On the error-bound in the nonuniform version of Esseen's inequality in the Lp-metric

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Summary. The aim of this paper is to investigate the known non-uniform version of ESSEEN's Inequality in the L_p - metric, $p \geq 1$, to get a numerical bound for the appearing constant L, see (2.1). For a long time the results given by several authors constate the impossibility of (2.1) in the most interesting case $\delta=1$, because the effect $L=O(\frac{1}{1-\delta})$, $\delta \to 1-0$, was observed, where $2+\delta$, $0<\delta\leq 1$, is the order of the assumed moments of the considered independent random variables X_k, $k=1,2,\ldots,n$.

Again making use of the method of conjugate distributions we improve the well-known technique to show in the most interesting case $\delta=1$ the finiteness of the absolute constant L and to prove

$$\left(\int_{-\infty}^{\infty} ((1+|x|^{3-1/p})|P(X_1+X_2+\ldots+X_n<xB_n)-\Phi(x)|)^p dx \right)^{\frac{1}{p}} \leq L\cdot L_{3,n} ,$$

where $L \leq 127.74 \cdot \sqrt[p]{7.31}$, $p \geq 1$.

In the case $\delta\varepsilon(0,1)$ we only give the analytical structure of L but omit numerical calculations.

Finally an example on normalapproximation of sums of l_2-valued random elements demonstrates the application of the nonuniform mean central limit bounds obtained here.

Zusammenfassung. Das Anliegen dieses Artikels besteht in der Untersuchung einer bekannten Variante der ESSEEN'schen Ungleichung in Form einer ungleichmäßigen Fehlerabschätzung in der L_p-Metrik, $p \geq 1$, mit dem Ziel, eine numerische Abschätzung für die in der Ungleichung (2.1) auftretende absolute Konstante L zu erhalten. Längere Zeit erweckten die Ergebnisse, die von verschiedenen Autoren angegeben wurden, den Eindruck, daß die Ungleichung (2.1) im interessantesten Fall $\delta=1$ nicht möglich wäre, weil auf Grund der geführten Beweisschritte der Einfluß von δ auf L in der Form $L=O(\frac{1}{1-\delta})$, $\delta \to 1-0$, beobachtet wurde, wobei $2+\delta$, $0<\delta\leq 1$, die Ordnung der vorausgesetzten Momente der betrachteten unabhängigen Zufallsgrößen X_k, $k=1,2,\ldots,n$, angibt.

Erneut wird die Methode der konjugierten Verteilungen angewendet und die gut bekannte Beweistechnik verbessert, um im interessantesten Fall $\delta=1$ die Endlichkeit der absoluten Konstanten L nachzuweisen und um zu zeigen, daß

$$\left(\int_{-\infty}^{\infty}((1+|x|^{3-1/p})|P(X_1+X_2+\ldots+X_n<xB_n)-\Phi(x)|)^p dx\right)^{\frac{1}{p}} \leq L\cdot L_{3,n}$$

mit $L \leq 127.74\cdot\sqrt[p]{7.31}$, $p \geq 1$, gilt.
Im Fall $\delta\in(0,1)$ wird nur die analytische Struktur von L herausgearbeitet, jedoch ohne numerische Berechnungen.
Schließlich wird mit einem Beispiel zur Normalapproximation von Summen von l_2-wertigen Zufallselementen die Anwendung der gewichteten Fehlerabschätzung im globalen zentralen Grenzwertsatz demonstriert.

Резюме. Задача настоящей работы состоит в исследовании одной известной версии неравенства Эссеена в виде неравномерной оценки остаточного члена в метрике пространства L_p, $p \geq 1$, с целью получить численную оценку для абсолютной постоянной L выступающей в неравенстве (2.1). До настоящего времени результаты различных авторов производили впечатление, что получить неравенство (2.1) в самом интересном случае $\delta=1$ является невозможным, так как было учтено влияние величины δ на постоянной L в виде $L=O(\frac{1}{1-\delta})$, $\delta\to 1-0$. $\delta=1$ соответствует случаю существования абсолютного момента 3-его порядка рассмотренных независимых случайных величин X_k, $k=1,2,\ldots$ Методом сопряженных распределений и улучением известной техники доказывается конечность постоянной L в случае $\delta=1$ и неравенство

$$\left(\int_{-\infty}^{\infty}((1+|x|^{3-1/p})|P(X_1+X_2+\ldots+X_n<xB_n)-\Phi(x)|)^p dx\right)^{\frac{1}{p}} \leq L\cdot L_{3,n},$$

где $L\leq 127.74\cdot\sqrt[p]{7.31}$, $p\geq 1$. В случае $\delta\in(0,1)$ приведется аналитическая структура постоянной L без численных результатов. С помощью одного примера о аппроксимации нормальным распределением для сумм l_2-значных случайных элементов демонстрируется применение неравномерных оценок в глобальной центральной предельной теореме.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables such that $EX_k=0$ and $E|X_k|^{2+\delta}<\infty$ for some fixed $\delta\varepsilon(0,1]$ and all k. Write $F_n(xB_n)=P(\sum_{k=1}^{n} X_k < xB_n)$, where $B_n^2=\sum_{k=1}^{n} EX_k^2 > 0$, and $\Phi(x)=(2\pi)^{-1/2}\int_{-\infty}^{x} e^{-u^2/2}\,du$, where Φ denotes the N(0,1)-distribution function. Let us put $L_{2+\delta,n}=B_n^{-(2+\delta)}\sum_{k=1}^{n} E|X_k|^{2+\delta}$ the LJAPUNOV - ratio of the order $2+\delta$.

The well-known ESSEEN Inequality, see PETROV (1975) p. 111, gives a result on the rate of convergence to zero of $D_n(x)=F_n(xB_n)-\Phi(x)$ in the following manner

(1.1) $$\sup_x |D_n(x)| \leq C_1 L_{3,n} \quad \text{for all } n\varepsilon N,$$

where $C_1>0$ is an "universal" constant.

In the last 45 years very much efforts has been given to the problem of estimating C_1. Up to now the best upper bound for C_1 is due to ŠIGANOV (1982), who has proved that $C_1 \leq 0.7915$.

Concerning the case $0<\delta<1$ it is remarkable that the constant C_1 in (1.1) depends on δ, i.e. $C_1=C_1(\delta)$, and numerical calculations show that $C_1(\delta)$ increases if δ decreases, see TYSIAK (1983). In this case we have in (1.1) the error-bound $C_1(\delta)L_{2+\delta,n}$, see PETROV (1975) p. 115. In the situation $\delta=0$, i.e. without additional assumptions about the existence of moments of the order $r>2$, we know an elementary estimation due to BHATTACHARYA|RAO (1976)

(1.2) $$\sup_x |D_n(x)| \leq C_o ,$$

where $C_o>0$ is an "universal" constant, and a result by PADITZ (1986)

(1.3) $$\sup_x |D_n(x)| \leq 3.51 \sum_{k=1}^{n} E\left(\frac{X_k^2}{B_n^2}\min\left(1, \frac{|X_k|}{B_n}\right)\right) ,$$

i.e. 3.51 is an upper bound for $C_1(\delta)$.

In MAI|THRUM (1987) the numerical estimation $C_0 \leq 0.5409366$ is given.

In contrast to the long history concerning the determination of the constant C_1 in (1.1) there are only some quantitative results on the corresponding constant in the nonuniform version of the ESSEEN Inequality that has been proved by BIKELIS (1966):

There exist a constant $K_1=K_1(\delta)>0$ such that for all $n\in N$ and all x

$$(1.4) \qquad (1+|x|^{2+\delta})|D_n(x)| \leq K_1 L_{2+\delta,n}\ , \quad 0<\delta\leq 1.$$

Up to now the best upper bound for $K_1(1)$ is due to PADITZ (1987):

$$K_1(1) \leq 31.935\ .$$

In the case of identically distributed random variables the estimate $K_1(1) \leq 30.54$ holds, cp. MICHEL (1981).

Concerning the case $0<\delta<1$ it is remarkable that the constant K_1 decreases if δ decreases, see TYSIAK (1983), i.e. $K_1(1)$ is an upper bound for $K_1(\delta)$ and thus the case $\delta=1$ is the most important case.

The main result in PADITZ (1986) is the so-called global form of the central limit theorem, i.e. an error-bound of type (1.3) for the L_p-norm of $D_n(x)$:

$$(1.5) \qquad \left(\int_{-\infty}^{\infty} |D_n(x)|^p\, dx\right)^{\frac{1}{p}} \leq 3.51\, \sqrt[p]{M} \sum_{k=1}^{n} E\left(\frac{X_k^2}{B_n^2} \min\left(1, \frac{|X_k|}{B_n}\right)\right),$$

where $M = \frac{33.40}{3.51} = 9.5157$ is an absolute constant.

Concerning the value of M see PADITZ|ŠARACHMETOV (1988).

2. The nonuniform version of ESSEEN's Inequality in the L_p-metric

The aim of this paper is to investigate the nonuniform version of ESSEEN's Inequality in the L_p-metric, i.e. to investigate the inequality

$$(2.1) \qquad \left(\int_{-\infty}^{\infty} ((1+|x|^{2+\delta-1/p})|D_n(x)|)^p dx\right)^{\frac{1}{p}} \leq L\cdot L_{2+\delta,n} \text{ for all } n\in N,$$

where $L=L(\delta,p)>0$ is an (unknown) absolute constant, only depending on $\delta\varepsilon(0,1]$ and $p\geq 1$.
The inequality (2.1) was obtained by MAEJIMA (1978) and AHMAD (1979). Note that from the proofs given in MAEJIMA (1978) or AHMAD (1979) the unboundedness of $L(\delta,p)$ follows if δ tends to 1, i.e. $L(\delta,p)=O(\frac{1}{1-\delta})$, $\delta \to 1-0$.

In an earlier paper on error-bounds, see VORONOVA (1972), we could remark the same unwished effect concerning the absolute constant in the error-bound. In an analog situation e.g. for m-dependent random variables, see HEINRICH (1985), also the case $\delta=1$ is excluded because of the above mentioned effect $L=O(\frac{1}{1-\delta})$, $\delta\to 1-0$.

The mean error-bound by SAKOJAN (1975), see also NAGAEV (1979), concerns the case $p=1$ and $\delta=1$ ($\delta\geq 1$) and the so-called "one-sided" version, i.e. we have the domain $(-\infty,\infty)$ of integration in (2.1) to substitute by $(0,\infty)$. However the proof in SAKOJAN (1975) is unclear in some steps so that we can not use this estimation to compute numerical bounds for L in (2.1).
To get numerical bounds for L we have to improve the known technique of proof given e.g. by TYSIAK (1983) or RYCHLIK (1983).
The basic idea in proving nonuniform central limit bounds consists in the partition of the range of x in an appropriate way.
Using the generalized partition given in PADITZ (1987) it is possible to get numerical bounds for L in (2.1) for all $\delta\varepsilon(0,1]$.
In the most important case $\delta=1$ we will prove the numerical bound

$$(2.2) \qquad L \leq 127.74 \sqrt[p]{7.31} \quad , \; p \geq 1 \; .$$

In general case $\delta\varepsilon(0,1]$ we only give the analytical structure of $L=L(\delta,p)$ but omit numerical calculations.

3. The partition of the domain of integration

According to PADITZ (1987) we use the generalized function

$$e_{n,\delta,a,\beta}(x) = 2\beta(\log|x|^{2+\delta} - \log(aL_{2+\delta,n})) \; ,$$

where $a>0$ and $\beta>1$ are certain parameters choosing later. Now all numerical estimations are obtained by the help of the following

partition (with $K>0$):

(3.1) $A_1 = \{x \mid 0 \leq x^2 \leq K^2\}$,

(3.2) $A_2 = \{x \mid K^2 \leq x^2 \leq c_{n,\delta,a,\beta}(x)\}$,

(3.3) $A_3 = \{x \mid c_{n,\delta,a,\beta}(x) \leq x^2 < \infty\}$,

i.e. using (1.5) we only have to estimate in (2.1) the parts

$$I_k = \int_{A_k} |x|^{1+\delta} |D_n(x)| dx \ , \ k=1,2,3 \ ,$$

where here we at first consider the case $p=1$.
We remark that the set A_2 is a so-called domain of moderate x, cp. e.g. the set A in MIRACHMEDOV (1985).
To estimate I_1 we use the uniform error-bound and get

(3.4) $$I_1 \leq 2 \int_0^K x^{1+\delta} C_1(\delta) L_{2+\delta,n} \, dx = \frac{2}{2+\delta} C_1(\delta) K^{2+\delta} L_{2+\delta,n} \ .$$

Next we consider large x, i.e. $x \in A_3$.

<u>Theorem 2.1.</u> Assume the condition

(3.5) $$L_{2+\delta,n} \leq \frac{1}{a} K^{2+\delta} \exp(-\frac{1}{2\beta} K^2) \ ,$$

where $K^2 > (2+\delta)\beta$. Then for all $x \in A_3$

(3.6) $$|D_n(x)| \leq G_n(\frac{1}{2\beta} |x| B_n) + a K^{(2+\delta)(\beta-1)} \exp(\frac{1}{a}(2\beta)^{2+\delta} - \frac{\beta-1}{2\beta} K^2) \frac{L_{2+\delta,n}}{|x|^{(2+\delta)\beta}} \ ,$$

where $G_n(y) = \sum_{k=1}^{n} P(|X_k|>y)$.

<u>Proof.</u> Without loss of generality let $x>0$ be. Obviously

$$|D_n(x)| \leq \max(1-F_n(xB_n), 1-\Phi(x)) \ .$$

Now by the help of the condition $K^2>(2+\delta)\beta$

$$1-\Phi(x) \leq (2\pi)^{-1/2} \frac{1}{x} e^{-x^2/2} \leq \exp(-\log \frac{x^{2+\delta}}{aL_{2+\delta,n}} - \frac{\beta-1}{2\beta} x^2) =$$

$$\frac{aL_{2+\delta,n}}{x^{(2+\delta)\beta}}\, x^{(2+\delta)(\beta-1)}\exp(-\frac{\beta-1}{2\beta}x^2) \leq \frac{aL_{2+\delta,n}}{x^{(2+\delta)\beta}}\, K^{(2+\delta)(\beta-1)} \cdot \exp(-\frac{\beta-1}{2\beta}K^2).$$

With a standard method we estimate $1-F_n(xB_n)$:

$$1-F_n(xB_n) \leq 1-F_n^y(xB_n) + F_n^y(xB_n)-F_n(xB_n) ,$$

where $F_n^y(xB_n) = P(\sum_{k=1}^{n} X_k \mathbb{1}\{|X_k|\leq y\} < xB_n)$ and $y=\frac{1}{2\beta}xB_n$.

Obviously

$$|F_n^y(xB_n) - F_n(xB_n)| \leq G_n(y) .$$

Write $h=\frac{1}{xB_n}c_{n,\delta,a,\beta}(x)$. Thus we get

$$1-F_n^y(xB_n)\leq \exp(-hxB_n+\frac{1}{2}h^2B_n^2 + y^{-(2+\delta)}e^{hy}\sum_{k=1}^{n}E|X_k|^{2+\delta})$$

$$\leq \exp(-c_{n,\delta,a,\beta}(x) + \frac{1}{2}x^{-2}c_{n,\delta,a,\beta}^2(x) + \frac{1}{a}(2\beta)^{2+\delta}).$$

Because of the condition (3.5) and $x\in A_3$ the estimate

$$1-F_n^y(xB_n) \leq \exp(-\frac{1}{2}c_{n,\delta,a,\beta}(x) + \frac{1}{a}(2\beta)^{2+\delta}) =$$

$$a\exp(\frac{1}{a}(2\beta)^{2+\delta}-\frac{\beta-1}{2\beta}K^2)K^{(2+\delta)(\beta-1)}\frac{L_{2+\delta,n}}{x^{(2+\delta)\beta}}$$

follows. ∎

By means of Theorem 2.1 we are able to estimate I_3:

$$(3.7)\quad I_3 \leq \int_{A_3}|x|^{1+\delta}G_n(\frac{1}{2\beta}|x|B_n)dx + 2aK^{(2+\delta)(\beta-1)}\exp(\frac{1}{a}(2\beta)^{2+\delta} - \frac{\beta-1}{2\beta}K^2)\int_K^\infty x^{1+\delta-(2+\delta)\beta}\,dx\, L_{2+\delta,n}$$

$$= \int_{A_3}|x|^{1+\delta}G_n(\frac{1}{2\beta}|x|B_n)\,dx + \frac{2a}{(2+\delta)(\beta-1)}\exp(\frac{1}{a}(2\beta)^{2+\delta} - \frac{\beta-1}{2\beta}K^2)\,L_{2+\delta,n} .$$

4. The domain of moderate x

Suppose again the condition (3.5) of Theorem 2.1. The most difficult problem is the estimation of I_2. Start with the fundamental inequality, cp. MICHEL (1981) or TYSIAK (1983),

$$(4.1)\quad |D_n(x)| \leq G_n(\tfrac{1}{2\beta}|x|B_n) + |\prod_{k=1}^{n} f_k(h) - e^{h^2B_n^2/2}|\, e^{-hxB_n} +$$

$$2\exp(h^2B_n^2/2 - hxB_n)\ \sup_u |P(S_n^* < uB_n) - \Phi(u - hB_n)|,$$

where $f_k(h) = E\exp(h\bar{\xi}_k)$, $\bar{\xi}_k = X_k\, \mathbb{1}\{|X_k|<y\}$, $S_n^* = \sum_{k=1}^{n} \xi_k^*$ and

$P(\xi_k^* < u) = f_k^{-1}(h) \int_{-\infty}^{u} e^{ht}\, dP(\bar{\xi}_k < t)$ (method of conjugate distributions). Here and further on put $h = (1-\gamma)x/B_n$ and $y = \gamma x B_n$, $\gamma = \frac{1}{2\beta}$. According to PADITZ (1987) we get

$$(4.2)\quad |\prod_{k=1}^{n} f_k(h) - e^{h^2B_n^2/2}|\, e^{-hxB_n} \leq \frac{1}{\alpha_1(K)}\, \alpha_2(x)\, x^{-(2+\delta)} L_{2+\delta,n},$$

where $\alpha_1(K) = 1 - \frac{1}{a}\gamma^{-(2+\delta)}\exp((\gamma(1-\gamma) - \frac{1}{2\beta})K^2)$ and $\alpha_2(x) =$

$$\tfrac{1}{4}(1-\gamma)^4 a^{-(2-\delta)/(2+\delta)} x^8 \exp(-\tfrac{x^2}{2\beta}(\tfrac{2-\delta}{2+\delta} + (1-\gamma^2)\beta)) + \gamma^{-(2+\delta)}\exp(-(1-\gamma)^2 \cdot x^2/2).$$

Thus we obtain

$$(4.3)\quad \int_{A_2} |x|^{1+\delta} |\prod_{k=1}^{n} f_k(h) - e^{h^2B_n^2/2}|\, e^{-hxB_n}\, dx \leq$$

$$\frac{2}{\alpha_1(K)}\Big(\frac{(1-\gamma)^4}{4}\, a^{-(2-\delta)/(2+\delta)} \int_0^{\infty} x^7 \exp(\frac{-x^2}{2\beta}(\frac{2-\delta}{2+\delta} + (1-\gamma^2)\beta))dx +$$

$$\gamma^{-(2+\delta)}\, \frac{1}{K} \int_0^{\infty} \exp(-\tfrac{1}{2}(1-\gamma)^2 x^2)dx\ \Big)\, L_{2+\delta,n} \leq$$

$$\frac{\alpha_1^{-1}(K)}{1-\gamma}\Big(\frac{\sqrt{2\pi}}{K}\, \gamma^{-(2+\delta)} + 24(1-\gamma)^5 a^{-(2-\delta)/(2+\delta)}(1-\gamma^2 + \frac{2-\delta}{(2+\delta)\beta})^{-4}\Big) \cdot L_{2+\delta,n}.$$

Next we estimate $\sup_u |P(S_n^* < uB_n) - \Phi(u - hB_n)|$ and get by the help of the condition $K^2 \geq 1.5(2+\delta)\beta$ the inequality, cp. PADITZ (1987),

$$(4.4)\quad \sup_u |P(S_n^* < uB_n) - \Phi(u - hB_n)| \leq \Big(\frac{1}{\sqrt{2\pi}}\alpha_{11}(x) + \frac{1}{\sqrt{8\pi e}}\alpha_7^{-1}(K)(\alpha_4(x) + \alpha_5(x) + \alpha_{12}(x)) + 0.7915\,\alpha_7^{-1.5}(K)\,\alpha_{10}(x)\Big)\, L_{2+\delta,n}\,,$$

where $\alpha_4(x) = \alpha_3^{-2}(K) x^{-\delta} \exp\big(-\frac{2-\delta}{2+\delta}\frac{x^2}{2\beta}\big) a^{-(2-\delta)/(2+\delta)}$.

$$((1-\gamma)x^2 + \gamma^{-(1+\delta)} a^{-\delta/(2+\delta)} \exp\big((\gamma(1-\gamma) - \frac{\delta}{(2+\delta)2\beta})x^2\big)^2\,,$$

$$\alpha_3(K) = 1 - (1-\gamma)\gamma^{-(1+\delta)}\frac{1}{a}K^2 \exp\big(-\frac{1}{2\beta}K^2\big)\,,$$

$$\alpha_5(x) = \gamma^{-(2+\delta)} a^{-2/(2+\delta)} x^{-\delta} \exp\big((\gamma(1-\gamma) - \frac{1}{(2+\delta)\beta})x^2\big) + \frac{1}{2}(1-\gamma)^2 a^{-(2-\delta)/(2+\delta)} x^{4-\delta} \exp\big(-\frac{2-\delta}{(2+\delta)2\beta}x^2\big)\,,$$

$$\alpha_7(K) = 1 - \frac{1}{a}K^{2+\delta}\exp\big(-\frac{1}{2\beta}K^2\big)\,\alpha_6(K)\,,$$

$$\alpha_6(K) = \alpha_4(K) + \alpha_5(K) + \alpha_3^{-1}(K)(\gamma K)^{-\delta}\max(1, \gamma(1-\gamma)K^2)\,,$$

$$\alpha_{10}(x) = \alpha_3^{-1}(K)(\gamma x)^{1-\delta}\exp(\gamma(1-\gamma)x^2) + \alpha_8(x) + \alpha_9(x)\,,$$

$$\alpha_8(x) = 3\alpha_3^{-2}(K) x^{-\delta}\Big((1-\gamma)a^{-(2-\delta)/(2+\delta)} x^3 \exp\big(-\frac{2-\delta}{(2+\delta)2\beta}x^2\big) + (1+x^2\gamma(1-\gamma))\gamma^{-(1+\delta)} a^{-2/(2+\delta)}\, x \exp\big((\gamma(1-\gamma) - \frac{1}{(2+\delta)\beta})x^2\big) + \gamma^{-(1+2\delta)} a^{-1}\, x \exp\big((2\gamma(1-\gamma) - \frac{1}{2\beta})x^2\big)\Big)\,,$$

$$\alpha_9(x) = \alpha_3^{-2.5}(K) a^{-(3-\delta)/(2+\delta)} x^{1-\delta} \exp\big((2.5\gamma(1-\gamma) - \frac{3}{4\beta})x^2\big)\cdot \Big(\gamma^{-\delta/2} a^{-\delta/(4+2\delta)} + \frac{1}{2}\gamma^{\delta/2} a^{\delta/(4+2\delta)} \exp\big((\frac{\delta}{(2+\delta)2\beta} - \gamma(1-\gamma))x^2\big)\Big)\cdot \Big(\gamma^{-(1+\delta)} a^{-\delta/(2+\delta)} + x^2(1-\gamma)\exp\big((\frac{\delta}{(2+\delta)2\beta} - \gamma(1-\gamma))x^2\big)^2\,,$$

$$\alpha_{11}(x) = \alpha_3^{-1}(K)\gamma^{-(1+\delta)} x^{-\delta}\Big(\frac{1}{x}\exp(\gamma(1-\gamma)x^2) + (1-\gamma)^2 x^3 a^{-2/(2+\delta)} \exp\big(-\frac{1}{(2+\delta)\beta}x^2\big)\Big)$$

and finally

$$\alpha_{12}(x) = \alpha_3^{-1}(K)(\gamma x)^{-\delta}\Big(\exp(\gamma(1-\gamma)x^2) + \frac{1-\gamma}{\gamma}x^2 a^{-2/(2+\delta)}\exp\big(\frac{-x^2}{(2+\delta)\beta}\big)\Big).$$

Using the identity $\exp(h^2B_n^2/2-hxB_n)=\exp(-(1-\gamma^2)x^2/2)$ we obtain according to (4.3)

$$(4.5)\quad \int_{A_2} |x|^{1+\delta} 2\exp(h^2B_n^2/2-hxB_n)\,\frac{1}{\sqrt{2\pi}}\,\alpha_{11}(x)\,dx \le$$

$$\frac{2}{1-\gamma}\,\gamma^{-(1+\delta)}\alpha_3^{-1}(K)(1+3(1-\gamma)^3 a^{-2/(2+\delta)}(1-\gamma^2+\frac{2}{(2+\delta)\beta})^{-2.5}),$$

$$(4.6)\quad \int_{A_2} |x|^{1+\delta} 2\exp(h^2B_n^2/2-hxB_n)\,\frac{1}{\sqrt{8\pi e}}\,\alpha_7^{-1}(K)(\alpha_4(x)+\alpha_5(x)+\alpha_{12}(x))dx$$

$$\le \frac{1}{\sqrt{2\pi e}}\,\alpha_7^{-1}(K)(\ 2\alpha_3^{-1}(K)(1-\gamma)^{-2}\gamma^{-\delta}\ +$$

$$16(\tfrac{1}{2}+\alpha_3^{-2}(K))(1-\gamma)^2 a^{-(2-\delta)/(2+\delta)}(1-\gamma^2+\frac{2-\delta}{(2+\delta)\beta})^{-3}\ +$$

$$2\gamma^{-(2+\delta)}a^{-2/(2+\delta)}((1-\gamma)^2+\frac{2}{(2+\delta)\beta})^{-1}\ +$$

$$2\alpha_3^{-2}(K)\,\frac{1}{a}\,\gamma^{-(2+2\delta)}((1-\gamma)(1-3\gamma)+\frac{1}{\beta})^{-1}\ +$$

$$8\alpha_3^{-2}(K)(1-\gamma)\gamma^{-(1+\delta)}a^{-2/(2+\delta)}((1-\gamma)^2+\frac{2}{(2+\delta)\beta})^{-2}\ +$$

$$4\alpha_3^{-1}(K)(1-\gamma)\gamma^{-(1+\delta)}a^{-2/(2+\delta)}(1-\gamma^2+\frac{2}{(2+\delta)\beta})^{-2}\ )$$

and

$$(4.7)\quad \int_{A_2} |x|^{1+\delta} 2\exp(h^2B_n^2/2-hxB_n)\ 0.7915\,\alpha_7^{-1.5}(K)\,\alpha_{10}(x)\,dx \le$$

$$6\cdot 0.7915\sqrt{2\pi}\,\alpha_7^{-1.5}(K)\,\alpha_3^{-2.5}(K)\cdot(\alpha_3^{1.5}(K)\gamma^{1-\delta}(1-\gamma)^{-3}\,\frac{1}{3}\ +$$

$$\alpha_3^{0.5}(K)\gamma^{-(1+2\delta)}\,\frac{1}{a}\,((1-\gamma)(1-3\gamma)+\frac{1}{\beta})^{-1.5}\ +$$

$$\alpha_3^{0.5}(K)\gamma^{-(1+\delta)}a^{-2/(2+\delta)}((1-\gamma)^2+\frac{2}{(2+\delta)\beta})^{-1.5}\ +$$

$$\frac{1}{3}\,\gamma^{-(2+2.5\delta)}a^{-(3+1.5\delta)/(2+\delta)}((1-\gamma)(1-4\gamma)+\frac{3}{2\beta})^{-1.5}\ +$$

$$\frac{1}{6}\,\gamma^{-(2+1.5\delta)}a^{-(3+0.5\delta)/(2+\delta)}(\frac{6+\delta}{(2+\delta)2\beta}+(1-\gamma)(1-2\gamma))^{-1.5}\ +$$

$$3\alpha_3^{0.5}(K)(1-\gamma)a^{-(2-\delta)/(2+\delta)}(1-\gamma^2+\frac{2-\delta}{(2+\delta)\beta})^{-2.5}\ +$$

$$3\alpha_3^{0.5}(K)(1-\gamma)\gamma^{-\delta}a^{-2/(2+\delta)}((1-\gamma)^2+\frac{2}{(2+\delta)\beta})^{-2.5}\ +$$

$$2(1-\gamma)\gamma^{-(1+1.5\delta)}a^{-(3+0.5\delta)/(2+\delta)}((1-\gamma)(1-2\gamma)+\frac{6+\delta}{(2+\delta)2\beta})^{-2.5}+$$

$$\gamma^{-(1+0.5\delta)}(1-\gamma)a^{-(3-0.5\delta)/(2+\delta)}(1-\gamma+\frac{6-\delta}{(2+\delta)2\beta})^{-2.5} +$$

$$5\gamma^{-\delta/2}(1-\gamma)^2 a^{-(3-0.5\delta)/(2+\delta)}(1-\gamma+\frac{6-\delta}{(2+\delta)2\beta})^{-3.5} +$$

$$2.5\gamma^{\delta/2}(1-\gamma)^2 a^{-(3-1.5\delta)/(2+\delta)}((1-\gamma)(1+2\gamma)+\frac{6-3\delta}{(2+\delta)2\beta})^{-3.5}\ ).$$

Concerning the choise of the parameters α, K, β and $\gamma=\frac{1}{2\beta}$ it is clear that we only consider such values of the parameters that fulfill the conditions $\alpha_1(K)>0$, $\alpha_3(K)>0$ and $\alpha_7(K)>0$. Furthermore we need the technical conditions $1<\beta\leq 1.5$, $K^2\geq\max(4\beta,\ 1.5(2+\delta)\beta)$ and

$$\max_{x\in A_2}\alpha(x) = \max_{x\in A_2}\Big(\frac{1}{2}\gamma^{-(2-\delta)}a^{-(2-\delta)/(2+\delta)}\exp(-\frac{2-\delta}{(2+\delta)2\beta}x^2) +$$

$$\gamma^{-4}(1-\gamma)^{-2}x^{-4}a^{-2/(2+\delta)}\exp((\gamma(1-\gamma)-\frac{1}{(2+\delta)\beta})x^2)\Big)$$

$$\leq\frac{1}{6},$$

to get the above mentioned estimate (4.4), cp. PADITZ (1987). Thus we have by the help of (4.1), (4.3), (4.5), (4.6) and (4.7) the estimation

$$(4.8)\quad \int_{A_2}|x|^{1+\delta}|D_n(x)|dx \leq \int_{A_2}|x|^{1+\delta}G_n(\frac{1}{2\beta}|x|B_n) + L\cdot L_{2+\delta,n}\ ,$$

where $L=L(a,K,\gamma,\beta,\delta)$ is the sum of the bounds in (4.3), (4.5), (4.6) and (4.7).

To illustrate the constant L appearing in (4.8) we consider the case $\delta=1$ and choose $a=17.5604$, $\beta=1.12981$, i.e. $\gamma=0.4428$, and $K=\frac{1}{\sqrt{\gamma(1-\gamma)}}=2.01322$. By means of a BASIC-programme we compute

$$(4.9)\qquad L\leq 282.700\ .$$

To compare with the constants appearing in (3.4) and (3.7) we compute in this case ($\delta=1$)

$$(4.10)\quad \frac{2}{2+\delta}C_1(\delta)K^{2+\delta}\leq 4.3056$$

and

$$(4.11)\quad \frac{2a}{(2+\delta)(\beta-1)}\exp(\frac{1}{a}(2\beta)^{2+\delta}-\frac{\beta-1}{2\beta}K^2)\leq 138.493\ .$$

It remains to estimate the term

$$\left(\int_{A_2} + \int_{A_3}\right)|x|^{1+\delta} G_n\left(\frac{1}{2\beta}|x|B_n\right) dx .$$

we get the bound

$$(4.12) \quad \int_{-\infty}^{\infty} |x|^{1+\delta} G_n\left(\frac{1}{2\beta}|x|B_n\right) dx = \frac{2}{2+\delta}\gamma^{-(2+\delta)} L_{2+\delta,n} ,$$

where in our example

$$\frac{2}{2+\delta}\gamma^{-(2+\delta)} \leqq 7.6787$$

holds.

Thus we obtain the following result (case $\delta=1$):

$$(4.13) \quad \int_{-\infty}^{\infty} |x|^{1+\delta} |D_n(x)| dx \leqq 433.178\, L_{2+\delta,n} ,$$

if we assume the condition (3.5).

5. An error bound for large values of $L_{2+\delta,n}$

Now we consider the case

$$(5.1) \quad L_{2+\delta,n} > \frac{1}{a} K^{2+\delta} \exp\left(-\frac{1}{2\beta} K^2\right)$$

and prove

Theorem 5.1. Suppose (5.1), where $a>0$, $K>0$, $\beta>1$ are appropriate parameters and $\delta\varepsilon(0,1]$. Then

$$(5.2) \quad \int_{-\infty}^{\infty} |x|^{1+\delta} |D_n(x)| dx \leqq M \cdot L_{2+\delta,n} ,$$

where $M = M(a,K,\beta,\delta) = \frac{1}{2+\delta}\left((1+\delta)2^{\delta/2}+D_\delta\right)aK^{-(2+\delta)}\exp\left(\frac{1}{2\beta}K^2\right) + (2+\frac{\delta}{2})^{2+\delta})$ and $D_\delta = 2e((2+\frac{\delta}{2})e)^{1+\delta/2}$.

To prove this theorem we need the following lemma:

Lemma (NAGAEV|PINELIS (1977), see NAGAEV (1979, p. 784)):

If $\delta\varepsilon(0,1]$ then

$$(5.3)\quad E\left|B_n^{-1}\sum_{k=1}^{n} X_k\right|^{2+\delta} \leq D_\delta + \left(2+\frac{\delta}{2}\right)^{2+\delta} L_{2+\delta,n} ,$$

where D_δ is the constant given above.

Taking $t=2+\delta$ and $c=2+\frac{\delta}{2}$ in the above mentioned result by NAGAEV|PINELIS we obtain (5.3). Let us continue the proof of Theorem 5.1. According to NAGAEV|ČEBOTAREV (1978), NAGAEV (1979) or ČEBOTAREV (1979) write

$$\int_{-\infty}^{\infty} |x|^{1+\delta}|D_n(x)|dx \leq \frac{1}{2+\delta} E\left|B_n^{-1}\sum_{k=1}^{n} X_k\right|^{2+\delta} + 2\int_0^{\infty} x^{1+\delta}(1-\Phi(x))dx$$

$$\leq \frac{1}{2+\delta} D_\delta + \frac{1}{2+\delta}\left(2+\frac{\delta}{2}\right)^{2+\delta} L_{2+\delta,n} + \frac{1}{2+\delta}(2\pi)^{-1/2}2^{(3+\delta)/2}\Gamma\left(\frac{3+\delta}{2}\right).$$

Taking into account the inequality (5.1) we obtain (5.2). ∎

To illustrate the constant M in (5.2) we consider the case $\delta=1$ and choose the same parameters already considered above: $a=17.5604$, $\beta=1.12918$ and $K=2.01322$. A simple calculation leads to the same bound given above in (4.13):

$$(5.4)\quad M \leq 433.178 .$$

In conclusion we note that the estimations (3.4), (3.7), (4.8) and on the other hand (5.2) give an upper bound of $I_1+I_2+I_3$, i.e. for all $\delta\varepsilon(0,1]$

$$(5.5)\quad \int_{-\infty}^{\infty} |x|^{1+\delta}|D_n(x)|dx \leq \inf_{\substack{a>0,K>0,\beta>1\\ (\gamma=\frac{1}{2\beta})}} \min\Big(M(a,K,\beta,\delta)\ ,\ \frac{2}{2+\delta}C_1(\delta)\cdot$$

$$K^{2+\delta} + \frac{2a}{(2+\delta)(\beta-1)}\exp\left(\frac{1}{a}(2\beta)^{2+\delta} - \frac{\beta-1}{2\beta}K^2\right) + \frac{2}{2+\delta}\gamma^{-(2+\delta)} +$$

$$L(a,K,\gamma,\beta,\delta)\Big)\, L_{2+\delta,n}$$

$$=: L_1(\delta)L_{2+\delta,n} .$$

6. The proof of the inequality (2.1)

We estimate the term on the left hand side in (2.1) in the following manner:

$$\left(\int_{-\infty}^{\infty}((1+|x|^{2+\delta-1/p})|D_n(x)|)^p dx\right)^{\frac{1}{p}} \leq \left(\int_{-\infty}^{\infty}((1+|x|)^{2+\delta-1/p}|D_n(x)|)^p \cdot dx\right)^{\frac{1}{p}}$$

$$\leq \left(\sup_x((1+|x|)^{2+\delta}|D_n(x)|)^{p-1}\right)^{\frac{1}{p}} \cdot \left(\int_{-\infty}^{\infty}(1+|x|)^{1+\delta}|D_n(x)|dx\right)^{\frac{1}{p}}$$

$$\leq 2^{1+\delta-1/p}\left(\sup_x(1+|x|^{2+\delta})|D_n(x)|\right)^{1-\frac{1}{p}}\left(\int_{-\infty}^{\infty}(1+|x|^{1+\delta})|D_n(x)|dx\right)^{\frac{1}{p}}$$

$$\leq 2^{1+\delta-1/p}\,(K_1(\delta))^{1-1/p}\,(33.40 + L_1(\delta))^{1/p}\,L_{2+\delta,n}\,,$$

where we used the inequalities (1.4), (1.5) and (5.5).
Remark that in (1.5) for all $\delta\varepsilon(0,1]$

$$\sum_{k=1}^{n} E\left(\frac{X_k^2}{B_n^2}\min\left(1, \frac{|X_k|}{B_n}\right)\right) \leq L_{2+\delta,n}\,.$$

In the case $\delta=1$ we obtain by the help of (4.13) and (5.4) the numerical estimation ($p\geq 1$)

$$2^{1+\delta-1/p}(K_1(\delta))^{1-1/p}(33.40+L_1(\delta))^{1/p} \leq$$

$$4\cdot 2^{-1/p}\cdot 31.935^{1-1/p}(33.40 + 433.178)^{1/p} \leq 127.74\,\sqrt[p]{7.31}\,.$$

7. An application to normalapproximation of sums of l_2-valued random elements

Let $X_k = (X_{k1}, X_{k2}, \ldots) = (X_{kj})_{j=1,2,\ldots,\infty}$, $k=1,2,\ldots,n$, be a sequence of independent random elements in the HILBERT space l_2, i.e. $E||X_k||^2 = E(\sum_{j=1}^{\infty} X_{kj}^2) < \infty$ for all k .

Assume for simplicity that the X_k , $k=1,2,\ldots,n$, are identically distributed with

(7.1) $EX_1 = 0$, $E||X_1||^2 = 1$ and $E||X_1||^3 < \infty$.

Write $EX_{1j}^2 = \alpha_j^2$, $E|X_{1j}|^3 = \beta_j$ and suppose $\alpha_j \geq \alpha_{j+1}$ for all j .

Now let $Z = (Z_1, Z_2, \ldots) = (Z_j)_{j=1,2,\ldots,\infty}$ be a gaussian random element in the space l_2 with $EZ=0$ and $EZ_j^2 = \alpha_j^2$ for all j.

We introduce the notation

$$S_n = n^{-1/2} \sum_{k=1}^{n} X_k \quad \text{and} \quad S_{nj} = n^{-1/2} \sum_{k=1}^{n} X_{kj} ,$$

i.e. S_{nj} is the j-th coordinate of S_n.

In the theory of summation of l_2-valued random elements the normalapproximation is considered e.g. by NAGAEV|ČEBOTAREV (1978) or by ČEBOTAREV (1979) in the sense of finding of upper bounds for

$$(7.2) \quad d_n = \sup_x |P(||S_n||<x) - P(||Z||<x)| .$$

To obtain useful central limit bounds for d_n let us additionally suppose the independence of the coordinates of X_k and of Z.

In order to formulate a result by NAGAEV|ČEBOTAREV (1978) we put

$$d_n^{(m)} = \sup_x |P(\sum_{j=1}^{m} S_{nj}^2 <x) - P(\sum_{j=1}^{m} Z_j^2 < x)| ,$$

i.e. $d_n = d_n^{(\infty)}$.

<u>Theorem 7.1</u> (NAGAEV|ČEBOTAREV (1978)): Let X_k, $k=1,2,\ldots,n$, be a sequence of independent and identically distributed random elements in l_2, where additionally the coordinates of X_k are independent. Assume the conditions (7.1), then

$$(7.3) \quad d_n \leq C\, n^{-1/2} \left(\prod_{j=1}^{4} \alpha_j\right)^{-3/4} \sum_{j=5}^{\infty} \beta_j + d_n^{(4)} ,$$

where C is an absolute constant.

The main idea of the proof of (7.3) consists in the representation of d_n by the help of the onedimensional variables S_{nj} and in the estimation of a so-called pseudomoment for S_{nj}:

$$(7.4) \quad \int_{-\infty}^{\infty} x^2 |P(S_{nj}<x\alpha_j) - \Phi(x)|\, dx ,$$

where $\Phi(x)$ is the $N(0,1)$-distribution function.
From (5.5) it follows that we have in (7.4) the upper bound

(7.5) $433.178\, n^{-1/2}\, \alpha_j^{-3}\, \beta_j$.

By the help of (7.5) a numerical estimation of C in (7.3) is possible in the following manner:

(7.6) $C \leq 2 \cdot 433.178 = 866.356$.

In order to better imagine which role plays the pseudomoment (7.4) in the proof of (7.3) and how to get the numerical bound (7.6) we sketch the proof of (7.3). Write

$$F_{nj}(x) = P(S_{nj}^2 < x) \quad \text{and} \quad G_j(x) = P(Z_j^2 < x) .$$

Taking into account the independence of the coordinates we obtain

$$d_n \leq d_n^{(m)} + D_n^{(m)} ,$$

where

$$D_n^{(m)} = \sup_x \Big| \mathop{\overline{\ast}}_{j=1}^{m} G_j \ast \Big(\mathop{\overline{\ast}}_{j=m+1}^{\infty} F_{nj} - \mathop{\overline{\ast}}_{j=m+1}^{\infty} G_j \Big)(x) \Big| .$$

Let f_{nj} and g_j be the characteristic functions of F_{nj} and G_j , respectively. Thus by the help of the inversion formulae

(7.7) $$D_n^{(m)} \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{|t|} \Big| \prod_{j=1}^{m} g_j(t) \Big| \cdot \Big| \prod_{j=m+1}^{\infty} f_{nj}(t) - \prod_{j=m+1}^{\infty} g_j(t) \Big| \, dt$$

and furthermore we use the known inequality

$$\Big| \prod_j f_{nj} - \prod_j g_j \Big| \leq \sum_j |f_{nj} - g_j| .$$

According to NAGAEV|ČEBOTAREV (1978) we get

(7.8) $$|f_{nj}(t) - g_j(t)| \leq 4|t|^{3/2} \int_{-\infty}^{\infty} x^2 |P(S_{nj} < x\, \alpha_j) - \Phi(x)| \, dx\, \alpha_j^3 ,$$

i.e. the pseudomoment (7.4) appears in the bound of $|f_{nj} - g_j|$.

Now we decompose the domain of integration in (7.7)

$$D_n^{(m)} \leq \frac{1}{\pi} \int_{|t| \leq T_n} \ldots \, dt + \frac{1}{\pi} \int_{|t| > T_n} \ldots \, dt = I_1 + I_2$$

and use the bounds (7.8) and (7.5) to estimate I_1 :

$$I_1 \leq \frac{4}{\pi} 433.178\, n^{-1/2} \sum_{j=m+1}^{\infty} \beta_j \int_{|t|\leq T_n} |t|^{1/2} |\prod_{j=1}^{m} g_j(t)|\, dt \,.$$

On the other hand for I_2 we get, choosing

$$T_n = \gamma n^{1/3} \left(\sum_{j=m+1}^{\infty} \beta_j \right)^{-2/3} , \quad \gamma > 0,$$

$$I_2 \leq \frac{2}{\pi} T_n^{-3/2} \int_{|t|>T_n} |t|^{1/2} |\prod_{j=1}^{m} g_j(t)|\, dt \,.$$

Therefore

$$D_n^{(m)} \leq \max\left(\frac{4}{\pi} 433.178, \frac{2}{\pi} \gamma^{-3/2}\right) \frac{1}{\sqrt{n}} \sum_{j=m+1}^{\infty} \beta_j \; 2 \int_0^{\infty} |t|^{1/2} \cdot |\prod_{j=1}^{m} g_j(t)|\, dt \,.$$

Now choosing γ as a solution of the equation

$$\frac{4}{\pi} 433.178 = \frac{2}{\pi} \gamma^{-3/2}$$

and using the following representation of $|g_j(t)|$

$$|g_j(t)| = (1+4\, \alpha_j^4\, t^2)^{-1/4}$$

we get in the case m=4 by the help of CAUCHY's inequality

$$\int_0^{\infty} |t|^{1/2} |\prod_{j=1}^{4} g_j(t)|\, dt \leq \frac{\pi}{4} \left(\prod_{j=1}^{4} \alpha_j^{3/4} \right)^{-1} . \blacksquare$$

The proof shows that the main part consists in the estimation of (5.5) to estimate (7.4). An analogous result of Theorem 7.1 is given by ČEBOTAREV (1979) in the case of not necessarily identically distributed random elements in the space l_p, $p \geq 2$, however without numerical computation of the appearing absolute constant. Numerical estimations will be possible by the help of the here demonstrated technique of the proof.

It remains to remark that in the case $p>2$ in (5.5) the parameter δ is equal to $p-1$ and in the bound appears the additional term $L_{3,n}$, cp. ČEBOTAREV (1979), RYCHLIK (1983) or MIRACHMEDOV (1985):

$$\int_{-\infty}^{\infty} |x|^p |D_n(x)|\, dx \leq L_1(p)\,(L_{p+1,n} + L_{3,n}) \;,\; p>2,$$

where $L_1(p)$ is an (unknown) absolute constant.

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