**Solution of Linear Equation Systems with Parameters**

cp.

http://www.informatik.htw-dresden.de/~paditz/Paditz_Beitrag_CEJ_2006.pdf

3x+2y+t*z=0⇒equ1

$$t\cdot z+3\cdot x+2\cdot y=0$$

0x+1y-4z=-1⇒equ2

$$y-4\cdot z=-1$$

1x+3y+0z=1⇒equ3

$$x+3\cdot y=1$$

-1x+0y+2z=1⇒equ4

$$-x+2\cdot z=1$$

$$\left\{\begin{array}{l} equ1 \\ equ2 \\ equ3 \\ equ4 \end{array}\right|_{x,y,z,u}$$

$$\left\{x=\frac{4\cdot t+8}{t-28},y=\frac{-(t+12)}{t-28},z=\frac{-10}{t-28},u=u\right\}$$

What happens with t=28?

$$\left\{\begin{array}{l} equ4 \\ equ2 \\ equ1 \\ equ3 \end{array}\right|_{x,y,z,u}$$

$$\left\{x=\frac{-(t+4)}{t+14},y=\frac{-(t-6)}{t+14},z=\frac{5}{t+14},u=u\right\}$$

What happens with t=-14?

$$\begin{cases} \text{equ4} \\ \text{equ2} \\ \text{equ3} \\ \text{equ1} \end{cases}\Bigg|_{x,y,z,u}$$

$$\left\{x=-\frac{2}{7}, y=\frac{3}{7}, z=\frac{5}{14}, u=u\right\}$$

Here it seems for all t we get an unique solution
(and parameter t disappears?)?

$$\begin{bmatrix} 3 & 2 & t & 0 \\ 0 & 1 & -4 & -1 \\ 1 & 3 & 0 & 1 \\ -1 & 0 & 2 & 1 \end{bmatrix} \Rightarrow \text{mat}$$

$$\begin{bmatrix} 3 & 2 & t & 0 \\ 0 & 1 & -4 & -1 \\ 1 & 3 & 0 & 1 \\ -1 & 0 & 2 & 1 \end{bmatrix}$$

ref(mat)

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{t}{3} & 0 \\ 0 & 1 & \frac{-t}{7} & \frac{3}{7} \\ 0 & 0 & 1 & \frac{5}{3\cdot t+14} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here it seems we have to study the special case
t=-3/14?

rref(mat)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here it seems the parameter t is without meaning (t disappears?)?

But in case t=0:

ref(mat|t=0)

$$\begin{bmatrix} 1 & \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & \frac{5}{14} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rref(mat|t=0)

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{2}{7} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & \frac{5}{14} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank(mat)

4

rank(mat|t=0)

3

The rank-function can't compute the rank in dependence on the parameter **t!**
If we use the "ref" or "rref" or "rank" functions the CP should give a message "no solution with parameters" (cp TI-Nspire CAS, OS 1.6)

**another exercise with complex numbers:**

$$\begin{bmatrix} j & 2 & t & 3+2j \\ 0 & 1 & 2j & 1+j \\ s & 0 & 4 & -1 \end{bmatrix} \Rightarrow mat$$

$$\begin{bmatrix} j & 2 & t & 3+2\cdot j \\ 0 & 1 & 2\cdot j & 1+j \\ s & 0 & 4 & -1 \end{bmatrix}$$

rank(mat)

3

rank(mat|s=-j and t=-4+4j)

2

Again, the rank is depending on the parameters s and t

## The exchange procedure:

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DelVar $a_{11}, a_{12}, a_{21}, a_{22}, \beta_1, \beta_2, x_1, x_2$, t

done

start with following system:

$\mathbf{x_1 \times a_{11} + x_2 \times a_{12} = \beta_1}$ (1)

$\mathbf{x_1 \times a_{21} + x_2 \times a_{22} = \beta_2}$ (2)

let be $a_{11} \neq 0$ (pivot element)

equivalent system:

$\mathbf{y_1 = 0 = x_1 \times a_{11} + x_2 \times a_{12} - \beta_1 \times 1}$ (3)

$\mathbf{y_2 = 0 = x_1 \times a_{12} + x_2 \times a_{22} - \beta_2 \times 1}$ (4)

solve($y_1 = x_1 \times a_{11} + x_2 \times a_{12} - \beta_1 \times 1, x_1$)

$$\left\{ x_1 = \frac{-a_{12}\cdot x_2}{a_{11}} + \frac{\beta_1}{a_{11}} + \frac{y_1}{a_{11}} \right\}$$

$$\text{expand}\left( y_2 = x_1 \times a_{21} + x_2 \times a_{22} - \beta_2 \times 1 \,\middle|\, x_1 = \frac{-a_{12}\cdot x_2}{a_{11}} + \frac{\beta_1}{a_{11}} + \frac{y_1}{a_{11}} \right)$$

$$y_2=\frac{-a_{12}\cdot a_{21}\cdot x_2}{a_{11}}+a_{22}\cdot x_2+\frac{\beta_1\cdot a_{21}}{a_{11}}-\beta_2+\frac{a_{21}\cdot y_1}{a_{11}}$$

new equivalent system:

$$x_1= \frac{1}{a_{11}}\cdot y_1 \quad + \quad \frac{a_{12}}{-a_{11}}\cdot x_2 \quad + \quad \frac{-\beta_1}{-a_{11}}\cdot 1$$

(5)

$$y_2= \frac{a_{12}}{a_{11}}\cdot y_1 + \left(a_{22}+a_{12}\cdot \frac{a_{21}}{-a_{11}}\right)\cdot x_2 + \left(-\beta_2+a_{12}\cdot \frac{-\beta_1}{-a_{11}}\right)\cdot 1$$

(6)

write (3), (4) in a table **ST** with matrix **matST**:

| **ST** | | $x_1$ | $x_2$ | 1 |
|---|---|---|---|---|
| $y_1$ | \| | $a_{11}$ | $a_{12}$ | $-\beta_1$ |
| $y_2$ | \| | $a_{21}$ | $a_{22}$ | $-\beta_2$ |
| K | | * | $\frac{a_{12}}{-a_{11}}$ | $\frac{-\beta_1}{-a_{11}}$ |

K is the bottom (a help row, put * in the pivot column and divide by -pivot otherwise.)

where **matST**$=\begin{bmatrix} a_{11} & a_{12} & -\beta_1 \\ a_{21} & a_{22} & -\beta_2 \end{bmatrix}$ - in matST are all informations on the system.

write (5), (6) in a table **T1** with matrix **matT1**
(an equivalent system, exchange $y_1 \leftrightarrow x_1$)

| **T1** | | $y_1$ | $x_2$ | 1 |
|---|---|---|---|---|
| $x_1$ | \| | $\frac{1}{a_{11}}$ | $\frac{a_{12}}{-a_{11}}$ | $\frac{-\beta_1}{-a_{11}}$ |

using

inverse value and K row respectively

$y_2$ | $\frac{a_{12}}{a_{11}}$ $\quad a_{22}+a_{12}\cdot\frac{a_{21}}{-a_{11}}$ $\quad -\beta_2+a_{12}\cdot\frac{-\beta_1}{-a_{11}}$ division by pivot (old pivot column) and

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"triangular rule" respectively

delete $y_1$ column (because $y_1=0$)

**T1** $y_1$ $x_2$ 1

| | | | |
|---|---|---|---|
| $x_1$ \| | ■ | $\frac{a_{12}}{-a_{11}}$ | $\frac{-\beta_1}{-a_{11}}$ |
| $y_2$ \| | ■ | $a_{22}+a_{12}\cdot\frac{a_{21}}{-a_{11}}$ | $-\beta_2+a_{12}\cdot\frac{-\beta_1}{-a_{11}}$ |

with **matT1**= $\begin{bmatrix} \frac{a_{12}}{-a_{11}} & \frac{-\beta_1}{-a_{11}} \\ a_{22}+a_{12}\cdot\frac{a_{21}}{-a_{11}} & -\beta_2+a_{12}\cdot\frac{-\beta_1}{-a_{11}} \end{bmatrix}$

in matT1 are all informations on the system.

The next pivot element could be $a_{22}+a_{12}\cdot\frac{a_{21}}{-a_{11}}$ $(\neq 0)$

to exchange $y_2 \leftrightarrow x_2$

Thus we get (if exchange $y_2 \leftrightarrow x_2$ possible)

**T2=ET** $y_1$ $y_2$ 1

| | | | |
|---|---|---|---|
| $x_1$ \| | ■ | ■ | □ |
| $x_2$ \| | ■ | ■ | □ |

unique solution

If $a_{22}+a_{12}\cdot\frac{a_{21}}{-a_{11}}=0$ than T1=ET (end table with followig decisions):

| **T1=ET** | $y_1$ | $x_2=t$ | 1 |
|---|---|---|---|
| $x_1$ | ■ | □ | □ |
| $y_2$ | ■ | 0 | [0] |

non-unique solution (with parameter t)

| **T1=ET** | $y_1$ | $x_2=t$ | 1 |
|---|---|---|---|
| $x_1$ | ■ | □ | □ |
| $y_2$ | ■ | 0 | [≠0] |

(no solution, contradiction in the y-row y≠0)

stop

**Now using the new created functions LinEqSys and AVRank in Main-menu**
(To use the LinEqSys and AVRank program in an eActivity the program must be stored in the Library-folder!)

| **ST** | $x_1$ | $x_2$ | $x_3$ | 1 |
|---|---|---|---|---|
| $y_1$ | $\mathbf{a_{11}}$ | $a_{12}$ | $a_{13}$ | $-\beta_1$ |
| $y_2$ | $a_{21}$ | $a_{22}$ | $a_{23}$ | $-\beta_2$ |
| $y_3$ | $a_{31}$ | $a_{32}$ | $a_{33}$ | $-\beta_3$ |
| $y_4$ | $a_{41}$ | $a_{42}$ | $a_{43}$ | $-\beta_4$ |

pivot $\mathbf{a_{11}\neq 0}$ to exchange exchange $y_1 \leftrightarrow x_1$

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$$\begin{bmatrix} 3 & 2 & t & 0 \\ 0 & 1 & -4 & 1 \\ 1 & 3 & 0 & -1 \\ -1 & 0 & 2 & -1 \end{bmatrix} \Rightarrow \text{matST}$$

$$\begin{bmatrix} 3 & 2 & t & 0 \\ 0 & 1 & -4 & 1 \\ 1 & 3 & 0 & -1 \\ -1 & 0 & 2 & -1 \end{bmatrix}$$

LinEqSys(matST,1,1)

done

**syntax: LinEqSys**(matrix, row index i of $a_{ik}$, column index k of $a_{ik}$), if $a_{ik}$ pivot

matnew⇒matT1

$$\begin{bmatrix} -\frac{2}{3} & \frac{-t}{3} & 0 \\ 1 & -4 & 1 \\ \frac{7}{3} & \frac{-t}{3} & -1 \\ \frac{2}{3} & \frac{t}{3}+2 & -1 \end{bmatrix}$$

| **T1** | | $y_1$ | $x_2$ | $x_3$ | 1 |
|---|---|---|---|---|---|
| $x_1$ | \| | ■ | □ | □ | □ |
| $y_2$ | \| | ■ | □ | □ | □ |

pivot $\mathbf{a_{22} \neq 0}$ to exchange exchange $y_2 \leftrightarrow x_2$

| | | | | | |
|---|---|---|---|---|---|
| $y_3$ | \| | ■ | □ | □ | □ |
| $y_4$ | \| | ■ | □ | □ | □ |

----------------------------

LinEqSys(matT1,2,1)

done

matnew⇒matT2

$$\begin{bmatrix} \frac{-(t+8)}{3} & \frac{2}{3} \\ 4 & -1 \\ \frac{-(t-28)}{3} & -\frac{10}{3} \\ \frac{t+14}{3} & -\frac{5}{3} \end{bmatrix}$$

| **T2** | | $y_1$ | $y_2$ | $x_3$ | 1 | |
|---|---|---|---|---|---|---|
| $x_1$ | \| | ■ | ■ | □ | □ | |
| $x_2$ | \| | ■ | ■ | □ | □ | |
| $y_3$ | \| | ■ | ■ | □ | □ | pivot $a_{33} \neq 0$ to |

exchange exchange $y_3 \Leftrightarrow x_3$ or

| | | | | | | |
|---|---|---|---|---|---|---|
| $y_4$ | \| | ■ | ■ | □ | □ | pivot $a_{43} \neq 0$ to |

exchange exchange $y_4 \Leftrightarrow x_3$

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If $t \neq 28$ than we have a third exchange-step $x_3 \Leftrightarrow y_3$

LinEqSys(matT2,3,1)

done

matnew⇒matET

$$\begin{bmatrix} \frac{4\cdot(t+2)}{t-28} \\ \frac{-40}{t-28}-1 \\ \frac{-10}{t-28} \\ \frac{-5\cdot t}{t-28} \end{bmatrix}$$

**T3=ET** $y_1$ $y_2$ $y_3$ 1

| | | | | | |
|---|---|---|---|---|---|
| $x_1$ | ■ | ■ | ■ | $\frac{4\cdot(t+2)}{t-28}$ | |
| $x_2$ | ■ | ■ | ■ | $\frac{-40}{t-28}-1$ | |
| $x_3$ | ■ | ■ | ■ | $\frac{-10}{t-28}$ | |
| $y_4$ | ■ | ■ | ■ | $\frac{-5\cdot t}{t-28}$ | $y_4=0$, if $t=0$ |

matET is the solution, if the last element $\frac{-5\cdot t}{t-28}$ equals 0, i.e. t=0

matET|t=0

$$\begin{bmatrix} -\frac{2}{7} \\ \frac{3}{7} \\ \frac{5}{14} \\ 0 \end{bmatrix}$$

**Thus x=-2/7, y=3/7, z=5/14**

If t≠14 than we have a third exchange-step $x_3 \leftrightarrow y_4$
We start again with matST:

LinEqSys(matST,1,1)

done

LinEqSys(matnew,2,1)

done

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LinEqSys(matnew,4,1)
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done

```
matnew⇒matET
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$$\begin{bmatrix} \frac{-(t+4)}{t+14} \\ \frac{20}{t+14}-1 \\ \frac{-5\cdot t}{t+14} \\ \frac{5}{t+14} \end{bmatrix}$$

**T3=ET** $y_1$ $y_2$ $y_4$ 1

| | | | | | |
|---|---|---|---|---|---|
| $x_1$ | ■ | ■ | ■ | $\frac{-(t+4)}{t+14}$ | |
| $x_2$ | ■ | ■ | ■ | $\frac{20}{t+14}-1$ | |
| $y_3$ | ■ | ■ | ■ | $\frac{-5\cdot t}{t+14}$ | $y_3=0$, if $t=0$ |
| $x_3$ | ■ | ■ | ■ | $\frac{5}{t+14}$ | |

matET is the solution, if the 3rd element $y_3$ equals 0, i.e. $t=0$

matET|t=0

$$\begin{bmatrix} -\frac{2}{7} \\ \frac{3}{7} \\ 0 \\ \frac{5}{14} \end{bmatrix}$$

**Thus x=-2/7, y=3/7, z=5/14**

stop

**Remark:** rank of matST is 3 for t=0 and 4 otherwise

AVRank(matST,1,1)

done

**syntax:**
**AVRank**(matrix, row index i of $a_{ik}$, column index k of $a_{ik}$), if $a_{ik}$ pivot
The rank of a matrix is the number of possible exchange steps

matnew⇒matT1

$$\begin{bmatrix} 1 & -4 & 1 \\ \frac{7}{3} & \frac{-t}{3} & -1 \\ \frac{2}{3} & \frac{t}{3}+2 & -1 \end{bmatrix}$$

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T1     y1    x2    x3    1
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x1  |  ■     ■     ■     ■     delete the old pivot
row and pivot column
y2  |  ■     □     □     □     pivot a11≠0 to
exchange exchange x2 ↔ y2
y3  |  ■     □     □     □
y4  |  ■     □     □     □
----------------------------
```

AVRank(matT1,1,1)

done

matnew⇒matT2

$$\begin{bmatrix} \frac{-(t-28)}{3} & -\frac{10}{3} \\ \frac{t+14}{3} & -\frac{5}{3} \end{bmatrix}$$

| **T2** | | $y_1$ | $y_2$ | $x_3$ | 1 |
|---|---|---|---|---|---|
| $x_1$ | \| | ■ | ■ | ■ | ■ |
| $x_2$ | \| | ■ | ■ | ■ | ■ |
| $y_3$ | \| | ■ | ■ | □ | □ |
| $y_4$ | \| | ■ | ■ | □ | □ |

delete the old pivot row and pivot column again

pivot $a_{11} \neq 0$ to exchange exchange $x_2 \leftrightarrow y_2$

If t≠28 we compute:

AVRank(matT2,1,1)

done

matnew⇒matT3

$$\begin{bmatrix} \frac{-5 \cdot t}{t-28} \end{bmatrix}$$

| **T3** | | $y_1$ | $y_2$ | $x_3$ | 1 |
|---|---|---|---|---|---|
| $x_1$ | \| | ■ | ■ | ■ | ■ |
| $x_2$ | \| | ■ | ■ | ■ | ■ |
| $y_3$ | \| | ■ | ■ | ■ | ■ |
| $y_4$ | \| | ■ | ■ | ■ | $\frac{-5 \cdot t}{t-28}$ |

delete the old pivot row and pivot column again

If $t \neq -14$ we compute:

AVRank(matT2,2,1)

done

matnew⇒matT3

$$\begin{bmatrix} \frac{-5\cdot t}{t+14} \end{bmatrix}$$

| **T3** | $y_1$ | $y_2$ | $x_3$ | 1 |
|---|---|---|---|---|
| $x_1$ | ■ | ■ | ■ | ■ |
| $x_2$ | ■ | ■ | ■ | ■ |
| $y_3$ | ■ | ■ | ■ | $\frac{-5\cdot t}{t+14}$ |
| $y_4$ | ■ | ■ | ■ | ■ |

delete the old pivot row and pivot column again

Thus we have 3 steps, i.e. rank equals 3, and for $t \neq 0$ we get rank equals 4

$$\operatorname{rank}\left(\begin{bmatrix} 3 & 2 & t & 0 \\ 0 & 1 & -4 & 1 \\ 1 & 3 & 0 & -1 \\ -1 & 0 & 2 & -1 \end{bmatrix}\right)$$

4

□

The rank(matrix) function can not differ between 3 or 4 in dependence of parameter t.