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Д и п л о м а
(thesis)

Тема:

Решаване на някои класи линейни диференциални уравнения с ClassPad 330

(Solving some classes of linear differential equations with ClassPad 330)

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Contents

1. Introduction to the ClassPad

The ClassPad is a natural, sophisticated form of a graphical calculator. Go with an impressive collection of applications and self-learning applications like 3D-Graph, Geometry, eActivity, SpreadSheet and much more.

The device allows adding text, both in English and in Bulgarian, and it appears in the way it is presented in the textbook. The authorization of texts, graphics, drawing, computing and other operations can be made quickly and authentically a large LCD screen. The ClassPad calculator offers several graphical tools necessary for plotting 3D-graphics and geometric objects that create a very favorable environment for education and training of students by themselves.

Interface of the calculator using pull-down menu format that simplifies even complex operations at first glance. Multiple sentences, phrases and other topics can be selected with the touch of a stylus. The ClassPad supports 'drag and drop', 'copy and paste' operations, and others who minimize the use of any other key operations.

Applications for presentations and tools for creating materials for lectures and to solve a wide range of tasks, allowing teachers to present these in a manner understandable to students and provides a good basis for understanding of mathematical concepts and problems in the process of solving them. The individual transaction can be captured and later added to the presentation and put in slideshow format.

Administration eActivity allows working with expressions, formulas, figures and other graphics, allowing students to learn and advance themselves using their personal methods for solving problems, utilizing well known algorithms for solving them. This is a cross (only) teaching tool that helps to study the various processes and sharing of mathematical solutions. ClassPad allows a comprehensive understanding of the problem by helping students to discover and develop their own capabilities.

The calculator can conveniently connect to computer to transfer information as is possible and implement the transfer of photos. The ClassPad Manager software is given in CD-ROM, which comes with a calculator and an additional cable. It can be performed to transfer programs and materials (e-Activities) of your ClassPad to a computer or by using the ClassPad Manager software to a ClassPad.

ClassPad is not just a graphical calculator, a next-generation calculator, for scientific research and educational purposes. With ClassPad students can manipulate complex mathematical expressions by translating them to simple and straightforward equations, allowing easy understanding of the connections between different mathematical fields (vector and geometry). Offers a new perspective on mathematics, bringing it to the other, easy to understand examples.

The ClassPad is different from other well known calculators because it offers:

- Input of informations with a touch of the stylus;
- Provides input / output natural mathematical expressions (e.g. Program Mode);
- Calculation with ClassPad is easily and quickly using fractions, limes, differentials, integrals, matrices, and many other built-in functions;
- Displaying the result in its natural mathematical form;
- Four new applications included as a supplement to the calculator: eActivity, Geometry, Presentations and 3D-graphic.

1.1 eActivity application

The eActivity and Presentation application are easy to manipulate, exploitation / exploration and sharing of solutions can be found without a large audience. This helps students to increase awareness and improve their personal opportunities in the field to mathematics.

The eActivity has four main advantages to using a sheet and a pen:

- Use this application to create a favorable environment for personal development as well as materials for lectures and presentations;

- Provides students work on tasks in their own pace and capability;

- Allows the detection of different ways to solve problems;

- Allows text expressions, shapes and graphics to be displayed on the screen of the ClassPad 330.

What is eActivity?

This is a new electronic tool that can be freely studied in ClassPad. With this application can benefit all the useful features built into the ClassPad 330. It is both a textbook and workbook. Teachers can use it to create electronic examples and exercises to solve problems using text, mathematical expressions, 2D- and 3D-graphics, geometric drawings and tables. The eActivity provides students with an environment for research and development, allowing them to save the results obtained, combined with observations on their part.

The eActivity application provides text input and mathematical expressions and Applied ClassPad informations, save the file as 'eActivity'. A standard eActivity file contains text informations placed in band or line. The order may be in the form of 'Row text', 'Geometric row' or 'Calculation row'. As a band / band takes the form of so-called. 'Application data strip' (main, geometry, graph, conics sequence, etc.)

Create an eActivity

Press the 'Menu' to visualize the 'Application menu', from which we select the icon for eActivity. It displays the window of the eActivity, as it was shown last time when it was used. When choosing a file / new will open a new window, and old

information will be deleted. There are four types of informations that can be incorporated into eActivity window: text, calculation expression, geometric object or process information for display. After eActivity is the way we want to be saved from the file / save. It is shown a file-dialog box, then select folder and file name (maximum 20 characters) is pressed to save. Saved files are sorted in an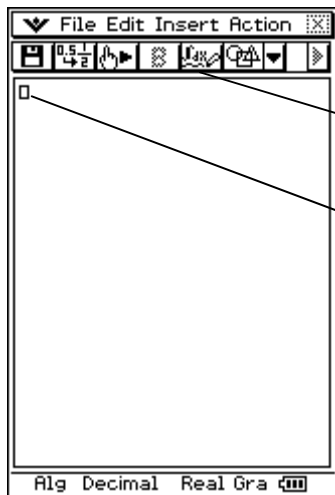 area of memory which is divided from the memory for other types of information. Therefore e.g., Managing eActivity file can not be accessed to the Variable Managing or Managing in a Geometry Application file.

The text row is used to add text and mathematical expressions in the normal state, as the text can thicken. The calculation row is used to bring computing operations which are valid in the Main application. The application data strip display window of any application ClassPad (e.g. Main, Graph and Table, Geometry) and use the window to create information that can be later incorporated into eActivity, Geometry link row is made to communicate with geometric shapes. They all add the menu 'Insert'.
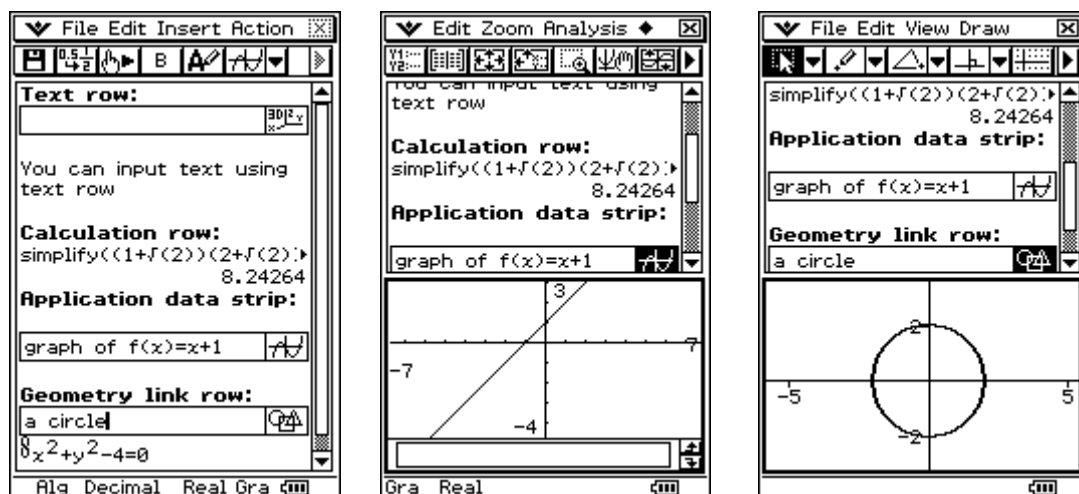
The text row inserted text and mathematical expressions, but which is not feasible in choosing the Execute button. All commands in the Text row are treated as text. When the text that is entered is too long, the cursor is automatically transferred to the next line. When entering mathematical expressions, the cursor stays on the same line, even if the expression is too long. Arrows appear to indicate that there is information to the left or right of it.

The calculation row allows the execution of any calculations of eActivity application. When we introduce the mathematical expression, the result is displayed aligned to the right of the screen. You can always edit the information entered, but never the result. In case of wrong entered expression result will appear.

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  ✔ File Edit Insert Action  ⊠
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This button shows that we are in the
Calculation input mode

The checkmark appears at the beginning of the
line when the calculation mode is selected

If the required result will not be displayed it should not be pressed Exe button from the built-in keyboard ClassPad. When you change the input, output is automatically changed. Even mathematical expressions, which are generally introduced into eActivity without calculation, are calculated and their results are displayed.

The application data strip can be used if required to add information from any other application into the eActivity. This element contains a field in which it may be heading information and a button that when pressed it takes us to the desired application. Often the screen is divided to two halves. Active is the half that is selected. Adding Application data is from the Insert menu and choose Strip. Right of the screen appears with a drop-down menu choices, which give access to applications such as ClassPad Graph: 2D, 3D, graphics of differential equations, tables and geometric objects, and many others.

E.g. to add Geometry data choose Insert → Strip → Geometry. A window similar to the previous-field in which it may be heading the project or development and button pressing of which screen is divided into two and visualize geometric object that is selected from the toolbar

```
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 └──────────────────────────────────┘
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In the same way you can add graphics. The main point here is that we need to prove equations needed for plotting graphs. The drawing may have more than one graph plotting each be made with different thickness of the line.
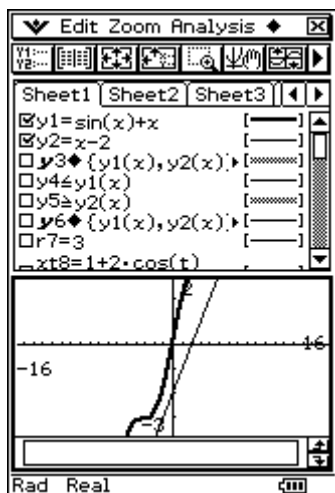
Adding this feature it is the button ⊞, you can add up to 100 functions.

You can use the Notes field to add any information. Notes is a text box for editing and styling comments or add explanations in complex mathematical expressions in eActivity.

When selecting Insert → Strip → Notes a window opens in the lower half of the screen. From the toolbar, inserted text may be adjusted in a way that we want. This application is only active in eActivity.

eActivity allows previewing the image (bitmap image). Adding this feature it is the button ⊞ choose it by Insert → Strip → Picture, in which the lower half of the screen window is opened for adding an image. When choosing File → Open menu opens the file chooser, but the main thing is, it must be of type PICT.

eActivity is both notebook and textbook, allowing to explore the world of mathematics. Almost any expression of eActivity may be included in any of the other applications as well be taken information from external applications and be included in eActivity.

The addition of **Geometry link row** is automatically added the information in the **Geometry window** by liaising with information eActivity. It can be visualized as lines and shapes values or mathematical expressions in Geometry link row. Dragging a line or geometrical figure of Geometry window to Geometry link row turns them automatically into their equivalent mathematical expression. This expression is related to the Geometry window, making change and who have been provided valid and visualization figure at the bottom of the screen. The change of expression in Geometry link automatically changes shape in Geometry window.
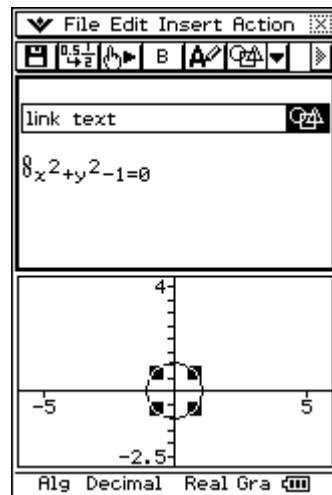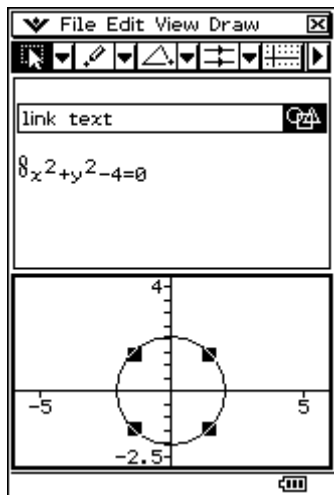
		• In the drag / drop function it is plotted quickly geometric shapes; • When you change the para-meters easily and effectively change the dimensions of the selected shape.

It is possible to transfer eActivity files to another ClassPad Unit, but it needs the host file part to support all types of Application data strip (Graph, Graph Editor, 3D Graph, 3D Graph Editor, Conics Graph, Conics Editor, Geometry, SpreadSheet, Stat Graph , Stat Editor, DiffEqGraph, DiffEqGraph Editor, Financial, Probability, NumSolve, Sequence Editor, Picture Viewer, Note, Main, Verify). If we transfer information to another ClassPad, which does not support some of these applications, the recipient can not open the file.

It is possible also a transfer between ClassPad and a computer. Here it could be used FA-CP1. It is desirable first to check the version of the FA-CP1, a computer loaded with the operating system of the ClassPad, to determine whether they are compatible. Another way to accomplish the transfer, which is much more convenient is using Exchange window on the ClassPad Manager (appears when you press the right mouse button). When that opens a dialog ClassPad Manager-Exchange. When ClassPad is connected to the computer the screen from left to become visible from computer files and ClassPad. Transfer is carried out in drag and drop the selected file.

The eActivity is also useful in the classroom and at home. Students can easily change their own mathematical expressions or numerical values and solve problems they themselves award. Based on the results obtained, it is possible to prepare presentations. For convenience, use Presentation application and RM-ClassPad. Features provide a margin presentation of information that appears on the LCD screen as a picture or in the form of slides. This is a very effective way of presenting the results of a study group or a major project. The presentation may be more effective if complemented by RM-ClassPad projection by OHP. It allows you to send to the date of the calculator screen or wall.

OH-ClassPad lets you send information management tool to the OHP projector for presentations OHP-screen!

In the ClassPad it is interesting and it must be addressed the organization of files. The system is divided into two directories, one that is outside eActivity, Main and others, and another which is included in eActivity, Main, etc. The two unrelated directories and files from one can not be read or processed by / in the other. However calls can be made and this became possible with the help directory **library**, which serves as a link between the other two.

1.2 Presentation application

Presentation application allows a picture to other Windows applications, organizing in the form of its presentation. With the application can realize and correct the contents when necessary. Presentation, for example, can show how they obtained the interim and final results step by step. They can be used in the classroom and / or other performance by connecting with ClassPad OHP-projector. Thus, teachers can submit their work before a large audience, doing tasks understandable and accessible to others. Students, on the other was using the application can gracefully make their tasks, essays and projects.

The application is started from Presentation application icon in the main menu. Opens the following dialog:

- When selecting a field disabled would lead to an automatic change of [Screen copy to] settings of the presentation and Communication dialogue boxes to the [Outer device];
- Files are numbered from P1 to P20. These numbers are fixed and can not be changed;
- When you created a new presentation file can be his name;
- The name of each file is visible and its size in pages;
- The soft keyboard appears automatically when you start the application.

Once the application is launched from file-list is selected the first row (from P1 to P20). The cursor appears on the line chosen by us, which allows to introduce eight byte file name. When ready press (EXE). You should make sure that the name that has just entered is selected. When you select Print screen, automatically adds the selected image in our presentation. Only when disable is selected process is terminated. The presentation runs from one of these buttons: ▢ ▢.

As for selection opens a dialog that allows to select the desired presentation. When selecting the first button presentation starts and ends only when all the slides be listed. When selecting the second button it is controlled by us.
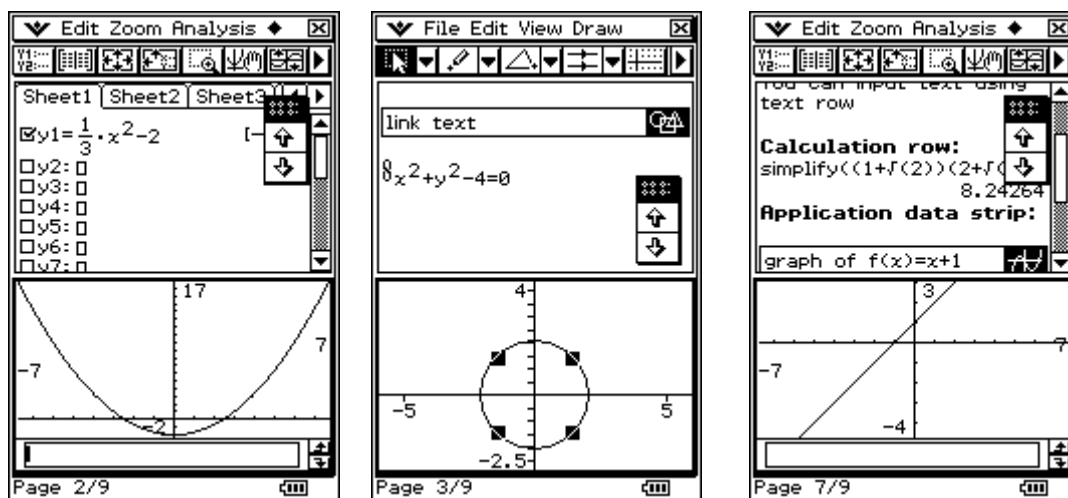
1.3 Geometry application

Geometry application allows to plot and analyze geometric figures. It can be plotted triangle and to determine accurately the relationship of the parties in the terms, such as 3:4:5, and check the size of the angles at each change.

It can be drawn a circle and then right, which is tangential to a particular point of the circle. This application also includes an option to animate shapes, which allows you to monitor how they change with each change introduced.

The application is launched when selecting the icon ▢. Draw menu provides plotting of points, polygons, straight, ellipses, circles and other geometric shapes.
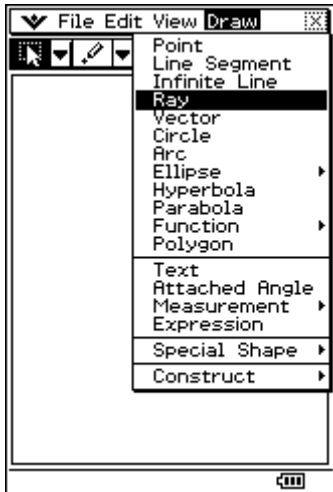
The application provides an opportunity to draw figures and functions.

Once plotted, the figure may, if desired, be moved or edited.

Draw menu contains submenus. It can be used to add the environment section, draw the perpendicular at another set point or other geometric shape, and testing of geometric theorems.
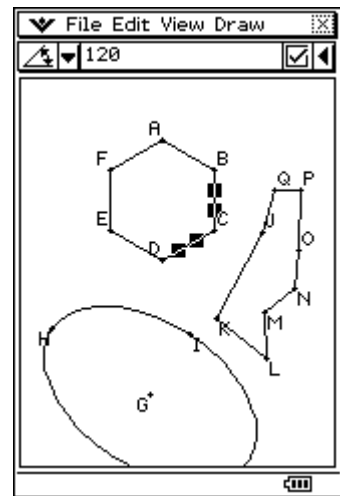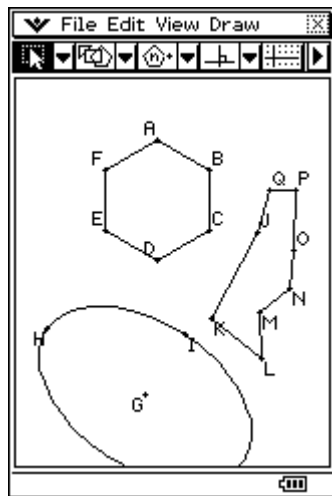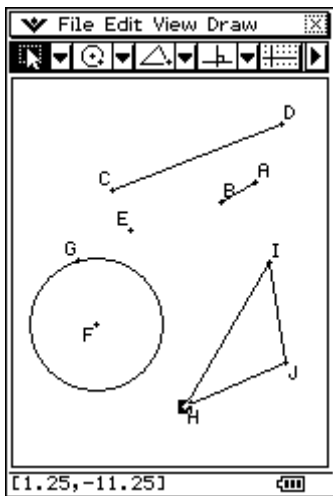
When you press the arrow to the right ▶ appears. Measurement box, where you can see or change the size selected by us. From the toolbar, choose the shape that we want to draw.

Point — Segment
Infinite loop — Beam
Vector — Circle
Arch — Ellipse
Ellipse — Hyperbola
Parabola — Function
Polygon
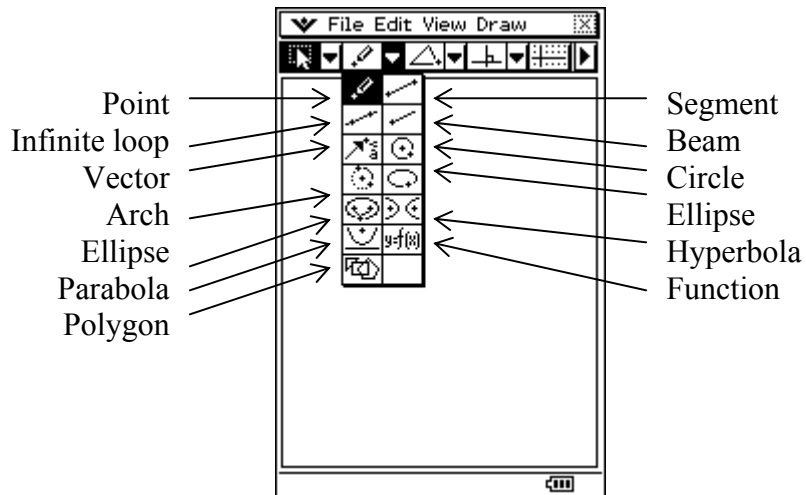
As already mentioned, Geometry application allows for plotting functions. This happens when selecting a Function button on the toolbar in which opens a dialog box that lets you choose the type of the function (Polar, Parametric, or f(x)) and a window, which has the desired function.
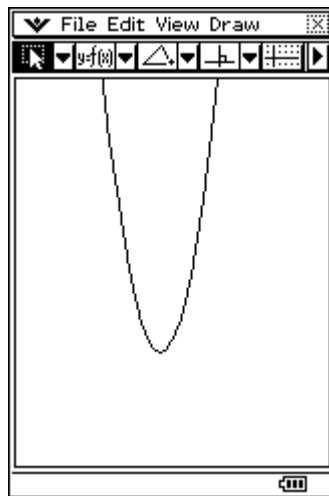
When choosing Polar screen appears Soft keyboard, a dialog box appears radius. In the example below shows the equation r = θ.
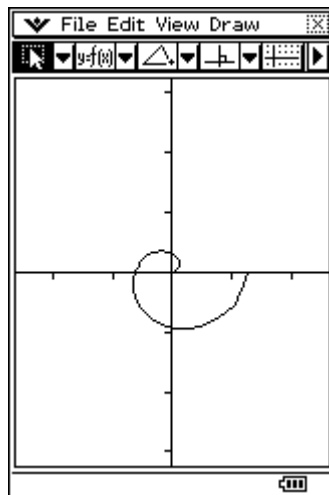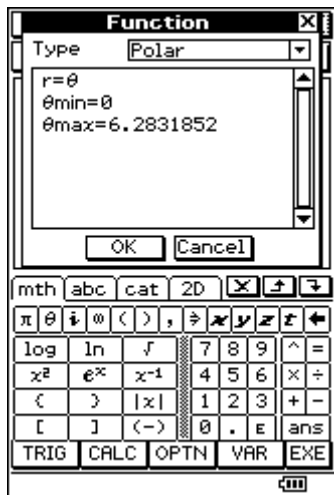
When choosing Parametric prove two functions, giving limits tmin and tmax. After drawing the figure may be included in the Main or eActivity. The equation appears as a text graphic that appears below.

It allows to add text while working with Geometry application. This is the menu Draw → Text in which opens a dialog box for entering text. It enables the introduction of both text and mathematical expressions.
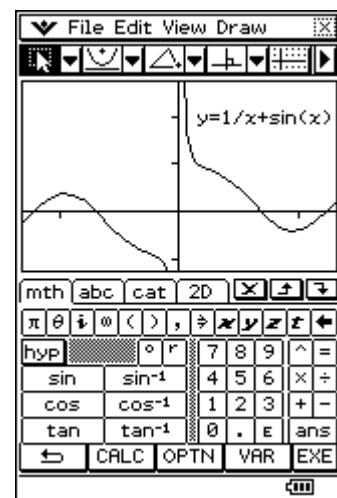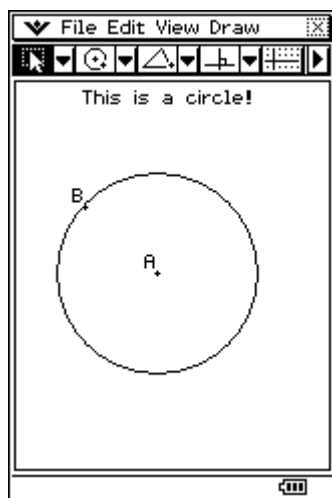
Easy and fast to make measurements of angles, and correcting them. In the example shown below is one of the measured angles in a triangle. For this purpose, countries are marked by choosing Draw Attachment angle.
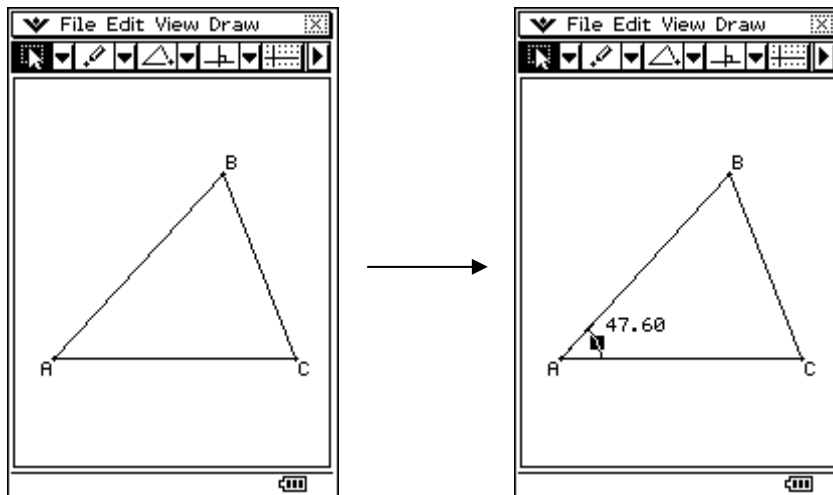
It is interesting to note that both sides of the figure formed not one but four corners. To assess which of them need only to use drag / drop and put the cursor where you want to measure.
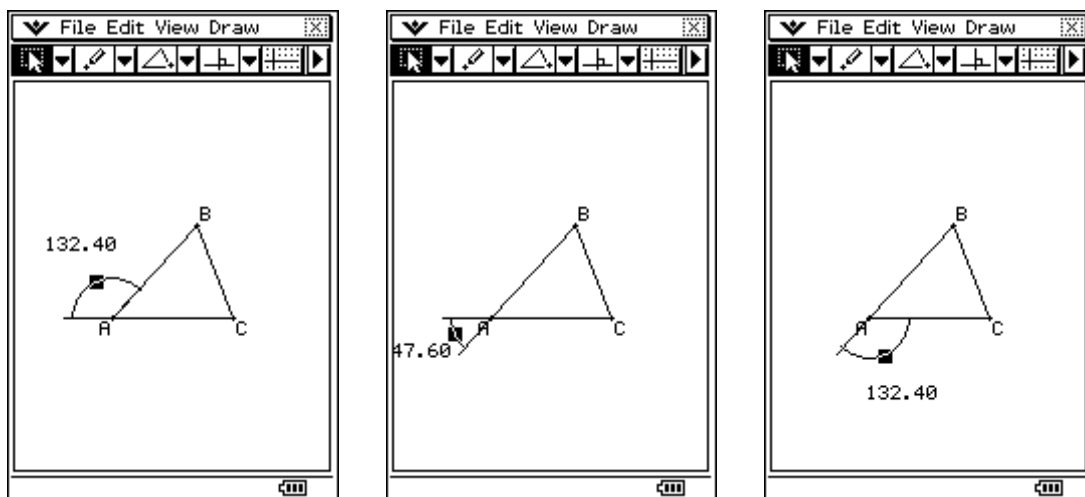
You can use the Expression command and commands from the submenu Measurement to meet a number of calculations using the size of angles, length countries face area or other measurement values rendered to the figure and results displayed on the working window of the Geometry application.

Example: Here are the segments AB and CD, as point C belongs rights AB. Find the sum of the angles ACD and BCD. To visualize the result on the screen.

- plotted are two lines;
- marked and the menu is killed attachment angle, which will automatically lead to image the size of angle ACD;
- mark up and drag (move) in the other corner;
- mark the two lines again and again chosen attachment angle. Thus displayed simultaneously and the two corners.

- On the Draw is choose Expression. In the Measurement window which enters the formula EXPR = @1 + @2. Thus, the two elements 1) and 2) will be selected and the result will be recorded is 3).
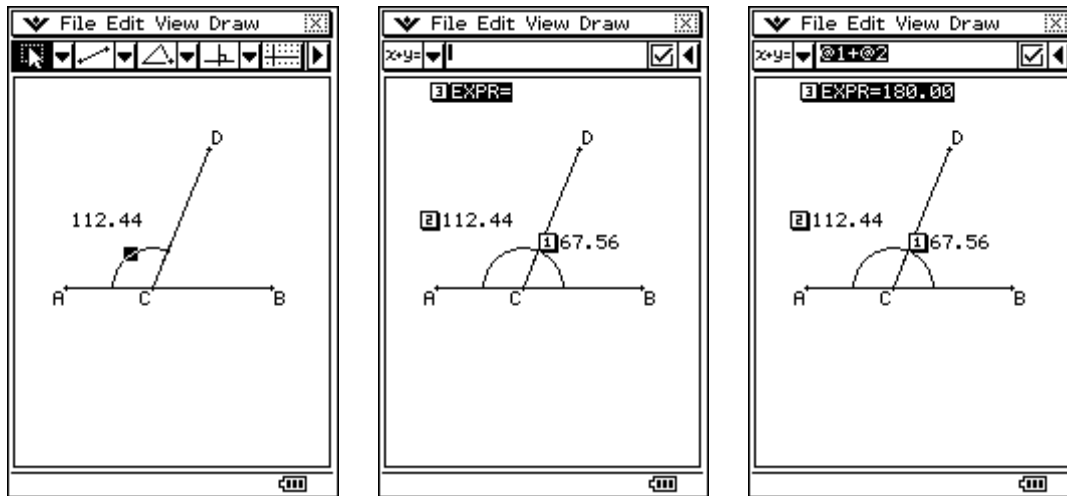
Built in the Geometry application

Construct submenu of the Draw menu assisted learning of geometric theorems. In addition to a menu included bisector, perpendicular, midpoint, section parallel lines tangent to the curve, as well as rotation, translation, transformation of shapes and reflex points.

For this purpose, before selecting any operation constructs first it must be drawn on the main items that will be designed. Only then from the Draw is selected Construct one of the options you want: Perpendicular Bisector, Perpendicular, Midpoint, Intersection, Angle bisector, Parallel, Tangent to curve, Reflection, Translation, Dilation or General transform.

In the construction of the section of two straight, the overall point can not be moved. The principle of finding a common point between a straight and a circle or two circles is similar to the section of two straight.

- In Translation (displacement) of the line defined values of a vector led to the emergence of a new line, which is consistent with the baseline parallel to and in accordance with the set values of the vector. It is also moving online by selecting the set vector. For that purpose must be drawn a further vector. When you open the dialog box is selected.
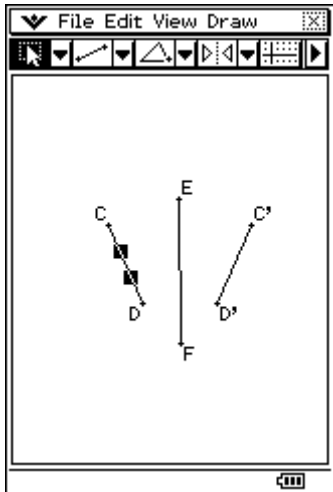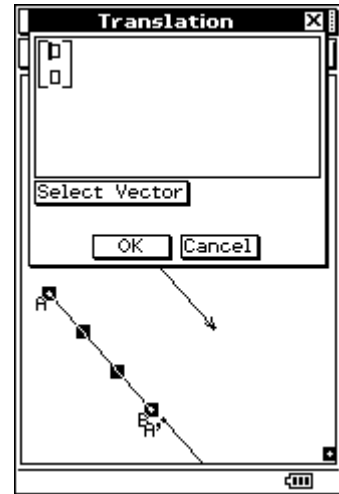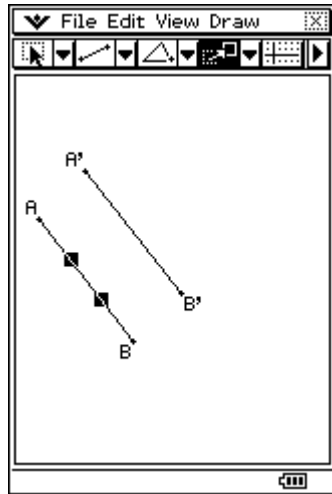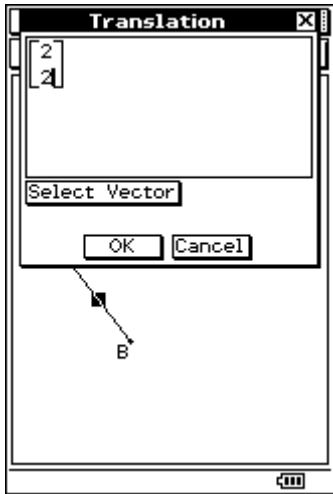
- to draw the mirror image of the item, the option Reflection. For this purpose, the line is drawn specimen and another to serve as axis. Then mark the base line and select Reflection from Draw menu.

  This automatically made active Reflection button in the toolbar. Select the axis of symmetry and the new line is displayed on the screen
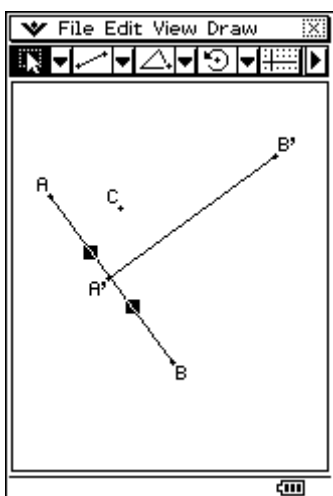
- Rotation line option is used, Rotate. Draw up the line and then marking and choose Rotate.

  You must be a point which will be considered a center of rotation.
  A dialog box appears indicating the angle of rotation. When you are ready press OK.

General Translation, using matrices or vectors allows, in introducing them to change shapes. Outcome of change is a new piece that appears on the screen. By Geometry application can be drawn to each other symmetrical shapes, as it turns to the abscissa and ordinate are symmetrical.

When the output figure is drawn are selected and all countries. From Draw → Construction → General Transform dialog box opens, divided in two. To occur so that the new figure is symmetric about x-axis of the initial figure is introduced in the first window [1, 0] [0, -1].

If you want can be done in parallel move by one unit in X and Y. For this purpose, writes [1, 1] in the second window, as shown in the example below.
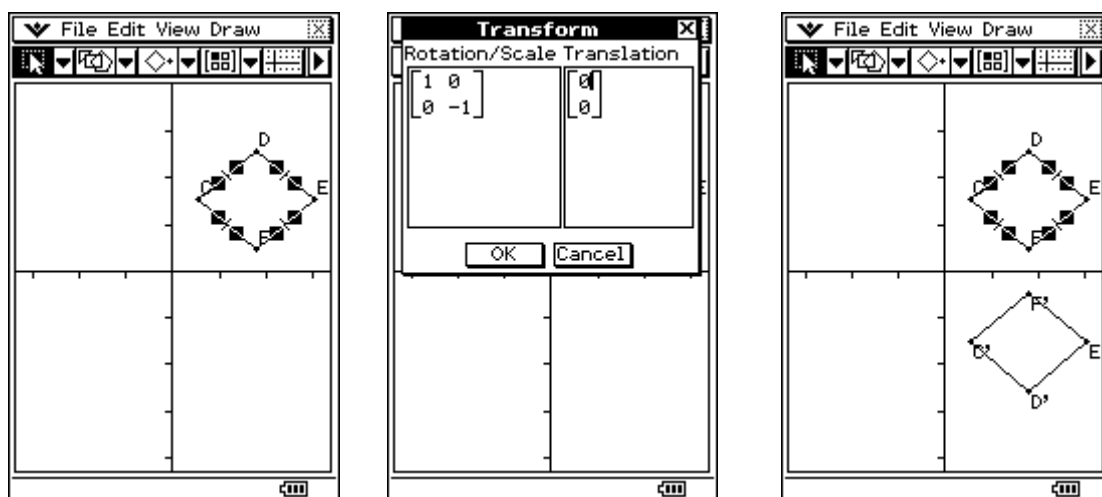
It is easier to understand how General Translation works if used together with or eActivity Main application, combined with the Geometry application. This makes it possible that the following transformations:

- In the Geometry application can choose a point on the figure obtained after transformation and point from the output figure. Dragging them in the Main application appears on the screen the words that led to transformation;

- You can choose a triangle, dragging it to Main, turn of the triangle as a matrix with two rows and three pillars, which depicts three coordinates. Is valid and reverse operation.

File Edit Insert Action

Geometry transformatic▸

File Edit Insert Action

Geometry transformatic▸

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
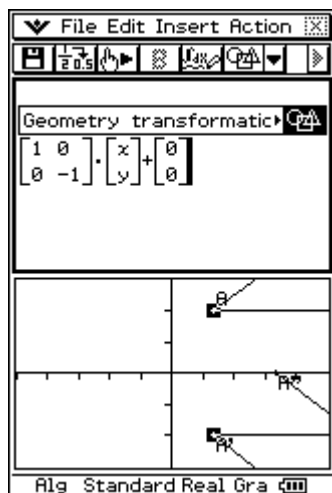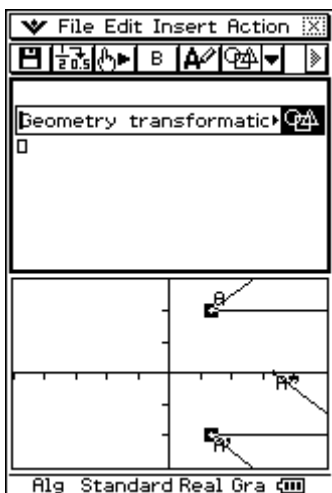
It should be noted that this operation is only possible if the points that are selected are the main figure and this correspondent.

Nothing will appear in the selection of two points of the new figure

Alg Standard Real Gra

File Edit Insert Action

$$\begin{bmatrix} -6 & -3 & -2 \\ -2 & 3 & -2 \end{bmatrix}$$

File Edit Insert Action

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} -6 & -3 & -2 \\ -2 & 3 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

File Edit View Draw

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} -6 & -3 & -2 \\ -2 & 3 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

Alg Standard Real Gra
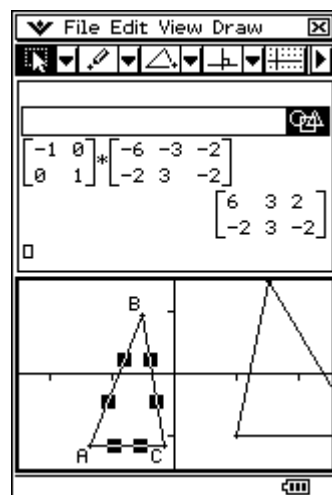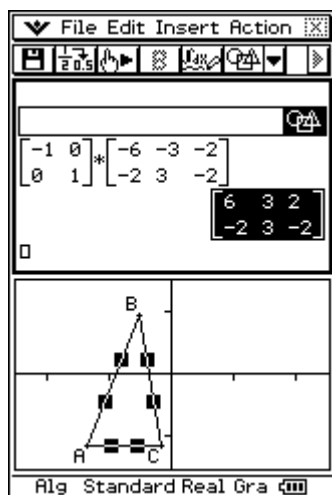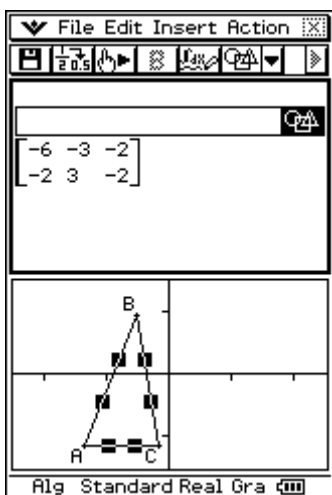
Geometry application and animation

The animation contains one or more points / curves like curves can be a line segment, circle, ellipse or function. The animation is built in the choice of point / curve and added to the Animation. Can happen in two ways. One is by selecting Edit → Animate, and the other is selected from the toolbar View → Animation UI

Draw is the first point and the curve at which it will move. Then marked from the main menu and choose Edit → Animate → Add animation, in which our chosen point automatically moves the curve. If after selecting Edit → Animate → Go (once) / Go (repeat) / Go (to and fro) appears to be how to move it. When choosing Stop animation stops.

Animation added

Steps   20
Animations
D   t0   0   Remove
     t1   1
Traces
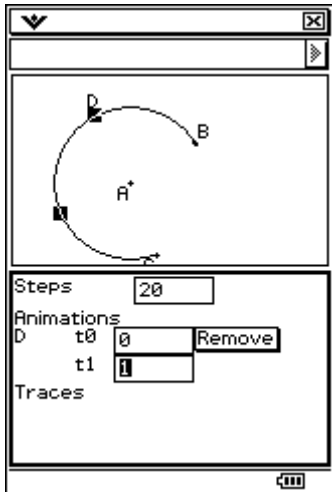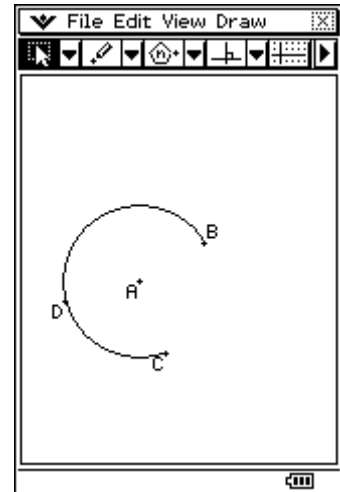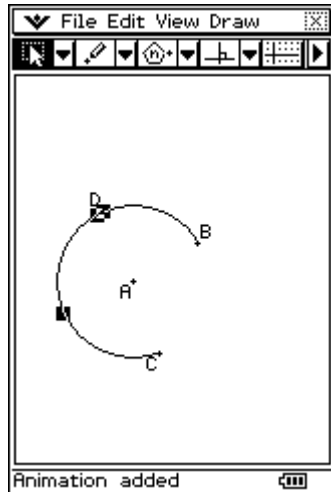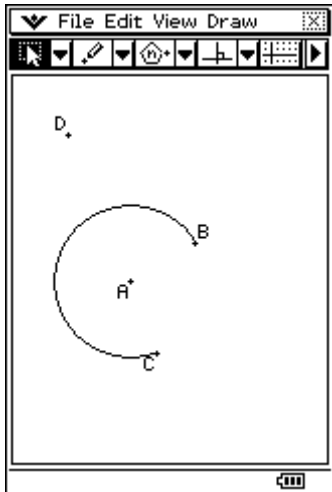
If you are required to change a set of us animation is elected Edit → Animate → Edit Animation. Appears next dialog.

It may prove what is the step of movement. These settings indicate steps that point D will be done in animation and AB. Sub menu Animations appear all points that are animated. In our case, point D. if you select Remove to remove the applied point. 't0' and 't1' indicate the curve AB. Example, if you change the settings of t0 from 0 to 0.5 will result in a change to the starting position and animated section will start its movement from the middle of the curve AB.

Working with files in the Geometry application

When choosing a fillet File → Save to open the following dialog box.

Select the directory where you want to save, as it turns out a name that should be a size 8 bytes.

Search files	Save the file under a different name	Rename files
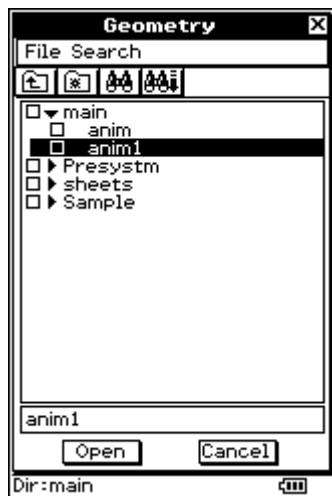
1.4 3D Dimensional Graph application

This application allows you to draw 3D graphics on the type of z = f (x,y) and parametric equations. It starts from the icon 3D graphics, in which the screen opens the 3D graph editor and the 3D graph window. Functions that are mapped in 3D graph editor is displayed in the 3D graph window.
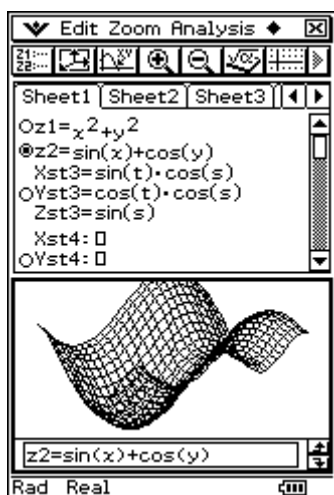
The 3D graph editor has 5 separate pages from sheet1 - sheet5.

Each page contains up to 20 functions.

This means that you can sort the 100 features in a 3D graphics.

You can select one of the functions and its image will appear in the graph window.
The figure can be rotated and seen from different angles.

At the bottom status bar is visible. Real indicates that the regime is Real mode (real number calculation). The other mode which ClassPad lets you select is working with complex numbers Cplx (Complex number calculation). Rad shows that work in radians. Other options are Gra and Deg.

As said you can enter the type of functions z = f (x,y) or entering parametric equations.

z = f(x,y)   parametric equations
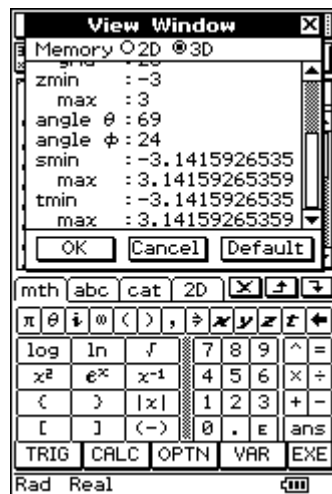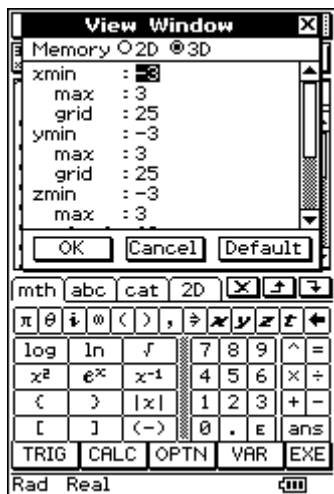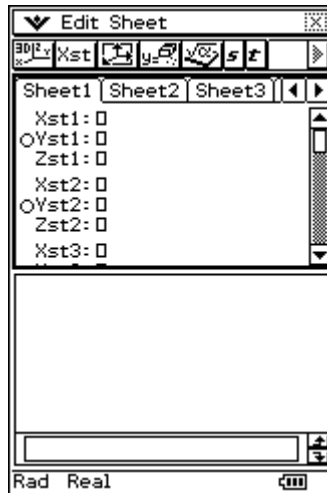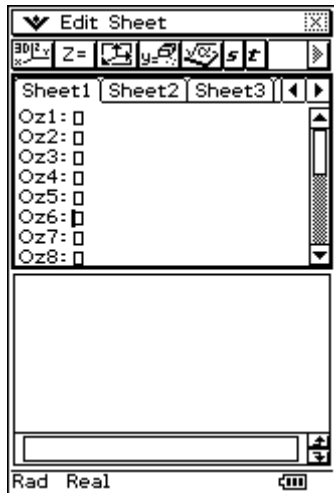
You can use the 3D Graph View Window to configure the minimum and maximum values for x-, y-, z-coordinates, and s and t variables used in the introduction of parametric equations.

The example below is a function z = xy, using different settings in the 3D Graph View Window.
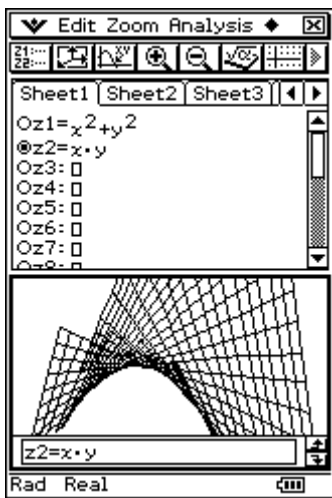
xmin/ ymin= -3, xmax/ ymax=3   xmin/ ymin= -1, xmax/ ymax= -1
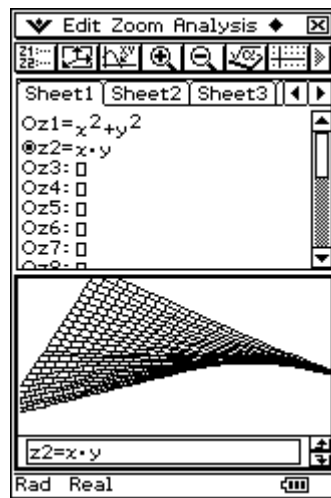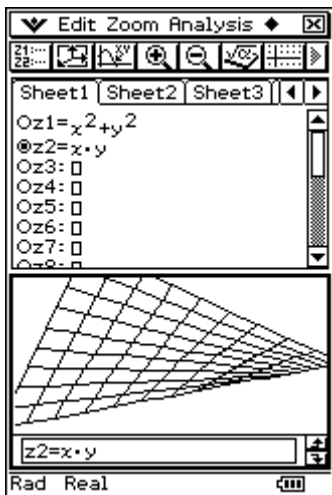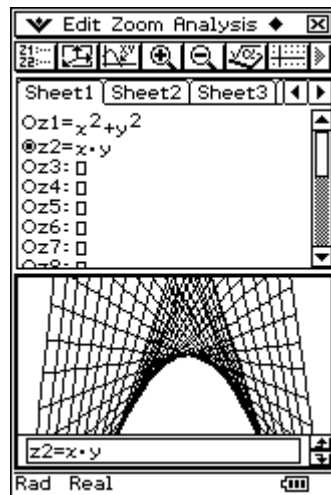
xgrid/ ygrid= 10   Initial default

Using Trace to read 3D graphics

Pressing ⌨ the screen as an exchange through which control the way that the mouse / cursor does appear as coordinates of each selected point.
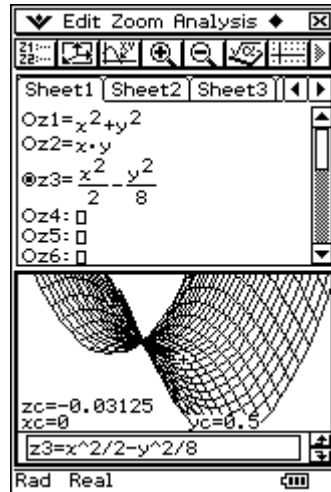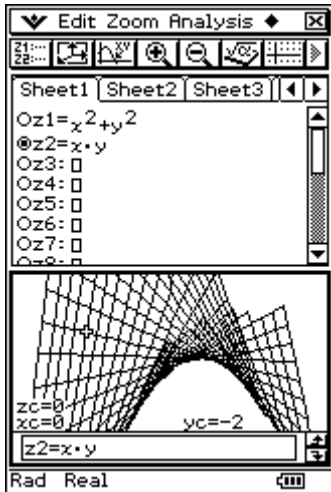
The calculation of the values of the z-coordinate in input values x and y is done by the main menu select Analysis → z-Cal.

Than that opens following dialog:
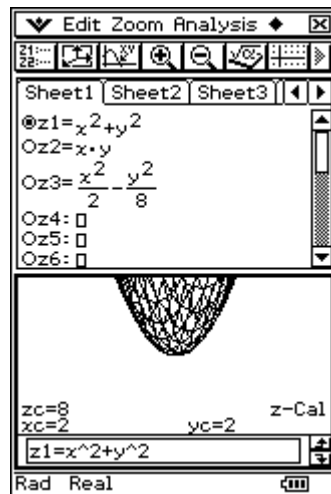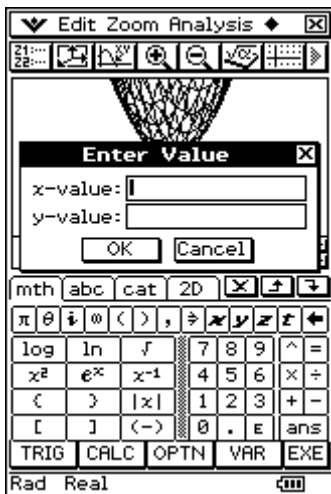
2. Differential equations and ClassPad
=====================================

2.1 Ordinary differential equations

An equation such as

$$(1)\quad F\left(x, y, y', y'', \ldots, \frac{d^n}{dx^n}(y)\right) = 0, \quad y = y(x),$$

where x is the independent variable, $y = y(x)$ is an unknown function,

and $y, y', y'', \ldots, \frac{d^n}{dx^n}(y)$ are its derivatives, is called

differential equation of n-order.
Each function $y = y(x)$ (in a type such as $\Phi(x, y) = 0$ and in parametrical type $x = x(t)$, $y = y(t)$), that convert the equation (1) into a identity is called particular solution (particular integral) of this equation.

Example 1: Find the particular solution of the differential equation $2 \cdot y \cdot y'' = (y')^2$ which solution contents the initial conditions $y(1) = 1$, $y'(1) = 2$.

Solution: The general integral of this equation is $y = C_1 \cdot (x + C_2)^2$.
Then we have $y(1) = C_1 \cdot (1 + C_2)^2 = 1$,
$y'(1) = 2C_1 \cdot (1 + C_2)^2 = 2$.
From these two equations we get $C_1 = 1$, $C_2 = 0$.
The particular solution of this differential equation is $y = x^2$.

ClassPad:

dSolve$\left(2 \cdot y \cdot y'' = (y')^2, x, y\right)$

$$\left\{ x \cdot \text{const}(1) + \int \frac{1}{\sqrt{|y|}} dy - \text{const}(2) = 0 \right\}$$

y>0: (condition $y(1)=1>0$) $\sqrt{|y|} = \sqrt{y}$

$x \cdot \text{const}(1) + \int_{\square}^{\square} \frac{1}{\sqrt{y}} dy - \text{const}(2) = 0$

$$x \cdot \text{const}(1) + 2 \cdot \sqrt{y} - \text{const}(2) = 0$$

solve(ans,y)

$$\left\{ y = \frac{(x \cdot \text{const}(1) - \text{const}(2))^2}{4} \right\}$$

Define $y(x) = C_1 \cdot (x + C_2)^2$

done

$2 \cdot y(x) \cdot \frac{d^2}{dx^2}(y(x)) = \left[\frac{d}{dx}(y(x)) \right]^2$

$$4 \cdot C_1^2 \cdot (x + C_2)^2 = 4 \cdot C_1^2 \cdot (x + C_2)^2$$

judge(ans)

TRUE

$\begin{cases} y(1) = 1 \\ \frac{d}{dx}(y(x)) = 2 \, | \, x = 1 \end{cases} \Bigg|_{C_1, C_2}$

$$\{C_1 = 1, C_2 = 0\}$$

$y(x) \, | \, \{C_1 = 1, C_2 = 0\}$

$$x^2$$

The wanted particular solution is $y = x^2$.

Example 2: Check if the function $y=\sin(x)$ is a particular solution of $y''+y=0$.

Solution: $y'=\cos(x)$, $y''=-\sin(x)$ therefore $y''+y=-\sin(x)+\sin(x)=0$.

ClassPad:
Define $y(x)=\sin(x)$

done

$$\frac{d^2}{dx^2}(y(x))+y(x)=0$$

$0=0$

The function $y=\phi(x,C_1,C_2,\ldots,C_n)$ contains n independent integration constants $C_1,C_2,\ldots,C_n$, is called **general solution** (**general integral**) of the equation of n-order.

2.2 Differential equations of first order

The differential equation of first order has the form

(2) $F(x,y,y')=0$.

The equation (2), solved by y', has the form $y'=f(x,y)$. The graphics of each particular solution (particular integral) is called **integrated curve.**

It is supposed that the function $f(x,y)$ is defined and continuous in some area D of the plane Oxy.

Normally for a given point in that area D one integrated curve passes exactly. If this is not valid, e.g. there are two or more integrated curves passing through some given point in that area, thus this point is called **singular point** of this differential equation.

A solution (integral) of the equation (2) is called singular if through every point from its graphic at least two integrated curves pass. The singular solution can be obtained from the general one if we have given some values (more than one) of the constants.

2.3 Differential equations with separated variables

An equation which can be written in that form

(3) $f_1(x) \cdot \phi_1(y)dx + f_2(x) \cdot \phi_2(y)dy = 0$,

is called **differential equation with separated variables.** If we divide the both sides of equation (3) by $f_2(x) \cdot \phi_1(y)$ we have an equation with separated variables.

(4) $\dfrac{f_1(x)}{f_2(x)}dx + \dfrac{\phi_2(y)}{\phi_1(y)}dy = 0$ $(f_2(x) \neq 0, \ \phi_1(y) \neq 0)$

The general solution of (4) therefore to equation (3) is

(5) $\displaystyle\int \dfrac{f_1(x)}{f_2(x)}dx + \int \dfrac{\phi_2(y)}{\phi_1(y)}dy = C$

It's necessary to check if the solution (5) contains a particular solution, for which we know $f_1(x) = 0$ or $\phi_1(y) = 0$, because if we divide by $f_1(x) \cdot \phi_1(y)$ they can be skipped.

2.4 Homogeneous differential equations

An equation that can be written in a way

(6) $P(x,y)dx + Q(x,y)dy = 0$,

where $P(x,y)$ и $Q(x,y)$ are homogeneous functions of the same degree, is called **homogeneous differential equation.**

Definition: The function $f(x,y)$ is called **homogeneous of degree m** if

$f(\lambda x, \lambda y) = \lambda^m f(x,y)$, $\lambda \neq 0$

The homogeneous equation (6) can be solved by using the substitution $y(x)=t(x) \cdot x$ or $x(y)=t(y) \cdot y$, and then it is transformed to an equation with separated variables.

Example 3: Find the solution of the equation $(x^2+2xy)dx+xydy = 0$

Solution: The functions $P(x,y)= x^2+2xy$ и $Q(x,y)= xy$ are homogeneous of degree 2. We make the substitution $y(x)=t(x) \cdot x$, $dy=xdt+tdx$, and then we have

$(x^2+2x^2t)dx + tx^2(xdt+tdx) = 0$
or
$x^2(t+1)^2dx + tx^3dt = 0,$

where is equation with separated variables.

Divide the both sides of the equation by $x^3(t+1)^2$ and then we have

$$\frac{dx}{x}+\frac{tdt}{(t+1)^2}=0 \Rightarrow \int\frac{dx}{x}+\int\frac{tdt}{(t+1)^2}=C \Rightarrow$$

$$\ln|x|+\int\frac{t+1-1}{(t+1)^2}dt = C \Rightarrow \ln|x|+\ln|t+1|+\frac{1}{t+1} = C$$

Substitute $t=\frac{y}{x}$ and finally we have

$$\ln|x+y| + \frac{x}{x+y} = C \text{ or } |x+y|\cdot e^{\frac{x}{x+y}} = C_1 \text{ or}$$

$$(x+y)\cdot e^{\frac{x}{x+y}} = C_2$$

ClassPad:
DelVar y

                done

dSolve$\left((x^2+2x\cdot y)+x\cdot y\cdot y'=0,x,y\right)$

$$\left\{\left|\frac{y}{x}+1\right|\cdot e^{\left(x^{-1}\cdot y+1\right)^{-1}}=\frac{const(1)}{|x|}\right\}$$

simplify$\left(x\cdot\left(\frac{y}{x}+1\right)\cdot e^{\left(x^{-1}\cdot y+1\right)^{-1}}=C\right)$

$$(x+y)\cdot e^{x\cdot(x+y)^{-1}}=C$$

2.5 Linear differential equations

A differential equation of first order is called **linear** if it contains the wanted function y and its first derivative y' in a linear way. It has the form

(7) $y' = P(x) \cdot y + Q(x)$.

With $Q(x) = 0$ equation (7) has the form

(8) $y' = P(x) \cdot y$

and is called **linear homogeneous** differential equation.

It is an equation with separated variables and has the general solution $y = C \cdot e^{\int P(x)dx}$

ClassPad:
dSolve($y' = P(x) \cdot y, x, y$)

$$\left\{ -\int P(x)dx + \ln(|y|) - \text{const}(1) = 0 \right\}$$

$$\text{solve}\left(-\int_{\square}^{\square} P(x)dx + \ln(|y|) - C_1 = 0, y \right)$$

$$\left\{ y = -e^{\int P(x)dx + C_1}, y = e^{\int P(x)dx + C_1} \right\}$$

$$y = C \cdot e^{\int_{\square}^{\square} P(x)dx}$$

The equation (7) with $Q(x) \neq 0$ is called **linear nonhomogeneous.** Its integration can be checked throughout the next methods

1) Variation of constants
we are looking for a solution of the equation (7) in a form

(9) $y = C(x) \cdot e^{\int P(x)dx}$,

which is a result of (8) if the constant C is replaced by the function $C(x)$. y is replaced from the (9) in the (7) and we have for the unknown function $C(x)$ an equation with separated variables

$y' = C'(x) \cdot e^{\int P(x)dx} + C(x) \cdot e^{\int P(x)dx} \cdot P(x)$

$= P(x) \cdot y + Q(x)$,

$C'(x) = Q(x) \cdot e^{-\int P(x)dx}$,

which has the general solution

(10) $C(x) = \int Q(x) \cdot e^{-\int P(x)dx} dx + C_1.$

We have in (10) a solution of $C(x)$. When we place it in (9) we find the general solution of (7).

ClassPad:
dSolve($y' = P(x) \cdot y + Q(x), x, y$)

$$\left\{ -\int e^{\int -P(x)dx} \cdot Q(x)dx \cdot e^{-\int -P(x)dx} - e^{-\int -P(x)dx} \cdot c \right.$$

has a general solution:

$$-\int e^{\displaystyle\int -P(x)dx}\cdot Q(x)dx\cdot e^{\displaystyle -\int -P(x)dx} \quad -$$

$$e^{\displaystyle -\int -P(x)dx}\cdot const(1)+y=0$$

$$solve\left(-\int e^{\displaystyle\int -P(x)dx}\cdot Q(x)dx\cdot e^{\displaystyle -\int -P(x)dx} \quad - \quad e^{\displaystyle -\int -P(x)}\right.$$

$$\left\{y=\left(\int e^{\displaystyle\int -P(x)dx}\cdot Q(x)dx+C\right)\cdot e^{\displaystyle -\int -P(x)dx}\right\}$$

2) Substitution

A substitution is made $y(x)=u(x)\cdot v(x)$ and the equation (7) has the form

$$y'=\frac{du}{dx}\cdot v+\frac{dv}{dx}\cdot u=P(x)\cdot u\cdot v+Q(x),$$

(11) $v\cdot\left(\dfrac{du}{dx}-P(x)\cdot u\right)+\left(\dfrac{dv}{dx}\cdot u-Q(x)\right)=0$

We choose the function $u(x)$ in a way that the content of the left side of the equation (11) will be equal to 0. For that purpose we integrate the equation with separated variables.

$$\frac{du}{dx}-P(x)\cdot u=0,$$

We choose one of its particular solutions $u=u_1(x)$, that is places to u in the left part of equation (11). We have then an equation with separated variables for $v(x)$:

$$\frac{dv}{dx} \cdot u_1(x) - Q(x) = 0$$

Its general solution is $v=v(x,C)$. According to the substitution the general solution of the equation (7) will be:

$$y(x) = u_1(x) \cdot v(x,C)$$

ClassPad:
dSolve($v' \cdot u_1(x) - Q(x) = 0, x, v$)

$$\left\{ -\int \frac{Q(x)}{u_1(x)}\, dx + v - \mathrm{const}(1) = 0 \right\}$$

$$\mathrm{solve}\left(-\int_{\square}^{\square} \frac{Q(x)}{u_1(x)}\, dx + v - C = 0, v \right)$$

$$\left\{ v = \int \frac{Q(x)}{u_1(x)}\, dx + C \right\}$$

2.6 Linear homogeneous differential equations with constant coefficients

Equation that has the form

(12) $y^{(n)} + p_1 \cdot y^{(n-1)} + \ldots + p_{n-1} \cdot y' + p_n \cdot y = 0$

where $p_1, p_2, \ldots, p_n$ are constants, $n \in \mathbf{N}$, is called **linear homogeneous differential equation with constant coefficients.** Its particular solution can be found with the help of the roots of the **characteristic equation.**

(13) $r^n + p_1 \cdot r^{n-1} + \ldots + p_{n-1} \cdot r + p_n = 0$

Every real root $r = a$ with multiplicity m of the equation (13) is in line with m particular solutions: $y_1 = e^{ax}$, $y_2 = xe^{ax}$, $\ldots$, $y_m = x^{m-1}e^{ax}$.

Every pair of complex roots $r = \alpha \pm \beta \cdot i$ with multiplicity m of the equation (13) is in line with m pairs of particular solutions:

$$e^{ax}\cos(\beta x), \qquad e^{ax}\sin(\beta x),$$
$$xe^{ax}\cos(\beta x), \qquad xe^{ax}\sin(\beta x),$$
$$\ldots$$
$$x^{m-1}e^{ax}\cos(\beta x), \quad x^{m-1}e^{ax}\sin(\beta x).$$

The general solution of (12) has the form

$$y = C_1 y_1 + C_2 y_2 + \ldots + C_n y_n,$$

where $y_1, y_2, \ldots, y_n$ are linear independent particular solutions of the equation (12) and C_1, C_2, $\ldots$, C_n are integration constants.

In that case we say that the functions y_1, y_2, $\ldots$, y_n form a **fundamental system** of basic solutions.

Definition: The functions $y_1(x), y_2(x), \ldots, y_n(x)$ are **linear dependent** if there is constants $\lambda_1, \lambda_2, \ldots, \lambda_n$ which are not all equal to zero with

$$\lambda_1 y_1(x) + \lambda_2 y_2(x) + \ldots + \lambda_n y_n(x) = 0,$$

with other words there is a nonzero linear combination of functions $y_1, y_2, \ldots, y_n$ equal to zero.

And vice versa, the functions $y_1(x), y_2(x), \ldots, y_n(x)$ are **linear independent** if there isn't their nonzero linear combination equal to zero.

Remark:
The necessary and sufficient condition the functions $y_1(x), y_2(x), \ldots, y_n(x)$ to be linear independent is the **Wronskian determinant**

$$W(x) = \det \begin{bmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{bmatrix} \neq 0$$

2.7 Linear homogeneous differential equations with nonconstant coefficients

An equation that has the form

$$(14) \quad y^{(n)} + p_1(x) \cdot y^{(n-1)} + \ldots + p_{n-1}(x) \cdot y' + p_n(x) \cdot y = 0$$

where $p_1(x), p_2(x), \ldots, p_n(x)$ are functions of x, continuous in some interval (a,b), $n \in \mathbf{N}$, is called **linear homogeneous differential equation with nonconstant coefficients.**

If $y_1(x), y_2(x), \ldots, y_n(x)$ is a fundamental system of basic solutions of the equation (14), its general solution has the form:
$$y = C_1 \cdot y_1(x) + C_2 \cdot y_2(x) + \ldots + C_n \cdot y_n(x)$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

1) if a particular solution $y_1(x)$ of the equation (14) is known, its order can be reduced by 1 (keeping the linear condition) using substitution:

$$y = y_1(x) \cdot \int u(x) dx$$

2) in a particular case when equation (14) has the form

(15)
$$(a \cdot x + b)^n \cdot y^{(n)} + p_1 \cdot (a \cdot x + b)^{(n-1)} \cdot y^{(n)} + \ldots$$
$$+ p_{n-1} \cdot (a \cdot x + b) \cdot y' + p_n \cdot y = 0$$

where a, b, p_1, p_2, p_n are constants, $n \in \mathbf{N}$, the **homogeneous differential equation** is called **Euler equation.**

If $a \cdot x + b > 0$ the solutions of (15) can be searched in the form

$y=(a \cdot x+b)^{\mu}$.

If we substitute
$y=(a \cdot x+b)^{\mu}$,
$y'=\mu \cdot a \cdot (a \cdot x+b)^{\mu-1}$,
$y''=\mu \cdot (\mu-1) \cdot a^2 \cdot (a \cdot x+b)^{\mu-2}$,
...
$y^{(n)}=\mu \cdot (\mu-1)...(\mu+1-n) \cdot a^{\mu} \cdot (a \cdot x+b)^{\mu-n}$

in the equation (15), we can find the **characteristic equation** to define the exponent μ:

(16) $\mu \cdot (\mu-1) \cdot ... \cdot (\mu+1-n) \cdot a^{\mu}+...$
$+p_{n-2} \cdot \mu \cdot (\mu-1) \cdot a^2+p_{n-1} \cdot \mu \cdot a+p_n=0$

If $\mu=\lambda$ is a real root with multiplicity m of this characteristic equation, λ is in line with m independent particular solutions:

$(a \cdot x+b)^{\lambda}$,
$(a \cdot x+b)^{\lambda} \ln(a \cdot x+b)$,
$(a \cdot x+b)^{\lambda} \ln(a \cdot x+b)^2$,
...
$(a \cdot x+b)^{\lambda} \ln(a \cdot x+b)^{m-1}$

If $\alpha \pm \beta \cdot i$ is a complex pair of roots with multiplicity m of the characteristic equation, $\alpha \pm \beta \cdot i$ is in line with m linear independent particular solutions:

$$\{(a \cdot x+b)^{\alpha} \cdot \cos(\beta \cdot \ln(a \cdot x+b)),$$
$$(a \cdot x+b)^{\alpha} \cdot \sin(\beta \cdot \ln(a \cdot x+b))\},$$
$$\{(a \cdot x+b)^{\alpha} \cdot \cos(\beta \cdot \ln(a \cdot x+b)) \cdot \ln(a \cdot x+b),$$
$$(a \cdot x+b)^{\alpha} \cdot \sin(\beta \cdot \ln(a \cdot x+b)) \cdot \ln(a \cdot x+b)\},$$
$$\ldots$$
$$\{(a \cdot x+b)^{\alpha} \cdot \cos(\beta \cdot \ln(a \cdot x+b)) \cdot (\ln(a \cdot x+b))^{m-1},$$
$$(a \cdot x+b)^{\alpha} \cdot \sin(\beta \cdot \ln(a \cdot x+b)) \cdot (\ln(a \cdot x+b))^{m-1}\}.$$

2.8 Linear nonhomogeneous differential equations with constant coefficients

An equation that has the form

$$(17) \quad y^{(n)}+p_1 \cdot y^{(n-1)}+\ldots+p_{n-1} \cdot y'+p_n \cdot y=f(x)$$

where $p_1, p_2, \ldots, p_n$ are constants, $n \in \mathbf{N}$, is called **linear nonhomogeneous differential equation with constant coefficients.** The general solution of (17) has the form

$$y=Y(x)+\eta(x)$$

where $Y(x)=C_1 \cdot y_1+C_2 \cdot y_2+\ldots+C_n \cdot y_n$ is the general solution of the homogeneous equation and $\eta(x)$ is a particular solution of the nonhomogeneous equation (17).
$\eta(x)$ can be searched using two methods:

1) The method of nondefined coefficients for equations with a special right part.

The right part can be appear in the form

$$f(x)=e^{\alpha \cdot x}((b_0 \cdot x^{m_1}+b_1 \cdot x^{m_1-1}+\ldots+b_{m_1}) \cdot \cos(\beta \cdot x)$$
$$+(d_0 \cdot x^{m_2}+d_1 \cdot x^{m_2-1}+\ldots+d_{m_2}) \cdot \sin(\beta \cdot x))$$

where α, β are constants, $\eta(x)$ must be found in the form with $m=\max(m_1, m_2)$

$$\eta(x)=e^{\alpha \cdot x} \cdot x^k((B_0 \cdot x^m+B_1 \cdot x^{m-1}+\ldots+B_m) \cdot \cos(\beta \cdot x)$$
$$+(D_0 \cdot x^{m_2}+D_1 \cdot x^{m-1}+\ldots+D_m) \cdot \sin(\beta \cdot x))$$

where $B_0, B_1, \ldots, B_m$, $D_0, D_1, \ldots, D_m$ are nondefined coefficients which should be found, k is the multiplicity of an eventual pair of complex roots $\alpha \pm \beta \cdot i$ (or one real root) of the characteristic equation

$$(18) \quad \mu^n+p_1 \cdot \mu^{n-1}+\ldots+p_{n-1} \cdot \mu+p_n=0$$

(If $\alpha \pm \beta \cdot i$ are not complex roots, or α, when $\beta=0$, is not a real root of equation (18), then $k=0$), and the m exponent is equal to the highest exponent among m_1 and m_2.

2) Lagrange method (variation of the arbitrary constants)

This method is used for equations with no special right part.

We must find $\eta(x)$ in the form

$$\eta(x)=C_1(x) \cdot y_1(x)+C_2(x) \cdot y_2(x)+\ldots+C_n(x) \cdot y_n(x)$$

where $y_1, y_2, \ldots, y_n$ is a fundamental system of solutions, and the functions $C_1(x), C_2(x), \ldots, C_n(x)$ are defined by the following system:

$$C_1'(x) \cdot y_1(x) + C_2'(x) \cdot y_2(x) + \ldots + C_n'(x) \cdot y_n(x) = 0$$

$$C_1'(x) \cdot y_1'(x) + C_2'(x) \cdot y_2'(x) + \ldots + C_n'(x) \cdot y_n'(x) = 0$$

$$C_1'(x) \cdot y_1^{(n-2)}(x) + C_2'(x) \cdot y_2^{(n-2)}(x) + \ldots$$
$$+ C_n'(x) \cdot y_n^{(n-2)}(x) = 0$$
$$\ldots$$

$$C_1'(x) \cdot y_1^{(n-1)}(x) + C_2'(x) \cdot y_2^{(n-1)}(x) + \ldots$$
$$+ C_n'(x) \cdot y_n^{(n-1)}(x) = f(x)$$

This system

$$
\begin{bmatrix}
y_1 & y_2 & \ldots & y_n \\
y_1' & y_2' & \ldots & y_n' \\
\ldots & \ldots & \ldots & \ldots \\
y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)}
\end{bmatrix}
\cdot
\begin{bmatrix}
C_1' \\
C_2' \\
\ldots \\
C_n'
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
\ldots \\
f
\end{bmatrix}
$$

has the solution

$$
\begin{bmatrix}
C_1' \\
C_2' \\
\ldots \\
C_n'
\end{bmatrix}
=
\begin{bmatrix}
y_1 & y_2 & \ldots & y_n \\
y_1' & y_2' & \ldots & y_n' \\
\ldots & \ldots & \ldots & \ldots \\
y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)}
\end{bmatrix}^{-1}
\cdot
\begin{bmatrix}
0 \\
0 \\
\ldots \\
f
\end{bmatrix}
$$

i.e.

$$\begin{cases} C_1'(x)=g_1(x) \\ C_2'(x)=g_2(x) \\ \ldots \\ C_n'(x)=g_n(x) \end{cases} \Bigg| C_1'(x),C_2'(x),\ldots,C_n'(x)$$

or

$$\begin{cases} C_1'(x)=\int g_1(x)dx+A_1 \\ C_2'(x)=\int g_2(x)dx+A_2 \\ \ldots \\ C_n'(x)=\int g_n(x)dx+A_n \end{cases} \Bigg| C_1(x),C_2(x),\ldots,C_n(x)$$

where $A_1,A_2,\ldots,A_n$ are arbitrary constants.

If the right part of equation (17) is formed by the sum of a few other functions

$$f(x)=f_1(x)+f_2(x)+\ldots+f_k(x)$$

and $\eta_i(x)$, $i=1,2,\ldots,k$, are some particular solutions of the equations

$$y^{(n)}+p_1\cdot y^{(n-1)}+\ldots+p_{n-1}\cdot y'+p_n\cdot y=f_i(x),$$
$$i=1,2,\ldots,k,$$

than the sum
$$\eta(x)=\eta_1(x)+\eta_2(x)+\ldots+\eta_k(x)$$
is a particular solution of equation (17).

2.9 Linear nonhomogeneous differential equations with nonconstant coefficients

An equation that has the form

(19)

$$y^{(n)}+p_1(x)\cdot y^{(n-1)}+...+p_{n-1}(x)\cdot y'+p_n(x)\cdot y=f(x)$$

where $p_1(x), p_2(x), ..., p_n(x)$ are functions of x, continuous in the given interval (a,b), $n \in \mathbf{N}$, is called **linear nonhomogeneous differential equation with nonconstant coefficients.**

The general solution of equation (19) has the form
$y=\bar{y}(x)+\bar{u}(x)$
where $\bar{y}(x)$ is the general solution of the homogeneous equation and $\bar{u}(x)$ is a particular solution of the nonhomogeneous equation.

There isn't a general method for solving equations such as (19).

1) If there is given in advance the general solution

$$\bar{y}(x)=C_1\cdot y_1+C_2\cdot y_2+...+C_n\cdot y_n$$

of the homogeneous equation, in order to find any particular solution $\bar{u}(x)$ of equation, in (19) can be used the **Lagrange method of variation the constants**

2) In a special case when the equation (19) has the form:

(20) $(a \cdot x + b)^n \cdot y^{(n)} + p_1 \cdot (a \cdot x + b)^{n-1} \cdot y^{(n-1)} + \ldots$

$$+ p_{n-1} \cdot (a \cdot x + b) \cdot y' + p_n \cdot y = f(x)$$

$f(x) \neq 0$ where $a, b, p_1, \ldots, p_n$ are constants, $n \in \mathbf{N}$, (20) is called nonhomogeneous differential equation – **Euler equation.** Using the substitution (when $a \cdot x + b > 0$)

$$a \cdot x + b = e^{t(x)}, \quad x = \frac{e^t - b}{a}, \quad t(x) = \ln(a \cdot x + b)$$

$$y(x) = y\left(\frac{e^t - b}{a} \right) = \varsigma(t),$$

$$y'(x) = \frac{d}{dt}(\varsigma) \cdot \frac{d}{dx}(t) = a \cdot e^{-t} \cdot \frac{d}{dt}(\varsigma),$$

$$y''(x) = \frac{d}{dx}\left(a \cdot e^{-t} \cdot \frac{d}{dt}(\varsigma) \right) = \frac{d}{dt}\left(a \cdot e^{-t} \cdot \frac{d}{dt}(\varsigma) \right) \cdot \frac{d}{dx}(t) =$$

$$= -a \cdot e^{-t} \cdot \frac{d}{dt}(\varsigma) \cdot a \cdot e^{-t} + a \cdot e^{-t} \cdot \frac{d^2}{dt^2}(\varsigma) \cdot a \cdot e^{-t}$$

$$= a^2 \cdot e^{-2t} \cdot \left(\frac{d^2}{dt^2}(\varsigma) - \frac{d}{dt}(\varsigma) \right),$$

$$y'''(x) = \frac{d}{dx}\left(a^2 \cdot e^{-2t} \cdot \left(\frac{d^2}{dt^2}(\varsigma) - \frac{d}{dt}(\varsigma) \right) \right)$$

$$= \frac{d}{dt}\left(a^2 \cdot e^{-2t} \cdot \left(\frac{d^2}{dt^2}(\varsigma) - \frac{d}{dt}(\varsigma) \right) \right) \cdot \frac{d}{dx}(t)$$

$$= a^3 \cdot e^{-3 \cdot t} \left(\frac{d^3}{dt^3}(\varsigma) - 3 \cdot \frac{d^2}{dt^2}(\varsigma) + 2 \cdot \frac{d}{dt}(\varsigma) \right), \ldots$$

equation (20) can be transformed into a nonhomogeneous differential equation with constant coefficients of the type (17) but with the independent variable t:

$$\frac{d^n}{dt^n}(\zeta)+P_1\cdot\frac{d^{n-1}}{dt^{n-1}}(\zeta)+\ldots+P_{n-1}\cdot\frac{d}{dt}(\zeta)+P_n\cdot\zeta=\phi(t)$$

2.10 Systems of ordinary differential equations

The system of k differential equations, giving the connection between the independent variable t and k functions $x_1(t),x_2(t),\ldots,x_k(t)$, and the derivatives of a higher range of these functions

$$x_1^{(P_1)}(t),x_2^{(P_2)}(t),\ldots,x_k^{(P_k)}(t),$$

it has the form

(21)

$$\left\{\begin{array}{l} x_1^{(P_1)}(t)=f_1(t,x_1,x_1',\ldots,x_1^{(P_1-1)}, \\ \qquad x_2,x_2',\ldots x_k,\ldots,x_k^{(P_k-1)}) \\ x_2^{(P_2)}(t)=f_2(t,x_1,x_1',\ldots,x_1^{(P_1-1)}, \\ \qquad x_2,x_2',\ldots x_k,\ldots,x_k^{(P_k-1)}) \\ \ldots \\ x_k^{(P_k)}(t)=f_k(t,x_1,x_1',\ldots,x_1^{(P_1-1)}, \\ \qquad x_2,x_2',\ldots x_k,\ldots,x_k^{(P_k-1)}) \end{array}\right| ,$$

The system is called **canonical** and has the order $n=p_1+p_2+\ldots+p_k$. If $p_1=p_2=\ldots=p_k=1$ the system is of **k-th order** and has the **normal form**:

$$(22) \quad \begin{cases} x_1'=f_1(t,x_1,\ldots,x_k) \\ x_2'=f_2(t,x_1,\ldots,x_k) \\ \ldots \\ x_k=f_k(t,x_1,\ldots,x_k) \end{cases} \bigg|_{x_1,x_2,\ldots,x_k}$$

A differential equation of k-order

$$(23) \quad x^{(k)}=f(t,x,x',\ldots,x^{(k-1)})$$

can be transformed into the system (22). And vice versa, the system (21) or (22) in most cases can be transformed into a differential equation (23). Solving this equation we can find the solution of the first given system. The easiest way to do so is by usingthe **method of the consecutive exception of the variables.** In some cases the system (21) or (22) can be obtained by more than one equation with one unknown function in each.

Let be $x_1,x_2,\ldots,x_k$, $x_1(t)=x(t)$. Now

$x_1'(t)=x_2(t)=x'(t)$,
$x_2'(t)=x_3(t)=x''(t)$,
$\ldots$
$x_{k-1}(t)=x_k(t)=x^{(k-1)}(t)$,
$x_k'(t)=f(t,x_1,x_2,\ldots,x_k)$,

is the system (22).

2.11 Linear homogeneous systems of n-th order with nonconstant coefficients and in a normal form

A system that has the form

$$(24) \begin{cases} x_1'=a_{11}(t) \cdot x_1 + a_{12}(t) \cdot x_2 + \ldots + a_{1n}(t) \cdot x_n \\ x_2'=a_{21}(t) \cdot x_1 + a_{22}(t) \cdot x_2 + \ldots + a_{2n}(t) \cdot x_n \\ \ldots \\ x_n'=a_{n1}(t) \cdot x_1 + a_{n2}(t) \cdot x_2 + \ldots + a_{nn}(t) \cdot x_n \end{cases},$$

or written in a matrix way

$$X'(t)=A(t) \cdot X(t)$$

where

$$A(t)=\begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{bmatrix}$$

$$X(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ \ldots \\ x_n(t) \end{bmatrix}, \quad X'(t)=\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \ldots \\ x_n'(t) \end{bmatrix}$$

and a_{ij}, $i-1,2,\ldots,n$, $j=1,2,\ldots,n$, functions of t, continuous in some interval (a,b), $n \in N$, is called differential equation of n-th order with nonconstant coefficients and in a normal form (the equations are solved concerning the derivatives)

If the determinant of the matrix

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_2 & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix},$$

gives n arbitrary particular solutions

$$(25) \quad X_k(t) = \begin{bmatrix} x_{1k}(t) \\ x_{2k}(t) \\ \cdots \\ x_{nk}(t) \end{bmatrix}, \quad k=1,2,\ldots,n,$$

of the system (24), is different from zero (these particular solutions are linear independent), $X_k(t)$ form the fundamental system of solutions.
The general solution of the system (24) is

$$(26) \quad \left\{ \begin{array}{l} x_1 = C_1 \cdot x_{11} + C_2 \cdot x_{12} + \ldots + C_n \cdot x_{1n} \\ x_2 = C_1 \cdot x_{21} + C_2 \cdot x_{22} + \ldots + C_n \cdot x_{2n} \\ \cdots \\ x_n = C_1 \cdot x_{n1} + C_2 \cdot x_{n2} + \ldots + C_n \cdot x_{nn} \end{array} \right|,$$

or $X(t) = \sum_{k=1}^{n} (C_k \cdot X_k(t))$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.
The integration of the system (24) normally is made by using the method of the consecutive exception of the unknown functions.

2.12 Linear homogeneous systems of n-th order with constant coefficients and in a normal form

System has the form

$$(27) \quad \begin{cases} x_1' = a_{11} \cdot x_1 + a_{12} \cdot x_2 + \ldots + a_{1n} \cdot x_n \\ x_2' = a_{21} \cdot x_1 + a_{22} \cdot x_2 + \ldots + a_{2n} \cdot x_n \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ x_n' = a_{n1} \cdot x_1 + a_{n2} \cdot x_2 + \ldots + a_{nn} \cdot x_n \end{cases},$$

or written in a matrix way

$$X'(t) = A \cdot X(t)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \ldots \\ x_n(t) \end{bmatrix}$$

and a_{ij}, $i-1,2,\ldots,n$, $j=1,2,\ldots,n$, are constants, $n \in N$, is called **linear homogeneous system of n-th order with constant coefficients and in a normal form.**

If (25) is a fundamental system of solutions of the system (27) then the general solution of this system has the form (26). To find this solution firstly it has to be solve the characteristic equation.

(28)

$$\det(A - r \cdot E) = \det \begin{bmatrix} a_{11}-r & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-r & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-r \end{bmatrix} = 0$$

where E is the unit matrix of n-order.

To each root of this equation concerning its multiplicity it should be found the particular solutions of this system:

1) If r_0 is a single real root then it is in line with only one particular solution

(29)

$$x_1 = A_{i1}(r_0) \cdot e^{r_0 \cdot t},$$

$$x_2 = A_{i2}(r_0) \cdot e^{r_0 \cdot t},$$

$$\cdots$$

$$x_n = A_{in}(r_0) \cdot e^{r_0 \cdot t}$$

where A_{ij}, $i \in \{1, \ldots, n\}, j = 1, \ldots, n$, is the cofactor (minor multiplied by the sign $(-1)^{i+j}$) of the element in a fixed row i and column j of the determinant (28).

2) If r_0 is a complex real root of (28), than another root of characteristic equation is the conjugate complex number $\acute{r}_0$ of r_0. Instead of having complex solutions $x_1, x_2, \ldots, x_n$ and $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ by (29), we take the real particular solutions.

$Re(x_1), Re(x_2), \ldots, Re(x_n), Im(x_1), Im(x_2), \ldots, Im(x_n)$

Remark: We use the Euler's formula.

3) If r_0 is a root of (28) with multiplicy $m \geq 2$, it is in line with m particular solutions.

$$x_{11} = A_{i1}(r_0) \cdot e^{r_0 \cdot t}, \quad \ldots, \quad x_{1m} = \frac{d^{m-1}}{dr^{m-1}} \left(A_{i1} \cdot e^{r \cdot t} \right) |_{r=r_0},$$

$$x_{21} = A_{i2}(r_0) \cdot e^{r_0 \cdot t}, \quad \ldots, \quad x_{2m} = \frac{d^{m-1}}{dr^{m-1}} \left(A_{i2} \cdot e^{r \cdot t} \right) |_{r=r_0},$$

$\ldots$

$$x_{n1} = A_{in}(r_0) \cdot e^{r_0 \cdot t}, \quad \ldots, \quad x_{nm} = \frac{d^{m-1}}{dr^{m-1}} \left(A_{in} \cdot e^{r \cdot t} \right) |_{r=r_0}.$$

2.13 Linear nonhomogeneous systems of n-th order and in a normal form

A system that has the form
(30)

$$\begin{cases} x_1' = a_{11}(t) \cdot x_1 + a_{12}(t) \cdot x_2 + \ldots + a_{1n}(t) \cdot x_n + f_1(t) \\ x_2' = a_{21}(t) \cdot x_1 + a_{22}(t) \cdot x_2 + \ldots + a_{2n}(t) \cdot x_n + f_2(t) \\ \ldots \\ x_n' = a_{n1}(t) \cdot x_1 + a_{n2}(t) \cdot x_2 + \ldots + a_{nn}(t) \cdot x_n + f_n(t) \end{cases},$$

or written in a matrix way

$$X'(t)=A(t)\cdot X(t)+F(t)$$

where

$$A(t)=\begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{bmatrix}$$

$$X(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ \ldots \\ x_n(t) \end{bmatrix}, \quad F(t)=\begin{bmatrix} f_1(t) \\ f_2(t) \\ \ldots \\ f_n(t) \end{bmatrix}$$

and a_{ij}, $i=1,2,..,n$, $j=1,2,...,n$, are functions of t, continuous in some interval (a,b), $n \in N$, is called **linear nonhomogeneous system of n-th order with nonconstant coefficients and in a normal form.**
The general solution of the system (30) has the form:

$$X(t)=X_0(t)+X_i(t), \text{ where } X_0(t)=\sum_{k=1}^{n}(C_k\cdot X_k(t)) \text{ is}$$

the general solution of the homogeneous system (24) of (30), and $X_i(t)$ is any particular solution of the system (30). Using the **Lagrange method**, $X_i(t)$ should be found in the way

$$X_i(t)=\sum_{k=1}^{n}(C_k(t)\cdot X_k(t)),$$

where the functions $C_k(t)$, $k=1,2,\ldots,n$, are defined correctly by using arbitrary functions fulfill the system.

$$\sum_{k=1}^{n} (C_k'(t) \cdot X_k(t)) = F(t).$$

The system has the form

$$\begin{cases} x_1' = a_{11} \cdot x_1 + a_{12} \cdot x_2 + \ldots + a_{1n} \cdot x_n + f_1(t) \\ x_2' = a_{21} \cdot x_1 + a_{22} \cdot x_2 + \ldots + a_{2n} \cdot x_n + f_2(t) \\ \ldots \\ x_n' = a_{n1} \cdot x_1 + a_{n2} \cdot x_2 + \ldots + a_{nn} \cdot x_n + f_n(t) \end{cases},$$

or written in a matrix way

$$X'(t) = A \cdot X(t) + F(t)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \ldots \\ x_n(t) \end{bmatrix}, \quad F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \ldots \\ f_n(t) \end{bmatrix}$$

and a_{ij}, $i=1,2,\ldots,n$, $j=1,2,\ldots,n$, are constants, $n \in \mathbf{N}$, is called **linear nonhomogeneous system of differential equation with constant coefficients and in a normal form.**

If the functions $f_1(t), f_2(t), \ldots, f_n(t)$ have the form

$$(31) \quad e^{\alpha \cdot t}((b_0 \cdot t^{m_1} + b_1 \cdot t^{m_1-1} + \ldots + b_{m_1}) \cdot \cos(\beta \cdot t) +$$
$$+ (d_0 \cdot t^{m_2} + d_1 \cdot t^{m_2-1} + \ldots + d_{m_2}) \cdot \sin(\beta \cdot t)),$$

where α and β are constants, then the particular solution $X_i(t)$ can be found using the methods of nondefined constants as we find in a way common to (31) and we should note the existence and nonexistance of an evental pair of complex roots $\alpha \pm \beta \cdot i$ (or one real root α when $\beta=0$) from multiplicity k of the characteristic equation.

If $m=\max(m_1, m_2)$ is equal to the higher exponent m_1 or m_2 and $\lambda = \alpha \pm \beta \cdot i$ is a root of the multiplicity k of the characteristic equation, then the particular equation $X_i(t)$ should be found in the form

$X_i(t)=$

$$\mathrm{Re}\left(\begin{bmatrix} B_{10} \cdot t^m + B_{11} \cdot t^{m-1} + \ldots + B_{1m} \\ B_{20} \cdot t^m + B_{21} \cdot t^{m-1} + \ldots + B_{2m} \\ \ldots \\ B_{n0} \cdot t^m + B_{n1} \cdot t^{m-1} + \ldots + B_{nm} \end{bmatrix} \cdot t^k \cdot e^{\lambda \cdot t}\right)$$

If λ is not a root of the characteristic equation, then $k=0$.

3. Solving problems using ClassPad
=====================================

3.1 Find the differential equations

3.1.1
(1) $y = a \cdot x$

In the equation above we have one variable ⇒ we have one integrated constant. We look for first derivative of $y(x)$

(2) $y'(x) = a$

(2)⇒(1) $y' - \dfrac{1}{x} \cdot y = 0$ is the wanted differential equation

ClassPad:

$\text{dSolve}\left(y' - \dfrac{1}{x} \cdot y = 0, x, y\right)$

$$\{y = |x| \cdot \text{const}(1)\}$$

⇒ $y = C \cdot x$, $C \in \mathbf{R}$.

3.1.2

(3) $y(x) = a \cdot x^2$

In the equation above we have one variable ⇒ we have one integrated constant. We look for first derivative of $y(x)$

(4) $y'(x) = 2 \cdot a \cdot x$

(3)⇒(4) $y' - \dfrac{2}{x} \cdot y = 0$ is the wanted differential equation

ClassPad:

Define $y(x)=a \cdot x^2$

done

$$\frac{d}{dx}(y(x))$$

$2 \cdot a \cdot x$

Check up:

$dSolve(y' - \frac{2}{x} \cdot y = 0, x, y)$

$$\left\{ y = x^2 \cdot const(1) \right\}$$

$\Rightarrow y = C \cdot x^2$, $C \in \mathbf{R}$.

3.1.3

(5) $x^2 + y^2 = a^2$, $y = y(x)$,

In the equation above we have one variable $\Rightarrow$ we have one integrated constant. We look for first derivative of $y(x)$

$$\frac{d}{dx}\left(x^2 + y^2 = a^2 \right) \Rightarrow 2x + 2yy' = 0$$

(6) $y'(x) = -\frac{x}{y} \Rightarrow y' + \frac{x}{y} = 0$

ClassPad:

DelVar y

done

$x^2 + (y(x))^2 = a^2 \Rightarrow$ Equation

$$(y(x))^2 + x^2 = a^2$$

$$\frac{d}{dx}(Equation)$$

$$2 \cdot \frac{d}{dx}(y(x)) \cdot y(x) + 2 \cdot x = 0$$

solve(2·z·y(x)+2·x=0,z)

$$\left\{z=\frac{-x}{y(x)}\right\}$$

⇒ $y'=\dfrac{-x}{y}$

Check up:

dSolve($y'+\dfrac{x}{y}=0,x,y$)

$$\left\{y=-\sqrt{-x^2+2\cdot const(1)}\ ,\ y=\sqrt{-x^2+2\cdot const(1)}\ \right\}$$

⇒ $y^2=-x^2+C$ ⇒ $x^2+y^2=C,\ C\geq 0.$

3.1.4

(7) $y(x)=a\cdot e^{\frac{x}{a}}$

In the equation above we have one variable⇒we have one integrated constant.We look for first derivative of $y(x)$

(8) $y'(x)=a\cdot e^{\frac{x}{a}}\cdot\dfrac{d}{dx}\left(\dfrac{x}{a}\right)$ ⇒ $y'(x)=e^{\frac{x}{a}}$

⇒ $a=\dfrac{y}{y'}$ ⇒ $y'(x)=e^{x\cdot\frac{y'}{y}}$ |logarithm ⇒

⇒ $\ln(y')=x\cdot\dfrac{y'}{y}$ ⇒ $y\cdot\ln(y')=x\cdot y'$

ClassPad:

$\dfrac{d}{dx}\left(y(x)=a\cdot e^{\frac{x}{a}}\right)$

$$\dfrac{d}{dx}(y(x))=e^{a^{-1}\cdot x}$$

$\text{solve}(z=e^{a^{-1} \cdot x}, a)$

$$\left\{ a = \frac{x}{\ln(z)} \right\}$$

$\Rightarrow a = \frac{x}{\ln(y')} \Rightarrow y = \frac{x}{\ln(y')} \cdot y'$

3.1.5

(9) $y + c^2 = (x-c)^2$

In the equation above we have one variable$\Rightarrow$we have one integrated constant$\Rightarrow$one derivative

(10) $y' = 2 \cdot x - 2 \cdot c \Rightarrow c = \frac{2 \cdot x - y'}{2}$

(10)$\Rightarrow$(9) $y + (\frac{2 \cdot x - y'}{2})^2 = (x - \frac{2 \cdot x - y'}{2})^2$

$\Rightarrow y = x^2 - x \cdot (2 \cdot x - y') \Rightarrow y' = \frac{y + x^2}{x}$

ClassPad:

DelVar y

done

Define $y(x) = (x-c)^2 - c^2$

done

$\frac{d}{dx}(y(x))$

$2 \cdot (x-c)$

Check up:

$\text{dSolve}(y' = \frac{y + x^2}{x}, x, y)$

$$\left\{ y = x^2 + |x| \cdot \text{const}(1) \right\}$$

$\Rightarrow y=x^2-2c\cdot x = (x-c)^2-c^2$

3.1.6

(11) $y=a\cdot x^2+b\cdot x$

two variables $\Rightarrow$ two integrated constants $\Rightarrow$ two derivatives

(12) $y'(x)=2\cdot a\cdot x+b$

(13) $y''(x)=2\cdot a \Rightarrow a=\dfrac{y''}{2}$, (13)$\Rightarrow$(12)

(14) $y'=2\cdot\dfrac{y''}{2}\cdot x+b \Rightarrow b=y'-y''\cdot x$

(14) и (13)$\Rightarrow$(11)

$$y=\frac{y''}{2}\cdot x^2+y'-2\cdot\frac{y''}{2}\cdot x)\cdot x \Rightarrow y''=2\cdot y'-\frac{2\cdot y}{x}$$

ClassPad:

DelVar y

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ done

Define $y(x)=a\cdot x^2+b\cdot x$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ done

$\dfrac{d}{dx}(y(x))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 2\cdot a\cdot x+b$

$\dfrac{d^2}{dx^2}(y(x))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2\cdot a$

$\Rightarrow$ a=y''/2, b=y'-y''$\cdot$x

$\Rightarrow y=\dfrac{y''}{2}\cdot x^2+(y'-y''\cdot x)\cdot x$

$\Rightarrow x^2 \cdot y'' - 2x \cdot y' + 2y = 0$ (Euler's equations)

dSolve($x^2 \cdot y'' - 2x \cdot y' + 2y = 0, x, y$)

$$\left\{ y = x^2 \cdot \text{const}(2) + x \cdot \text{const}(1) \right\}$$

3.2 Differential equations with seperate variables

3.2.1 $y' = x$

$\dfrac{dy}{dx} = x \Rightarrow dy = xdx$ |we integrate $\Rightarrow \int dy = \int xdx$

$y = \dfrac{x^2}{2} + C_1$, $C_1 \in \mathbf{R}$.

ClassPad:
dSolve($y' = x, x, y$)

$$\left\{ y = \dfrac{x^2}{2} + \text{const}(1) \right\}$$

3.2.2 $y' = \dfrac{y}{x}$

$\dfrac{dy}{dx} = \dfrac{y}{x} \Rightarrow \dfrac{dy}{y} = \dfrac{dx}{x}$ |we integrate $\Rightarrow \int \dfrac{dy}{y} = \int \dfrac{dx}{x}$

$\Rightarrow \ln|y| = \ln|x| + \ln(C_1) \Rightarrow y = |x| \cdot C_1 \Rightarrow y = x \cdot C_2$, $C_2 \in \mathbf{R}$.

ClassPad:
dSolve($y' = y/x, x, y$)

$$\{ y = |x| \cdot \text{const}(1) \}$$

3.2.3 $y'=-\dfrac{y}{x}$

$\dfrac{dy}{dx}=-\dfrac{y}{x}$ ⇒ $\dfrac{dy}{y}=-\dfrac{dx}{x}$ |we integrate ⇒ $\int\dfrac{dy}{y}=-\int\dfrac{dx}{x}$

$\ln|y|=-\ln|x|+\ln(C_1)$ ⇒ $y=\dfrac{C_1}{x}$, $C_1\in\mathbf{R}$.

ClassPad:
dSolve(y'=-y/x,x,y)

$$\left\{y=\dfrac{const(1)}{|x|}\right\}$$

3.2.4 $y'=y-x$, subst. $y-x=z(x)$ ⇒
$y'(x)-x'=z'(x)$ ⇒ $y'(x)=z'(x)+1$ ⇒
$z'(x)+1=z(x)$ ⇒ $z'=z-1$

$\dfrac{dz}{dx}=z-1$ |we integrate ⇒ $\int\dfrac{dz}{z-1}=-\int dx$

$\ln|z-1|=x+C_1$ ⇒ $z=e^x\cdot C_1+1$

⇒ $y=e^x\cdot C_1+x+1$, $C_1\in\mathbf{R}$.

ClassPad:
dSolve(y'=y-x,x,y)

$$\left\{y=e^x\cdot const(1)+x+1\right\}$$

3.2.5 $y'=1+y^2$ ⇒ $\dfrac{dy}{dx}=1+y^2$ |we integrate ⇒

$\int\dfrac{dy}{1+y^2}=\int dx$ ⇒

$\arctan(y)=x+C_1$ ⇒ $y=\tan(x+C_1)$, $C_1\in\mathbf{R}$.

ClassPad:

dSolve($y'=1+y^2$,x,y)

$$\{y=\tan(x+\text{const}(1))\}$$

3.2.6 $y'=\sqrt{x+y}$, subst. $x+y=z$

$z'=1+y'$ ⇒ $y'=z'-1$

⇒ $z'-1=\sqrt{z}$ ⇒ $\dfrac{dz}{\sqrt{z}+1}=dx$ |we integrate

$\displaystyle\int\dfrac{dz}{\sqrt{z}+1}=\int dx$

finding the solution by the help of ClassPad:

dSolve($y'=\sqrt{x+y}$,x,y)

$$\left\{\frac{e^{2\cdot\sqrt{x+y}}}{\left(\left|\sqrt{x+y}+1\right|\right)^2}=e^x\cdot\text{const}(1)\right\}$$

$\displaystyle\int_{\square}^{\square}\dfrac{1}{\sqrt{z}+1}\,dz=x+C$

$$-2\cdot\ln\left(\left|\sqrt{z}+1\right|\right)+2\cdot\sqrt{z}=x+C$$

ans|z=x+y

$$-2\cdot\ln\left(\left|\sqrt{x+y}+1\right|\right)+2\cdot\sqrt{x+y}=x+C$$

т.е. $-2\cdot\ln\left(\sqrt{x+y}+1\right)+2\cdot\sqrt{x+y}=x+C$, $C\in\mathbf{R}$.

3.2.7 $y'=x^2+1$ ⇒ $\dfrac{dy}{dx}=x^2+1$

⇒ $dy=(x^2+1)dx$ |we integrate

⇒ $y=\dfrac{x^3}{3}+x+C_1$, $C_1\in\mathbf{R}$.

ClassPad:

dSolve($y'=x^2+1,x,y$)

$$\left\{y=\frac{x^3}{3}+x+\text{const}(1)\right\}$$

3.2.8 $x \cdot y' = \dfrac{1}{\ln(y)}$

$\Rightarrow \ln(y) \cdot \dfrac{dy}{dx} = \dfrac{1}{x}$ $\Rightarrow \ln(y) \cdot dy = \dfrac{1}{x} \cdot dx$ |we integrate

$\Rightarrow \int(\ln(y) \cdot dy) = \int(\dfrac{1}{x} \cdot dx)$ |we integrate

$\Rightarrow y \cdot \ln(y) - \int y \cdot \dfrac{1}{y} dy = \ln|x| + C_1$

$\Rightarrow y \cdot \ln(y) - y + C_2 = \ln|x| + C_1$

$\Rightarrow \ln(y)^y - y = \ln|x| + C$, $C = C_1 + C_2$

$\Rightarrow y^y \cdot e^{-y} = x \cdot C$, $C \in \mathbf{R}$.

ClassPad:

dSolve($x \cdot y' = \dfrac{1}{\ln(y)}, x, y$)

$$\left\{y^y \cdot e^{-y} = |x| \cdot \text{const}(1)\right\}$$

3.2.9 $y' + y \cdot \tan(x) = 0$

$y' = -y \cdot \tan(x)$ $\Rightarrow$ $\dfrac{dy}{dx} = -y \cdot \tan(x)$

$\Rightarrow \int \dfrac{dy}{y} = -\int \dfrac{\sin(x)}{\cos(x)} dx$ $\Rightarrow$ $\int \dfrac{dy}{y} = \int \dfrac{d(\cos(x))}{\cos(x)}$

$\Rightarrow \ln|y| = \ln|\cos(x)| + C_1$

$\Rightarrow y = |\cos(x)| \cdot C_1$ $\Rightarrow$ $y = \cos(x) \cdot C_2$, $C_2 \in \mathbf{R}$.

ClassPad:

dSolve(y'+y·tan(x)=0,x,y)

$$\{y=|\cos(x)|\cdot const(1)\}$$

3.2.10 ln(y')=x−ln(y)

ln(y')+ln(y)=x ⇨ ln(y'·y)=x |logarithm

⇨ y'·y=e^x ⇨ $\dfrac{dy}{dx}$·y=e^x |we integrate

⇨ $\int y\cdot dy=\int e^x\cdot dx$ ⇨ $\dfrac{y^2}{2}=e^x+C_1$

⇨ $y_{1,2}=\pm\sqrt{2\cdot(e^x+C_1)}=\pm\sqrt{2\cdot e^x+C_2}$, $C_2\in\mathbf{R}$.

dSolve(ln(y')=x−ln(y),x,y)

$$\left\{y=-\sqrt{2\cdot\left(e^x+const(1)\right)}\,,y=\sqrt{2\cdot\left(e^x+const(1)\right)}\,\right\}$$

3.3 Solve the following differential equations with their initial conditions

3.3.1 $y'=\dfrac{2\cdot x}{x^2+1}$, y(0)=1

$\dfrac{dy}{dx}=\dfrac{2\cdot x}{x^2+1}$ ⇨ $dy=\dfrac{2\cdot x}{x^2+1}\cdot dx$ |we integrate

$\int dy=2\cdot\int\dfrac{x}{x^2+1}\cdot dx$ ⇨ $y+C_1=\int\dfrac{d(x^2+1)}{x^2+1}$

⇨ $y+C_1=\ln|x^2+1|+C_2$,$C=-C_1+C_2$

⇨ $y=\ln|x^2+1|+C$ при y(0)=1 ⇨ C=1

⇨ $y=\ln|x^2+1|+1=\ln(x^2+1)+1$

ClassPad:

dSolve($y'=\dfrac{2\cdot x}{x^2+1}$,x,y,x=0,y=1)

$$\left\{y=\ln\left(x^2+1\right)+1\right\}$$

3.3.2 $y'=\sinh(x)$, $y(0)=1$

$\dfrac{dy}{dx}=\sinh(x)$ |we integrate

$\Rightarrow$

$\int dy=\int\sinh(x)dx$ $\Rightarrow$ $y=\cosh(x)+C_1$

при $y(0)=1$ $\Rightarrow$ $C_1=0$,

т.е. $y=\cosh(x)$

ClassPad:

dSolve($y'=\sinh(x)$,x,y)

$$\{y=\cosh(x)+\text{const}(1)\}$$

dSolve($y'=\sinh(x)$,x,y,x=0,y=1)

$$\{y=\cosh(x)\}$$

3.3.3 $y'=\dfrac{1}{1+\sqrt{x}}$ (ср. 3.2.7)

$\dfrac{dy}{dx}=\dfrac{1}{1+\sqrt{x}}$ |we integrate

$\Rightarrow$

(15) $y=\int\dfrac{1}{1+\sqrt{x}}dx$

we solve it noticing this $\int\dfrac{1}{1+\sqrt{x}}dx$ as we do the

substitution $x=t^2$, $dx=2\cdot t\cdot dt$,

⇒

(16) $\int \dfrac{1}{1+\sqrt{t^2}} \cdot 2 \cdot t \, dt = \int \dfrac{2t}{1+t} \cdot dt = 2 \cdot \int (1 - \dfrac{1}{t+1}) \, dt$

$= 2 \cdot \int 1 \, dt - 2 \int \dfrac{1}{t+1} \, dt = 2 \cdot t - 2 \cdot \ln|t+1| + C$

$= 2 \cdot \sqrt{x} - 2 \cdot \ln|\sqrt{x} + 1| + C$

(16)⇒(15)

⇒ $y = 2 \cdot \sqrt{x} - 2 \cdot \ln|\sqrt{x} + 1| + C$

$\quad = 2 \cdot \sqrt{x} - 2 \cdot \ln(\sqrt{x} + 1) + C, \quad C \in \mathbf{R}.$

ClassPad:

$\mathrm{dSolve}\left(y' = \dfrac{1}{1+\sqrt{x}}, x, y\right)$

$\left\{ y = -2 \cdot \ln\left(\left|\sqrt{x} + 1\right|\right) + 2 \cdot \sqrt{x} + \mathrm{const}(1) \right\}$

3.3.4 $y' = e^y$

$\dfrac{dy}{dx} = e^y \;\Rightarrow\; \int dx = \int \dfrac{dy}{e^y} \;\Rightarrow\; \int dx = -\int e^{-y} \, d(-y)$

$\Rightarrow\; x = -\dfrac{1}{e^y} + C_1 \;\Rightarrow\; e^y = \dfrac{-1}{x - C_1} \quad |\text{logarithm}$

$\Rightarrow\; y = \ln\left(\dfrac{-1}{x - C_1}\right) = \ln\left(\dfrac{1}{C_1 - x}\right) = -\ln(C_1 - x), \quad C_1 \in \mathbf{R}.$

ClassPad:

$\mathrm{dSolve}(y' = e^y, x, y)$

$\left\{ y = \ln\left(\dfrac{-1}{x - \mathrm{const}(1)}\right) \right\}$

3.3.5 $y' = 2^{-y}$

$$\frac{dy}{dx} = 2^{-y} \;\Rightarrow\; dx = \frac{dy}{2^{-y}} \quad |\text{we integrate}$$

$$\Rightarrow\; x = \int \frac{dy}{2^{-y}} = \int 2^y \, dy \Rightarrow\; x = \frac{2^y}{\ln 2} + C$$

$$\Rightarrow\; y = \frac{\ln(\ln(2) \cdot (x+C))}{\ln(2)}, \quad C \in \mathbf{R}.$$

ClassPad:

dSolve($y' = 2^{-y}, x, y$)

$$\left\{ y = \frac{\ln(x \cdot \ln(2) + \ln(2) \cdot \text{const}(1))}{\ln(2)} \right\}$$

3.3.6 $y' = \frac{y-1}{x-1}$

$$\frac{dy}{dx} = \frac{y-1}{x-1} \;\Rightarrow\; \frac{dy}{y-1} = \frac{dx}{x-1} \quad |\text{we integrate}$$

$$\Rightarrow\; \ln|y-1| = \ln|x-1| + C_1$$

$y = (x-1) \cdot C_1 + 1$, при $y(0) = 1 \Rightarrow C_1 = 0$,

т.е. $y = 1$

ClassPad:

dSolve$\left(y' = \frac{y-1}{x-1}, x, y \right)$

$$\{ y = |x-1| \cdot \text{const}(1) + 1 \}$$

dSolve$\left(y' = \frac{y-1}{x-1}, x, y, x=0, y=1 \right)$

$$\{ y = 1 \}$$

3.3.7 $y' \cdot (x^2 - x) = y - 1$

$y' = \dfrac{y-1}{x^2-x} \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{y-1}{x^2-x}$ |we integrate

$\Rightarrow \displaystyle\int \dfrac{dy}{y-1} = \int \dfrac{dx}{x^2+x} \;\Rightarrow\; \ln(y-1) = \int \dfrac{dx}{x \cdot (x-1)}$

$\Rightarrow \ln(|y-1|) = -\displaystyle\int \dfrac{dx}{x} + \int \dfrac{dx}{x-1}$

$\Rightarrow \ln(|y-1|) = -\ln|x| + \ln|x-1| + C_1$

$\Rightarrow y-1 = \dfrac{|x-1| \cdot C_2}{|x|} \;\Rightarrow\; y = \dfrac{(x-1) \cdot C_3}{x} + 1, \; C_3 \in \mathbf{R}.$

ClassPad:

dSolve($y' \cdot (x^2 - x) = y - 1, x, y$)

$$\left\{ y = \dfrac{|x-1| \cdot \text{const}(1)}{|x|} + 1 \right\}$$

3.3.8 $y' \cdot y^2 = \sin(3 \cdot x), \; y(0) = 2$

$y' = \dfrac{\sin(3 \cdot x)}{y^2} \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{\sin(3 \cdot x)}{y^2}$ |we integrate

$\Rightarrow \displaystyle\int y^2 \cdot dy = \dfrac{1}{3} \cdot \int \sin(3 \cdot x) \, d(3 \cdot x)$

$\Rightarrow \dfrac{y^3}{3} = \dfrac{1}{3} \cdot (-\cos(3 \cdot x))$

$\Rightarrow y = \sqrt[3]{-\cos(3 \cdot x) + 3 \cdot C_1}$, when $y(0) = 2 \;\Rightarrow\; C_1 = 3$,

т.е. $y = \sqrt[3]{-\cos(3 \cdot x) + 9}$

ClassPad:

dSolve($y' \cdot y^2 = \sin(3 \cdot x), x, y$)

$$\left\{ y = (-\cos(3 \cdot x) + 3 \cdot \text{const}(1))^{\frac{1}{3}} \right\}$$

$\text{dSolve}(y' \cdot y^2 = \sin(3 \cdot x), x, y, x=0, y=2)$

$$\left\{ y = (-\cos(3 \cdot x) + 9)^{\frac{1}{3}} \right\}$$

3.3.9 $y' = \dfrac{2 \cdot x \cdot y}{1 + x^2}$

$(1 + x^2)dy = (2 \cdot x \cdot y)dx$

$\Rightarrow \dfrac{dy}{y} = \dfrac{2 \cdot x \, dx}{1 + x^2} \Rightarrow$ |we integrate

$\Rightarrow \int \dfrac{dy}{y} = \int \dfrac{2 \cdot x \, dx}{1 + x^2} \Rightarrow \ln|y| = \dfrac{2}{2} \cdot \int \dfrac{d(x^2 + 1)}{x^2 + 1}$

$\Rightarrow \ln(|y|) = \ln(x^2 + 1) + C$

$\Rightarrow y = (x^2 + 1) \cdot C_1, \quad C_1 \in \mathbf{R}.$

ClassPad:

$\text{dSolve}(y' = \dfrac{2 \cdot x \cdot y}{1 + x^2}, x, y)$

$$\left\{ y = \left(x^2 + 1 \right) \cdot \text{const}(1) \right\}$$

3.3.10 $(1 + e^x) \cdot y \cdot y' = e^x, \quad y(1) = 1$

$\Rightarrow y \cdot y' = \dfrac{e^x}{1 + e^x} \Rightarrow y \cdot \dfrac{dy}{dx} = \dfrac{e^x}{1 + e^x}$ |we integrate

$\Rightarrow \int y \, dy = \int \dfrac{e^x}{1 + e^x} dx \Rightarrow \dfrac{y^2}{2} = \int \dfrac{d(e^x + 1)}{e^x + 1}$

$\Rightarrow y^2 = 2 \cdot \ln(e^x + 1) + 2 \cdot C_1$

$\Rightarrow y = \pm \sqrt{2 \cdot \ln(e^x + 1) + 2 \cdot C_1} \quad$, when $y(1) = 1$

$\Rightarrow C_1 = \dfrac{1}{2} - \ln(e + 1), \quad y = +\sqrt{2 \cdot \ln(e^x + 1) - 2 \cdot \ln(e + 1) + 1}$

ClassPad:

dSolve$\left(y \cdot y' = \dfrac{e^x}{1+e^x}, x, y\right)$

$\left\{y = -\sqrt{2 \cdot \left(\ln\left(e^x+1\right)+const(1)\right)}, y = \sqrt{2 \cdot \left(\ln\left(e^x+1\right)+const\blacktriangleright\right.}\right.$

dSolve$\left(y \cdot y' = \dfrac{e^x}{1+e^x}, x, y, x=1, y=1\right)$

$\left\{y = \sqrt{2 \cdot \ln\left(e^x+1\right)-2 \cdot \ln(e+1)+1}\right\}$

3.3.11 $y' = \dfrac{x}{y} + \dfrac{y}{x}$

$y = \dfrac{1}{\dfrac{y}{x}} + \dfrac{y}{x}$ subst. $\dfrac{y}{x} = z$ ⇒ $y = z \cdot x$

⇒ $y' = z + z' \cdot x$

⇒ $z + z' \cdot x = \dfrac{1}{z} + z$ ⇒ $z dz = \dfrac{dx}{x}$ |we integrate

⇒ $\int z dz = \int \dfrac{dx}{x}$ ⇒ $\dfrac{z^2}{2} = \ln|x| + C_1$

⇒ $z^2 = 2 \cdot \ln|x| + 2 \cdot C_1$

⇒ $z = \pm\sqrt{2 \cdot \ln|x| + 2 \cdot C_1}$ ⇒ $\dfrac{y}{x} = \pm\sqrt{2 \cdot \ln|x| + 2 \cdot C_1}$

⇒ $y = \pm x \cdot \sqrt{2 \cdot \ln|x| + 2 \cdot C_1} = \pm x \cdot \sqrt{\ln(x^2) + C}$, $C \in \mathbf{R}$.

ClassPad:

dSolve$\left(y' = \dfrac{x}{y} + \dfrac{y}{x}, x, y\right)$

$\left\{y = -x \cdot \sqrt{2 \cdot \ln(|x|)+const(1)}, y = x \cdot \sqrt{2 \cdot \ln(|x|)+const(1)}\blacktriangleright\right.$

3.3.12 $y'=(x+y+1)^2$

subst. $x+y+1=z \Rightarrow y=z-x-1$

$\Rightarrow y'=z'-1 \Rightarrow z'-1=z^2$

$\Rightarrow \dfrac{dz}{dx}=z^2+1 \Rightarrow \int\dfrac{dz}{z^2+1}=\int dx$

$\Rightarrow \arctan(z)=x+C_1$

$\Rightarrow z=\tan(x+C_1) \Rightarrow x+y+1=\tan(x+C_1)$

$\Rightarrow y=\tan(x+C_1)-x-1, \; C_1\in\mathbf{R.}$

ClassPad:
dSolve($y'=(x+y+1)^2,x,y$)

$$\{y=\tan(x+const(1))-x-1\}$$

3.3.13 $y'-y\cdot\tan(x)=0$

$\dfrac{y'}{y}=\tan(x) \Rightarrow \dfrac{dy}{y}=\tan(x)dx$ |we integrate

$\Rightarrow \ln(|y|)=-\ln(|\cos(x)|)+\ln(|C|)$

$\Rightarrow \ln(|y|)=\ln\left(\left|\dfrac{C}{\cos(x)}\right|\right)$

$\Rightarrow y=\dfrac{C}{\cos(x)}, \; C\in\mathbf{R.}$

ClassPad:
dSolve($y'-y\cdot\tan(x)=0,x,y$)

$$\left\{y=\dfrac{const(1)}{|\cos(x)|}\right\}$$

3.3.14 $y' \cdot x + y = 0$, $y(1) = 4$

$\Rightarrow y' + \dfrac{1}{x} \cdot y = 0 \Rightarrow \dfrac{y'}{y} = -\dfrac{1}{x}$ |we integrate

$\Rightarrow \ln(|y|) = \ln\left(\left|\dfrac{C}{x}\right|\right) \Rightarrow y(1) = 4$

$\Rightarrow 4 = \dfrac{C}{1} \Rightarrow C = 4 \Rightarrow y = \dfrac{4}{x}$, $x \neq 0$.

ClassPad:
dSolve(y'·x+y=0,x,y)

$$\left\{ y = \dfrac{\text{const}(1)}{|x|} \right\}$$

dSolve(y'·x+y=0,x,y,x=1,y=4)

$$\left\{ y = \dfrac{4}{|x|} \right\}$$

3.3.15 $y'' - y = 0$, $y(0) = 0$, $y'(0) = 2$

the characteristical equation is: $r^2 - 1 = 0$
$\Rightarrow r_1 = 1$, $r_2 = -1$
$\Rightarrow y = C_1 \cdot e^{1 \cdot x} + C_2 \cdot e^{-1 \cdot x} \Rightarrow C_1 = ?$, $C_2 = ?$
when $y(0) = 0 \Rightarrow C_1 = -C_2$
when $y'(0) = 2 \Rightarrow C_1 - C_2 = 2 \Rightarrow 2 \cdot C_1 = 2$
$\Rightarrow C_1 = 1$, $C_2 = -1$
$\Rightarrow y = e^x - e^{-x}$

ClassPad:
solve($r^2 - 1 = 0$,r)

$\{r = -1, r = 1\}$

dSolve(y"-y=0,x,y)

$$\{y=e^x \cdot const(2)+e^{-x} \cdot const(1)\}$$

Define y(x)=$e^x \cdot C2+e^{-x} \cdot C1$

done

$$\begin{cases} y(0)=0 \\ \dfrac{d}{dx}(y(x))=2|x=0 \end{cases}\Bigg|_{C1,C2}$$

$$\{C1=-1,C2=1\}$$

dSolve(y"-y=0,x,y,x=0,y=0,x=0,y'=2)

$$\{y=e^x-e^{-x}\}$$

3.3.16 y"-4·y'-5·y=0

the characteristical equation is: $r^2-4 \cdot r-5=0$

$\Rightarrow$ $r_{1,2}=\dfrac{+16\pm\sqrt{16+20}}{2}$ $\Rightarrow$ $r_1=5$, $r_2=-1$

$\Rightarrow$ y=$C_1 \cdot e^{5 \cdot x}+C_2 \cdot e^{-x}$, $C_1,C_2 \in \mathbb{R}$.

ClassPad:

solve($r^2-4 \cdot r-5=0$,r)

$$\{r=-1,r=5\}$$

dSolve(y"-4·y'-5·y=0,x,y)

$$\{y=e^{5 \cdot x} \cdot const(2)+e^{-x} \cdot const(1)\}$$

3.3.17 $y'-\dfrac{y}{x+1}=\dfrac{1}{(x+1)^2}$

$$P(x) = \frac{1}{x+1}, \quad Q(x) = \frac{1}{(x+1)^2}$$

$$y = C_1 \cdot e^{\int P(x)dx} \quad \Rightarrow \quad C_1 = C(x) = ?$$

$$C'(x) = \int Q(x) \cdot e^{-\int P(x)dx} dx$$

$$P(x) = \frac{y}{x+1}, \quad Q(x) = \frac{1}{(x+1)^2}$$

$$y = C \cdot e^{\int \frac{1}{x+1}dx} \quad \Rightarrow \quad y = C \cdot e^{\ln|x+1|} \quad \Rightarrow \quad C = C(x) = ?$$

$$C'(x) = \int \left(\frac{1}{(x+1)^2} \cdot \frac{1}{x+1} \right) dx$$

$$C'(x) = \int \left(\frac{1}{(x+1)^3} \right) dx \quad \Rightarrow$$

$$y = -\frac{1}{2(x+1)^2} + C$$

$$= C \cdot (x+1) - \frac{1}{2} \cdot \frac{1}{x+1}, \quad C \in \mathbf{R}.$$

ClassPad:

$$\text{dSolve}\left(y' - \frac{y}{x+1} = \frac{1}{(x+1)^2}, x, y \right)$$

$$\left\{ -\int \frac{1}{|x+1| \cdot (x+1)^2} dx \cdot |x+1| - |x+1| \cdot \text{const}(1) + y = 0 \right\}$$

$$\text{dSolve}\left(y' - \frac{y}{x+1} = \frac{1}{(x+1)^2}, x, y \right) | x > 1$$

$$\left\{ y = x \cdot \text{const}(1) - \frac{1}{2 \cdot (x+1)} + \text{const}(1) \right\}$$

3.3.18 $y' = \frac{y}{x} + 2$

$P(x) = \frac{1}{x}, \quad Q(x) = 2$

$y = C \cdot e^{\int P(x)dx} \Rightarrow y = C \cdot e^{\ln|x|} = C \cdot x$

$\Rightarrow C = C(x) = ?$

$C'(x) = \int Q(x) \cdot e^{-\int P(x)dx} dx$

$C'(x) = \int \left(2 \cdot \frac{1}{x} \right) dx \Rightarrow C'(x) = 2\ln|x| + C_1$

$y = \left(\ln(x^2) + C_1 \right) \cdot x$

ClassPad:

$\text{dSolve}(y' = \frac{y}{x} + 2, x, y)$

$$\{ y = 2 \cdot x \cdot \ln(|x|) + |x| \cdot \text{const}(1) \}$$

3.4 Linear homogeneous/nonhomogeneous differential equations with constant/ nonconstant coefficients

3.4.1 $y'''+y'=e^x$

$y=y_0+\eta$ we should find $y_0=?,\eta=?$

the characteristical equation is: $r^3+r=0$

$r \cdot (r^2+1)=0 \Rightarrow r_1=0,\ r_2=i \cdot 1,\ r_3=-i \cdot 1$

$y_0=C_1 e^{0 \cdot x}+C_2 \cdot \cos(x)+C_3 \cdot \sin(-x)$

$\eta \Rightarrow e^x \Rightarrow \eta=A \cdot e^x \cdot x^m,\ m=0$

$\eta=A \cdot e^x$

$\eta'=A \cdot e^x \qquad \Rightarrow \eta'''+\eta'=e^x$

$\eta''=A \cdot e^x \qquad A \cdot e^x+A \cdot e^x=e^x \Rightarrow A=\dfrac{1}{2}$

$\eta'''=A \cdot e^x$

$y=C_1+C_2 \cdot \cos(x)+C_3 \cdot \sin(-x)+\dfrac{1}{2} \cdot e^x,\ \ C_1,C_2,C_3 \in \mathbf{R}.$

ClassPad:

dSolve($y'''+y'=e^x$,x,y)

$$\left\{ y=\frac{e^x}{2}+\cos(x) \cdot const(2)+\sin(x) \cdot const(3)+const(1) \right\}$$

3.4.2 $y''+13 \cdot y'+40 \cdot y=0$

the characteristical equation is: $r^2+13 \cdot r+40=0$

$$r_{1,2}=\frac{-13 \pm \sqrt{169-160}}{2}$$

$r_1=-5$, $r_2=-8$

$\Rightarrow y=C_1 \cdot e^{-5 \cdot x}+C_2 \cdot e^{-8 \cdot x}$, $C_1,C_2 \in \mathbb{R}$.

ClassPad:

solve(r^2+13·r+40=0,r)

$\{r=-8,r=-5\}$

dSolve(y''+13·y'+40·y=0,x,y)

$\left\{y=e^{-5 \cdot x} \cdot \text{const}(2)+e^{-8 \cdot x} \cdot \text{const}(1)\right\}$

3.4.3 y''−6·y'+34·y=0

the characteristical equation is: r^2−6·r+34=0

$$r_{1,2}=\frac{-6 \pm \sqrt{36-136}}{2}$$

$r_1=-3+i \cdot 5$, $r_2=-5-i \cdot 5$

$\Rightarrow y=C_1 \cdot \cos(5 \cdot x) \cdot e^{3 \cdot x}+C_2 \cdot \sin(5 \cdot x) \cdot e^{3 \cdot x}$, $C_1,C_2 \in \mathbb{R}$.

ClassPad:

dSolve(y''−6·y'+34·y=0,x,y)

$\left\{y=\cos(5 \cdot x) \cdot e^{3 \cdot x} \cdot \text{const}(1)+\sin(5 \cdot x) \cdot e^{3 \cdot x} \cdot \text{const}(2)\right\}$

3.4.4 y''+4·y'+4·y=0

the characteristical equation is: r^2+4·r+4=0

$$r_{1,2}=\frac{-4 \pm \sqrt{16-16}}{2} \Rightarrow r_1=r_2=-2$$

$y=(C_1+C_2 \cdot x) \cdot e^{-2 \cdot x}$, $C_1,C_2 \in \mathbb{R}$.

ClassPad:

solve($r^2+4 \cdot r+4=0,r$)

$$\{r=-2\}$$

dSolve($y''+4 \cdot y'+4 \cdot y=0,x,y$)

$$\left\{y=x \cdot e^{-2 \cdot x} \cdot \text{const}(2)+e^{-2 \cdot x} \cdot \text{const}(1)\right\}$$

3.4.5 $y'''+y'=\sin(x)$

$y=y_0+\eta$ we should find $y_0=?, \eta=?$

the characteristical equation is: $r^3+r=0$

$r \cdot (r^2+1)=0 \Rightarrow r_1=0, \; r_2=+i \cdot 1, \; r_3=-i \cdot 1$

$y_0=C_1 \cdot e^{0 \cdot x}+C_2 \cdot \cos(x)+C_3 \cdot \sin(x)$

$\sin(x) \Rightarrow \eta \; \Rightarrow \; \eta=e^{0 \cdot x}(A \cdot \sin(x)+B \cdot \cos(x)) \cdot x^m, \; m=1.$

$\eta'=A \cdot x \cdot \cos(x)-B \cdot x \cdot \sin(x)+B \cdot \cos(x)+A \cdot \sin(x)$
$\eta''=-B \cdot \cos(x)-A \cdot \sin(x)$
$\eta'''=-A \cdot x \cdot \cos(x)+B \cdot x \cdot \sin(x)-3 \cdot B \cdot \cos(x)-3 \cdot A \cdot \sin(x)$
$\eta'''+\eta'=\sin(x)$

when $x=0, x=1$ we have the following system

$$\left. \begin{cases} -2 \cdot B=0 \\ -2 \cdot A \cdot \sin(1)-2 \cdot B \cdot \cos(1)=\sin(1) \end{cases} \right|_{A,B}$$

$\Rightarrow A=-\dfrac{1}{2}, \; B=0$

$y=C_1+C_2 \cdot \cos(x)+C_3 \cdot \sin(x)-\dfrac{x}{2} \cdot \sin(x), \; C_1, C_2, C_3 \in \mathbf{R}.$

ClassPad:
Define $\eta(x)=(A \cdot \sin(x)+B \cdot \cos(x)) \cdot x$

done

$$\frac{d^3}{dx^3}(\eta(x)) + \frac{d}{dx}(\eta(x)) = \sin(x) \Rightarrow eq3$$

$$-2 \cdot B \cdot \cos(x) - 2 \cdot A \cdot \sin(x) = \sin(x)$$

eq3|x=0

$$-2 \cdot B = 0$$

eq3|x=1

$$-2 \cdot A \cdot \sin(1) - 2 \cdot B \cdot \cos(1) = \sin(1)$$

$$\left\{ \begin{array}{l} eq3|x=0 \\ eq3|x=1 \end{array} \right|_{A,B}$$

$$\left\{ A = -\frac{1}{2}, B = 0 \right\}$$

dSolve(y'''+y'=sin(x),x,y)

$$\left\{ y = \frac{-x \cdot \sin(x)}{2} + \cos(x) \cdot const(2) - \frac{3 \cdot \cos(x)}{4} + \sin(x) \cdot cons \blacktriangleright \right.$$

т.е.

$$y = \frac{-x \cdot \sin(x)}{2} + \cos(x) \cdot const(2) - \frac{3 \cdot \cos(x)}{4}$$
$$+ \sin(x) \cdot const(3) + const(1)$$

$$\cos(x) \cdot const(2) - \frac{3 \cdot \cos(x)}{4} = C2 \cdot \cos(x)$$

ClassPad solution:

$$y(x) = \frac{-x \cdot \sin(x)}{2} + C1 + C2 \cdot \cos(x) + C3 \cdot \sin(x)$$

3.4.6 $y'' - 3 \cdot y' + 2 \cdot y = x - e^x$

$y = y_0 + \eta_1 + \eta_2$ we should find $y_0 = ?, \eta_1 = ?, \ \eta_2 = ?$

the characteristical equation is: $r^2 - 3 \cdot r + 2 = 0$

$\Rightarrow r_1 = 1, \ r_2 = 2$

$\Rightarrow y_0 = C_1 \cdot e^x + C_2 \cdot e^{2 \cdot x}$

$x \nrightarrow \eta_1 \Rightarrow \eta_1 = A + B \cdot x$

$\eta_1' = B, \quad \eta_1'' = 0 \Rightarrow \eta_1'' + 3 \cdot \eta_1' + 2 \cdot \eta_1 = x$

we have the following system

$$\begin{cases} 2 \cdot A - 3 \cdot B = 0 \\ 2 \cdot (A+B) - 3 \cdot B = 1 \end{cases}\bigg|_{A,B} \Rightarrow A = \frac{3}{4}, \quad B = \frac{1}{2} \Rightarrow \eta_1 = \frac{3}{4} + \frac{1}{2}x$$

$-e^x \nrightarrow \eta_2 \Rightarrow \eta_2 = A \cdot e^x \cdot x^m, m=1.$

$\eta_2' = A \cdot e^x \cdot x + A \cdot e^x = A \cdot (x+1) \cdot e^x$

$\eta_2'' = A \cdot (x+2) \cdot e^x \Rightarrow \eta_2'' - 3 \cdot \eta_1' + 2 \cdot \eta_1 = -e^x$

$A \cdot (x+2) - 3 \cdot A \cdot (x+1) + 2 \cdot A \cdot x = -1, \quad A = 1$

$y = y_0 + \eta_1 + \eta_2 \Rightarrow y = C_1 \cdot e^x + C_2 \cdot e^{2 \cdot x} + \frac{3}{4} + \frac{x}{2} + e^x \cdot x, \quad C_1, C_2 \in \mathbb{R}.$

ClassPad:

solve($r^2 - 3 \cdot r + 2 = 0, r$)

$\{r=1, r=2\}$

Define $\eta_1(x) = A + B \cdot x$

done

$\dfrac{d^2}{dx^2}(\eta_1(x)) - 3 \cdot \dfrac{d}{dx}(\eta_1(x)) + 2 \cdot \eta_1(x) = x \nrightarrow$ eq4

$2 \cdot (B \cdot x + A) - 3 \cdot B = x$

eq4|x=0

$2 \cdot A - 3 \cdot B = 0$

eq4|x=1

$2 \cdot (A+B) - 3 \cdot B = 1$

$\begin{cases} eq4|x=0 \\ eq4|x=1 \end{cases}\bigg|_{A,B}$

$\left\{ A = \frac{3}{4}, B = \frac{1}{2} \right\}$

Define $\eta_2(x)=A\cdot e^x\cdot x$

$\text{factorOut}\left(\dfrac{d^2}{dx^2}(\eta_2(x))-3\cdot\dfrac{d}{dx}(\eta_2(x))+2\cdot\eta_2(x),A\right)=-e^x$

$-A\cdot e^x=-e^x$

$\text{dSolve}(y''-3\cdot y'+2\cdot y=x-e^x,x,y)$

$$\left\{y=e^{2\cdot x}\cdot\text{const}(2)+x\cdot e^x+e^x\cdot\text{const}(1)+e^x+\dfrac{x}{2}+\dfrac{3}{4}\right\}$$

3.4.7 $y'''-y''-2\cdot y'=x\cdot e^x$

$y=y_0+\eta$ we should find $y_0=?,\eta=?$

the characteristical equation is: $r^3-r^2-2\cdot r=0$

$r\cdot(r^2-r-2)=0 \Rightarrow r=-1,\ r=0,\ r=2$

$y_0=C_1\cdot e^{0\cdot x}+C_2\cdot e^{-x}+C_3\cdot e^{2\cdot x}$

$x\cdot e^x\Rightarrow\eta \Rightarrow \eta=(A+B\cdot x)\cdot e^x$

$\eta'=B\cdot x\cdot e^x+A\cdot e^x+B\cdot e^x$

$\eta''=B\cdot x\cdot e^x+A\cdot e^x+2\cdot B\cdot e^x$

$\eta'''=B\cdot x\cdot e^x+A\cdot e^x+3\cdot B\cdot e^x$

$\dfrac{d^3}{dx^3}(\eta(x))-\dfrac{d^2}{dx^2}(\eta(x))-2\cdot\dfrac{d}{dx}(\eta(x))=x\cdot e^x$

$\Rightarrow B\cdot e^x-2\cdot\left(B\cdot x\cdot e^x+A\cdot e^x+B\cdot e^x\right)=x\cdot e^x$

when $x=0,x=1$ we have the following system

$\left.\begin{cases}-2\cdot(A+B)+B=0\\-2\cdot(A\cdot e+2\cdot B\cdot e)+B\cdot e=e\end{cases}\right|_{A,B} \Rightarrow A=\dfrac{1}{4},\ B=-\dfrac{1}{2}$

$\Rightarrow\eta=\left(\dfrac{1}{4}-\dfrac{1}{2}\cdot x\right)\cdot e^x$

$$y = C_1 + C_2 \cdot e^{-x} + C_3 \cdot e^{2 \cdot x} + \left(\frac{1}{4} - \frac{1}{2} \cdot x \right) \cdot e^x, \quad C_1, C_2, C_3 \in \mathbf{R}.$$

ClassPad:

Solve($r^3 - r^2 - 2 \cdot r = 0, r$)

$$\{r = -1, r = 0, r = 2\}$$

Define $\eta(x) = (A + B \cdot x) \cdot e^x$

done

$$\frac{d^3}{dx^3}(\eta(x)) - \frac{d^2}{dx^2}(\eta(x)) - 2 \cdot \frac{d}{dx}(\eta(x)) = x \cdot e^x \Rightarrow eq5$$

$$B \cdot e^x - 2 \cdot \left(B \cdot x \cdot e^x + A \cdot e^x + B \cdot e^x \right) = x \cdot e^x$$

eq5|x=0

$$-2 \cdot (A + B) + B = 0$$

eq5|x=1

$$-2 \cdot (A \cdot e + 2 \cdot B \cdot e) + B \cdot e = e$$

$$\begin{cases} -2 \cdot (A + B) + B = 0 \\ -2 \cdot (A \cdot e + 2 \cdot B \cdot e) + B \cdot e = e \end{cases} \Bigg|_{A, B}$$

$$\left\{ A = \frac{1}{4}, B = -\frac{1}{2} \right\}$$

dSolve($y''' - y'' - 2 \cdot y' = x \cdot e^x, x, y$)

$$\left\{ y = e^{2 \cdot x} \cdot const(3) - \frac{x \cdot e^x}{2} + \frac{e^x}{4} + e^{-x} \cdot const(1) + const(2) \right.$$

т.е. $y = e^{2 \cdot x} \cdot const(3) - \dfrac{x \cdot e^x}{2} + \dfrac{e^x}{4}$

$\qquad + e^{-x} \cdot const(1) + const(2)$

3.4.8 $y'' - 7 \cdot y' + 6 \cdot y = 12 \cdot x - 2$
$y = y_0 + \eta$ we should find $y_0 = ?, \eta = ?$

the characteristical equation is: $r^2-7\cdot r+6=0$

$$r_{1,2}=\frac{7+\sqrt{49-24}}{2} \;\Rightarrow\; r_1=6,\;\; r_2=1$$

$y_0=C_1\cdot e^{6\cdot x}+C_2\cdot e^x$
$\eta \Rightarrow 12\cdot x-2 \;\Rightarrow\; \eta=A\cdot x+B$
$\eta'=A, \eta''=0 \;\;\Rightarrow\; \eta''-7\cdot \eta'+6\cdot \eta=12\cdot x-2$

when $x=0, x=1$ we have the following system

$$\begin{cases} -7\cdot A+6\cdot B=-2 \\ 6\cdot(A+B)-7\cdot A=10 \end{cases}\Bigg|_{A,B} \;\Rightarrow\; A=2,\;\; B=2$$

$\Rightarrow\; y=C_1\cdot e^{6\cdot x}+C_2\cdot e^x+2\cdot x+2,\;\; C_1, C_2 \in \mathbf{R}.$

ClassPad:

solve($r^2-7\cdot r+6=0, r$)

{$r=1, r=6$}

Define $\eta(x)=A\cdot x+B$

done

$$\frac{d^2}{dx^2}(\eta(x))-7\cdot \frac{d}{dx}(\eta(x))+6\cdot \eta(x)=12\cdot x-2 \Rightarrow eq2$$

$6\cdot(A\cdot x+B)-7\cdot A=12\cdot x-2$

eq2|x=0

$-7\cdot A+6\cdot B=-2$

eq2|x=1

$6\cdot(A+B)-7\cdot A=10$

$$\begin{cases} eq2|x=0 \\ eq2|x=1 \end{cases}\Bigg|_{A,B}$$

{A=2, B=2}

dSolve(y"-7·y'+6·y=12·x-2,x,y)
$$\left\{y=e^{6\cdot x}\cdot const(2)+e^{x}\cdot const(1)+2\cdot x+2\right\}$$

3.4.9 $y''-6\cdot y'+6\cdot y=0$

the characteristical equation is: $r^2-6\cdot r+6=0$

$$r_{1,2}=\frac{6\pm\sqrt{36-24}}{2} \Rightarrow$$

$r_1=3+\sqrt{3}$ $r_2=3-\sqrt{3}$

$\Rightarrow$ $y=C_1\cdot e^{(3+\sqrt{3})\cdot x}+C_2\cdot e^{(3-\sqrt{3})\cdot x}$, $C_1,C_2\in\mathbb{R}$.

ClassPad:
dSolve(y"-6·y'+6·y=0,x,y)

$$\left\{y=e^{\sqrt{3}\cdot x+3\cdot x}\cdot const(2)+e^{-\sqrt{3}\cdot x+3\cdot x}\cdot const(1)\right\}$$

3.4.10 $y'-y=e^{-x}$

the characteristical equation is: $r-1=0$

$\Rightarrow r=1$ $\Rightarrow y=y_0+\eta$

$\Rightarrow y_0=C_1\cdot e^{x}$,

$\eta\Rightarrow e^{-x}$ $\Rightarrow A\cdot e^{-x}\cdot x^{m}$, $m=0$

$\Rightarrow \eta=A\cdot e^{-x}$

$\eta'=-A\cdot e^{-x}$ $\Rightarrow \eta'-\eta=e^{-x}$

$\Rightarrow -A\cdot e^{-x}-A\cdot e^{-x}=e^{-x}$

$\Rightarrow -2A=1$ $\Rightarrow A=-\frac{1}{2}$

от $y=y_0+\eta$ ⇒ $y=C_1 \cdot e^x - \dfrac{1}{2} \cdot e^{-x}$, $C_1 \in \mathbf{R}$.

ClassPad:
solve(r-1=0,r)
$$\{r=1\}$$

dSolve(y'-y=e^{-x},x,y)
$$\left\{ y=e^x \cdot \text{const}(1) - \dfrac{e^{-x}}{2} \right\}$$

3.4.11 $y''-4 \cdot y'+13 \cdot y=0$

the characteristical equation is: $r^2-4 \cdot r+13=0$

$r_{1,2}=\dfrac{4 \pm \sqrt{16-52}}{2}$

⇒$r_1=2+i \cdot 3$, $r_2=2-i \cdot 3$

⇒ $y=C_1 \cdot e^{2 \cdot x} \cdot \cos(3 \cdot x)+C_2 \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x)$, $C_1,C_2 \in \mathbf{R}$.

ClassPad:
dSolve(y''-4·y'+13·y=0,x,y)
$$\left\{ y=\cos(3 \cdot x) \cdot e^{2 \cdot x} \cdot \text{const}(1)+\sin(3 \cdot x) \cdot e^{2 \cdot x} \cdot \text{const}(2) \right\}$$

3.4.12 $y''-4 \cdot y'+13 \cdot y=\sin(x)$

$y=y_0+\eta$ търсим y_0=?,η=?

от 3.4.11 ⇒ $y_0=C_1 \cdot e^{2 \cdot x} \cdot \cos(3 \cdot x)+C_2 \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x)$

η⇒$\sin(x)$ ⇒ $\eta=e^0(A \cdot \cos(x)+B \cdot \sin(x))$

$\eta'(x)=B \cdot \cos(x)-A \cdot \sin(x)$

$\eta''(x)=-A \cdot \cos(x)-B \cdot \sin(x)$ ⇒ $A=\dfrac{1}{40}$,$B=\dfrac{3}{40}$

$\Rightarrow$ $y = C_1 \cdot e^{2 \cdot x} \cdot \cos(3 \cdot x) + C_2 \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x) +$

$\frac{1}{40} \cdot \cos(x) + \frac{3}{40} \cdot \sin(x)$, $C_1, C_2 \in \mathbf{R}$.

ClassPad:

dSolve(y''−4·y'+13·y=sin(x),x,y)

$\left\{ y = \cos(3 \cdot x) \cdot e^{2 \cdot x} \cdot \text{const}(1) + \sin(3 \cdot x) \cdot e^{2 \cdot x} \cdot \text{const}(2) + \blacktriangleright \right.$

Define η(x)=A·cos(x)+B·sin(x)

$\hspace{10cm}$ done

$\frac{d^2}{dx^2}(\eta(x)) - 4 \cdot \frac{d}{dx}(\eta(x)) + 13 \cdot \eta(x) = \sin(x) \Rightarrow eq$

$\quad$ 13·(A·cos(x)+B·sin(x))−4·(B·cos(x)−A·sin(x))−A·cos($\blacktriangleright$

eq|x=0

$\hspace{10cm}$ 12·A−4·B=0

eq|x=1

$\quad$ 13·(A·cos(1)+B·sin(1))+4·(A·sin(1)−B·cos(1))−A·cos($\blacktriangleright$

$\left\{ \begin{array}{l} eq|x=0 \\ eq|x=1 \end{array} \right| _{A,B}$

$\hspace{8cm} \left\{ A = \frac{1}{40}, B = \frac{3}{40} \right\}$

3.4.13 $\quad$ y'''−3·y'+2·y=0

the characteristical equation is: $r^3 - 3 \cdot r + 2 = 0$

$r^3 - r - 2 \cdot r + 2 = 0$ $\Rightarrow$ $r \cdot (r^2 - 1) + 2 \cdot (r - 1) = 0$

$(r-1) \cdot (r^2 - 1 + r + 2) = 0$ $\Rightarrow$ $r_1 = 1$, $r_2 = 1$, $r_3 = -2$

$y = (C_1 + C_2 \cdot x) \cdot e^x + C_3 \cdot e^{-2 \cdot x}$, $C_1, C_2, C_3 \in \mathbf{R}$.

ClassPad:

solve($r^3-3 \cdot r+2=0,r$)

$$\{r=-2,r=1\}$$

dSolve($y'''-3 \cdot y'+2 \cdot y=0,x,y$)

$$\left\{y=x \cdot e^x \cdot \text{const}(3)+e^x \cdot \text{const}(2)+e^{-2 \cdot x} \cdot \text{const}(1)\right\}$$

3.4.14 $y''+y=\sin(2 \cdot x)$

$y=y_0+\eta$ we should find $y_0=?,\eta=?$

the characteristical equation is: $r^2+1=0$

$\Rightarrow r_1=-i, \quad r_2=i$

$y_0=C_1 \cdot \cos(x)+C_2 \cdot \sin(x)$

$\sin(2 \cdot x)\Rightarrow\eta \Rightarrow \eta=A \cdot \sin(2 \cdot x)+B \cdot \cos(2 \cdot x)$

$\eta'=2 \cdot A \cdot \cos(2 \cdot x)-2 \cdot B \cdot \sin(2 \cdot x)$

$\eta''=-4 \cdot A \cdot \sin(2 \cdot x)-4 \cdot B \cdot \cos(2 \cdot x)$

$\eta''+\eta=\sin(2 \cdot x) \Rightarrow$

when $x=0,x=1$ we have the following system

$$\left. \begin{cases} -3 \cdot B=0 \\ -3 \cdot A \cdot \sin(2)-3 \cdot B \cdot \cos(2)=\sin(2) \end{cases} \right|_{A,B} \Rightarrow$$

$A=-\dfrac{1}{3}, \quad B=0 \Rightarrow$

$\eta=-\dfrac{1}{3} \cdot \sin(2 \cdot x)$

$y=C_1 \cdot \cos(x)+C_2 \cdot \sin(x)-\dfrac{1}{3} \cdot \sin(2 \cdot x), \quad C_1,C_2 \in \mathbf{R}.$

ClassPad:

Define $\eta(x)=A \cdot \sin(2 \cdot x)+B \cdot \cos(2 \cdot x)$

done

$$\frac{d^2}{dx^2}(\eta(x))+\eta(x)=\sin(2 \cdot x) \Rightarrow eq6$$

eq6|x=0

$$-3 \cdot B=0$$

eq6|x=1

$$-3 \cdot A \cdot \sin(2)-3 \cdot B \cdot \cos(2)=\sin(2)$$

$$\begin{cases} eq6|x=0 \\ eq6|x=1 \end{cases} \Big|_{A,B}$$

$$\left\{A=-\frac{1}{3}, B=0\right\}$$

dSolve(y''+y=sin(2·x),x,y)

$$\left\{y=\cos(x) \cdot const(1)+\sin(x) \cdot const(2)-\frac{\sin(2 \cdot x)}{3}\right\}$$

3.4.15 $y'''-3 \cdot y''+3 \cdot y'-y=e^{-x} \cdot \cos(2 \cdot x)$

$y=y_0+\eta$ we should find $y_0=?,\eta=?$

the characteristical equation is: $r^3-3 \cdot r^2+3 \cdot r-1=0$

$\Rightarrow r_1=r_2=r_3=1 \Rightarrow$

$y_0=(C_1+C_2 \cdot x+C_3 \cdot x^2) \cdot e^x$

$e^{-x} \cdot \cos(2 \cdot x) \Rightarrow \eta$ $\eta=e^{-x} \cdot (A \cdot \cos(2 \cdot x)+B \cdot \sin(2 \cdot x))$

$\eta'=-(A \cdot \cos(2 \cdot x)+2 \cdot A \cdot \sin(2 \cdot x)-2 \cdot B \cdot \cos(2 \cdot x)+B \cdot \sin(2 \cdot$ ▸

$\eta''=-(3 \cdot A \cdot \cos(2 \cdot x)-4 \cdot A \cdot \sin(2 \cdot x)+4 \cdot B \cdot \cos(2 \cdot x)+3 \cdot B \cdot s$ ▸

$\eta'''=-(A \cdot \cos(2 \cdot x)+B \cdot \sin(2 \cdot x)) \cdot e^{-x}+3 \cdot (4 \cdot A \cdot \cos(2 \cdot x)+$ ▸

$\eta'''-3 \cdot \eta''+3 \cdot \eta'-\eta=e^{-x} \cdot \cos(2 \cdot x)$

when x=0,x=1 we have the following system

$$\begin{cases} 3 \cdot (3 \cdot A + 4 \cdot B) - 3 \cdot (A - 2 \cdot B) + 10 \cdot A - 2 \cdot B = 1 \\ -3 \cdot (A \cdot \cos(2) + 2 \cdot A \cdot \sin(2) - 2 \cdot B \cdot \cos(2) + B \cdot \sin(2)) \cdot e^{-1} \cdots \end{cases}$$

$\Rightarrow A = \dfrac{1}{32}, \quad B = \dfrac{1}{32}$

$$y = (C_1 + C_2 \cdot x + C_3 \cdot x^2) \cdot e^x + \frac{e^{-x}}{32} \cdot (\cos(2 \cdot x) + \sin(2 \cdot x)),$$

$C_1, C_2, C_3 \in \mathbf{R}.$

ClassPad:

Solve($r^3 - 3 \cdot r^2 + 3 \cdot r - 1 = 0, r$)

$$\{r = 1\}$$

Define $\eta(x) = e^{-x} \cdot (A \cdot \cos(2 \cdot x) + B \cdot \sin(2 \cdot x))$

done

$\dfrac{d^3}{dx^3}(\eta(x)) - 3 \cdot \dfrac{d^2}{dx^2}(\eta(x)) + 3 \cdot \dfrac{d}{dx}(\eta(x)) - \eta(x) = e^{-x} \cdot \cos($ ▸

$\quad -3 \cdot (A \cdot \cos(2 \cdot x) + 2 \cdot A \cdot \sin(2 \cdot x) - 2 \cdot B \cdot \cos(2 \cdot x) + B \cdot \sin(2 \cdot x$ ▸

eq7|x=0

$$3 \cdot (3 \cdot A + 4 \cdot B) - 3 \cdot (A - 2 \cdot B) + 10 \cdot A - 2 \cdot B = 1$$

eq7|x=1

$\quad -3 \cdot (A \cdot \cos(2) + 2 \cdot A \cdot \sin(2) - 2 \cdot B \cdot \cos(2) + B \cdot \sin(2)) \cdot e^{-1} - 2$ ▸

$\begin{cases} eq7|x=0 \\ eq7|x=1 \end{cases}\bigg|_{A,B}$

$$\left\{ A = \frac{1}{32}, B = \frac{1}{32} \right\}$$

dSolve($y''' - 3 \cdot y'' + 3 \cdot y' - y = e^{-x} \cdot \cos(2 \cdot x), x, y$)

$\begin{cases} y = x^2 \cdot e^x \cdot \text{const}(3) + x \cdot e^x \cdot \text{const}(2) + e^x \cdot \text{const}(1) + \dfrac{\cos}{} \end{cases}$ ▸

Answer:

$$y = x^2 \cdot e^x \cdot \text{const}(3) + x \cdot e^x \cdot \text{const}(2) +$$
$$e^x \cdot \text{const}(1) + \frac{\cos(2 \cdot x) \cdot e^{-x}}{32} + \frac{\sin(2 \cdot x) \cdot e^{-x}}{32}$$

3.4.16 $y'' + 13 \cdot y' + 40 \cdot y = 5 \cdot x$

$y = y_0 + \eta$ we should find $y_0 = ?, \eta = ?$

the characteristical equation is: $r^2 + 13 \cdot r + 40 = 0$

$$r_{1,2} = \frac{-13 \pm \sqrt{169 - 160}}{2} \;\Rightarrow\; r_1 = -5, \; r_2 = -8$$

$$y_0 = C_1 \cdot e^{-5 \cdot x} + C_2 \cdot e^{-8 \cdot x}$$

$5 \cdot x \Rightarrow \eta \;\Rightarrow\; \eta = A + B \cdot x$
$\eta' = B, \; \eta'' = 0$
$\Rightarrow \eta'' + 13 \cdot \eta' + 40 \cdot (A + B \cdot x) = 5 \cdot x$

$$\left\{ \begin{array}{l} 40 \cdot B \cdot x = 5 \cdot x \\ 40 \cdot A = -13 \cdot B \end{array} \right|_{A,B}$$

$$A = \frac{-13}{320}, \quad B = \frac{1}{8}$$

$$y = C_1 \cdot e^{-5 \cdot x} + C_2 \cdot e^{-8 \cdot x} - \frac{13}{320} + \frac{1}{8} \cdot x, \quad C_1, C_2 \in \mathbf{R}.$$

ClassPad:

Solve$(r^2 + 13 \cdot r + 40 = 0, r)$

$\{r = -8, r = -5\}$

Define $\eta(x)=A+B\cdot x$

done

$$\frac{d}{dx}(\eta(x))\Rightarrow op1$$

B

$13\cdot op1+40\cdot\eta(x)=5\cdot x\Rightarrow eq8$

$$40\cdot(B\cdot x+A)+13\cdot B=5\cdot x$$

$eq8|x=0$

$$40\cdot A+13\cdot B=0$$

$eq8|x=1$

$$40\cdot(A+B)+13\cdot B=5$$

$$\left\{\begin{array}{l} eq8|x=0 \\ eq8|x=1 \end{array}\right|_{A,B}$$

$$\left\{A=-\frac{13}{320}, B=\frac{1}{8}\right\}$$

dSolve($y''+13\cdot y'+40\cdot y=5\cdot x,x,y$)

$$\left\{y=e^{-5\cdot x}\cdot const(2)+e^{-8\cdot x}\cdot const(1)+\frac{x}{8}-\frac{13}{320}\right\}$$

3.4.17 $y'''-3\cdot y'+2\cdot y=2\cdot e^{x}+\sin(x)$

$y=y_0+\eta_1+\eta_2$ търсим $y_0=?$, $\eta_1=?$, $\eta_2=?$

the characteristical equation is:$r^3-3\cdot r+2=0$

$r^3-r-2\cdot r+2=0 \Rightarrow r\cdot(r^2-1)+2\cdot(r-1)=0$

$(r-1)\cdot(r^2-1+r+2)=0$

$\Rightarrow r_1=1$, $r_2=1$, $r_3=-2$

$y_0=(C_1+C_2\cdot x)\cdot e^{x}+C_3\cdot e^{-2\cdot x}$

$2\cdot e^{x}\Rightarrow\eta_1 \Rightarrow \eta_1=A\cdot e^{x}\cdot x^{m}$, $m=2$

−90−

$n_1 = A \cdot e^x \cdot x^2$

$n_1' = A \cdot x^2 \cdot e^x + 2 \cdot A \cdot x \cdot e^x$

$n_1'' = A \cdot x^2 \cdot e^x + 4 \cdot A \cdot x \cdot e^x + 2 \cdot A \cdot e^x$

$n_1''' = A \cdot x^2 \cdot e^x + 6 \cdot A \cdot x \cdot e^x + 6 \cdot A \cdot e^x$

$n_1''' - 3 \cdot n_1' + 2 \cdot n_1 = 2 \cdot e^x$

$\Rightarrow A = \dfrac{1}{3} \Rightarrow n_1 = \dfrac{1}{3} \cdot e^x \cdot x^2$

$\sin(x) \Rightarrow n_2 \Rightarrow n_2 = A \cdot \sin(x) + B \cdot \cos(x)$

$n_2' = A \cdot \cos(x) - B \cdot \sin(x)$

$n_2'' = -B \cdot \cos(x) - A \cdot \sin(x)$

$n_2''' = -A \cdot \cos(x) + B \cdot \sin(x)$

$n_1''' - 3 \cdot n_1' + 2 \cdot n_1 = \sin(x)$

when $x=0, x=1$ we have the following system

$$\left.\begin{cases} -4 \cdot A + 2 \cdot B = 0 \\ -3 \cdot (A \cdot \cos(1) - B \cdot \sin(1)) + 2 \cdot (A \cdot \sin(1) + B \cdot \cos(1)) \\ \qquad -A \cdot \cos(1) + B \cdot \sin(1) = \sin(1) \end{cases}\right| A, B$$

$A = \dfrac{1}{10}, \quad B = \dfrac{1}{5}$

$\Rightarrow n_2 = \dfrac{1}{10} \cdot \sin(x) + \dfrac{1}{5} \cdot \cos(x) \Rightarrow$

$y = (C_1 + C_2 \cdot x) \cdot e^x + C_3 \cdot e^{-2 \cdot x} + \dfrac{1}{3} \cdot e^x \cdot x^2 +$

$\qquad \dfrac{1}{10} \cdot \sin(x) + \dfrac{1}{5} \cdot \cos(x), \quad C_1, C_2, C_3 \in \mathbf{R}.$

ClassPad:

Solve($r^3 - 3 \cdot r + 2 = 0, r$)

$\{r=-2, r=1\}$

Define $\eta_1(x) = A \cdot e^x \cdot x^2$

                  done

$$\frac{d^3}{dx^3}(\eta_1(x)) - 3 \cdot \frac{d}{dx}(\eta_1(x)) + 2 \cdot \eta_1(x) = 2 \cdot e^x \Rightarrow eq9$$

   $3 \cdot A \cdot x^2 \cdot e^x + 6 \cdot A \cdot x \cdot e^x + 6 \cdot A \cdot e^x - 3 \cdot (A \cdot x^2 \cdot e^x + 2 \cdot A \cdot x \cdot e^x \blacktriangleright$

т.е.

$eq9 \Rightarrow 3 \cdot A \cdot x^2 \cdot e^x + 6 \cdot A \cdot x \cdot e^x + 6 \cdot A \cdot e^x$
   $-3 \cdot (A \cdot x^2 \cdot e^x + 2 \cdot A \cdot x \cdot e^x) = 2 \cdot e^x$

$eq9 | x=0$

                  $6 \cdot A = 2$

Define $\eta_2(x) = A \cdot \sin(x) + B \cdot \cos(x)$

                  done

$$\frac{d^3}{dx^3}(\eta_2(x)) - 3 \cdot \frac{d}{dx}(\eta_2(x)) + 2 \cdot \eta_2(x) = \sin(x) \Rightarrow eq10$$

   $-3 \cdot (A \cdot \cos(x) - B \cdot \sin(x)) + 2 \cdot (B \cdot \cos(x) + A \cdot \sin(x)) - A \cdot \cos \blacktriangleright$

т.е.

$eq10 \Rightarrow -3 \cdot (A \cdot \cos(x) - B \cdot \sin(x))$
    $+2 \cdot (B \cdot \cos(x) + A \cdot \sin(x))$
    $-A \cdot \cos(x) + B \cdot \sin(x) = \sin(x)$

$eq10 | x=0$

                $-4 \cdot A + 2 \cdot B = 0$

$eq10 | x=1$

   $-3 \cdot (A \cdot \cos(1) - B \cdot \sin(1)) + 2 \cdot (A \cdot \sin(1) + B \cdot \cos(1)) - A \cdot \cos \blacktriangleright$

$\left\{ \begin{array}{l} eq10 | x=0 \\ eq10 | x=1 \end{array} \right| {}_{A,B}$

             $\left\{ A = \dfrac{1}{10}, B = \dfrac{1}{5} \right\}$

dSolve(y'''−3·y'+2·y=2·e^x+sin(x),x,y)

$$\begin{cases} y=\dfrac{x^2 \cdot e^x}{3}+x \cdot e^x \cdot const(3)-\dfrac{2 \cdot x \cdot e^x}{9}+\dfrac{2 \cdot e^x}{27}+e^x \cdot const(\blacktriangleright \end{cases}$$

$$x \cdot e^x \cdot const(3)-\dfrac{2 \cdot x \cdot e^x}{9}=C3 \cdot x \cdot e^x \quad \text{and}$$

$$\dfrac{2 \cdot e^x}{27}+e^x \cdot const(2)=C2 \cdot e^x$$

Answer:

$$y=\dfrac{x^2 \cdot e^x}{3}+x \cdot e^x \cdot C3+e^x \cdot C2+$$

$$e^{-2 \cdot x} \cdot C1+\dfrac{\cos(x)}{5}+\dfrac{\sin(x)}{10}, \quad C1,C2,C3 \in \mathbf{R}.$$

3.4.18 $y'''−3 \cdot y''+4 \cdot y'−2 \cdot y=x \cdot e^x \cdot \sin(x)$

$y=y_0+\eta$ we should find $y_0=?,\eta=?$

the characteristical equation is: $r^3-3 \cdot r^2+4 \cdot r-2=0$
$\Rightarrow r_1=1, \quad r_2=1-i, \quad r_3=1+i$

$y_0=C_1 \cdot e^x+C_2 \cdot e^x \cdot \cos(x)+C_3 \cdot e^x \cdot \sin(x)$

$x \cdot e^x \cdot \sin(x) \Rightarrow \eta$

$\eta=x \cdot (A1 \cdot x+A2) \cdot e^x \cdot \sin(x)+x \cdot (B1 \cdot x+B2) \cdot e^x \cdot \cos(x)$

ClassPad:

dSolve(y'''−3·y''+4·y'−2·y=x·e^x·sin(x),x,y)

$$\begin{cases} y=\dfrac{-x^2 \cdot \sin(x) \cdot e^x}{4}-\dfrac{3 \cdot x \cdot \cos(x) \cdot e^x}{4}+\cos(x) \cdot e^x \cdot const \blacktriangleright \end{cases}$$

we have

$$y = \frac{-x^2 \cdot \sin(x) \cdot e^x}{4} - \frac{3 \cdot x \cdot \cos(x) \cdot e^x}{4} +$$

$$\cos(x) \cdot e^x \cdot \text{const}(2) + \frac{7 \cdot \sin(x) \cdot e^x}{8} +$$

$$\sin(x) \cdot e^x \cdot \text{const}(3) + e^x \cdot \text{const}(1)$$

$$\frac{7 \cdot \sin(x) \cdot e^x}{8} + \sin(x) \cdot e^x \cdot \text{const}(3) = \sin(x) \cdot e^x \cdot C3$$

$$y = \frac{-x^2 \cdot \sin(x) \cdot e^x}{4} - \frac{3 \cdot x \cdot \cos(x) \cdot e^x}{4}$$

$$+ \cos(x) \cdot e^x \cdot C2 + \sin(x) \cdot e^x \cdot C3 + e^x \cdot C1,$$

$$C1, C2, C3 \in \mathbf{R}.$$

$\text{Solve}(r^3 - 3 \cdot r^2 + 4 \cdot r - 2 = 0, r)$

$$\{r=1, r=1-i, r=1+i\}$$

$\text{rFactor}(r^3 - 3 \cdot r^2 + 4 \cdot r - 2)$

$$(r-1) \cdot (r-1+i) \cdot (r-1-i)$$

$\text{Define } \eta(x) = x \cdot (A1 \cdot x + A2) \cdot e^x \cdot \sin(x) + x \cdot (B1 \cdot x + B2) \cdot e^x \blacktriangleright$

$$\text{done}$$

in other words the equation we defined is:

$\eta(x) = x \cdot e^x \cdot ((A1 \cdot x + A2) \cdot \sin(x) + (B1 \cdot x + B2) \cdot \cos(x))$

$\frac{d^3}{dx^3}(\eta(x)) - 3 \cdot \frac{d^2}{dx^2}(\eta(x)) + 4 \cdot \frac{d}{dx}(\eta(x)) - 2 \cdot \eta(x) = x \cdot e^x \cdot \text{s} \blacktriangleright$

$2 \cdot x \cdot \cos(x) \cdot (A1 \cdot x + A2) \cdot e^x + 6 \cdot \cos(x) \cdot (A1 \cdot x + A2) \cdot e^x - 2 \cdot \blacktriangleright$

eq11 is:

$(-4 \cdot B1 \cdot x + 6 \cdot A1 - 2 \cdot B2) \cdot \cos(x) \cdot e^x +$

$(-4 \cdot A1 \cdot x - 2 \cdot A2 - 6 \cdot B1) \cdot \sin(x) \cdot e^x = x \cdot \sin(x) \cdot e^x$

eq11|x=0

$$6 \cdot A1 - 3 \cdot (2 \cdot A2 + 2 \cdot B1 + 2 \cdot B2) + 6 \cdot A2 + 6 \cdot B1 + 4 \cdot B2 = 0$$

eq11|x=π/2

$$4 \cdot \left(\left(\frac{A1 \cdot \pi}{2} + A2 \right) \cdot e^{\frac{\pi}{2}} + \frac{\left(\frac{A1 \cdot \pi}{2} + A2 \right) \cdot \pi \cdot e^{\frac{\pi}{2}}}{2} + \frac{A1 \cdot \pi \cdot e^{\frac{\pi}{2}}}{2} - \left(\frac{B1}{\cdot} \right. \right.$$

eq11|x=−π/2

$$-4 \cdot \left(\frac{\left(\frac{A1 \cdot \pi}{2} - A2 \right) \cdot \pi \cdot e^{\frac{-\pi}{2}}}{2} - \left(\frac{A1 \cdot \pi}{2} - A2 \right) \cdot e^{\frac{-\pi}{2}} - \frac{A1 \cdot \pi \cdot e^{\frac{-\pi}{2}}}{2} \cdot \right.$$

eq11|x=π

$$-4 \cdot \left((A1 \cdot \pi + A2) \cdot \pi \cdot e^\pi + (B1 \cdot \pi + B2) \cdot e^\pi + (B1 \cdot \pi + B2) \cdot \pi \cdot e^\pi + | \right.$$

$$\left\{ \begin{array}{l} \text{eq11|x=0} \\ \text{eq11|x=π/2} \\ \text{eq11|x=−π/2} \\ \text{eq11|x=π} \end{array} \right| A1, A2, B1, B2$$

$$\left\{ A1 = -\frac{1}{4}, A2 = 0, B1 = 0, B2 = -\frac{3}{4} \right\}$$

3.4.19 $y''+4 \cdot y'+4 \cdot y=2 \cdot e^{-x}+\sin(x)+\dfrac{1}{3} \cdot \sin(2 \cdot x)$

$y=y_0+\eta_1+\eta_2+\eta_3$, $y_0=?$, $\eta_1=?$, $\eta_2=?$, $\eta_3=?$

the characteristical equation is: $r^2+4 \cdot r+4=0$

$(x+2)^2=0 \Rightarrow r_{1,2}=-2$

$y_0=(C_1+C_2 \cdot x) \cdot e^{-2 \cdot x}$

$2 \cdot e^{-x} \Rightarrow \eta_1 \Rightarrow \eta_1=A \cdot e^{-x} \cdot x^m$, $m=0$

$\eta_1=A \cdot e^{-x}$

$\eta_1'=-A \cdot e^{-x}$

$\eta_1''=A \cdot e^{-x}$

$\eta_1''+4 \cdot \eta_1'+4 \cdot \eta_1=2 \cdot e^{-x} \Rightarrow$

$A=2$, $\eta_1=e^{-x}$

$\sin(x) \Rightarrow \eta_2 \Rightarrow \eta_2=e^{0 \cdot x} \cdot (A \cdot \sin(x)+B \cdot \cos(x))$

$\eta_2=A \cdot \sin(x)+B \cdot \cos(x)$

$\eta_2'=A \cdot \cos(x)-B \cdot \sin(x)$

$\eta_2''=-B \cdot \cos(x)-A \cdot \sin(x)$

$\eta_2''+4 \cdot \eta_2'+4 \cdot \eta_2=\sin(x)$

when $x=0, x=1$ we have the following system

$\begin{cases} 4 \cdot A+3 \cdot B=0 \\ 4 \cdot (A \cdot \cos(1)-B \cdot \sin(1))+4 \cdot (A \cdot \sin(1)+B \cdot \cos(1)) \\ \quad -A \cdot \sin(1)-B \cdot \cos(1)=\sin(1) \end{cases} \Bigg|_{A,B}$

$\Rightarrow A=\dfrac{3}{25}$, $B=-\dfrac{4}{25} \Rightarrow$

$\eta_2=\dfrac{3}{25} \cdot \sin(x)-\dfrac{4}{25} \cdot \cos(x)$

$$\frac{1}{3} \cdot \sin(2 \cdot x) \Rightarrow \eta_3 \Rightarrow$$

$\eta_3 = A \cdot \sin(2 \cdot x) + B \cdot \cos(2 \cdot x)$

$\eta_3' = 2 \cdot A \cdot \cos(2 \cdot x) - 2 \cdot B \cdot \sin(2 \cdot x)$

$\eta_3'' = -4 \cdot A \cdot \sin(2 \cdot x) - 4 \cdot B \cdot \cos(2 \cdot x)$

$\eta_3'' + 4 \cdot \eta_3' + 4 \cdot \eta_3 = \dfrac{1}{3} \cdot \sin(2 \cdot x)$

when $x=0, x=1$ we have the following system

$$\left\{ \begin{array}{l} 8 \cdot A = 0 \\ 4(2A \cdot \cos(2) - 2B \cdot \sin(2)) + 4(A \cdot \sin(2) + B \cdot \cos(2)) \\ \qquad -4 \cdot A \cdot \sin(2) - 4 \cdot B \cdot \cos(2) = \dfrac{\sin(2)}{3} \end{array} \right| A, B$$

$A = 0, \quad B = -\dfrac{1}{24}$

$$\Rightarrow \eta_3 = -\frac{1}{24} \cos(2 \cdot x)$$

$y = y_0 + \eta_1 + \eta_2 + \eta_3$

$y = (C_1 + C_2 \cdot x) \cdot e^{-2 \cdot x} + e^{-x} + \dfrac{3}{25} \cdot \sin(x) - \dfrac{4}{25} \cdot \cos(x) -$

$\qquad \dfrac{1}{24} \cos(2 \cdot x), \quad C_1, C_2 \in \mathbf{R}.$

ClassPad:

Solve$(r^2 + 4 \cdot r + 4 = 0, r)$

$\{r = -2\}$

Define $\eta_1(x) = A \cdot e^{-x}$

done

$$\frac{d^2}{dx^2}(\eta_1(x))+4\cdot\frac{d}{dx}(\eta_1(x))+4\cdot\eta_1(x)=2\cdot e^{-x}\ \mathord{\scriptstyle\triangleright}\ eq12$$

eq12|x=0

$$A=2$$

Define $\eta_2(x)=A\cdot\sin(x)+B\cdot\cos(x)$

done

$$\frac{d^2}{dx^2}(\eta_2(x))+4\cdot\frac{d}{dx}(\eta_2(x))+4\cdot\eta_2(x)=\sin(x)\ \mathord{\scriptstyle\triangleright}\ eq13$$

$$4\cdot(A\cdot\cos(x)-B\cdot\sin(x))+4\cdot(B\cdot\cos(x)+A\cdot\sin(x))-B\cdot\cos(x\ \blacktriangleright$$

we have
$$4\cdot(A\cdot\cos(x)-B\cdot\sin(x))+4\cdot(B\cdot\cos(x)+A\cdot\sin(x))$$
$$-B\cdot\cos(x)-A\cdot\sin(x)=\sin(x)$$

eq13|x=0

$$4\cdot A+3\cdot B=0$$

eq13|x=1
$$4\cdot(A\cdot\cos(1)-B\cdot\sin(1))+4\cdot(A\cdot\sin(1)+B\cdot\cos(1))-A\cdot\sin(1\ \blacktriangleright$$

in other words
$$4\cdot(A\cdot\cos(1)-B\cdot\sin(1))+4\cdot(A\cdot\sin(1)+B\cdot\cos(1))$$
$$-A\cdot\sin(1)-B\cdot\cos(1)=\sin(1)$$

$$\begin{cases}eq13|x=0\\eq13|x=1\end{cases}\Bigg|_{A,B}$$

$$\left\{A=\frac{3}{25},B=-\frac{4}{25}\right\}$$

Define $\eta_3(x)=A\cdot\sin(2\cdot x)+B\cdot\cos(2\cdot x)$

done

$$\frac{d^2}{dx^2}(\eta_3(x)) + 4 \cdot \frac{d}{dx}(\eta_3(x)) + 4 \cdot \eta_3(x) = \frac{1}{3} \cdot \sin(2 \cdot x) \Rightarrow eq14$$

$4 \cdot (2 \cdot A \cdot \cos(2 \cdot x) - 2 \cdot B \cdot \sin(2 \cdot x)) + 4 \cdot (A \cdot \sin(2 \cdot x) + B \cdot \cos(\blacktriangleright$

so we have

$4 \cdot (2 \cdot A \cdot \cos(2 \cdot x) - 2 \cdot B \cdot \sin(2 \cdot x)) +$
  $4 \cdot (A \cdot \sin(2 \cdot x) + B \cdot \cos(2 \cdot x)) -$
    $4 \cdot A \cdot \sin(2 \cdot x) - 4 \cdot B \cdot \cos(2 \cdot x) = \dfrac{\sin(2 \cdot x)}{3}$

eq14|x=0

$$8 \cdot A = 0$$

eq14|x=1

$4 \cdot (2 \cdot A \cdot \cos(2) - 2 \cdot B \cdot \sin(2)) + 4 \cdot (A \cdot \sin(2) + B \cdot \cos(2)) - 4 \cdot \blacktriangleright$

we have

$4 \cdot (2 \cdot A \cdot \cos(2) - 2 \cdot B \cdot \sin(2)) + 4 \cdot (A \cdot \sin(2) + B \cdot \cos(2))$
  $-4 \cdot A \cdot \sin(2) - 4 \cdot B \cdot \cos(2) = \dfrac{\sin(2)}{3}$

$\begin{cases} eq14|x=0 \\ eq14|x=1 \end{cases} \Big|_{A,B}$

$$\left\{ A = 0, B = -\frac{1}{24} \right\}$$

dSolve$(y'' + 4 \cdot y' + 4 \cdot y = 2 \cdot e^{-x} + \sin(x) + \frac{1}{3} \cdot \sin(2 \cdot x), x, y)$

$\left\{ y = 2 \cdot e^{-x} + x \cdot e^{-2 \cdot x} \cdot const(2) + e^{-2 \cdot x} \cdot const(1) - \dfrac{4 \cdot \cos(}{25} \blacktriangleright \right.$

Answer:

$$y = 2 \cdot e^{-x} + x \cdot e^{-2 \cdot x} \cdot \text{const}(2) + e^{-2 \cdot x} \cdot \text{const}(1)$$
$$- \frac{4 \cdot \cos(x)}{25} + \frac{3 \cdot \sin(x)}{25} - \frac{\cos(2 \cdot x)}{24}$$

3.4.20 $y'' - 2 \cdot y' + y = \frac{1}{x} \cdot e^x$

$y = y_0 + \eta, \quad y_0 = ?, \quad \eta = ?$

the characteristical equation is: $r^2 - 2 \cdot r + 1 = 0$

$(r-1)^2 = 0 \Rightarrow r_{1,2} = 1$

$y_0 = (C_1 + C_2 \cdot x) \cdot e^x$

$\frac{1}{x} \cdot e^x \Rightarrow \eta \Rightarrow \eta(x) = (C1(x) + C2(x) \cdot x) \cdot e^x$

Define $\eta(x) = (C1(x) + C2(x) \cdot x) \cdot e^x$

done

$\frac{d}{dx}(\eta(x))$

$\ln(|x|) \cdot e^x + x \cdot \ln(|x|) \cdot e^x - x \cdot e^x$

$\frac{d}{dx}(C1(x)) \cdot e^x + x \cdot \frac{d}{dx}(C2(x)) \cdot e^x = 0$

$\frac{d}{dx}\left(e^x \cdot C1(x) + e^x \cdot C2(x) + x \cdot e^x \cdot C2(x)\right) = \frac{1}{x} \cdot e^x$

$\frac{x^2 \cdot \ln(|x|) \cdot e^x - x^2 \cdot e^x + 2 \cdot x \cdot \ln(|x|) \cdot e^x + e^x}{x} = \frac{e^x}{x}$

$$
\left\{
\begin{array}{l}
CS1 \cdot e^x + x \cdot CS2 \cdot e^x = 0 \\
CS1 \cdot e^x + CS2 \cdot e^x + x \cdot CS2 \cdot e^x = \dfrac{e^x}{x}
\end{array}
\right|_{CS1,CS2}
$$

$$\left\{CS1=-1, CS2=\frac{1}{x}\right\}$$

Define $C1(x) = \displaystyle\int_{\Box}^{\Box} -1\,dx$

done

Define $C2(x) = \displaystyle\int_{\Box}^{\Box} \frac{1}{x}\,dx$

done

$C1(x)$

$-x$

$C2(x)$

$\ln(|x|)$

Define $\eta(x) = (-x + \ln(x) \cdot x) \cdot e^x$

done

$\dfrac{d^2}{dx^2}(\eta(x)) - 2 \cdot \dfrac{d}{dx}(\eta(x)) + \eta(x)$

$(x \cdot \ln(x) - x) \cdot e^x + \dfrac{x^2 \cdot \ln(x) \cdot e^x - x^2 \cdot e^x + 2 \cdot x \cdot \ln(x) \cdot e^x + e}{x} \blacktriangleright$

simplify(ans)

$\dfrac{e^x}{x}$

ClassPad:

Solve($r^2-2 \cdot r+1=0,r$)

$$\{r=1\}$$

factor($r^2-2 \cdot r+1$)

$$(r-1)^2$$

dSolve($y''-2 \cdot y'+y=\frac{1}{x} \cdot e^x, x, y$)

$$\left\{y=x \cdot \ln(|x|) \cdot e^x+x \cdot e^x \cdot \text{const}(2)-x \cdot e^x+e^x \cdot \text{const}(1)\right\}$$

$y(x)=x \cdot \ln(|x|) \cdot e^x+x \cdot e^x \cdot C2+e^x \cdot C1, \quad C1,C2 \in \mathbf{R}.$

3.4.21 $y'''-3 \cdot y''+3 \cdot y'-y=e^{-x} \cdot \cos(2 \cdot x)$

$y=y_0+\eta, \quad y_0=?, \quad \eta=?$

the characteristical equation is: $r^3-3 \cdot r^2+3 \cdot r-1=0$

$r_1=r_2=r_3=1$

$y_0=(C_1+C_2 \cdot x+C_3 \cdot x^2) \cdot e^x$

$e^{-x} \cdot \cos(2 \cdot x) \nrightarrow \eta \Rightarrow$

$\eta=e^{-x} \cdot (A \cdot \cos(2 \cdot x)+B \cdot \sin(2 \cdot x))$

$\eta'=-(A \cdot \cos(2 \cdot x)+2 \cdot A \cdot \sin(2 \cdot x)-2 \cdot B \cdot \cos(2 \cdot x)+B \cdot \sin(2 \cdot$ ▸

$\eta''=-(3 \cdot A \cdot \cos(2 \cdot x)-4 \cdot A \cdot \sin(2 \cdot x)+4 \cdot B \cdot \cos(2 \cdot x)+3 \cdot B \cdot s$ ▸

$\eta'''=-(A \cdot \cos(2 \cdot x)+B \cdot \sin(2 \cdot x)) \cdot e^{-x}+$

$\quad 3 \cdot (4 \cdot A \cdot \cos(2 \cdot x)+4 \cdot B \cdot \sin(2 \cdot x)) \cdot e^{-x}+$

$\quad (8 \cdot A \cdot \sin(2 \cdot x)-8 \cdot B \cdot \cos(2 \cdot x)) \cdot e^{-x}-$

$\quad 3 \cdot (2 \cdot A \cdot \sin(2 \cdot x)-2 \cdot B \cdot \cos(2 \cdot x)) \cdot e^{-x}$

$\eta'''-3 \cdot \eta''+3 \cdot \eta'-\eta=e^{-x} \cdot \cos(2 \cdot x)$

when x=0,x=1 we have the system

$A=\dfrac{1}{32}$, $B=\dfrac{1}{32}$ ⇒

$\eta=e^{-x}\cdot(\dfrac{1}{32}\cdot\cos(2\cdot x)+\dfrac{1}{32}\cdot\sin(2\cdot x))$

$y=(C_1+C_2\cdot x+C_3\cdot x^2)\cdot e^x+e^{-x}\cdot(\dfrac{1}{32}\cdot\cos(2\cdot x)+$

$\dfrac{1}{32}\cdot\sin(2\cdot x)$, $C_1,C_2,C_3\in\mathbf{R}$.

ClassPad:

Solve($r^3-3\cdot r^2+3\cdot r-1=0,r$)

{r=1}

Define $\eta(x)=e^{-x}\cdot(A\cdot\cos(2\cdot x)+B\cdot\sin(2\cdot x))$

done

$\dfrac{d^3}{dx^3}(\eta(x))-3\cdot\dfrac{d^2}{dx^2}(\eta(x))+3\cdot\dfrac{d}{dx}(\eta(x))-\eta(x)=e^{-x}\cdot\cos(▸$

$-3\cdot(A\cdot\cos(2\cdot x)+2\cdot A\cdot\sin(2\cdot x)-2\cdot B\cdot\cos(2\cdot x)+B\cdot\sin(2\cdot x▸$

eq16|x=0

$3\cdot(3\cdot A+4\cdot B)-3\cdot(A-2\cdot B)+10\cdot A-2\cdot B=1$

eq16|x=1

$-3\cdot(A\cdot\cos(2)+2\cdot A\cdot\sin(2)-2\cdot B\cdot\cos(2)+B\cdot\sin(2))\cdot e^{-1}-2▸$

$\left\{\begin{array}{l}\text{eq16|x=0}\\\text{eq16|x=1}\end{array}\right|_{A,B}$

$\left\{A=\dfrac{1}{32},B=\dfrac{1}{32}\right\}$

dSolve(y'''−3·y''+3·y'−y=e^{−x}·cos(2·x),x,y)

$$\begin{cases} y=x^2 \cdot e^x \cdot \text{const}(3)+x \cdot e^x \cdot \text{const}(2)+e^x \cdot \text{const}(1)+\frac{\cos}{} \end{cases}$$

Answer:

$$y=x^2 \cdot e^x \cdot \text{const}(3)+x \cdot e^x \cdot \text{const}(2)+e^x \cdot \text{const}(1)+$$

$$\frac{\cos(2 \cdot x) \cdot e^{−x}}{32}+\frac{\sin(2 \cdot x) \cdot e^{−x}}{32}$$

3.4.22 $y'''−4 \cdot y'=x \cdot e^{3 \cdot x}$

$y=y_0+\eta$, y_0=?, η=?

the characteristical equation is: $r^3−4 \cdot r=0$

$r \cdot (r^2−4)=0 \Rightarrow r=−2, r=0, r=2$

$y_0=C_1 \cdot e^{0 \cdot x}+C_2 \cdot e^{−2 \cdot x}+C_3 \cdot e^{2 \cdot x}$

$x \cdot e^{3 \cdot x} \Rightarrow \eta$

$\eta=(A+B \cdot x) \cdot e^{3 \cdot x}$

$\eta'=3 \cdot B \cdot x \cdot e^{3 \cdot x}+3 \cdot A \cdot e^{3 \cdot x}+B \cdot e^{3 \cdot x}$

$\eta''=9 \cdot B \cdot x \cdot e^{3 \cdot x}+9 \cdot A \cdot e^{3 \cdot x}+6 \cdot B \cdot e^{3 \cdot x}$

$\eta'''=27 \cdot B \cdot x \cdot e^{3 \cdot x}+27 \cdot A \cdot e^{3 \cdot x}+27 \cdot B \cdot e^{3 \cdot x}$

$\eta'''−4 \cdot \eta'=x \cdot e^{3 \cdot x}$

when x=0,x=1 we have the following system

−104−

$$\begin{cases} -4 \cdot \left(3 \cdot A \cdot e^3 + 4 \cdot B \cdot e^3\right) + 27 \cdot A \cdot e^3 \\ \qquad\qquad\qquad + 54 \cdot B \cdot e^3 = e^3 \\ -4 \cdot (3 \cdot A + B) + 27 \cdot A + 27 \cdot B = 0 \end{cases} \Bigg|_{A,B}$$

$$A = -\frac{23}{225}, \quad B = \frac{1}{15}$$

$$\eta = \left(-\frac{23}{225} + \frac{1}{15} \cdot x\right) \cdot e^{3 \cdot x}$$

$$y = C_1 + C_2 \cdot e^{-2 \cdot x} + C_3 \cdot e^{2 \cdot x} + \left(-\frac{23}{225} + \frac{1}{15} \cdot x\right) \cdot e^{3 \cdot x},$$

$$C_1, C_2, C_3 \in \mathbf{R}.$$

ClassPad:

Solve($r^3 - 4 \cdot r = 0, r$)

$$\{r = -2, r = 0, r = 2\}$$

Define $\eta(x) = (A + B \cdot x) \cdot e^{3 \cdot x}$

done

$$\frac{d^3}{dx^3}(\eta(x)) - 4 \cdot \frac{d}{dx}(\eta(x)) = x \cdot e^{3 \cdot x} \Rightarrow \text{eq18}$$

$$27 \cdot B \cdot x \cdot e^{3 \cdot x} + 27 \cdot A \cdot e^{3 \cdot x} + 27 \cdot B \cdot e^{3 \cdot x} - 4 \cdot \left(3 \cdot B \cdot x \cdot e^{3 \cdot x} + \blacktriangleright\right.$$

eq18|x=1

$$-4 \cdot \left(3 \cdot A \cdot e^3 + 4 \cdot B \cdot e^3\right) + 27 \cdot A \cdot e^3 + 54 \cdot B \cdot e^3 = e^3$$

eq18|x=0

$$-4 \cdot (3 \cdot A + B) + 27 \cdot A + 27 \cdot B = 0$$

$$\begin{cases} \text{eq18}|x=0 \\ \text{eq18}|x=1 \end{cases} \Bigg|_{A,B}$$

$$\left\{A = -\frac{23}{225}, B = \frac{1}{15}\right\}$$

dsolve(y'''−4·y'=x·e^(3·x),x,y)

$$\left\{y=\dfrac{x \cdot e^{3 \cdot x}}{15}-\dfrac{23 \cdot e^{3 \cdot x}}{225}+e^{2 \cdot x} \cdot \text{const}(3)+e^{-2 \cdot x} \cdot \text{const}(1\blacktriangleright\right.$$

3.4.23 $y'''-y''+4y'-4y=x \cdot (x+\sin(x))$, $y(0)=\dfrac{1}{18}$,

$y'(0)=-\dfrac{1}{3}$, $y''(0)=-\dfrac{7}{9}$

$y=y_0+\eta_1+\eta_2$, $y_0=?$, $\eta_1=?$, $\eta_2=?$

the characteristical equation is: $r^3-r^2+4 \cdot r-4=0$

$r^2 \cdot (r-1)+4 \cdot (r-1)=0$

$(r-1) \cdot (r^2+4)=0$

$r_1=1$, $r_2=-i \cdot 2$, $r_3=i \cdot 2$

$y_0=C_1 \cdot e^x+C_2 \cdot \cos(2 \cdot x)+C_3 \cdot \sin(2 \cdot x)$

$x \cdot \sin(x) \doteq \eta_1$

$\eta_1=(A+B \cdot x) \cdot \sin(x)+(C+D \cdot x) \cdot \cos(x)$

$\eta_1'=B \cdot x \cdot \cos(x)-D \cdot x \cdot \sin(x)+A \cdot \cos(x)+$
    $D \cdot \cos(x)+B \cdot \sin(x)-C \cdot \sin(x)$

$\eta_1''=-D \cdot x \cdot \cos(x)-B \cdot x \cdot \sin(x)+2 \cdot B \cdot \cos(x)-$
    $C \cdot \cos(x)-A \cdot \sin(x)-2 \cdot D \cdot \sin(x)$

$\eta_1'''=-B \cdot x \cdot \cos(x)+D \cdot x \cdot \sin(x)-A \cdot \cos(x)-$
    $3 \cdot D \cdot \cos(x)-3 \cdot B \cdot \sin(x)+C \cdot \sin(x)$

$\eta_1'''-\eta_1''+4 \cdot \eta_1'-4 \cdot \eta_1=x \cdot \sin(x)$

$A=-\dfrac{1}{9}$, $B=-\dfrac{1}{6}$, $C=-\dfrac{1}{18}$, $D=-\dfrac{1}{6}$

$$\eta_1 = \left(-\frac{1}{9} - \frac{1}{6}x\right) \cdot \sin(x) + \left(-\frac{1}{18} - \frac{1}{6}x\right) \cdot \cos(x)$$

$$x^2 \Rightarrow \eta_2$$

$$\eta_2 = A + B \cdot x + C \cdot x^2$$

$$\eta_2' = B + C \cdot x$$

$$\eta_2'' = C, \quad \eta_2''' = 0$$

$$\eta_2''' - \eta_2'' + 4 \cdot \eta_2' - 4 \cdot \eta = x^2$$

when $x=0, x=1, x=-1$ we have the following system

$$\begin{cases} -4 \cdot (A+B+C) + 4 \cdot (B+2 \cdot C) - 2 \cdot C = 1 \\ -4 \cdot A + 4 \cdot B - 2 \cdot C = 0 \\ -4 \cdot (A-B+C) + 4 \cdot (B-2 \cdot C) - 2 \cdot C = 1 \end{cases} \Big| A, B, C$$

$$A = -\frac{3}{8}, \quad B = -\frac{1}{2}, \quad C = -\frac{1}{4}$$

$$\eta_2 = -\frac{3}{8} - \frac{1}{2} \cdot x - \frac{1}{4} \cdot x^2$$

$$y = C_1 \cdot e^x + C_2 \cdot \cos(2 \cdot x) + C_3 \cdot \sin(2 \cdot x) + \left(-\frac{1}{9} - \frac{1}{6} \cdot x\right)$$

$$\cdot \sin(x) + \left(-\frac{1}{18} - \frac{1}{6} \cdot x\right) \cdot \cos(x) - \frac{3}{8} - \frac{1}{2} \cdot x - \frac{1}{4} \cdot x^2,$$

$$C_1, C_2, C_3 \in \mathbf{R}.$$

ClassPad:

ClassPad provides two different methods for solving special type of linear differential equations with constant coefficients. One is direct way with dSolve and indirect, using formulars for Laplace transformations

laplace(y'''−y''+4y'−4y=$x \cdot (x+\sin(x))$,x,y,s)

$$-s^2 \cdot y(0) - s \cdot y'(0) - y''(0) + Lp \cdot s^3 + s \cdot y(0) + y'(0) - Lp \cdot s^2 - 4$$

so we have

$$-s^2 \cdot y(0) - s \cdot y'(0) - y''(0) + Lp \cdot s^3 + s \cdot y(0) + y'(0) -$$
$$Lp \cdot s^2 - 4 \cdot (y(0) - Lp \cdot s) - 4 \cdot Lp = \frac{2 \cdot s}{\left(s^2 + 1\right)^2} + \frac{2}{s^3}$$

ans|y''(0)=-7/9 and y'(0)=-1/3 and y(0)=1/18

$$Lp \cdot s^3 - Lp \cdot s^2 - \frac{s^2}{18} + 4 \cdot \left(Lp \cdot s - \frac{1}{18}\right) - 4 \cdot Lp + \frac{7 \cdot s}{18} + \frac{4}{9} = \frac{2 \cdot s}{\left(s^2 + 1\right.}$$ ▸

Solve(ans,Lp)

$$\left\{ Lp = \frac{s^9 - 7 \cdot s^8 - 2 \cdot s^7 - 14 \cdot s^6 - 7 \cdot s^5 + 65 \cdot s^4 - 4 \cdot s^3 + 72 \cdot s^2}{18 \cdot s^3 \cdot \left(s^3 - s^2 + 4 \cdot s - 4\right) \cdot \left(s^2 + 1\right)^2} \right.$$ ▸

invLaplace$\left(\dfrac{s^9 - 7 \cdot s^8 - 2 \cdot s^7 - 14 \cdot s^6 - 7 \cdot s^5 + 65 \cdot s^4 - 4 \cdot s^3 \cdot}{18 \cdot s^3 \cdot \left(s^3 - s^2 + 4 \cdot s - 4\right) \cdot \left(s^2 + 1\right)^2} \right.$ ▸

$$-\left(18 \cdot x^2 + 12 \cdot x \cdot \cos(x) + 12 \cdot x \cdot \sin(x) - 28 \cdot \cosh(x) - 28 \cdot \sin\right.$$ ▸

expand(ans)

$$\frac{-x^2}{4} - \frac{x \cdot \cos(x)}{6} - \frac{x \cdot \sin(x)}{6} + \frac{7 \cdot \cosh(x)}{18} + \frac{7 \cdot \sinh(x)}{18} - \frac{\cos(}{18}$$ ▸

Define $\eta_2(x) = (A + B \cdot x) \cdot \sin(x) + (C + D \cdot x) \cdot \cos(x)$

<div align="right">done</div>

$$\frac{d^3}{dx^3}(\eta_2(x)) - \frac{d^2}{dx^2}(\eta_2(x)) + 4 \cdot \frac{d}{dx}(\eta_2(x)) - 4 \cdot \eta_2(x) = x \cdot \sin($$ ▸

$$-4 \cdot (\sin(x) \cdot (B \cdot x + A) + \cos(x) \cdot (D \cdot x + C)) + 4 \cdot (B \cdot x \cdot \cos(x) - D$$ ▸

eq17|x=0

<div align="right">$4 \cdot (A + D) - A - 2 \cdot B - 3 \cdot C - 3 \cdot D = 0$</div>

eq17|x=1

$$-4 \cdot ((A + B) \cdot \sin(1) + (C + D) \cdot \cos(1)) + 4 \cdot (A \cdot \cos(1) + B \cdot \cos(1$$ ▸

eq17|x=-1

  4·((A-B)·sin(1)-(C-D)·cos(1))+4·(A·cos(1)-B·cos(1)▸

eq17|x=-π

  -4·(A-B·π+D)+A-B·π+2·B+4·(C-D·π)-C+D·π+3·D=0

$$\begin{cases} eq17|x=0 \\ eq17|x=1 \\ eq17|x=-1 \\ eq17|x=-\pi \end{cases}_{A,B,C,D}$$

$$\left\{ A=-\frac{1}{9}, B=-\frac{1}{6}, C=-\frac{1}{18}, D=-\frac{1}{6} \right\}$$

dsolve(y'''-y''+4y'-4y=x·(x+sin(x)),x,y)

$$\left\{ y=e^x \cdot const(1) - \frac{x^2}{4} - \frac{x \cdot cos(x)}{6} - \frac{x \cdot sin(x)}{6} - \frac{cos(x)}{18} - \frac{sin}{} \right. ▸$$

Answer:

$$y = e^x \cdot const(1) - \frac{x^2}{4} - \frac{x \cdot cos(x)}{6} - \frac{x \cdot sin(x)}{6} -$$

$$\frac{cos(x)}{18} - \frac{sin(x)}{9} - \frac{x}{2} + cos(2 \cdot x) \cdot const(2)$$

$$+ sin(2 \cdot x) \cdot const(3) - \frac{3}{8}$$

3.5 System of differential equations

3.5.1

$$\begin{cases} y_1'=2 \cdot y_1 + 4 \cdot y_2 + cos(x) \\ y_2'=-y_1 - 2 \cdot y_2 + sin(x) \end{cases}_{y_1(x), y_2(x)}$$

$$\begin{cases} (1)y_1'=2 \cdot y_1 + 4 \cdot y_2 + cos(x) \\ (2)y_1=-y_2' - 2 \cdot y_2 + sin(x) \end{cases}_{y_1(x), y_2(x)}$$

(2)⇒(1)

$-y_2''-2y_2'+\cos(x)=-2\cdot y_2'-4\cdot y_2+2\cdot\sin(x)$

$y_2''=-2\cdot\sin(x)$

the characteristical equation is: $r^2=0$

$\Rightarrow r_1=r_2=0 \Rightarrow y_{2.1}(x)=(C_1+C_2\cdot x)\cdot e^{0\cdot x}$

The second part of the equation $y_2(x)$ can be found from:

$-2\cdot\sin(x)\Rightarrow\eta$

$\eta=A\cdot\sin(x)+B\cdot\cos(x)$

$\eta'=A\cdot\cos(x)-B\cdot\sin(x)$

$\eta''=-A\cdot\sin(x)-B\cdot\cos(x)$

$\Rightarrow A=2,\ B=0$

$y_{2.2}(x)=2\cdot\sin(x)$

$y_2(x)=y_{2.1}(x)+y_{2.2}(x)$

$y_2(x)=(C_1+C_2\cdot x)\cdot e^{0\cdot x}+2\cdot\sin(x),\ C_1,C_2\in\mathbb{R}.$

$y_2'(x)=C_2+2\cdot\cos(x)$

we replace in (2) and then we have:

$y_1(x)=-C_2-2\cdot\cos(x)-2\cdot C_1-2\cdot C_2\cdot x-4\cdot\sin(x)+\sin(x)$

$y_1(x)=-2\cdot C_1-C_2(1+2\cdot x)-2\cdot\cos(x)-3\cdot\sin(x),\ C_1,C_2\in\mathbb{R}.$

ClassPad:
dSolve($\{y_1'=2\cdot y_1+4\cdot y_2+\cos(x),y_2'=-y_1-2\cdot y_2+\sin(x)\},x$ ▸

$\left\{y_1=-2\cdot\cos(x)-3\cdot\sin(x)+x\cdot\text{const}(2)+\text{const}(1), y_2=\dfrac{8\cdot\text{si}}{}\right.$ ▸

Answer:
$y_1=-2\cdot\cos(x)-3\cdot\sin(x)+x\cdot\text{const}(2)+\text{const}(1),$

$$y_2=\frac{8\cdot\sin(x)-2\cdot x\cdot\text{const}(2)+\text{const}(2)-2\cdot\text{const}(1)}{4}$$

3.5.2

$$\begin{cases} y_1'=3 \cdot y_1+y_2 \\ y_2'=8 \cdot y_1+5 \cdot y_2 \end{cases} \Big| \, y_1(t), y_2(t)$$

$$\begin{cases} (3)\, y_2=y_1'-3 \cdot y_1 \\ (4)\, y_2'=8 \cdot y_1+5 \cdot y_2 \end{cases} \Big| \, y_1(t), y_2(t)$$

$(3) \Rightarrow (4)$

$y_1''-3 \cdot y_1'=8 \cdot y_1+5 \cdot y_1'-15 \cdot y_1$

$y_1''-8 \cdot y_1'+7 \cdot y_1=0$

the characteristical equation is:

$r^2-8 \cdot r+7=0$

$$r_{1,2}=\frac{8 \pm \sqrt{64-28}}{2}$$

$r_1=7 \Rightarrow y_{1.1}=e^{7 \cdot t}$

$r_2=1 \Rightarrow y_{1.2}=e^{t}$

$\Rightarrow y_1(t)=C_1 \cdot e^{7 \cdot t}+C_2 \cdot e^{t}, \quad C_1, C_2 \in \mathbf{R}.$

$y_1'(t)=7 \cdot C_1 \cdot e^{7 \cdot t}+C_2 \cdot e^{t}$

To find $y_2(t)$, we place $y_1(t)$ and $y_1'(t)$ in equation (3)

$y_2(t)=7 \cdot C_1 \cdot e^{7 \cdot t}+C_2 \cdot e^{t}-3 \cdot C_1 \cdot e^{7 \cdot t}-3 \cdot C_2 \cdot e^{t}$

$y_2(t)=4 \cdot C_1 \cdot e^{7 \cdot t}-2C_2 \cdot e^{t}, \quad C_1, C_2 \in \mathbf{R}.$

ClassPad:

dSolve({y_1'=3 \cdot y_1+y_2, y_2'=8 \cdot y_1+5 \cdot y_2}, t, {y_1, y_2})

$\{y_1=e^{7 \cdot t} \cdot const(2)+e^{t} \cdot const(1), y_2=4 \cdot e^{7 \cdot t} \cdot const(2)$-▸

Answer:

$y_1 = e^{7 \cdot t} \cdot \text{const}(2) + e^t \cdot \text{const}(1)$,

$y_2 = 4 \cdot e^{7 \cdot t} \cdot \text{const}(2) - 2 \cdot e^t \cdot \text{const}(1)$

3.5.3

$$\begin{cases} y_1' + 5 \cdot y_1 - 2 \cdot y_2 = e^x \\ y_2' - y_1 + 6 \cdot y_2 = e^{2 \cdot x} \end{cases} \Bigg|_{y_1(x), y_2(x)}$$

$$\begin{cases} (5)\, y_1' = e^x + 2 \cdot y_2 - 5 \cdot y_1 \\ (6)\, y_1 = y_2' + 6 \cdot y_2 - e^{2 \cdot x} \end{cases} \Bigg|_{y_1(x), y_2(x)}$$

$(6) \Rightarrow (5)$

$y_2'' + 6 \cdot y_2' - 2 \cdot e^{2 \cdot x} = e^x + 2 \cdot y_2 - 5 \cdot y_2' - 30 \cdot y_2 - 5 \cdot e^{2 \cdot x}$

$y_2'' + 11 \cdot y_2' + 28 \cdot y_2 = -3 \cdot e^{2 \cdot x} + e^x$

$r^2 + 11 \cdot r + 28 = 0$

$r_{1,2} = \dfrac{-11 \pm \sqrt{121 - 112}}{2}$

$r_1 = -7, \quad r_2 = -4$

$y_{2.1}(x) = C_1 \cdot e^{-7 \cdot x} + C_2 \cdot e^{-4 \cdot x}$

$-3 \cdot e^{2 \cdot x} \Rightarrow \eta_1$

$\eta_1 = A \cdot e^{2 \cdot x}$

$\eta_1' = 2 \cdot A \cdot e^{2 \cdot x}$

$\eta_1'' = 4 \cdot A \cdot e^{2 \cdot x}$

$\eta_1'' + 11 \cdot \eta' + 28 \cdot \eta = 7 \cdot e^{2 \cdot x}$

$54 \cdot A = 7 \Rightarrow A = \dfrac{7}{54}$

$e^x \Rightarrow \eta_2$

$\eta_2 = B \cdot e^x$

$\eta_2' = B \cdot e^x$

$\eta_2'' = B \cdot e^x \Rightarrow$

$40 \cdot B = 1 \Rightarrow B = \dfrac{1}{40}$

$y_{2.2}(x) = \dfrac{7}{54} \cdot e^{2 \cdot x} + \dfrac{1}{40} e^x$

$y_2(x) = C_1 \cdot e^{-7 \cdot x} + C_2 \cdot e^{-4 \cdot x} + \dfrac{7}{54} \cdot e^{2 \cdot x} + \dfrac{1}{40} e^x$

$y_2'(x) = -7 \cdot C_1 \cdot e^{-7 \cdot x} - 4 \cdot C_2 \cdot e^{-4 \cdot x} + \dfrac{7}{27} \cdot e^{2 \cdot x} + \dfrac{1}{40} \cdot e^x$

To find $y_1(x)$, we place $y_2(x)$ and $y_2'(x)$ in equation (6)

$y_1(x) = -7 \cdot C_1 \cdot e^{-7 \cdot x} - 4 \cdot C_2 \cdot e^{-4 \cdot x} + \dfrac{7}{27} \cdot e^{2 \cdot x} + \dfrac{1}{40} \cdot e^x +$

$\qquad 6 \cdot C_1 \cdot e^{-7 \cdot x} + 6 \cdot C_2 \cdot e^{-4 \cdot x} - \dfrac{6}{54} \cdot e^{2 \cdot x} +$

$\qquad \dfrac{6}{40} \cdot e^x - e^{2 \cdot x}$

$y_1(x) = C_1 \cdot e^{-7 \cdot x} + 2 \cdot C_2 \cdot e^{-4 \cdot x} + \dfrac{28}{27} e^{2 \cdot x} + \dfrac{4}{40} \cdot e^x$

ClassPad:

dSolve($\{y_1' + 5 \cdot y_1 - 2 \cdot y_2 = e^x, y_2' - y_1 + 6 \cdot y_2 = e^{2 \cdot x}\}, x, \{y_1, y \blacktriangleright$

$\begin{cases} y_1 = \dfrac{e^{2 \cdot x}}{27} + \dfrac{7 \cdot e^x}{40} + e^{-4 \cdot x} \cdot \text{const}(2) + e^{-7 \cdot x} \cdot \text{const}(1), y_2 \blacktriangleright \end{cases}$

Answer:

$$y_1 = \frac{e^{2 \cdot x}}{27} + \frac{7 \cdot e^x}{40} + e^{-4 \cdot x} \cdot \text{const}(2) + e^{-7 \cdot x} \cdot \text{const}(1),$$

$$y_2 = \frac{7 \cdot e^{2 \cdot x}}{54} + \frac{e^x}{40} + \frac{e^{-4 \cdot x} \cdot \text{const}(2)}{2} - e^{-7 \cdot x} \cdot \text{const}(1)$$

4. Numerical solutions of differential equations

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Very often equations that describes mathematical models in engineer analysis and desine involve derivatives and integrals.

Equations that include derivatives are colled differential equations and those that include integrals are called integrated equations or integro-differential equations. These equations are more difficult that only the simple one differential equations.

There are methods and technics for analytical solutions for ordinary differential equations. But there are some that cannot be solved analytically. In that case a numerical solution for ordinary differential equations are in charge. But this methods has many defaults as well.
It is e.g. not that simple, when it has to be showen how the change of the constant and parameters in the equations affects the result at the end.

There are of course a lot of softwares that help finding of these numerical solutions of ordinary differential equations.
ClassPad has functions for analytical and numerical solution of differential equations.

The software can be useful to write programs for solving differential equations numerically. It doesn't matter if the engineer uses ready software for these solving or he writes himself the program. It is required to understand and estimate himself how the numerical solutions of differential equations is achieved mathematically.

There are two methods for findning these solutions. The first is Euler's method and the second is Runge-Kutta method.

Euler method

For a differential equation from first order $\frac{dx}{dt}$=f(t,x) can be defined a direction field. This field is such a field that has two-dimensional vector field, where the vector in each point (t,x) has slope $\frac{dx}{dt}$.

If we start from some starting point (t_0,x_0) a line $f(t_0,x_0)$ is drawn, and it is parallel to the direction of the field at that point. This line continue to the point with x-axes t_0+h. y-axes in that point is x_0+h·$f(t_0,x_0)$, which we will call x_1. The value of the field at that point is calculated and other streight line is drawn and another slope is created. This line continute to the point with x-axis t_0+2·h. This prosses can continue many times and the curve in (t,x) plane that contains the little streight lines is created.

$$t_1 = t_0 + h \qquad x_1 = x_0 + h \cdot f(t_0, x_0)$$
$$t_2 = t_1 + h \qquad x_2 = x_1 + h \cdot f(t_1, x_1)$$
$$t_3 = t_2 + h \qquad x_3 = x_2 + h \cdot f(t_2, x_2)$$
$$\dots \qquad\qquad \dots$$
$$t_{n+1} = t_n + h \qquad x_{n+1} = x_n + h \cdot f(t_n, x_n)$$

These equations define, mathematically, the simpliest method for integrating a differential equation from first order, known as well as an **Euler method.**

Step by step, the solutions are built starting from some initial point (t_0, x_0). For a given t_0 every different x_0 will describe a curve of solutions. This curve for every solution, but every each of them corespond with a different initial condition.

These solution curves, built by the help of this method are not the exact solutions but only their approximate one. Between these points the curves are approximately tangential to the direction field. When the distance is reduced, the curves that we drow will get better and better approximately to the exact solutions.

Increasing the h in every independent variable t together with the straight line is called **size of the step** which is later using in the solution.

Euler method can be seen throughout Taylor series.

$$x(t+h)=x(t)+h\cdot\frac{dx(t)}{dt}+\frac{h^2}{2!}\cdot\frac{d^2x(t)}{d^2t}+\frac{h^3}{3!}\cdot\frac{d^3x(t)}{d^3t}+\ldots$$

Using these formulas we can on theory to give a value to $x(t)$ and all derivatives to $x(t)$ as well as computing the values of $x(t+h)$ of every given h. If we choose a small step h we will provide a better approximation to the values of $x(t+h)$. The Euler method can be interpretated with Taylor series as follow:

$$x(t+h)=x(t)+h\cdot\frac{dx(t)}{dt}=x(t)+h\cdot f(t,x)$$

It starts from point (t_0,x_0), and showing $t_0+n\cdot h$ from t_n, we have:

$$x(t_1)=x(t_0+h)=x(t_0)+h\cdot f(t_0,x_0)$$
$$x(t_2)=x(t_1+h)=x(t_1)+h\cdot f(t_1,x_1)$$
$$x(t_3)=x(t_2+h)=x(t_2)+h\cdot f(t_2,x_2)$$

and so on.

Runge-Kutta method

Compering to the Euler method, Runge-Kutta method is more exact. But mathematically this method if more complicated.
The method follows the princip of using a slope at some point in an interval in order to estimate the average slope of all solution in this interval.

The most common usage of this method is from 4th order, as it is showen below:

$$c_1 = h \cdot f(t_n, x_n)$$

$$c_2 = h \cdot f\left(t_n + \frac{1}{2} \cdot h, x_n + \frac{1}{2} \cdot c_1\right)$$

$$c_3 = h \cdot f\left(t_n + \frac{1}{2} \cdot h, x_n + \frac{1}{2} \cdot c_2\right)$$

$$c_4 = h \cdot f(t_n + h, x_n + c_3)$$

$$x_{n+1} = x_n + \frac{1}{6} \cdot (c_1 + 2 \cdot c_2 + 2 \cdot c_3 + c_4)$$

Geometrically, this can be estimate with the **figure 4.4.**

The line **AB** has the same gradient as the direction field of the equation with this point **A**.

The coordinates of this line $t_n + \frac{1}{2}h$ define point **B**.

The line **AC** has gradient equal to the direction field at this point **C**.
The line **AD** with gradient equal to the direction field at point **C** define point **D**.
The average gradient of the solution in the interval (t_n, t_{n+1}) is then estimate from the found average values of the gradient at points **A, B, C** and **D**.
It is accepted to think that suck a process is able to give an exact estimation of the average gradient in some interval.

Figure 4.3
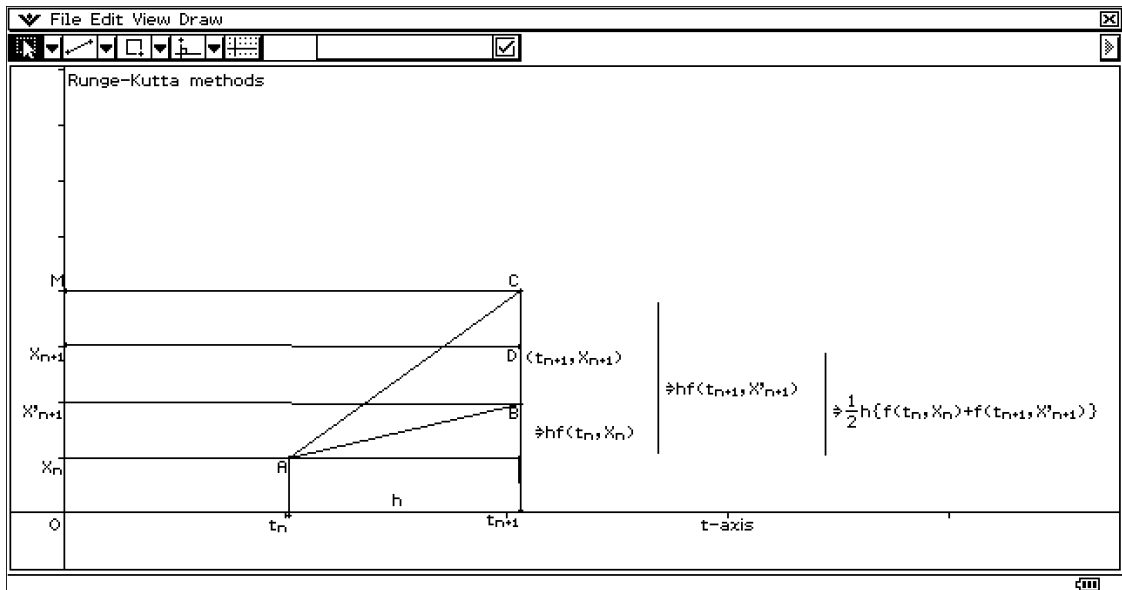
Figure 4.4
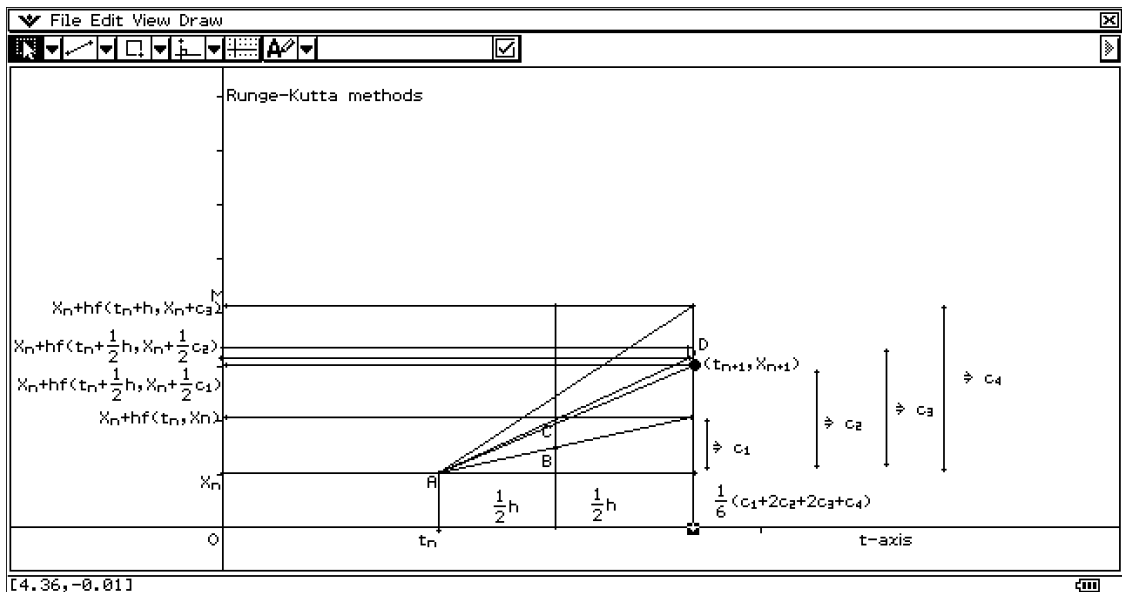
Example 1
The example below shows the closeness of the approximation x_a of the following differential equation $\mathbf{dx/dt = x^2/(t+1)}$, $\mathbf{x(0)=1}$, using the Euler method with step $h = 0.10$ and x_b, which point was obtained using step $h = s = 0.05$, and the calculated values of $x_a(t)$ and $x_b(t)$ for $t = 0.1, 0.2, 0.3 ,..., 1.0$.
Values obtained are compared with the exact solution $x(t)$:

```
V File Edit Graph Calc                                                    X
[toolbar]                                                                 ▷
      A          B          C          D          E          F       G    H    I
1     t          xa         xb         x(t)       |x-xa|     xb of s
2     0.000000   1.000000   1.000000   1.000000   0.000000   1.050000
3     0.100000   1.100000   1.102500   1.105351   0.005351   1.157750
4     0.200000   1.210000   1.216028   1.222975   0.012975   1.277641
5     0.300000   1.332008   1.342936   1.355683   0.023674   1.412301
6     0.400000   1.468489   1.486174   1.507096   0.038607   1.565057
7     0.500000   1.622522   1.649519   1.681987   0.059465   1.740216
8     0.600000   1.798027   1.837905   1.886805   0.088779   1.943465
9     0.700000   2.000083   2.057921   2.130507   0.130424   2.182481
10    0.800000   2.235397   2.318573   2.425928   0.190532   2.467900
11    0.900000   2.513008   2.632509   2.792157   0.279149   2.814880
12    1.000000   2.845387   3.018048   3.258891   0.413504   3.245763
=A2+h                                                                    ✓X
A3  0.1                                                                   ▥
```

The exact solution to a differential equation is **x=x(t)=1/(1-ln(t+1)).**
The numerical solutions x_a and x_b and their errors are shown in the table.
Below you can see the formulas by which it is calculated each row.

For x_a:


```
V File Edit Graph Calc                                                    X
[toolbar]                                                                 ▷
      A          B          C          D          E          F       G    H    I
1     t          xa         xb         x(t)       |x-xa|     xb of s
2     0.000000   1.000000   1.000000   1.000000   0.000000   1.050000
3     0.100000   1.100000   1.102500   1.105351   0.005351   1.157750
4     0.200000   1.210000   1.216028   1.222975   0.012975   1.277641
5     0.300000   1.332008   1.342936   1.355683   0.023674   1.412301
6     0.400000   1.468489   1.486174   1.507096   0.038607   1.565057
7     0.500000   1.622522   1.649519   1.681987   0.059465   1.740216
8     0.600000   1.798027   1.837905   1.886805   0.088779   1.943465
9     0.700000   2.000083   2.057921   2.130507   0.130424   2.182481
10    0.800000   2.235397   2.318573   2.425928   0.190532   2.467900
11    0.900000   2.513008   2.632509   2.792157   0.279149   2.814880
12    1.000000   2.845387   3.018048   3.258891   0.413504   3.245763
=B2+h·B2^2/(A2+1)                                                        ✓X
B3  1.1                                                                   ▥
```

For x_b: visible that x_b is used in smaller step (s = 0.05), values are calculated in the last column:

	A	B	C	D	E	F	G	H	I
1	t	xa	xb	x(t)	|x-xa|	xb of s			
2	0.000000	1.000000	1.000000	1.000000	0.000000	1.050000			
3	0.100000	1.100000	1.102500	1.105351	0.005351	1.157750			
4	0.200000	1.210000	1.216028	1.222975	0.012975	1.277641			
5	0.300000	1.332008	1.342936	1.355683	0.023674	1.412301			
6	0.400000	1.468489	1.486174	1.507096	0.038607	1.565057			
7	0.500000	1.622522	1.649519	1.681987	0.059465	1.740216			
8	0.600000	1.798027	1.837905	1.886805	0.088779	1.943465			
9	0.700000	2.000083	2.057921	2.130507	0.130424	2.182481			
10	0.800000	2.235397	2.318573	2.425928	0.190532	2.467900			
11	0.900000	2.513008	2.632509	2.792157	0.279149	2.814880			
12	1.000000	2.845387	3.018048	3.258891	0.413504	3.245763			

=F2+s·F2^2/(A2+s+1)

C3 1.1025

For the exact solution:

	A	B	C	D	E	F	G	H	I
1	t	xa	xb	x(t)	|x-xa|	xb of s			
2	0.000000	1.000000	1.000000	1.000000	0.000000	1.050000			
3	0.100000	1.100000	1.102500	1.105351	0.005351	1.157750			
4	0.200000	1.210000	1.216028	1.222975	0.012975	1.277641			
5	0.300000	1.332008	1.342936	1.355683	0.023674	1.412301			
6	0.400000	1.468489	1.486174	1.507096	0.038607	1.565057			
7	0.500000	1.622522	1.649519	1.681987	0.059465	1.740216			
8	0.600000	1.798027	1.837905	1.886805	0.088779	1.943465			
9	0.700000	2.000083	2.057921	2.130507	0.130424	2.182481			
10	0.800000	2.235397	2.318573	2.425928	0.190532	2.467900			
11	0.900000	2.513008	2.632509	2.792157	0.279149	2.814880			
12	1.000000	2.845387	3.018048	3.258891	0.413504	3.245763			

=1/(1-ln(A3+1))

D3 1.105351224

For the calculation of each line it is taken from the previous value:

File Edit Graph Calc

	A	B	C	D	E	F	G	H	I
	t	xa	xb	x(t)	|x-xa|	xb of s			
1									
2	0.000000	1.000000	1.000000	1.000000	0.000000	1.050000			
3	0.100000	1.100000	1.102500	1.105351	0.005351	1.157750			
4	0.200000	1.210000	1.216028	1.222975	0.012975	1.277641			
5	0.300000	1.332008	1.342936	1.355683	0.023674	1.412301			
6	0.400000	1.468489	1.486174	1.507096	0.038607	1.565057			
7	0.500000	1.622522	1.649519	1.681987	0.059465	1.740216			
8	0.600000	1.798027	1.837905	1.886805	0.088779	1.943465			
9	0.700000	2.000083	2.057921	2.130507	0.130424	2.182481			
10	0.800000	2.235397	2.318573	2.425928	0.190532	2.467900			
11	0.900000	2.513008	2.632509	2.792157	0.279149	2.814880			
12	1.000000	2.845387	3.018048	3.258891	0.413504	3.245763			

=1/(1-ln(A4+1))

D4 1.22297464

|x-x$_a$| shows the error which we have worked:

$|x-x_a|$ shows the error which we have worked:

File Edit Graph Calc

	A	B	C	D	E	F	G	H	I
	t	xa	xb	x(t)	|x-xa|	xb of s			
1									
2	0.000000	1.000000	1.000000	1.000000	0.000000	1.050000			
3	0.100000	1.100000	1.102500	1.105351	0.005351	1.157750			
4	0.200000	1.210000	1.216028	1.222975	0.012975	1.277641			
5	0.300000	1.332008	1.342936	1.355683	0.023674	1.412301			
6	0.400000	1.468489	1.486174	1.507096	0.038607	1.565057			
7	0.500000	1.622522	1.649519	1.681987	0.059465	1.740216			
8	0.600000	1.798027	1.837905	1.886805	0.088779	1.943465			
9	0.700000	2.000083	2.057921	2.130507	0.130424	2.182481			
10	0.800000	2.235397	2.318573	2.425928	0.190532	2.467900			
11	0.900000	2.513008	2.632509	2.792157	0.279149	2.814880			
12	1.000000	2.845387	3.018048	3.258891	0.413504	3.245763			

=abs(D4-B4)

E4 0.01297464035

	A	B	C	D	E	F	G	H	I
1	t	xa	xb	x(t)	|x-xa|	xb of s			
2	0.000000	1.000000	1.000000	1.000000	0.000000	1.050000			
3	0.100000	1.100000	1.102500	1.105351	0.005351	1.157750			
4	0.200000	1.210000	1.216028	1.222975	0.012975	1.277641			
5	0.300000	1.332008	1.342936	1.355683	0.023674	1.412301			
6	0.400000	1.468489	1.486174	1.507096	0.038607	1.565057			
7	0.500000	1.622522	1.649519	1.681987	0.059465	1.740216			
8	0.600000	1.798027	1.837905	1.886805	0.088779	1.943465			
9	0.700000	2.000083	2.057921	2.130507	0.130424	2.182481			
10	0.800000	2.235397	2.318573	2.425928	0.190532	2.467900			
11	0.900000	2.513008	2.632509	2.792157	0.279149	2.814880			
12	1.000000	2.845387	3.018048	3.258891	0.413504	3.245763			
13									
14									
15									
16									
17									
18									
19									
20									
21									
22									
23									

=C2+s·C2^2/(A2+1)

F2 1.05

Example 2
The problem has been solved using the **Runge-Kutta method**. As already mentioned, the method is more complex to implement, but is more accurate than the Euler method.
For example is taken non-linear differential equation
$4y'=x^2+y^2$, $y(0)=1$. The values for x_1, x_2 and x_3 respectively are 0.8, 1.6, 2.4 ($h = r = 0.8$).

	A	B	C	D	E	F	G	H	I
1	x	ya	c1	c2	c3	c4			
2	0.000000	1.000000	0.200000	0.274000	0.290554	0.461106			
3	0.800000	1.298369	0.465152	0.756759	0.850297	1.435353			
4	1.600000	2.150805	1.437192	2.446693	3.076979	6.617946			
5	2.400000	5.334552	6.843489	16.902545	39.577791	405.471709			
6									
7									
8									
9									
10									
11									
12									
13									
14									
15									
16									
17									
18									
19									
20									
21									
22									
23									

x

A1 x

As clearly seen to find any single point is necessary first to find the constants c_1, c_2, c_3 and c_4. Each of them is using the formulas described above.

	A	B	C	D	E	F	G	H	I
1	x	ya	c1	c2	c3	c4			
2	0.000000	1.000000	0.200000	0.274000	0.290554	0.461106			
3	0.800000	1.298369	0.465152	0.756759	0.850297	1.435353			
4	1.600000	2.150805	1.437192	2.446693	3.076979	6.617946			
5	2.400000	5.334552	6.843489	16.902545	39.577791	405.471709			

=r·(A2^2+B2^2)/4

C2 0.2

	A	B	C	D	E	F	G	H	I
1	x	ya	c1	c2	c3	c4			
2	0.000000	1.000000	0.200000	0.274000	0.290554	0.461106			
3	0.800000	1.298369	0.465152	0.756759	0.850297	1.435353			
4	1.600000	2.150805	1.437192	2.446693	3.076979	6.617946			
5	2.400000	5.334552	6.843489	16.902545	39.577791	405.471709			

=r·((A2+r/2)^2+(B2+C2/2)^2)/4

D2 0.274

	A	B	C	D	E	F	G	H	I
1	x	ya	c1	c2	c3	c4			
2	0.000000	1.000000	0.200000	0.274000	0.290554	0.461106			
3	0.800000	1.298369	0.465152	0.756759	0.850297	1.435353			
4	1.600000	2.150805	1.437192	2.446693	3.076979	6.617946			
5	2.400000	5.334552	6.843489	16.902545	39.577791	405.471709			

=r·((A2+r/2)^2+(B2+D2/2)^2)/4

E2 0.2905538

=r·((A2+r)^2+(B2+E2)^2)/4

F2 0.4611058221

```
=B2+(C2+2·D2+2·E2+F2)/6
B3 1.298368904
```

	A	B	C	D	E	F	G	H	I
1	x	ya	c1	c2	c3	c4			
2	0.000000	1.000000	0.200000	0.274000	0.290554	0.461106			
3	0.800000	1.298369	0.465152	0.756759	0.850297	1.435353			
4	1.600000	2.150805	1.437192	2.446693	3.076979	6.617946			
5	2.400000	5.334552	6.843489	16.902545	39.577791	405.471709			

For greater accuracy and precision, the task can be solved by using a small step (not here as h = 0.8).

The values for x are in 0.05, reaching a value 2.4:

Then each of the constants are calculated consistently with other step by step starting from 0.05 and lead it step 0.8, as shown below.

File Edit Insert Action

Define $f(x,y)=\dfrac{x^2+y^2}{4}$ ..... done

$0.1 \Rightarrow h$ 0.1

$0.05 \Rightarrow s$ 0.05

$0.2 \Rightarrow p$ 0.2

$0.8 \Rightarrow r$ 0.8

$0.4 \Rightarrow m$ 0.4

Alg Decimal Real Rad
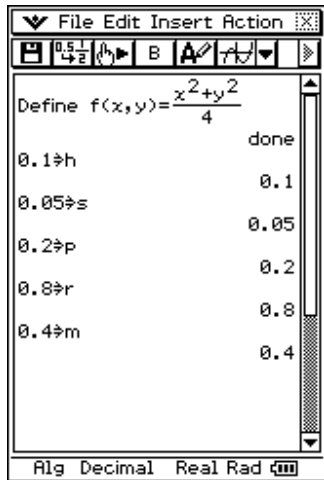
When we work with, these values can be set in the Main or eActivity and can be used in SpreadSheet application.

File Edit Graph Calc

	G	H	I	J	K	L	M	N	O
1	c1 with s	c2 with s	c3 with s	c4 with s	c1 with h	c2 with h	c3 with h	c4 with h	
2	0.01250	0.01266	0.01267	0.01285	0.02500	0.02569	0.02571	0.02655	
3	0.01285	0.01305	0.01305	0.01328	0.02655	0.02755	0.02758	0.02874	
4	0.01328	0.01352	0.01352	0.01378	0.02874	0.03006	0.03010	0.03159	
5	0.01378	0.01406	0.01407	0.01437	0.03160	0.03327	0.03332	0.03518	
6	0.01437	0.01469	0.01470	0.01504	0.03518	0.03723	0.03729	0.03955	
7	0.01504	0.01540	0.01541	0.01580	0.03955	0.04201	0.04208	0.04477	
8	0.01580	0.01621	0.01621	0.01665	0.04477	0.04768	0.04777	0.05093	
9	0.01665	0.01710	0.01711	0.01759	0.05093	0.05434	0.05445	0.05814	
10	0.01759	0.01809	0.01810	0.01863	0.05814	0.06211	0.06224	0.06653	
11	0.01863	0.01918	0.01919	0.01977	0.06653	0.07113	0.07129	0.07626	
12	0.01977	0.02038	0.02039	0.02102	0.07625	0.08158	0.08178	0.08753	
13	0.02102	0.02168	0.02169	0.02238	0.08752	0.09370	0.09394	0.10060	
14	0.02238	0.02310	0.02311	0.02386	0.10060	0.10777	0.10807	0.11583	
15	0.02386	0.02464	0.02465	0.02547	0.11582	0.12419	0.12456	0.13365	
16	0.02547	0.02631	0.02632	0.02720	0.13364	0.14346	0.14393	0.15465	
17	0.02720	0.02811	0.02812	0.02907	0.15464	0.16627	0.16687	0.17964	
18	0.02907	0.03005	0.03007	0.03109	0.17963	0.19355	0.19433	0.20972	
19	0.03109	0.03215	0.03217	0.03326	0.20970	0.22658	0.22762	0.24643	
20	0.03326	0.03440	0.03442	0.03561	0.24641	0.26719	0.26860	0.29200	
21	0.03561	0.03683	0.03685	0.03813	0.29197	0.31805	0.32000	0.34973	
22	0.03813	0.03945	0.03947	0.04084	0.34967	0.38312	0.38591	0.42462	
23	0.04084	0.04226	0.04229	0.04376	0.42453	0.46859	0.47275	0.52467	

=s·f(A2,F2)

G2 0.0125

For easier data entry and formulas outside SpreadSheet application declare a function with its variables. Every time you want to enter the equation **4y'=x²+y²** is not enough to write out **f(x,y)**. As indicated above table it is shown how the calculated c_1, c_2, c_3 and c_4 with step 0.05 and 0.1 respectively, s and h. It is seen that to find any formula used to disclose a function f (x, y). A2 and F2 are x and y.

	G	H	I	J	K	L	M	N	O
	c1 with s	c2 with s	c3 with s	c4 with s	c1 with h	c2 with h	c3 with h	c4 with h	
2	0.01250	0.01266	0.01267	0.01285	0.02500	0.02569	0.02571	0.02655	
3	0.01285	0.01305	0.01305	0.01328	0.02655	0.02755	0.02758	0.02874	
4	0.01328	0.01352	0.01352	0.01378	0.02874	0.03006	0.03010	0.03159	
5	0.01378	0.01406	0.01407	0.01437	0.03160	0.03327	0.03332	0.03518	
6	0.01437	0.01469	0.01470	0.01504	0.03518	0.03723	0.03729	0.03955	
7	0.01504	0.01540	0.01541	0.01580	0.03955	0.04201	0.04208	0.04477	
8	0.01580	0.01621	0.01621	0.01665	0.04477	0.04768	0.04777	0.05093	
9	0.01665	0.01710	0.01711	0.01759	0.05093	0.05434	0.05445	0.05814	
10	0.01759	0.01809	0.01810	0.01863	0.05814	0.06211	0.06224	0.06653	
11	0.01863	0.01918	0.01919	0.01977	0.06653	0.07113	0.07129	0.07626	
12	0.01977	0.02038	0.02039	0.02102	0.07625	0.08158	0.08178	0.08753	
13	0.02102	0.02168	0.02169	0.02238	0.08752	0.09370	0.09394	0.10060	
14	0.02238	0.02310	0.02311	0.02386	0.10060	0.10777	0.10807	0.11583	
15	0.02386	0.02464	0.02465	0.02547	0.11582	0.12419	0.12456	0.13365	
16	0.02547	0.02631	0.02632	0.02720	0.13364	0.14346	0.14393	0.15465	
17	0.02720	0.02811	0.02812	0.02907	0.15464	0.16627	0.16687	0.17964	
18	0.02907	0.03005	0.03007	0.03109	0.17963	0.19355	0.19433	0.20972	
19	0.03109	0.03215	0.03217	0.03326	0.20970	0.22658	0.22762	0.24643	
20	0.03326	0.03440	0.03442	0.03561	0.24641	0.26719	0.26860	0.29200	
21	0.03561	0.03683	0.03685	0.03813	0.29197	0.31805	0.32000	0.34973	
22	0.03813	0.03945	0.03947	0.04084	0.34967	0.38312	0.38591	0.42462	
23	0.04084	0.04226	0.04229	0.04376	0.42453	0.46859	0.47275	0.52467	

=s·f(A2+s,F2+I2)

J2 0.0128499210̄6

Each subsequent value of the constants is taking value from the previous row.

	G	H
	c1 with s	c2 with s
2	0.01250	0.01266
3	0.01285	0.01305
4	0.01328	0.01352
5	0.01378	0.01406
6	0.01437	0.01469
7	0.01504	0.01540
8	0.01580	0.01621
9	0.01665	0.01710
10	0.01759	0.01809
11	0.01863	0.01918
12	0.01977	0.02038
13	0.02102	0.02168
14	0.02238	0.02310
15	0.02386	0.02464

=s·f(A3,F3)

G3 0.01284997397

	G	H
	c1 with s	c2 with s
2	0.01250	0.01266
3	0.01285	0.01305
4	0.01328	0.01352
5	0.01378	0.01406
6	0.01437	0.01469
7	0.01504	0.01540
8	0.01580	0.01621
9	0.01665	0.01710
10	0.01759	0.01809
11	0.01863	0.01918
12	0.01977	0.02038
13	0.02102	0.02168
14	0.02238	0.02310
15	0.02386	0.02464

=s·f(A4,F4)

G4 0.0132764083

	G	H
38	0.12320	0.12834
39	0.13395	0.13969
40	0.14598	0.15243
41	0.15952	0.16679
42	0.17483	0.18309
43	0.19227	0.20171
44	0.21226	0.22313
45	0.23536	0.24797
46	0.26226	0.27703
47	0.29389	0.31135
48	0.33146	0.35234
49	0.37663	0.40190
50	0.43163	0.46266
51		
52		

=s·f(A50,F50)

G50 0.4316324453

The values of the points calculated using these constants.

File Edit Graph Calc

	F	G
1	**ya for 0.05**	**c1 with**
2	**1.00000**	0.0125
3	1.01267	0.0128
4	1.02573	0.0132
5	1.03925	0.0137
6	1.05332	0.0143
7	1.06801	0.0150
8	1.08342	0.0158
9	1.09964	0.0166
10	1.11675	0.0175
11	1.13485	0.0186
12	1.15404	0.0197
13	1.17443	0.0210
14	1.19612	0.0223

```
=F2+(G2+2·H2+2·I2+J2)
F3 1.012668711
```


File Edit Graph Calc

	F	G
1	**ya for 0.05**	**c1 with**
2	**1.00000**	0.0125
3	1.01267	0.0128
4	1.02573	0.0132
5	1.03925	0.0137
6	1.05332	0.0143
7	1.06801	0.0150
8	1.08342	0.0158
9	1.09964	0.0166
10	1.11675	0.0175
11	1.13485	0.0186
12	1.15404	0.0197
13	1.17443	0.0210
14	1.19612	0.0223

```
=F3+(G3+2·H3+2·I3+J3)
F4 1.025725433
```


File Edit Graph Calc

	F	G
1	**ya for 0.05**	**c1 with**
2	**1.00000**	0.0125
3	1.01267	0.0128
4	1.02573	0.0132
5	1.03925	0.0137
6	1.05332	0.0143
7	1.06801	0.0150
8	1.08342	0.0158
9	1.09964	0.0166
10	1.11675	0.0175
11	1.13485	0.0186
12	1.15404	0.0197
13	1.17443	0.0210
14	1.19612	0.0223

```
=F4+(G4+2·H4+2·I4+J4)
F5 1.039247811
```

The same is done for the remaining steps p, r and m.

File Edit Graph Calc

	H	I	J	K	L	M	N	O	P
27	0.05588	0.05592	0.05791	**c1 with p**	**c2 with p**	**c3 with p**	**c4 with p**		
28	0.05998	0.06003	0.06219	0.05000	0.05303	0.05319	0.05746		
29	0.06443	0.06448	0.06682	0.05747	0.06304	0.06334	0.07035		
30	0.06925	0.06931	0.07185	0.07036	0.07885	0.07934	0.08953		
31	0.07449	0.07456	0.07732	0.08954	0.10149	0.10223	0.11629		
32	0.08020	0.08027	0.08329	0.11628	0.13250	0.13360	0.15252		
33	0.08643	0.08652	0.08981	0.15251	0.17422	0.17586	0.20123		
34	0.09326	0.09336	0.09697	0.20120	0.23038	0.23288	0.26735		
35	0.10076	0.10087	0.10485	0.26728	0.30726	0.31123	0.35939		
36	0.10903	0.10915	0.11355	0.35926	0.41600	0.42265	0.49308		
37	0.11818	0.11832	0.12321	0.49281	0.57773	0.58979	0.69991		
38	0.12834	0.12851	0.13396	0.69934	0.83645	0.86075	1.05039		
39	0.13969	0.13989	0.14599	1.04903	1.29601	1.35286	1.73005		
40	0.15243	0.15266	0.15952						
41	0.16679	0.16707	0.17484	**c1 with m**	**c2 with m**	**c3 with m**	**c4 with m**		
42	0.18309	0.18343	0.19228	0.10000	0.11425	0.11575	0.14049		
43	0.20171	0.20212	0.21227	0.14071	0.17692	0.18125	0.23248		
44	0.22313	0.22363	0.23537	0.23257	0.30012	0.30979	0.40261		
45	0.24797	0.24858	0.26227	0.40241	0.52314	0.54534	0.71947		
46	0.27703	0.27779	0.29390	0.71853	0.95396	1.01444	1.40178		
47	0.31135	0.31232	0.33149	1.39866	1.97357	2.20372	3.45312		
48	0.35234	0.35359	0.37666						
49	0.40190	0.40353	0.43168	**c1 with r**	**c2 with r**	**c3 with r**	**c4 with r**		

```
=p·f(A2,D2)
K28 0.05
```

	H	I	J	K	L	M	N	O	P
27	0.05588	0.05592	0.05791	c1 with p	c2 with p	c3 with p	c4 with p		
28	0.05998	0.06003	0.06219	0.05000	0.05303	0.05319	0.05746		
29	0.06443	0.06448	0.06682	0.05747	0.06304	0.06334	0.07035		
30	0.06925	0.06931	0.07185	0.07036	0.07885	0.07934	0.08953		
31	0.07449	0.07456	0.07732	0.08954	0.10149	0.10223	0.11629		
32	0.08020	0.08027	0.08329	0.11628	0.13250	0.13360	0.15252		
33	0.08643	0.08652	0.08981	0.15251	0.17422	0.17586	0.20123		
34	0.09326	0.09336	0.09697	0.20120	0.23038	0.23288	0.26735		
35	0.10076	0.10087	0.10485	0.26728	0.30726	0.31123	0.35939		
36	0.10903	0.10915	0.11355	0.35926	0.41600	0.42265	0.49308		
37	0.11818	0.11832	0.12321	0.49281	0.57773	0.58979	0.69991		
38	0.12834	0.12851	0.13396	0.69934	0.83645	0.86075	1.05039		
39	0.13969	0.13989	0.14599	1.04903	1.29601	1.35286	1.73005		
40	0.15243	0.15266	0.15952						
41	0.16679	0.16707	0.17484	c1 with m	c2 with m	c3 with m	c4 with m		
42	0.18309	0.18343	0.19228	0.10000	0.11425	0.11575	0.14049		
43	0.20171	0.20212	0.21227	0.14071	0.17692	0.18125	0.23248		
44	0.22313	0.22363	0.23537	0.23257	0.30012	0.30979	0.40261		
45	0.24797	0.24858	0.26227	0.40241	0.52314	0.54534	0.71947		
46	0.27703	0.27779	0.29390	0.71853	0.95396	1.01444	1.40178		
47	0.31135	0.31232	0.33149	1.39866	1.97357	2.20372	3.45312		
48	0.35234	0.35359	0.37666						
49	0.40190	0.40353	0.43168	c1 with r	c2 with r	c3 with r	c4 with r		

=m·f(A2,C2)

K42 0.1

	H	I	J	K	L	M	N	O	P
34	0.09326	0.09336	0.09697	0.20120	0.23038	0.23288	0.26735		
35	0.10076	0.10087	0.10485	0.26728	0.30726	0.31123	0.35939		
36	0.10903	0.10915	0.11355	0.35926	0.41600	0.42265	0.49308		
37	0.11818	0.11832	0.12321	0.49281	0.57773	0.58979	0.69991		
38	0.12834	0.12851	0.13396	0.69934	0.83645	0.86075	1.05039		
39	0.13969	0.13989	0.14599	1.04903	1.29601	1.35286	1.73005		
40	0.15243	0.15266	0.15952						
41	0.16679	0.16707	0.17484	c1 with m	c2 with m	c3 with m	c4 with m		
42	0.18309	0.18343	0.19228	0.10000	0.11425	0.11575	0.14049		
43	0.20171	0.20212	0.21227	0.14071	0.17692	0.18125	0.23248		
44	0.22313	0.22363	0.23537	0.23257	0.30012	0.30979	0.40261		
45	0.24797	0.24858	0.26227	0.40241	0.52314	0.54534	0.71947		
46	0.27703	0.27779	0.29390	0.71853	0.95396	1.01444	1.40178		
47	0.31135	0.31232	0.33149	1.39866	1.97357	2.20372	3.45312		
48	0.35234	0.35359	0.37666						
49	0.40190	0.40353	0.43168	c1 with r	c2 with r	c3 with r	c4 with r		
50	0.46266	0.46483	0.49970	0.20000	0.27400	0.29055	0.46111		
51				0.46515	0.75676	0.85030	1.43535		
52				1.43719	2.44669	3.07698	6.61795		
53									
54									
55									
56									

=r·f(A2,B2)

K50 0.2

	A	B	C	D	E	F	G	H	I	
1	x	ya for 0.8	ya for 0.4	ya for 0.2	ya for 0.1	ya for 0.05	c1 with s	c2 with s	c3 with s	c4
2	0.00	1.00000	1.00000	1.00000	1.00000	1.00000	0.01250	0.01266	0.01267	0
3	0.050					1.01267	0.01285	0.01305	0.01305	0
4	0.100				1.02573	1.02573	0.01328	0.01352	0.01352	0
5	0.150					1.03925	0.01378	0.01406	0.01407	0
6	0.200			1.05332	1.05332	1.05332	0.01437	0.01469	0.01470	0
7	0.250					1.06801	0.01504	0.01540	0.01541	0
8	0.300				1.08342	1.08342	0.01580	0.01621	0.01621	0
9	0.350					1.09964	0.01665	0.01710	0.01711	0
10	0.400		1.11675	1.11675	1.11675	1.11675	0.01759	0.01809	0.01810	0
11	0.450					1.13485	0.01863	0.01918	0.01919	0
12	0.500				1.15404	1.15404	0.01977	0.02038	0.02039	0
13	0.550					1.17443	0.02102	0.02168	0.02169	0
14	0.600			1.19612	1.19612	1.19612	0.02238	0.02310	0.02311	0
15	0.650					1.21924	0.02386	0.02464	0.02465	0
16	0.700				1.24389	1.24389	0.02547	0.02631	0.02632	0
17	0.750					1.27021	0.02720	0.02811	0.02812	0
18	0.80	1.29837	1.29834	1.29834	1.29834	1.29834	0.02907	0.03005	0.03007	0
19	0.850					1.32840	0.03109	0.03215	0.03217	0
20	0.900				1.36057	1.36057	0.03326	0.03440	0.03442	0
21	0.950					1.39499	0.03561	0.03683	0.03685	0
22	1.000			1.43184	1.43184	1.43184	0.03813	0.03945	0.03947	0
23	1.050					1.47130	0.04084	0.04226	0.04229	0

=E2+(K2+2·L2+2·M2+N2)/6

E4 1.025725434

	A	B	C	D	E	F	G	H	I	
17	0.750					1.27021	0.02720	0.02811	0.02812	0
18	0.80	1.29837	1.29834	1.29834	1.29834	1.29834	0.02907	0.03005	0.03007	0
19	0.850					1.32840	0.03109	0.03215	0.03217	0
20	0.900				1.36057	1.36057	0.03326	0.03440	0.03442	0
21	0.950					1.39499	0.03561	0.03683	0.03685	0
22	1.000			1.43184	1.43184	1.43184	0.03813	0.03945	0.03947	0
23	1.050					1.47130	0.04084	0.04226	0.04229	0
24	1.100				1.51359	1.51359	0.04376	0.04529	0.04532	0
25	1.150					1.55890	0.04691	0.04856	0.04859	0
26	1.200		1.60750	1.60749	1.60749	1.60749	0.05030	0.05208	0.05211	0
27	1.250					1.65959	0.05396	0.05588	0.05592	0
28	1.300				1.71550	1.71550	0.05791	0.05998	0.06003	0
29	1.350					1.77552	0.06219	0.06443	0.06448	0
30	1.400			1.84000	1.84000	1.84000	0.06682	0.06925	0.06931	0
31	1.450					1.90930	0.07185	0.07449	0.07456	0
32	1.500				1.98384	1.98384	0.07732	0.08020	0.08027	0
33	1.550					2.06410	0.08329	0.08643	0.08652	0
34	1.600	2.15080	2.15064	2.15061	2.15060	2.15060	0.08981	0.09326	0.09336	0
35	1.650					2.24394	0.09697	0.10076	0.10087	0
36	1.700				2.34479	2.34479	0.10485	0.10903	0.10915	0
37	1.750					2.45392	0.11355	0.11818	0.11832	0
38	1.800			2.57221	2.57221	2.57221	0.12320	0.12834	0.12851	0
39	1.850					2.70069	0.13395	0.13969	0.13989	0

=E48+(K25+2·L25+2·M25+N25)/6

E50 5.363815206

	A	B	C	D	E	F	G	H	I	
34	1.600…	2.15080	2.15064	2.15061	2.15060	2.15060	0.08981	0.09326	0.09336	0
35	1.650…					2.24394	0.09697	0.10076	0.10087	0
36	1.700…				2.34479	2.34479	0.10485	0.10903	0.10915	0
37	1.750…					2.45392	0.11355	0.11818	0.11832	0
38	1.800…			2.57221	2.57221	2.57221	0.12320	0.12834	0.12851	0
39	1.850…					2.70069	0.13395	0.13969	0.13989	0
40	1.900…				2.84054	2.84054	0.14598	0.15243	0.15266	0
41	1.950…					2.99316	0.15952	0.16679	0.16707	0
42	2.000…		3.16016	3.16018	3.16017	3.16017	0.17483	0.18309	0.18343	0
43	2.050…					3.34353	0.19227	0.20171	0.20212	0
44	2.100…				3.54556	3.54556	0.21226	0.22313	0.22363	0
45	2.150…					3.76909	0.23536	0.24797	0.24858	0
46	2.200…			4.01753	4.01754	4.01754	0.26226	0.27703	0.27779	0
47	2.250…					4.29518	0.29389	0.31135	0.31232	0
48	2.300…				4.60729	4.60730	0.33146	0.35234	0.35359	0
49	2.350…					4.96063	0.37663	0.40190	0.40353	0
50	2.400…	5.33455	5.36122	5.36367	5.36382	5.36382	0.43163	0.46266	0.46483	0
51										
52										
53										
54										
55										
56										

=E48+(K25+2·L25+2·M25+N25)/6

E50 5.363815206

5. Create programs with ClassPad

ClassPad allows the creation of programs when it is necessary to perform the same calculations repeatedly. Use the **Program Application**. The menu contains a **Program Editor** for entering data and a **Program Loader** for loading existing programs.
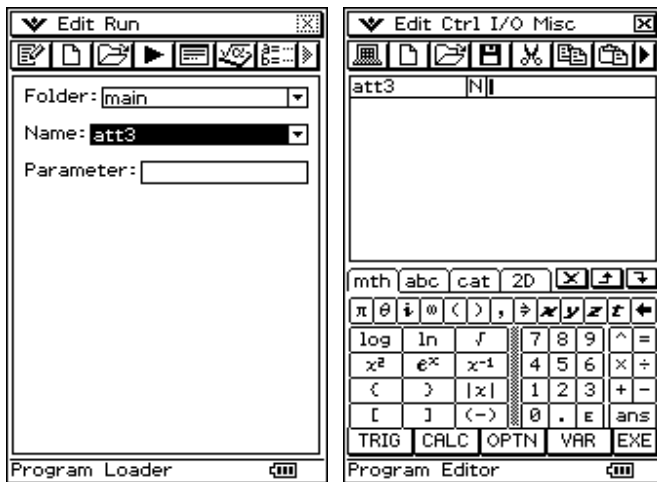
The **Program Loader** opens automatically when loading the application.
The **Program Editor** is used to introduce new programs or to edit existing ones.
With N turns out to be the type of file that you create.
It can be a text (T), programming (N) or defined function file (F)

In the next field parameters themselves if they occur. And yelling at each program, if such a set must not be missed.
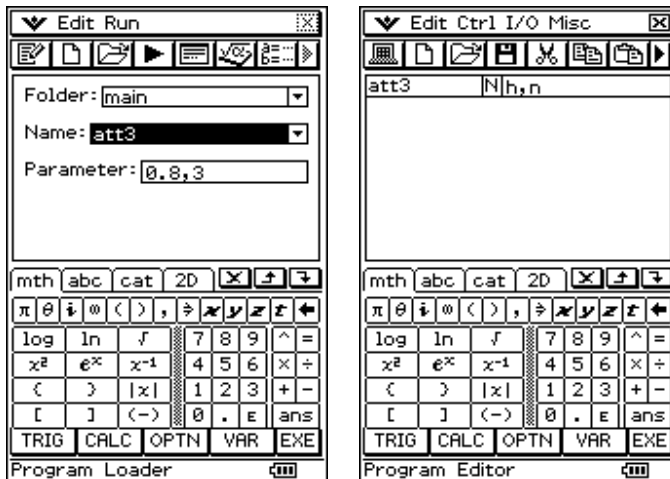
Example, establishing a program rests on the basis of the Runge-Kutta method. Made program, which introduces the steps and points x_1, …, x_n, which the program calculates the numerical solutions of differential equation $4y'=x^2+y^2$.

```
Edit Ctrl I/O Misc                                        ⊠
▣ ◻ ◿ 🖫 ✄ ▣ ▣ ♠♠ ♠♠ ▤ ▤                                  ▷
att3      N h,n
DelVar x,y
seq(0,y,0,n-1,1)⇒Listy
seq(0,c,0,n-1,1)⇒ListC1
seq(0,c,0,n-1,1)⇒ListC2
seq(0,c,0,n-1,1)⇒ListC3
seq(0,c,0,n-1,1)⇒ListC4
approx(seq(x,x,0,(n-1)*h,h))⇒Listx
1⇒y
1⇒Listy[1]
For 2⇒k To n Step 1
Listx[k-1]⇒x
approx(h*f(x,y))⇒ListC1[k]:ListC1[k]⇒C1
approx(h*f(x+h/2,y+C1/2))⇒ListC2[k]:ListC2[k]⇒C2
approx(h*f(x+h/2,y+C2/2))⇒ListC3[k]:ListC3[k]⇒C3
approx(h*f(x+h,y+C3))⇒ListC4[k]
approx(y+(ListC1[k]+2*ListC2[k]+2*ListC3[k]+ListC4[k])/6)⇒y
approx(y)⇒Listy[k]
Next
DispListEditor




Program Editor                                            ▥
```

Code:

DelVar x,y
seq(0,y,0,n-1,1)⇒Listy
seq(0,c,0,n-1,1)⇒ListC1
seq(0,c,0,n-1,1)⇒ListC2
seq(0,c,0,n-1,1)⇒ListC3
seq(0,c,0,n-1,1)⇒ListC4
approx(seq(x,x,0,(n-1)*h,h))⇒Listx
1⇒y
1⇒Listy[1]
For 2⇒k To n Step 1
Listx[k-1]⇒x
approx(h*f(x,y))⇒ListC1[k]:ListC1[k]⇒C1
approx(h*f(x+h/2,y+C1/2))⇒ListC2[k]:ListC2[k]⇒C2
approx(h*f(x+h/2,y+C2/2))⇒ListC3[k]:ListC3[k]⇒C3
approx(h*f(x+h,y+C3))⇒ListC4[k]
approx(y+(ListC1[k]+2*ListC2[k]+2*ListC3[k]+ListC4[k])/6)⇒y
approx(y)⇒Listy[k]
Next
DispListEditor

Edit Calc SetGraph

Folder: main
Name: att3
Parameter: 0.8,3

	Listx	Listy	ListC1	ListC2	ListC3	ListC4				
1	0	1	0	0	0	0				
2	0.8	1.29837	0.2	0.274	0.29055	0.46111				
3	1.6	2.1508	0.46515	0.75676	0.8503	1.43535				
4										
5										
6										
7										
8										
9										
10										

Cal▶

[1]=0

Rad Auto Standard

Edit Calc SetGraph

Folder: main
Name: att3
Parameter: 0.4,7

	Listx	Listy	ListC1	ListC2	ListC3	ListC4				
1	0	1	0	0	0	0				
2	0.4	1.11675	0.1	0.11425	0.11575	0.14049				
3	0.8	1.29834	0.14071	0.17692	0.18125	0.23248				
4	1.2	1.6075	0.23257	0.30012	0.30979	0.40261				
5	1.6	2.15064	0.40241	0.52314	0.54534	0.71947				
6	2	3.16016	0.71853	0.95396	1.01444	1.40178				
7	2.4	5.36122	1.39866	1.97357	2.20372	3.45312				
8										
9										
10										

Cal▶

[1]=0

Rad Auto Standard

```
▼ Edit Calc SetGraph                                                    ☒
[toolbar icons]                                                         ▷
Folder: main                                                           ▼
Name: att3                                                             ▼
Parameter: 0.2,13




    Listx    Listy    ListC1   ListC2   ListC3   ListC4              ▲
  4    0.6  1.19612  0.07036  0.07885  0.07934  0.08953              █
  5    0.8  1.29834  0.08954  0.10149  0.10223  0.11629
  6      1  1.43184  0.11628   0.1325   0.1336  0.15252
  7    1.2  1.60749  0.15251  0.17422  0.17586  0.20123
  8    1.4     1.84   0.2012  0.23038  0.23288  0.26735
  9    1.6  2.15061  0.26728  0.30726  0.31123  0.35939
 10    1.8  2.57221  0.35926    0.416  0.42265  0.49308
 11      2  3.16018  0.49281  0.57773  0.58979  0.69991
 12    2.2  4.01753  0.69934  0.83645  0.86075  1.05039
 13    2.4  5.36367  1.04903  1.29601  1.35286  1.73005
Cal▶                                                                   ▼
◀                                                      ▒▒▒▒▒         ▶
[   1]=0
Rad  Auto  Standard                                                   ▥
```

By creating programs quickly and easily change values to obtain the satisfying results.

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Erklärung:

Die vorliegende Arbeit wurde selbständig und ohne fremde Hilfe im SS2010 an der HTW Dresden, Fakultät Informatik/Mathematik unter der fachlichen Betreuung von Prof. Dr. Ludwig Paditz angefertigt.

Ралица Иванова, 20.07.2010   (korrigierte Version: 22.12.2010)