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# Solving Problems in Algebra and Analysis with the CAS-Calculator


In this paper we want to discuss on new possibilities and bounds using the CAS-graphics-calculator ClassPad300PLUS in the educational process of our students.

The CP300 was introduced in 2003 with the OS ver. 1.00. In 2005 an improved hardware (the CP300 PLUS) and the OS 2.20 appear and in 2006 we will get the OS ver. 3.00.

Let us start with a problem on the solution of a linear system of equations with parameters. Essentially there are two reasons why the solution of systems of linear equations and the development of algorithms are a basic problem.

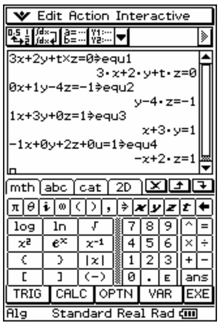
First of all today nearly every linear problem in applied mathematics, e.g. a boundary problem for linear ordinary or partial differenttial equations, is reduced by appropriate techniques to a system of linear equations, especially when electronic computers are to be employed for its solution.

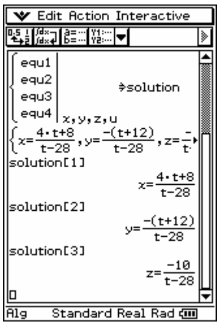
Secondly nonlinear problems can in most cases be solved only by approximating them by linear problems.

Consider the following linear problem with the parameter *t*, cp. [6]:

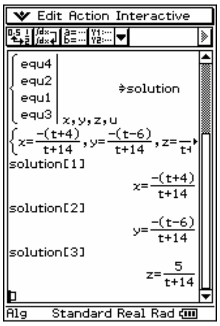
$$\begin{array}{rcl} 3\,x + 2\,y + \ t * z = \ 0 & & \text{(equation 1)} \\ 0\,x + 1\,y \ - \ 4\,z \ = -1 & & \text{(equation 2)} \\ 1\,x + 3\,y \ + \ 0\,z \ = \ 1 & & \text{(equation 3)} \\ -1\,x + 0\,y \ + \ 2\,z \ = \ 1 & & \text{(equation 4)} \end{array}$$

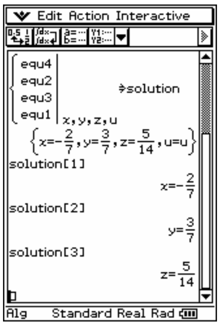
The solution depends on the parameter *t*. For which *t* we get no solutions? For which *t* we get a unique solution or infinite many solutions? We want to solve the problem by the help of the CAS in the main menu of the graphics calculator. At first we store the equations in equ1, equ2, equ3 and equ4 respectively and we use a dummy variable *u*. Thus we have 4 equations and 4 unknowns.



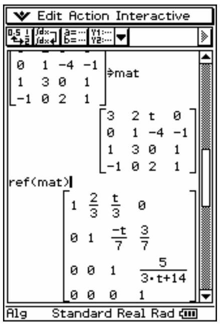

It seems, for $t = 28$ we get no solution. What happens, if we change the order of the equations?

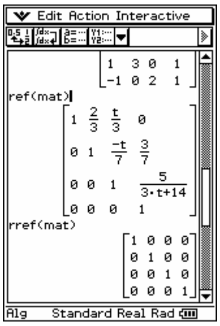


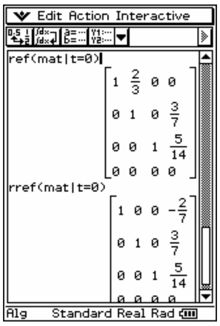

It seems, for $t = -14$ we get no solution. Finally we get the unique solution $x = -2/7$, $y = 3/7$, $z = 5/14$ and the parameter *t* disappears?

Now we try another way of solution by means of the ref- or rref-function (row reduced echelon form) and get the following result:





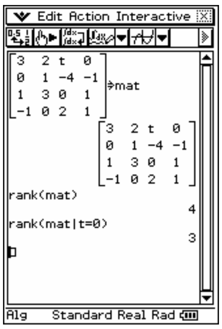

The ref-matrix gives no answer for $t$ = -14/3 and with the rref-matrix we get no solutions for all $t$, because $t$ disappears in the result.

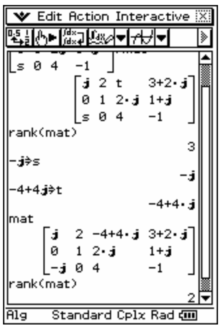



Now let us see what happens, if we choose $t$ = 0. Here we get the right result for $t$ = 0: the unique solution
$x$ = -2/7, $y$ = 3/7, $z$ = 5/14.

However we have the open problem:
What happens in the case $t \neq 0$ ?

It seems, that we can't solve our problem with the CAS-commands ref(mat) or rref(mat) or the at first considered solver for linear systems with equ1, equ2, equ3, equ4. We need a new matrix-command for the CAS-graphics-calculator.
Now have a look on the rank-command of the CAS-calculator.





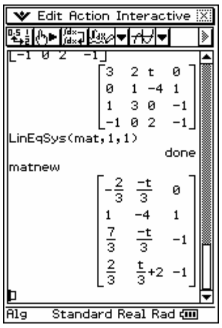

The rank-function can't compute the rank of a matrix in dependence on the choice of a parameter, appearing in the matrix.

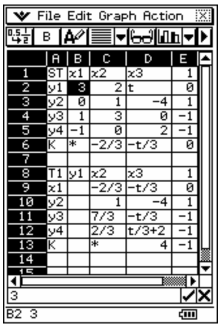
In this paper we introduce a one-step-procedure to transform a matrix (with parameters) in another one. Each of in such a manner created matrix is the matrix of an equivalent system, i.e. the system with the transformed matrix has the same behaviour concerning the solutions. It is clear, that we consider the augmented matrix with the coefficients and the right hand side of the linear system.

We hope, that the new commands will be included in one of the next updates of the OS of the here considered CAS-calculator.

The idea for the new **LinEqSys(mat,i,k)**-function and the new **AVRank(mat,i,k)**-function respectively is the exchange-algorithm by the Swiss Prof. Dr. math. Dr. h. c. Eduard Stiefel of the ETH Zurich (*1909, †1978), cp. [1] – [4] and [7].

Every step of the EX-algorithm (EX=exchange) works with a so-called **pivot** (pivot element) in the row *i* and the column *k* of the considered matrix *mat*. The start-matrix *mat* is the augmented matrix (**A**,-**b**), if we want to solve the linear system **A*****x** = **b** or **A*****x** - **b** = **o**. Here **o** denotes the zero-vector and vector **x** contains the unknown variables. Now we check the new **LinEqSys(mat,i,k)** with CP300:



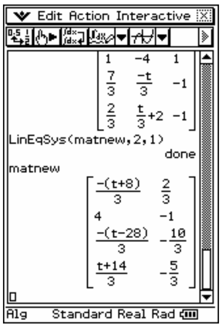

Here we use the test version of the new OS 3.00 to realize the program **LinEqSys(mat,1,1)** in the main-menu of the CP300.

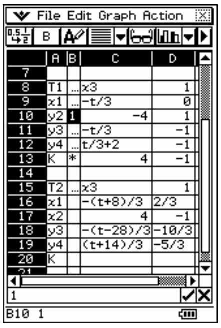
The right screenshot shows the start table ST with the augmented matrix *mat* and the help-row **K** for computation the next table T1.

The pivot is 3 and we exchange x1 against y1. We don't fill the column y1 in T1, i.e. T1 is a shorter equivalent representation of the considered linear system. The EX-rules to get T1 we can summarize:

1. Compute the help-row K: Put * in the pivot column, the other elements of the pivot row are to be divided by the pivot with the opposite sign and put in K.

2. In T1 the pivot is replaced by its reciprocal value.
3. The other elements of the pivot column are to be divided by the pivot. (We omit rule 2. and 3., i.e. no new y1-column.)
4. The other elements of the pivot row are now the elements of the help-row K
5. Elements in the remaining part of the new table or matrix are transformed by means of the old pivot column and help-row K in the following manner (triangle-rule): add to the considered element the product of the element of the help-row K and in the pivot column, where you have to go in vertical and horizontal direction respectively.



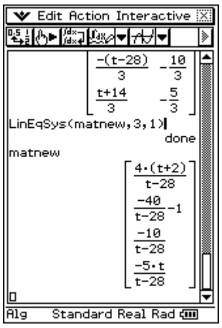



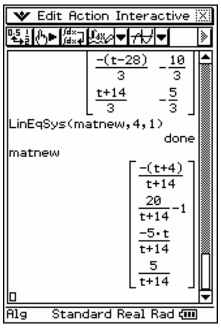

Now we do the next EX-step with the same rules in the matrix *matnew*:

**LinEqSys(matnew, 2,1)**

The next step depends on the parameter $t$: If $t$ = -14 then we exchange x3 with y3 and otherwise we could exchange x3 with y4.

In the case $t$ = -14 we get in the last row of *matnew* the value y4 = $-5t/(t-28)$. In the start table all help-variables y1, …, y4 have to be 0. Thus we get a contradiction in y4 and we can decide: no solution in the case $t$ = -14.




If we exchange x3 with y4 ($t \neq -14$), we get only a solution for $t = 0$: y3 = $-5t/(t+14) = 0$. This one-step-procedure shows that we have a good possibility to discover the behaviour of a linear system with parameters.

This one-step-procedure is helpful for the learning process of our students. It is an interactive work with the CAS-calculator. The decision on the choice of the pivot will be done by the student and the transformation of the matrix or table will be done by means of the tool. Thus we get in a good manner all cases of the parameter *t*, which we have to consider.

**Remark:**

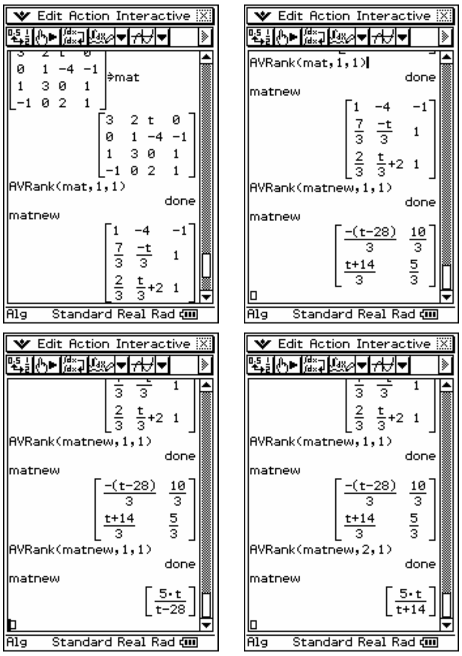
We know that the number of EX-steps is the rank of the considered matrix. Thus we have:

$$\text{rank}(\underline{\mathbf{A}}) = 3 = \text{rank}(\underline{\mathbf{A}},-\underline{\mathbf{b}}) = \text{rank}(mat) \quad \text{for } t = 0$$

and

$$\text{rank}(\underline{\mathbf{A}}) = 3 < \text{rank}(\underline{\mathbf{A}},-\underline{\mathbf{b}}) = \text{rank}(mat) = 4 \quad \text{for } t \neq 0.$$

If only we want to get the rank of a matrix (and not the solution of the linear system, connected with the matrix), then we should use the new created **AVRank(mat,i,k)**-function, which omits step by step the old pivot column and pivot row in *matnew*.




Thus we have a fast procedure to count the number of EX-steps to get the rank of a matrix in dependence on the values of the parameters.

The denotation **AV-**Rank comes from the German word “**A**ustausch**v**erfahren”.

Let us consider the example mentioned above:
In the screenshots you see that we have done two EX-steps.

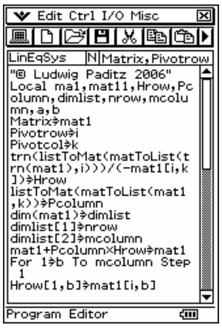
A third step is possible, if we choose a pivot in the first column. Here we need only $t \neq -14$ or $t \neq 28$, cp. the last pictures.

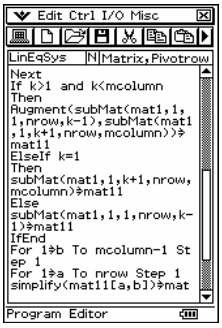
Thus we have rank($\underline{\mathbf{A}}$,$\underline{\mathbf{b}}$) = 3, if and only if $t = 0$.

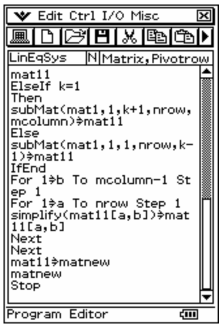
For $t \neq 0$ we can do a $4^{th}$ EX-step to get rank($\underline{\mathbf{A}}$,$\underline{\mathbf{b}}$) = 4.

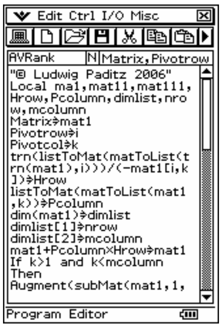
The definition of the rank is the number of linear independent vectors in the considered matrix, i.e. if we can do formally 4 EX-steps (the last EX-step choosing the pivot in the 1-column with $t \neq 0$) then the corresponding linear system has no solutions.

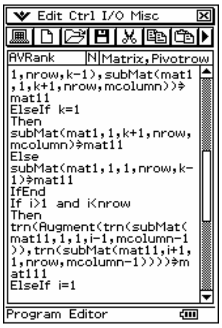
Let us consider the program-text of the new created **LinEqSys**- and **AVRank**-functions:

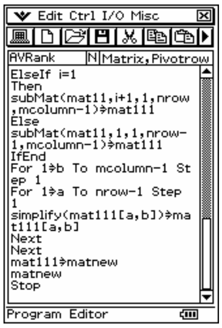


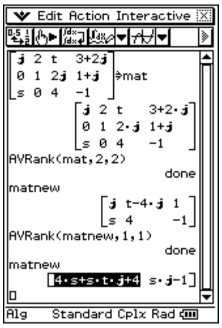



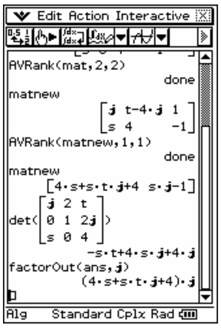




As another example, let us consider the three linear equations with the three unknowns $x$, $y$ and $z$ and the two parameters $s$ and $t$, cp. [5]:

$$\begin{array}{ll} j\ x + 2\,y + \ t * z = \ 3 + 2j & \text{(equation 1)} \\ 0\,x + 1\,y + \ 2j\,z \ = \ 1 + j & \text{(equation 2)} \\ s * x + 0\,y + 4\,z \ = \ -1 & \text{(equation 3)} \end{array}$$

For which parameters ***s*** and ***t*** we have a unique solution or infinite many solutions or no solution? In the case of solution, what is the result in the complex plane? At first we consider the rank of the augmented matrix of the given linear system in dependence on the parameters *s* and *t*. Two EX-steps are possible in every case and the 3rd EX-step only for **$4 * s + s * t * j + 4 \neq 0$ :**




By the way, the pivot for the 3rd EX-step is similar to the determinant of the coefficient matrix of our considered system.

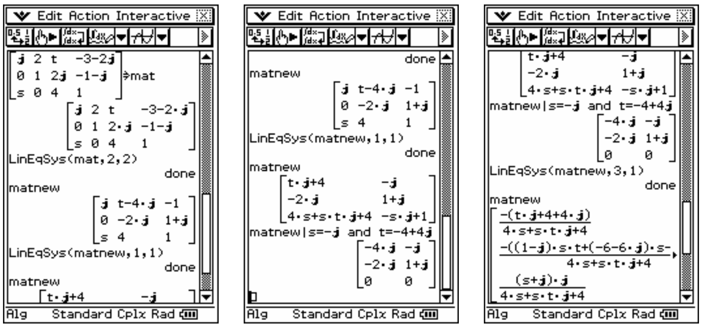
Now we conclude from the row- and column-reduced *matnew*:

rank(**A**) = rank(**A**,**b**) = 2, if and only if **$4 * s + s * t *$ j $+4 = 0$, *s* = -j,** i.e. ***s*=-j** and ***t*=-4+4j** and det(**A**)=0. We have infinite many solutions.

rank(**A**) = 2 < rank(**A**,**b**) = 3, if and only if **$4 * s + s * t *$ j $+ 4 = 0$ ,** ***s* ≠ -j,** i.e. ***s*≠-j** and ***t* ≠ -4+4j** and det(**A**) = 0. We have no solution.

rank(**A**) = rank(**A**,**b**) = 3, if and only if **$4 * s + s * t *$ j $+ 4 \neq 0$**, i.e. det(**A**) ≠ 0 and **_s_ = -j** and ***t* ≠ -4 + 4j** or ***s* ≠ -j** and ***t* = -4 + 4j**.
Here we have a unique solution.

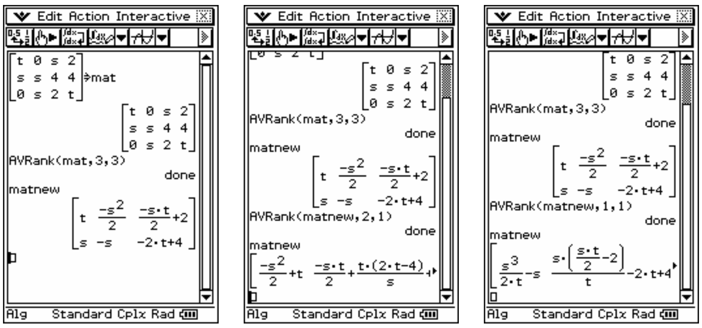
Thus we get in a quick manner by the help of the new **AVRank**-function all information on the considered complex system. Now we want to state the result by the help of the **LinEqSys**-function:




In the case ***s*=-j** and ***t*=-4+4j** we get **$x$ = -4j*c*-j, $y$ = -2j*c*+1+j, $z$ = *c*,** where *c* is an arbitrary complex number.

In the case det(**A**) ≠ 0 we get the unique solution:
$x$ = -($t$*j + 4 + 4*j) / (4*$s$ + $s$ * $t$ *j + 4),
$y$ = ( ((-1+j)* $t$ + 6 + 6*j)* $s$ + 4 + 6*j ) / (4* $s$ + $s$ * $t$ *j + 4),
$z$ = ($s$ + j)*j / (4* $s$ + $s$ * $t$ *j + 4).

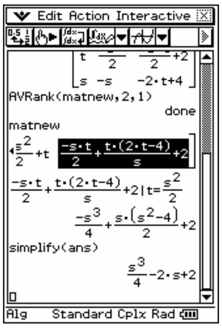
Another application of the new created functions is the following real linear problem with two real parameters. The augmented matrix *mat* you see in the screenshot:

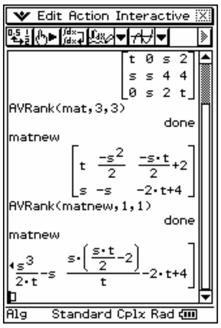



In the case $s = t = 0$ we get rank(**A**) = 1 < rank(**A**,**b**) = 2 (1 EX-step). In the case $s \neq 0$ or t ≠ 0 we can do a 2nd EX-step and thus get with **AVRank(matnew,2,1)** and **AVRank(matnew,1,1)** respectively:
rank(**A**) = 2 for $s \neq 0$ and $t = s$^2/2 and in the other case
rank(**A**) = 2 for $t \neq 0$ and $s = 0$ or $t = s$^2/2.

Finally we can establish that a 3rd EX-step exist in the case $s \neq 0$ and $t \neq s$^2/2 and thus rank(**A**) = 3 = rank(**A**,**b**).

Now we discuss the open question on the rank(**A**,**b**) in the case rank(**A**) = 2. Again have a look in the screenshots after 2nd EX-step:

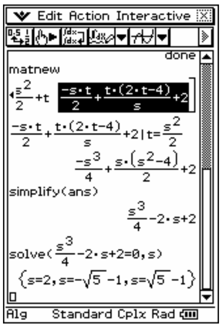


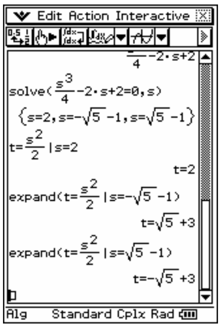




We have to consider the 2nd column in the last *matnew*.

In the case $s \neq 0$, $t$ = $s$^2/2 we consider $s$^3/4 – 2$s$ + 2.

In the case $t \neq 0$ we have to discuss $s$*($st$/2-2)/$t$ – 2$t$ + 4.




Thus in case $s \neq 0$, $t$ = $s$^2/2 we get rank(**A**) = 2 = rank(**A**,**b**) for the following pairs of parameters:
($s$, $t$) = (2, 2) or
($s$, $t$) = (-√5-1, √5+3) or
($s$, $t$) = (√5-1, -√5+3).

Furthermore in case $s \neq 0$, $t$ = $s$^2/2 we get rank(**A**) =2 < rank(**A**,**b**) = 3 for $s \neq 0$, $t$ = $s$^2/2 and $s$ ^3 – 8 $s$ + 8 ≠ 0.

Now in the case $t \neq 0$ (rank(**A**) = 2) we consider $s$*($st$/2-2)/$t$ – 2$t$ + 4. At first with $s$ = 0 we get

rank(**A**,**b**) = 2 for $t$ = 2 and rank(**A**,**b**) = 3 for $t \neq 2$.

Now with $t$ = $s$^2/2 ≠ 0 we get in the 2nd column of matrix *matnew*: $s$*($st$/2–2)/$t$ – 2$t$ + 4 = – ($s$^3 – 8*$s$ + 8) / (2 $s$) and have the same situation considered already above with $s$^3 – 8*$s$ + 8.

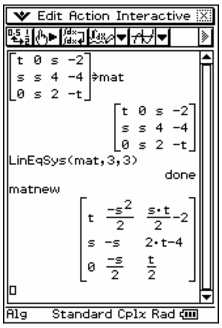
**Summary:**

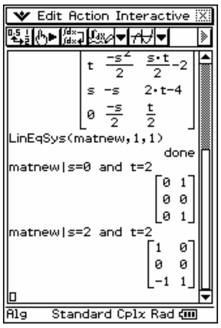
The discussion on the rank shows:

A unique solution: only for $s \neq 0$ and $t \neq s$^2/2

Infinite many solutions: only for ($s$, $t$) = (0, 2) or ($s$, $t$) = (2, 2) or ($s$, $t$) = (-√5-1, √5+3) or ($s$, $t$) = (√5-1, -√5+3).

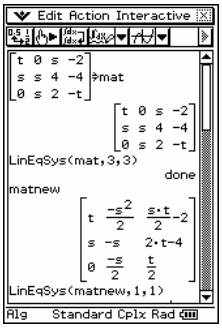
By the help of the **LinEqSys**-function we get the following results:

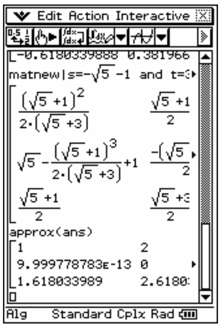


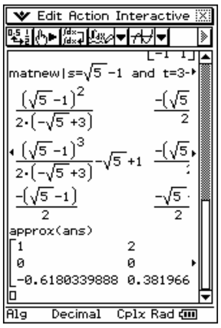




($s$, $t$) = (0, 2):
$x = z = 1$, $y = c$, where $c$ an arbitrary real number.

($s$, $t$) = (2, 2):
$x = y = c$, $z = 1 - c$, where $c$ an arbitrary real number.



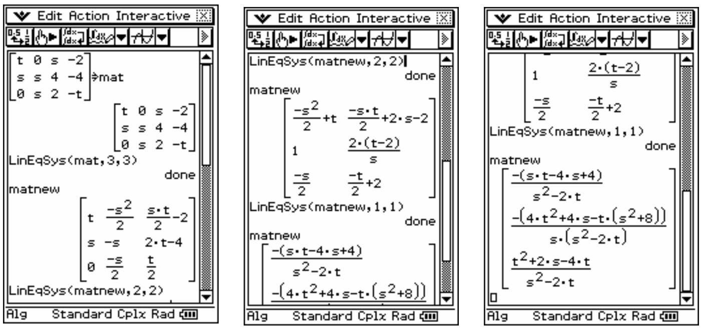




($s$, $t$) = (-√5-1, √5+3):
$x = c + 2$, $y = c$, $z$ = (√5+1)*$c$/2 + (√5+3)/2, $c$ arbitrary real number.

($s$, $t$) = (√5-1, -√5+3):
$x = c + 2$, $y = c$, $z$ = (-√5+1)*$c$/2 + (-√5+3)/2, $c$ arbitrary real number.

Finally the unique solution you see in the last screenshot:




This examples show, how to work with the useful tool, to solve linear systems with parameters or to get the rank of the matrix or augmented matrix. Thus it is good work in algebra with CP300Plus.

In the following considerations we want to show some examples in analysis: Discussion on functions and the graphical representation. Let us start with following function:

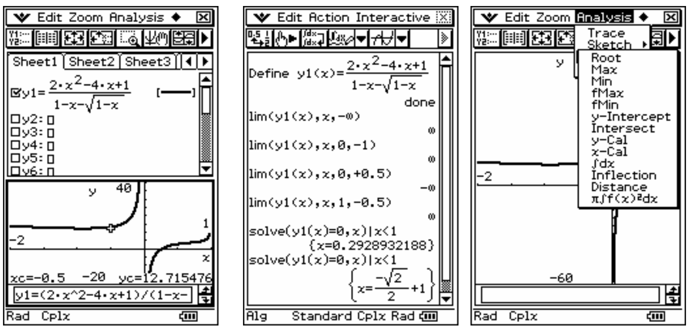
$$y = \mathrm{f}(x) = ( 2x\hat{}2 - 4x + 1) / ( 1\text{-}x - \sqrt{(1\text{-}x)}\ ) .$$

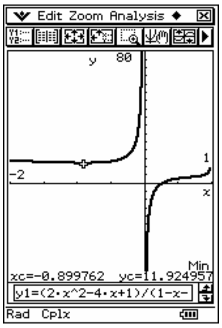
What is the domain of f($x$) and what is the range of f($x$)?

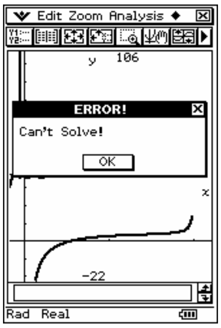
Where are the extreme values and where are the inflection points? What are the roots and the poles? What are the one-sided limits?

By the help of the graphics calculator we have very good possibilities to explore functions with analytical or graphical tools.

Let us begin to answer on all questions: domain $x \in (-\infty, 0) \cup (0, 1)$. Thus we have two parts in the graphical representation of f($x$).

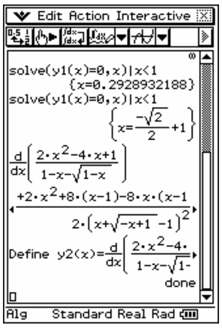


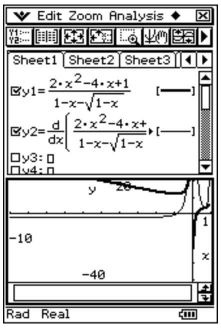


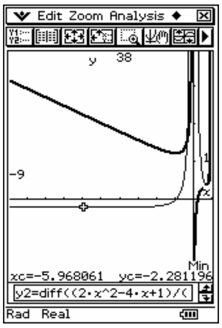

The CP300PLUS computes in the graphics-menu the Minimum but not the inflection point.

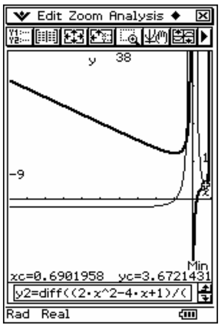
Thus we consider the 1st deviation:




The extreme-values of the 1st deviation give a hint on the inflection points of f($x$)!




Thus we discover two inflection points!

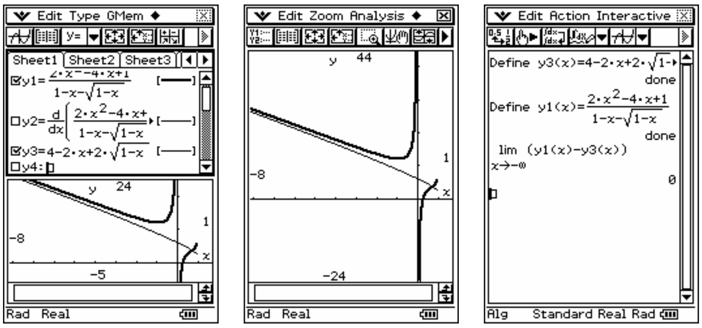
P1(-5.97, f(-5.97)) and
P2(0.69, f(0.69)).

Without such a tool it is impossible to get these points!

Finally the range of f(x): all real numbers.

**Additional exercise 1:**

Show that the function $y = 4-2x+2*\sqrt{(1-x)}$ is an asymptotic function of f($x$) for $x \to -\infty$.




The graphical solution is no analytical proof!

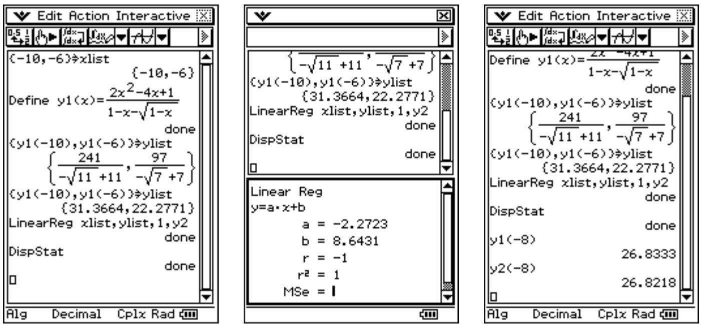
Thus we compute a limit with the both functions to get the analytical solution: lim( f($x$) - $y$ ) = 0.

**Additional exercise 2:**

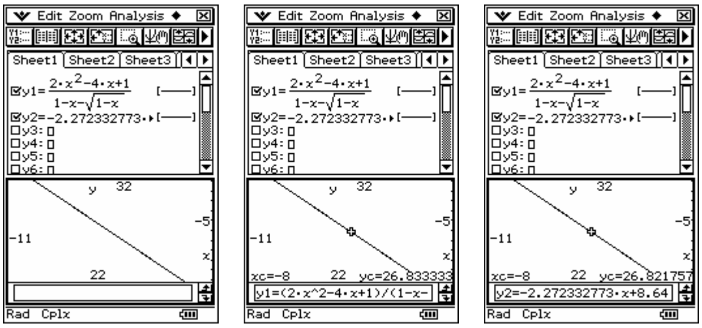
Show that in the $x$-interval (-10, -6) the considered function is a concave function in the following manner:

Compute the line, which connected the points P1(-10, f(-10)) and P2(-6, f(-6)) and show, that the point of the line at $x$ = -8 is situated lower then the point P0(-8, f(-8)). Compute with tree decimals!

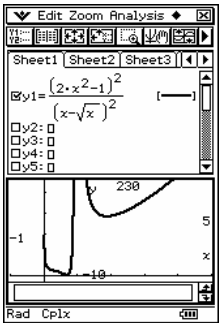
Without a calculator it is impossible to solve this exercise. At first we compute the line using the linear regression function:




Thus we get f(-8) = 26.8333 > y2(-8) = 26.8218, i.e. f is concave.



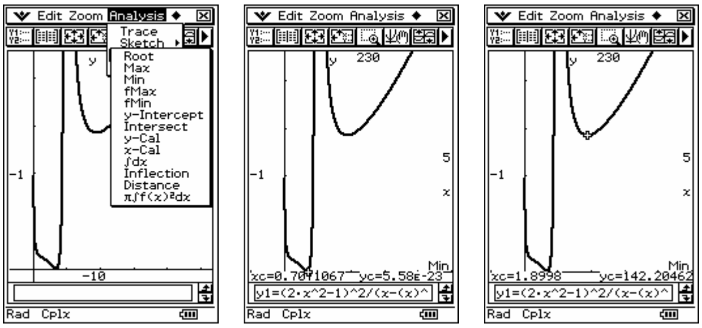

This example shows that we can’t see the solution in the graphics window – we need the numerical calculation.



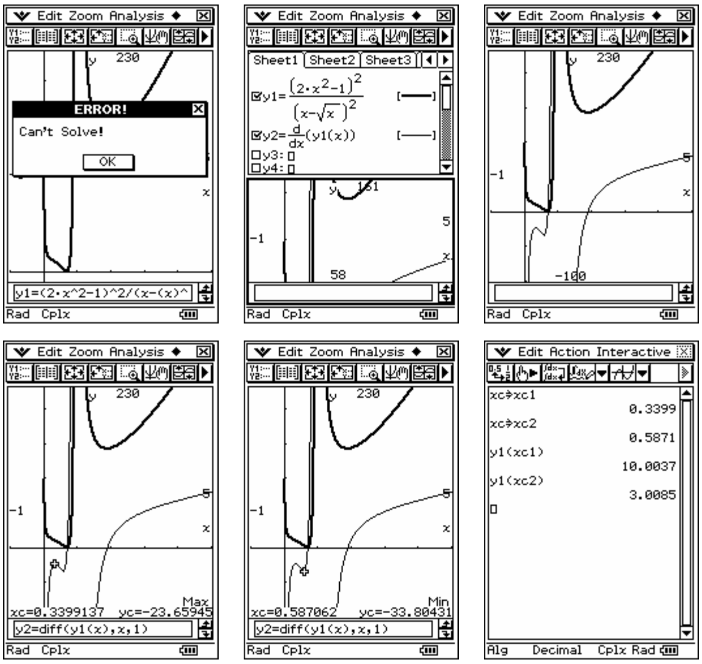

Now let us consider the following function:

f($x$) = (2 $x$^2 - 1)^2 / ( $x$ - √$x$ )^2
with $x \in (0, 1) \cup (1, \infty)$.

The first task is get an appropriate view window to see all parts of the graphical representation. The question is to find out the inflection points.

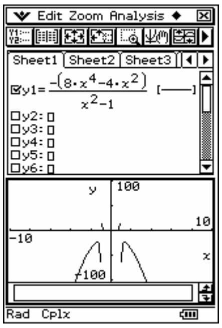



Again the problem appears concerning computation the inflection points. We study the 1st derivative:




Thus we could get the inflection points by means of our tool.

Now consider the rational function $f(x) = -(8x^4 - 4x^2) / (x^2 - 1)$. Again we want to discuss on the graphical representation.

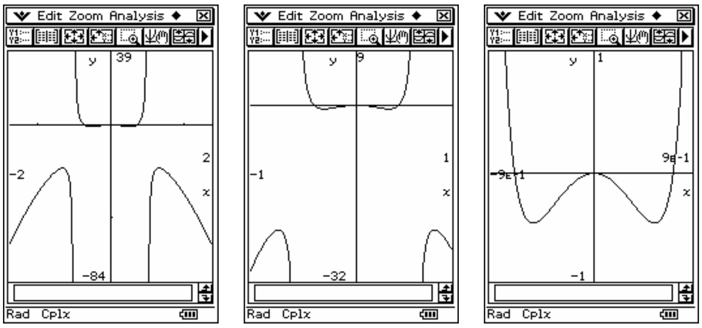


This is a typical situation:

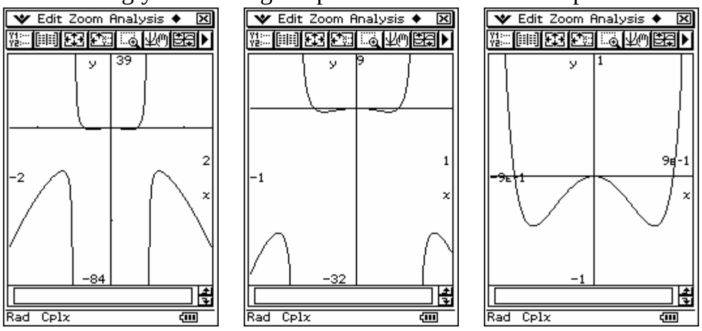
The student can't see in the display what happens.

Now he has to explore the graphical representation.

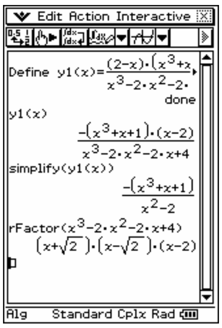
He should zoom in to get a better picture:

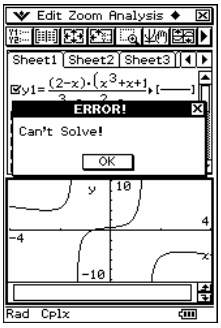


The following function f($x$) = (2- $x$)($x$^3 +$x$ +1) / ($x$^3 -2$x$^2 -2$x$ +4) is interestingly concerning the position of the inflection point.

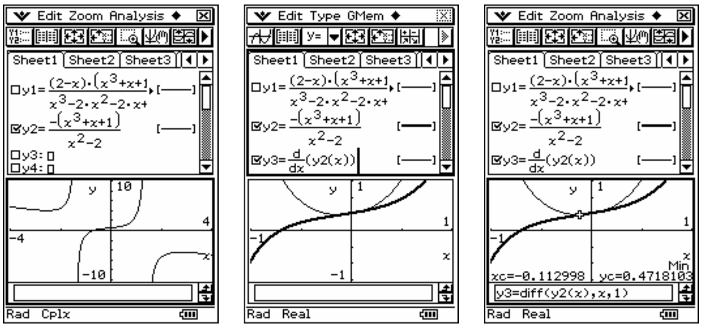


What do you think? Are the inflection point and the intersection with the $y$-axis the same points? Why we get no value f(2)? By means of the graphics calculator we try to answer on the questions.
At first we compute something in the main-menu:



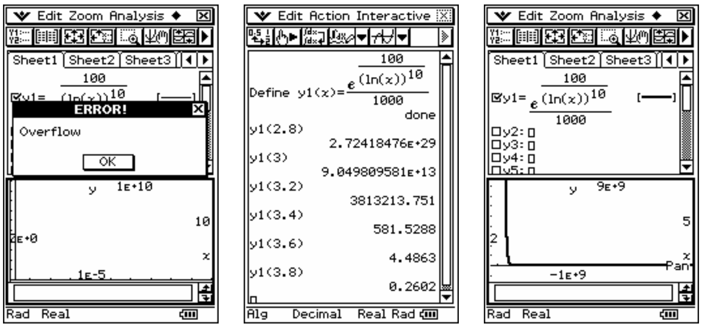




In the left picture you see that for $x \neq 2$ we can get a more simple representation of our function.
The domain of f($x$) is $x \in (-\infty, -\sqrt{2}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, 2) \cup (2, \infty)$.

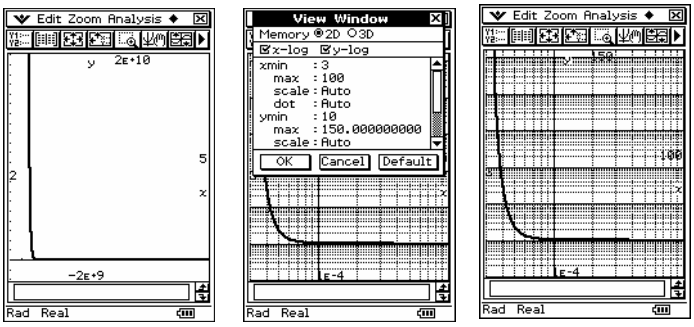



The right picture shows that the Min of f'($x$) appears at $x$ = -0.113. Thus the infection point is not the intersection of f($x$) with the $y$-axis.

Finally we consider a special exponential function:
f($x$) = 0.001 * exp( 100 * ( ln($x$) )^(-10) ), $x$ > 1.
Try to graph this function! Have a look on the range of the function!




It seems that it would be difficult to draw this L-shaped function!



By the help of a log-log-scaled view window we have good view on the graphical representation of the considered function.

Thus by the help of our tool we can draw functions with a very strong growth in a good manner to explore the range of the function.

**Summary of the above considered explorations:**

The solving of systems of linear equations or the exploration of non-linear functions are problems in mathematical education, which have to solve our students in examinations. Often they can use electronic tools, e.g. a graphical CAS-calculator. Thus we need another kind of tasks in examinations. The students should show in the examina what they have learnt and they should use the allowed tools.

The examples on the parametric linear systems are appropriate to see, which abilities and qualifications the students have reached during a lecture on linear systems. Systems only with numbers and without parameters are too simple to solve with a graphics calculator. However the solving of systems with parameters needs knowledge on the several cases, which can appear during the computation. Thus we think the new created **LinEqSys-** and **AVRank-**functions are a useful supplement for the operating system of each graphical CAS-calculator. At the beginning we could see, that the actual OS with the functions for linear systems does not work with symbolic variables (parameters) in general. This is a problem with many hand-held computers and we hope, the software developer will include the here published one-step-functions for matrix-calculation in one of the next updates of the OS.

The other kind of here considered exercises are concerned with the discussion of functions. Here we could observe that the possibilities of the calculator are bounded concerning e.g. the computation of inflection points. Thus the student needs background knowledge, to go other ways by the help of the graphic calculator to get finally the wished result.

Especially the log-log-scaled view-window is here a useful tool, to check more difficult functions with a strong growth.

**References**

[1] Stiefel, Eduard L.: *An Introduction to Numerical Mathematics*, 1963, Academic Press Inc., New York.

[2] Stiefel, Eduard L.: *Einführung in die numerische Mathematik*, 5. erw. Aufl., 1976, Teubner-Verl. Stuttgart, ISBN 3-519-12039-9

[3] Labuch, Dieter: *Aufgaben zur Linearen Algebra*, 3. neubearb. Aufl., 1998, Teubner-Verl. Stuttgart, ISBN 3-519-00240-X

[4] Brauch, Wolfgang; Dreyer, Hans-Joachim; Haacke, Wolfhart: *Mathematik für Ingenieure*, 11. Aufl., 2006, Teubner-Verl. Stuttgart, ISBN 3-8351-0073-4

[5] Paditz, Ludwig: *Rechnen und graphische Darstellungen mit komplexen Zahlen*, Anwendungsbeispiele aus Schule und Studium für den ALGEBRA FX 2.0, 1. Aufl., 2001, Hrg. v. CASIO Europe GmbH Norderstedt

[6] Paditz, Ludwig: *Mathematische Modelle und wissenschaftlich-technische Anwendungen,* Beispiele aus Schule und Studium mit dem grafikfähigen Symbol-Taschenrechner ClassPad300, 1. Aufl., 2004, Hrg. v. CASIO Europe GmbH Norderstedt im Bildungsverlag EINS

[7] Geise, Gerhard: *Grundkurs lineare Algebra – Einführung in die Theorie endlichdimensionaler Vektorräume*, 1. Aufl., 1979, Teubner-Verl. Leipzig