the functional relationship. Once they accepted these errors, then they were convinced about their formula found.

Buoyancy experiment

S23 [9:24]: so, I'm 90 percent sure, since all results are similar.

Boyle's Law experiment

S28 [3:26]: Yes, the other values are pretty close. There you see, sometimes it was 6.5 here 6.51 and there 6.75. That was pretty close, that could be equal as well.

This means that a good introduction about measurement errors is essential. Besides that are students more convinced about their result, if they are measuring by their own.

Buoyancy experiment

I2 [4:43]: Mmhmm, if a mathematics teacher said, that your formula would be wrong, are you sure about it anyhow?

S35 [4:53]: Uhm, yes.

S37 [4:54]: Actually yes.

S36 [4:55]: We have done, we have done it by our own.

S37 [4:58]: Yes.

Further potential these experiments have

In the experiments variables are used in a functional relationship. So the same experiment can also be used to introduce the concept of function and especially proportional and inversely proportional functions in addition to the concept of variable. Then the worksheet could be supplemented by a graph to see the relationship between the different representations of a function. Höfer (2008) has shown that experiments have a great potential to promote functional thinking. Besides the concept of function the concept of equivalence can be taught. By giving open tasks different groups can come up with equivalent formulas, all describing the same phenomenon. After the experiments students' results would be a good basis to discuss this principle. Some students even came up with all three equivalent formulas during the treatment.

Thermal expansion experiment

S17 [5:19]: Uhm, we had to find formulas. That was height times x is difference of temperature and difference of temperature divided by x equals height and difference of temperature divided by height equals x then.

Boyle's law experiment

S38 [3:09]: Well, we always multiplied the cm times the air pressure and the result was always about 14 and then it is logical that for example dividing cm by 14 is equal to the air pressure. So it is possible for example to calculate the air pressure without such a device.

Students didn't think about a certain rule. They just found different formulas describing their phenomenon.

Another competency touched by the experiments is modelling. While working on the experiments and the worksheet students go through the whole modelling cycle. By experiments especially the reflection and validation process can be emphasized because measurement errors always occur so that the constant doesn't reflect the measuring values exactly and due to changes of the formula in other setting one has to consider the validity of the formula found. In the interviews students were asked if their formula is valid, since their quotients respectively products weren't exact. It made them think about their model.

Boyle's law experiment

S2 [7:22]: It (measuring) isn't that exact.

I1 [7:27]: But your formula you have written down, is correct, isn't it?

S2 [7:32]: Well, not that exact either. It's like. No it isn't that exact. […] So it could be also 7.1 instead of seven.

I1 [7:48]: mmhmm. Would you say your formula is valid or false?

S2 [7:57]: I would say, the formula is valid, because with this device you cannot determine a value exactly for sure. So my numbers, I have written down, are actually as exact as possibly can be done with this device.

That doesn't mean that results could be falsified.

Buoyancy experiment

S3 [6:59]: And if we did that again and would get other results, we would be in a fix and wouldn't know what would be right.

So experiments have a great potential to go through the modelling cycle and especially promotes the reflection and validation process which can be overseen easily.

Besides the other concepts which could be covered in addition to the concept of variable this interdisciplinary approach could be deepened. This approach presented here used physical contents and methods in mathematics lessons to study mathematical content. Another approach would be if mathematics teacher and physics teacher would teach the same phenomenon at the same time. For example the physics teacher would teach buoyancy using forces in physics class and at the same time the math teacher would use the buoyancy experiment to introduce the concept of variable in mathematics lesson. It would be also possible to teach in common class for a short time to even better see common and subject-specific aspects. An overview of cooperation forms can be found in Beckmann (2009).

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Hypothesis Aided Approach to the Instruction of the Limit of a Function

Ivan Mezník
Faculty of Business and Management
Brno University of Technology,Czech Republic
meznik@fbm.vutbr.cz

Abstract
The concept of the limit of a function is undoubtedly the key to higher mathematics. With a view to very fine mathematical essence of the notion mathematics educators permanently deliberate what didactic method to take in order to reach relatively satisfactory level of its understanding. The paper presents an approach based on the aid of hypothesis that is put forward by means of calculator support.

Introduction
The concept of the limit of a function is a crucial issue of higher mathematics. Its extremely delicate mathematical essence brings about a lot of difficult didactic questions for the mathematics instructors when commencing calculus. We do not intend to open the problem from the viewpoint of timing of calculus teaching. We respect the diversified reality that calculus is taught both on secondary and tertiary level. On the other hand we consistently take into account that the entire matter concerns instructing of non-mathematicians who need the concept of a limit particularly for the purpose of introducing derivatives to investigate rates of changes (Strang [8]). The portfolio of such students is formed mostly by secondary school students and undergraduate university students of programs where a standard course of calculus is prescribed. Today´s undergraduate students do not have the background and experience in rigorous thinking. They are unaccustomed to proofs and to the strict rules of mathematical logic. Our task is not just teaching these students some advanced mathematics, but also teaching them how to think. Despite a certain categorization of teachers on traditionalists, that seem to prefer the drill first and then developing understanding and reformers, favoring the reverse dogma (see Krantz [4]), mathematics instructors constitute a heterogeneous group. This heterogeneity is even more expressive when discussing our topic. In any case we will join the group of reformers favoring (also) drill, but that drill should be built atop a bedrock of understanding. Notice, that the problem of didactics of limit has not a unique solution even in case of instructing of mathematics specialists, when the full rigorousness is claimed. The discussions on whether to start with Heine definition using sequences and then to continue with Cauchy ε-δ definition or vice versa will probably never end. Although some modernizers tried to use this way of presenting the limit also for non-mathematicians, the resulting positive pedagogical outcome did not turn up. Moreover, it is time consuming and as we know, there is no spare time in teaching hours for mathematics, the subject belonging to the theoretical foundations of the program. Profile disciplines usually need derivatives as early as possible. Our approach (Meznik[5]) may be characterized as possessing the orientation “from Cauchy to Heine” , ie. we will start with the limit of a function while the limit of a sequence becomes a special case of the limit at infinity, yet the support of working with sequences of numbers will be evident due to intuitive reasoning. As a matter of fact, we will formulate the notion of a limit of a function in a verbal form containing the moment of approaching or closeness respectively leading to the ability of putting the hypothesis about the limit or about its nonexistence and consequently to the intuitive understanding of the concept. Besides to put a hypothesis requires the use of calculators which is extremely contributing symbiosis. The ability to carry out the hypothesis proved to be a basic test of understanding the concept. Most of students in the first year at the

university have some conception of a limit and possess certain skill in calculation of limits including L´Hospital´s rule. We found that many of them have difficulties to associate their knowledge with the mechanism of making hypothesis. It signalizes the absence of comprehension. Our experience with the hypothesis method is very positive from the viewpoint of the understanding of the limit. The idea of guessing the limit is not quite original (see for instance Buck[1]). We will insert it in the algorithm of limit calculations, which will be one of the main objectives of our further considerations.

Hypothesis about the limit of a function

In the sequel the following formulation for the limit of a function will be employed:
A function f(x) has the limit a at the point c, if for the points close (from both sides) to c, but different from c, the corresponding values of function f(x) are close to the unique value a.
Then we write $\lim_{x \to c} f(x) = a$. Alternatively we write $f(x) \to a$ as $x \to c$ and say that $f(x)$ approaches a as x approaches *c*.
With a view to the fact that our formulation is not "ε-δ definition" of limits several aspects should be recalled. Partly, approaching of *x* the number *c* is from both sides of *c*, partly approaching of *f(x)* the number *a* as to a unique value. Further, the mechanism of approaching of *x* the number *c* means that the distance of *x* from *c* goes to zero, more precisely that the distance of *x* from *c* eventually goes below an arbitrarily small positive number ε and stays there. All the mentioned seeming details should be at least occasionally stressed. Also it is important to realize that no definition of the limit provides the algorithm of its calculation and that the limit of the function at a point ignores whether or not *f(x)* is defined at this point itself.
We put a hypothesis about a limit of the function *f(x)* at the point *c* in the way that we substitute values of *x* closer and closer (from both sides of *c*) to c into *f(x)* and from the trend of the resulting values of *f(x)* we judge the value of limit or its nonexistence.

Example We put a hypothesis about $\lim_{x \to 1} \frac{x^2-1}{x-1}$. Denote $f(x) = \frac{x^2-1}{x-1}$. Choose numbers close to 1, eg 1.1, 0.9, 1.01, 0.99. Computation gives $f(1.1) = 2.1$, $f(0.9) = 1.9$, $f(1.01) = 2.01$, $f(0.99) = 1.99$. From the resulting values we put number 2 forward as a hypothesis about the limit.

Example We put a hypothesis about $\lim_{x \to 0} \frac{\sin x}{x}$. In a similar way as in Example 1 denoting $f(x) = \frac{\sin x}{x}$ computation gives $f(0.1) = f(-0.1) = 0.998$, $f(0.05) = f(-0.05) = 0.995$, $f(0.001) = f(-0.001) = 0.9998$. Now, we may put number 1 forward as a hypothesis about the limit.

Example We put a hypothesis about $\lim_{x \to o} \frac{1}{x}$. Denoting $f(x) = \frac{1}{x}$ we get $f(0.1) = 10$, $f(-0.1) = -10$, $f(0.01) = 100$, $f(-0.01) = -100$. We obviously come to the conclusion, that approaching 0 from the right the resulting values of the function are increasing positive numbers and approaching 0 from the left the resulting values are decreasing negative numbers. So the resulting values can not approach any unique value. Hence we put a hypothesis that the limit does not exist.

Obviously, putting a hypothesis about the limit requires the use of a calculator. We must be careful of possible breakdowns caused by representation of numbers and numerical aspects of calculations with approximate numbers. We will comment on it in the sequel.

Calculation of limits

We have already pointed out that the formulation of the limit does not provide a straightforward way how to find it. Since we have resigned to use standard mathematical tools, our minimal task is to suggest the way of correct calculations and handling with limits of "well-behaved" functions given as elementary functions, that are mostly emerging in current applications. In other words, the aim is to achieve certain rigor without rigor mortis. For this purpose we propose two algorithms- *Algorithm 1* and *Algorithm 2*, while from practical reasons Algorithm 2 is used when Algorithm 1 is not applicable.

Algorithm 1 This algorithm is applied in case if the function satisfies the condition of continuity. The concept of continuity is usually (but not necessarily) introduced in terms of the limit, i.e. after limits. Since we work with elementary functions we may at this stage state the rule:

The limits of elementary functions at the points of their standard domains are calculated by substituting the point into the function. In symbols, under given conditions

$$\lim_{x\to c} f(x) = f(c).$$

Algorithm 2 This algorithm has a general use but if algorithm 1 may be used, then by its application the calculation of limit is much simpler, as mentioned above. It proceeds in two steps, while Step 1 may be omitted, though it is from several reasons valuable.

Step 1 We put a hypothesis about the limit as described in the previous paragraph. We may skip this step when the calculation of the limit is quite evident (experience, knowledge of formulas, etc.). The hypothesis is very important if it signalizes the nonexistence of the limit. In that case the finding of the limit is unreasonable and the negative result follows. In the affirmative the hypothesis may control the correctness of further calculations performed in the next step. Besides, in the event that our calculation in the next step is not successful, the hypothesis may serve as an orientation in the possible result.

Step 2 We state three basic theorems on limits describing the behavior of limits with respect to arithmetic operations, namely the limit of the sum, the product and the ratio of functions and give the list of important formulas on limits. Usually the following formulas are mentioned:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \lim_{x\to 0}\frac{1-\cos x}{x} = 0, \lim_{x\to 0}\frac{e^x - 1}{x} = 1.$$

We refer to the fact, that the given limits can not be calculated using Algorithm 1 because although they are elementary functions, they are not defined at the limiting point *0.* Also draw the attention to the frequent situation making the use of Algorithm 1 impossible- after substituting the limiting point (in our case *0*) into *f* we come to the undefined expression. In our case we formally get $\frac{0}{0}$; notice of intention is desirable that a gross mistake is often made, namely that the result is 1. It seems to be sensible at this moment to present more examples leading to similar undefined expressions. Further, for a given $\lim_{x\to c} f(x)$ we present a method (or a trick) of replacing a function *f* by another function, say *g*, that agrees with it near a point *c*, but which is elementary and defined at *c*, which makes the calculation easy applying Algorithm 1. It is absolutely necessary to give convincing argument of the correctness of such replacement to find the limit, namely to stress the fact that according to our definition the limit at *c* does not depend on the value $f(c)$ or on its nonexistence, respectively. Typical instances of such functions are rational functions that may be simplified by means of factorization in such a way that Algorithm 1 is applicable. As we know, not a few instructors like this method, which in any case contributes to the flourishing of creativity. It is questionable how much time should be devoted to drill this technique.

Finally, let us make a note that in the event that when using Algorithm 2 step 1 was omitted and step 2 did not yield a correct result, it is necessary to return back to step 1, to put a hypothesis about the limit, advance again to step 2 and try a new attempt to find the limit.

Other types of limits

There is a strong analogy between limits and their mostly treated variants-limits at infinity, infinite limits and one-sided limits. For the above specified purpose seems to be sufficient only to modify the formulation of the limit admitting that real numbers *c* and *a* may be replaced by the symbols $\pm\infty$ explaining intuitively their meaning and similarly for one-sided limits changing approaching from both sides by approaching from either eft or right side only. Symbols $\pm\infty$ evoke at students an impression of mystery and the possibility to get over the borders of finity. In fact this infinite version fills the concept of a limit with full sense. To show that we do not abandon the world of reality some convincing motivation from real processes is desirable. “Economics for all” supplies a grateful suggestion as the following example illustrates.

Example Total cost TC of a firm is given by $TC = TC(Q) = 10 + 4Q,$ where Q is a production. Number 10 represents so called fixed cost and number 4 variable cost per unit of production. Average cost AC is the ratio of total cost and production, $AC = AC(Q) = \frac{10}{Q} + 4.$ It is legitimate to investigate what happens with average cost when the production grows to large amount. In terms of the limit we ask for the limit of AC when Q approaches to infinity. It is quite evident that the larger Q is then $\frac{10}{Q}$ is closer to 0, ie. as Q increases then $\frac{10}{Q}$ approaches 0 and consequently AC approaches 4 (the graph of AC is very helpful). In practical formulation in economics it anticipates the trend of behavior of average cost as production grows, i.e. if production is sufficiently large, then average cost approximately equals variable cost per unit of production.

The advance from infinite type limits to one-sided limits is more convenient for one-sided limits have also infinite variants. Of course, some further calculation formulas and also rules (best in symbolic form) for “calculations” with $\pm\infty$ symbols are to be added. We did not get into difficulties with students comprehension when employing for other types of limits the above mentioned didactic approach for limits. For limits of infinite types the support of geometric tools is strongly recommended. Of an extraordinary importance is the instance of the limit at infinity leading to the irrational number *e* = 2,71828…, the base of the natural logarithm, $e = \lim_{x\to\infty}(1+\frac{1}{x})^x$. It is also a quite exceptional case when putting a hypothesis about the limit. There is no other way then to take this limit for granted.

As a special case of the limit at infinity *the limit of a sequence* may be considered for a sequence is a special type of function defined on the set of natural numbers. With a view to existing theorem, we formally express the sequence as a function and the result of calculation of the limit at infinity we assign to the sequence.

Computing breakdowns

Working with hypothesis we may encounter computing problems that are caused by the representation of numbers in computers. It is a good opportunity to release students objective complications stemming from the use of calculators or more sophisticated computing devices. Many of them currently use mathematical software and they should be aware of possible so called legitimate (irremovable) errors that are theoretically very complicated. The realization of mathematical software has variety of faces and may perform calculations on numbers differently. Particularly those based on the principle of symbolic calculations process numbers with a fixed number of digits after the decimal point. All those mentioned facts may considerably negatively influence the results as the following example illustrates.

Example We intend to put forward a hypothesis about $\lim_{x\to\infty}(1+\frac{1}{x})^x$. Rearranging we get $\lim_{x\to\infty}(\frac{x+1}{x})^x$. Computation (with a standard concrete calculator) gives (for $f(x)=(\frac{x+1}{x})^x$) $f(100)=2.70481...$, $f(1000)=2.71692...$, $f(10^6)=2.71828...$, $f(10^{10})=1$. The obvious reason is that starting from some (sufficiently large) value of x the ratio $\frac{x+1}{x}$ is very close to 1 and due to technical limitations of a calculator is identified as 1 and therefore for 1^x yields 1. We realize that such hypothesis is out of use.

Conclusions

The objective of the paper is to propose an approach to in many ways discussed the problem of instruction of the limit. The following boundary conditions were regarded:(1)The instruction concerns non-mathematicians (2) Limited time is at the disposal for the issue (3) The understanding has the priority over drill calculations (4) The main purpose is to set up the tool to define a derivative. The stress is laid on putting forward a hypothesis about the limit which requires the use of calculators. This proved to be an active element in grasping the concept of the limit. We do not claim a definite way how to proceed to instruction of the limit. Didactic is still and would last forever an open and turbulent discipline.

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