Mathematik II, 5.Übung, **C8.**11a

$$\int_{\square}^{\square} \frac{1}{\sqrt{-x^2+4x+5}}\,dx$$

$$\sin^{-1}\left(\frac{x}{3}-\frac{2}{3}\right)$$

(Taschenrechnerergebnis)

Eulersubstitutionen:

factor($-x^2+4x+5$)

$$-(x+1)\cdot(x-5)$$

1. Substitution:

solve$\left(t=\frac{\sqrt{-x^2+4x+5}}{x-5},x\right)$

$$\left\{x=\frac{5\cdot t^2-1}{t^2+1}\right\}$$

diff$\left(\frac{5\cdot t^2-1}{t^2+1},t\right)$

$$\frac{12\cdot t}{\left(t^2+1\right)^2}$$

$$\int_{\square}^{\square} \frac{1}{\sqrt{-x^2+4x+5}}\,dx=\int_{\square}^{\square} \frac{x-5}{\sqrt{-x^2+4x+5}}\times\frac{1}{x-5}\,dx=\int_{\square}^{\square} \frac{1}{t}\times\frac{1}{\frac{5\cdot t^2-}{t^2+1}}$$

$$\text{simplify}\left(\frac{1}{t}\times\frac{1}{\frac{5\cdot t^2-1}{t^2+1}-5}\times\frac{12\cdot t}{\left(t^2+1\right)^2}\right)$$

$$\frac{-2}{t^2+1}$$

$$\int_{\square}^{\square}\frac{-2}{t^2+1}\,dt$$

$$-2\cdot\tan^{-1}(t)$$

Endergebnis: $-2\cdot\tan^{-1}\left(\frac{\sqrt{-x^2+4x+5}}{x-5}\right)+C$

2. Substitution:

$$\text{solve}\left(t=\frac{\sqrt{-x^2+4x+5}}{x+1},x\right)$$

$$\left\{x=\frac{-\left(t^2-5\right)}{t^2+1}\right\}$$

$$\text{diff}\left(\frac{-\left(t^2-5\right)}{t^2+1},t\right)$$

$$\frac{-12\cdot t}{\left(t^2+1\right)^2}$$

$$\int_{\square}^{\square}\frac{1}{\sqrt{-x^2+4x+5}}\,dx=\int_{\square}^{\square}\frac{x+1}{\sqrt{-x^2+4x+5}}\times\frac{1}{x+1}\,dx=\int_{\square}^{\square}\frac{1}{t}\times\frac{1}{\frac{-\left(t^2-\right.}{t^2+1}}$$

$$\text{simplify}\left(\frac{1}{t}\times\frac{1}{\frac{-(t^2-5)}{t^2+1}+1}\times\frac{-12\cdot t}{(t^2+1)^2}\right)$$

$$\frac{-2}{t^2+1}$$

$$\int_{□}^{□}\frac{-2}{t^2+1}\,dt$$

$$-2\cdot\tan^{-1}(t)$$

Endergebnis: $-2\cdot\tan^{-1}\left(\frac{\sqrt{-x^2+4x+5}}{x+1}\right)+C$

3. Substitution:

$$\text{solve}\left(t\times x+\sqrt{5}=\sqrt{-x^2+4x+5},x\right)$$

$$\left\{x=0,\ x=\frac{-2\cdot(\sqrt{5}\cdot t-2)}{t^2+1}\right\}$$

$$\text{diff}\left(\frac{-2\cdot(\sqrt{5}\cdot t-2)}{t^2+1},t\right)$$

$$\frac{2\cdot\sqrt{5}\cdot t^2-8\cdot t-2\cdot\sqrt{5}}{(t^2+1)^2}$$

$$\int_{□}^{□}\frac{1}{\sqrt{-x^2+4x+5}}\,dx=\int_{□}^{□}\frac{1}{t\times x+\sqrt{5}}\times\frac{2\cdot\sqrt{5}\cdot t^2-8\cdot t-2\cdot\sqrt{5}}{(t^2+1)^2}\,dt$$

$$\text{simplify}\left(\frac{1}{t\times\frac{-2\cdot(\sqrt{5}\cdot t-2)}{t^2+1}+\sqrt{5}}\times\frac{2\cdot\sqrt{5}\cdot t^2-8\cdot t-2\cdot\sqrt{5}}{(t^2+1)^2}\right)$$

$$\frac{-2}{t^2+1}$$

$$\int_{□}^{□} \frac{-2}{t^2+1}\,dt$$

$$-2\cdot\tan^{-1}(t)$$

□

Endergebnis: $-2\cdot\tan^{-1}\left(\frac{\sqrt{-x^2+4x+5}-\sqrt{5}}{x}\right)+C$

trigonometrische Substitution:

$$\int_{□}^{□} \frac{1}{\sqrt{-x^2+4x+5}}\,dx = \frac{1}{3}\times\int_{□}^{□} \frac{1}{\sqrt{-\left(\frac{x-2}{3}\right)^2+1}}\,dx$$

$\frac{x-2}{3}=\sin(t)$

$dx=3\times\cos(t)\times dt$

$$\frac{1}{3}\times\int_{□}^{□} \frac{3\times\cos(t)}{\sqrt{-(\sin(t))^2+1}}\,dt$$

$$t$$

Endergebnis: $\arcsin(\frac{x-2}{3})+C$

hyperbolische Substitution:

$$\frac{x-2}{3}=\tanh(t)$$

$$dx=\frac{3}{(\cosh(t))^2}\times dt$$

Es gilt $1-(\tanh(t))^2=\frac{1}{(\cosh(t))^2}$

$$\frac{1}{3}\times\int_{\square}^{\square}\frac{1}{\sqrt{-\left(\frac{x-2}{3}\right)^2+1}}\,dx=\frac{1}{3}\times\int_{\square}^{\square}\frac{1}{\sqrt{-(\tanh(t))^2+1}}\times\frac{3}{(\cosh(}$$

$$\text{simplify}\left(\frac{1}{\sqrt{-(\tanh(t))^2+1}}\times\frac{3}{(\cosh(t))^2}\right)$$

$$\frac{3}{(\cosh(t))^2\cdot\sqrt{-(\tanh(t))^2+1}}$$

$$\int_{\square}^{\square}\frac{1}{\cosh(t)}\,dt$$

$$2\cdot\tan^{-1}\left(\tanh\left(\frac{t}{2}\right)\right)$$

$$\text{diff}\left(2\cdot\tan^{-1}\left(\tanh\left(\frac{t}{2}\right)\right),t\right)$$

$$\frac{1}{\left(\left(\tanh\left(\frac{t}{2}\right)\right)^2+1\right)\cdot\left(\cosh\left(\frac{t}{2}\right)\right)^2}$$

$$\text{simplify}\left(\frac{1}{\left(\left(\frac{\sinh\left(\frac{t}{2}\right)}{\cosh\left(\frac{t}{2}\right)}\right)^2+1\right)\cdot\left(\cosh\left(\frac{t}{2}\right)\right)^2}\right)$$

$$\frac{2\cdot e^{t}}{e^{2\cdot t}+1}$$

$$\int_{\square}^{\square} \frac{2\cdot e^{t}}{e^{2\cdot t}+1}\,dt$$

$$2\cdot \tan^{-1}\left(e^{t}\right)$$

Endergebnis: $2\cdot \operatorname{arctan}\left(\tanh\left(\frac{\operatorname{artanh}\left(\frac{x-2}{3}\right)}{2}\right)\right)+C$

oder $2\cdot \operatorname{arctan}\left(e^{\operatorname{artanh}\left(\frac{x-2}{3}\right)}\right)+C$

Anmerkung:

Es gilt die Umformung: Bartsch S. 421:

$$\operatorname{artanh}\left(\frac{x-2}{3}\right)=\frac{1}{2}\cdot \ln\left(\frac{1+\frac{x-2}{3}}{1-\frac{x-2}{3}}\right)=\frac{1}{2}\cdot \ln\left(\frac{x+1}{x-5}\right)$$

Endergebnis: $2\cdot \operatorname{arctan}\left(\sqrt{\frac{x+1}{x-5}}\right)+C$

Lösung der Aufgabe C8.8d mithilfe der Integraltafel, Bartsch S. 772, Nr. (450)

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geg. $f(x)=\frac{1}{5-4\cdot \sin(x)+3\cdot \cos(x)}$

ges. $\int_{\Box}^{\Box} f(x)dx$

Lösung: Integral (450) lautet mit a=1, b=5, c=3, d=-4

$$\int_{\Box}^{\Box} \frac{1}{b+c\cdot\cos(ax)+d\cdot\sin(ax)}\,dx =$$

$$\int_{\Box}^{\Box} \frac{1}{b+\sqrt{c^2+d^2}\,\sin(ax+\tau)}\,d(x+\tau/a)$$

mit $\tan(\tau)=\frac{c}{d}$,

d.h. $\tau = \arctan\left(\frac{c}{d}\right) = \arctan\left(\frac{3}{-4}\right) = -\arctan\left(\frac{3}{4}\right) >$
$-\arctan(1) = -45°$

und $\sin(\tau)=\frac{c}{\sqrt{c^2+d^2}}$, d.h. $\sin(\tau) < 0 <$
$\frac{3}{\sqrt{3^2+(-4)^2}} = \frac{3}{5}$

Damit ist die sin(τ)-Bedingung im Integral (450) nicht erfüllt!

Ausweg: $\int_{\Box}^{\Box} f(x)dx =$

$$(-1)\times\int_{\square}^{\square}\frac{1}{-5+4\cdot\sin(x)-3\cdot\cos(x)}\,dx \quad \text{betrachten:}$$

mit a=1, b=-5, c=-3, d=4

$$\tau = -\arctan\left(\frac{3}{4}\right),\quad \sin(\tau) = \frac{c}{\sqrt{c^2+d^2}} = -\frac{3}{5}$$

Subst.: $t=\boldsymbol{x}+\tau/a=\boldsymbol{x}+\tau$, $dt=d\boldsymbol{x}$

$$(-1)\times\int_{\square}^{\square}\frac{1}{-5+4\cdot\sin(x)-3\cdot\cos(x)}\,dx =$$

$$(-1)\times\int_{\square}^{\square}\frac{1}{-5+5\cdot\sin(t)}\,dt = \frac{1}{5}\times\int_{\square}^{\square}\frac{1}{1-\sin(t)}\,dt$$

Weiter mit Bartsch S. 764, Nr. (337):

$$\frac{1}{5}\times\int_{\square}^{\square}\frac{1}{1-\sin(t)}\,dt = \frac{1}{5}\times\tan\left(\frac{\pi}{4}+\frac{t}{2}\right) =$$

$$\frac{1}{5}\times\tan\left(\frac{\pi}{4}+\frac{x}{2}-\frac{\arctan(3/4)}{2}\right)+C$$

Probe:

$$\text{simplify}\left(\text{tExpand}\left(\frac{d}{dx}\left(\frac{1}{5}\times\tan\left(\frac{\pi}{4}+\frac{x}{2}-\frac{\tan^{-1}(3/4)}{2}\right)\right)\right)\right)$$

$$\frac{1}{3\cdot\cos(x)-4\cdot\sin(x)+5}$$