3.3.9 Solution of Linear Equation Systems with Parameters

cp. http://www.informatik.htw-dresden.de/

~paditz/Paditz_Beitrag_CEJ_2006.pdf

(bearbeitet 31.10.2021, Ludwig Paditz)

3x+2y+t*z=0➜equ1

$$t \cdot z + 3 \cdot x + 2 \cdot y = 0$$

0x+1y−4z=−1➜equ2

$$y - 4 \cdot z = -1$$

1x+3y+0z=1➜equ3

$$x + 3 \cdot y = 1$$

−1x+0y+2z=1➜equ4

$$-x + 2 \cdot z = 1$$

$$\left[\begin{matrix} equ1 \\ equ2 \\ equ3 \\ equ4 \end{matrix} \right|_{x, y, z, u}$$

$$\left\{ x = \frac{4 \cdot t + 8}{t - 28}, \ y = \frac{-(t+12)}{t-28}, \ z = \frac{-10}{t-28}, \ u = u \right\}$$

What happens with t=28?

$$\left[\begin{matrix} equ4 \\ equ2 \\ equ1 \\ equ3 \end{matrix} \right|_{x, y, z, u}$$

$$\left\{ x = \frac{-(t+4)}{t+14}, y = \frac{-(t-6)}{t+14}, z = \frac{5}{t+14}, u = u \right\}$$

What happens with t=−14?

$$\begin{bmatrix} equ4 \\ equ2 \\ equ3 \\ equ1 \end{bmatrix}_{x,\,y,\,z,\,u}$$

$$\left\{ x = -\frac{2}{7}, y = \frac{3}{7}, z = \frac{5}{14}, u = u \right\}$$

Here it seems for all t we get an unique solution (and parameter t disappears?)?

$$\begin{bmatrix} 3 & 2 & t & 0 \\ 0 & 1 & -4 & -1 \\ 1 & 3 & 0 & 1 \\ -1 & 0 & 2 & 1 \end{bmatrix} \Rightarrow mat$$

$$\begin{bmatrix} 3 & 2 & t & 0 \\ 0 & 1 & -4 & -1 \\ 1 & 3 & 0 & 1 \\ -1 & 0 & 2 & 1 \end{bmatrix}$$

ref(mat)

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{t}{3} & 0 \\ 0 & 1 & \frac{-t}{7} & \frac{3}{7} \\ 0 & 0 & 1 & \frac{5}{3 \cdot t + 14} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here it seems we have to study the special case t=−3/14?

rref(mat)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here it seems the parameter t is without meaning (t disappears?)?

But in case t=0:

ref(mat|t=0)

$$\begin{bmatrix} 1 & \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & \frac{5}{14} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rref(mat|t=0)

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{2}{7} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & \frac{5}{14} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank(mat)

$$4$$

rank(mat|t=0)

$$3$$

The rank–function can't compute the rank in dependence on the parameter t!

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If we use the "ref" or "rref" or "rank" functions the CP should give a message "no solution with parameters" (cp TI–Nspire CAS, OS 1.6)

another exercise with complex numbers:

$$\begin{bmatrix} j & 2 & t & 3+2j \\ 0 & 1 & 2j & 1+j \\ s & 0 & 4 & -1 \end{bmatrix} \Rightarrow \text{mat}$$

$$\begin{bmatrix} j & 2 & t & 3+2\cdot j \\ 0 & 1 & 2\cdot j & 1+j \\ s & 0 & 4 & -1 \end{bmatrix}$$

rank(mat)

$$3$$

rank(mat|s=$-j$ and t=$-4+4j$)

$$2$$

Again, the rank is depending on the parameters s and t

The exchange procedure:
========================

DelVar $a_{11}, a_{12}, a_{21}, a_{22}, \beta_1, \beta_2, x_1, x_2, t$

done

start with following system:

$x_1 \times a_{11} + x_2 \times a_{12} = \beta_1$ (1)

$x_1 \times a_{21} + x_2 \times a_{22} = \beta_2$ (2)

let be $a_{11} \neq 0$ (pivot element)

equivalent system:

$y_1=0=x_1 \times a_{11}+x_2 \times a_{12}-\beta_1 \times 1$ （3）

$y_2=0=x_1 \times a_{12}+x_2 \times a_{22}-\beta_2 \times 1$ （4）

solve$(y_1=x_1 \times a_{11}+x_2 \times a_{12}-\beta_1 \times 1, x_1)$

$$\left\{ x_1=\frac{-a_{12} \cdot x_2}{a_{11}}+\frac{\beta_1}{a_{11}}+\frac{y_1}{a_{11}} \right\}$$

expand$\left(y_2=x_1 \times a_{21}+x_2 \times a_{22}-\beta_2 \times 1 \mid x_1=\frac{-a_{12} \cdot x_2}{a_{11}}+\frac{\beta_1}{a_{11}}+\frac{y_1}{a_{11}} \right)$

$$y_2=\frac{-a_{12} \cdot a_{21} \cdot x_2}{a_{11}}+a_{22} \cdot x_2+\frac{\beta_1 \cdot a_{21}}{a_{11}}-\beta_2+\frac{a_{21} \cdot y_1}{a_{11}}$$

new equivalent system:

$$x_1=\frac{1}{a_{11}} \cdot y_1 \quad + \quad \frac{a_{12}}{-a_{11}} \cdot x_2 \quad + \quad \frac{-\beta_1}{-a_{11}} \cdot 1 \quad （5）$$

$$y_2=\frac{a_{12}}{a_{11}} \cdot y_1 \quad + \quad \left(a_{22}+a_{12} \cdot \frac{a_{21}}{-a_{11}} \right) \cdot x_2 \quad + \quad \left(-\beta_2+a_{12} \cdot \frac{-\beta_1}{-a_{11}} \right) \cdot 1 \quad （6）$$

write （3）, （4） in a table **ST** with matrix **matST:**

ST	x_1	x_2	1
y_1	a_{11}	a_{12}	$-\beta_1$
y_2	a_{21}	a_{22}	$-\beta_2$

| K | $*$ | $\frac{a_{12}}{-a_{11}}$ | $\frac{-\beta_1}{-a_{11}}$ |

K is the bottom （a help row,

put $*$ in the pivot column

and

divide by $-$pivot otherwise.）

where **matST**$=\begin{bmatrix} a_{11} & a_{12} & -\beta_1 \\ a_{21} & a_{22} & -\beta_2 \end{bmatrix}$ $-$ in matST are all

informations on the system.

write (5), (6) in a table **T1** with matrix **matT1**
(an equivalent system, exchange $y_1 \leftrightarrow x_1$)

T1	y_1	x_2	1	
x_1	$\dfrac{1}{a_{11}}$	$\dfrac{a_{12}}{-a_{11}}$	$\dfrac{-\beta_1}{-a_{11}}$	using
		inverse value and K row respectively		
y_2	$\dfrac{a_{12}}{a_{11}}$	$a_{22}+a_{12}\cdot\dfrac{a_{21}}{-a_{11}}$	$-\beta_2+a_{12}\cdot\dfrac{-\beta_1}{-a_{11}}$	division by
			pivot (old pivot column) and	
		"triangular rule" respectively		

delete y_1 column (because $y_1=0$)

T1	y_1	x_2	1
x_1	■	$\dfrac{a_{12}}{-a_{11}}$	$\dfrac{-\beta_1}{-a_{11}}$
y_2	■	$a_{22}+a_{12}\cdot\dfrac{a_{21}}{-a_{11}}$	$-\beta_2+a_{12}\cdot\dfrac{-\beta_1}{-a_{11}}$

with **matT1** $=\begin{bmatrix} \dfrac{a_{12}}{-a_{11}} & \dfrac{-\beta_1}{-a_{11}} \\ a_{22}+a_{12}\cdot\dfrac{a_{21}}{-a_{11}} & -\beta_2+a_{12}\cdot\dfrac{-\beta_1}{-a_{11}} \end{bmatrix}$

in matT1 are all informations on the system.

The next pivot element could be $a_{22}+a_{12}\cdot\dfrac{a_{21}}{-a_{11}}$ ($\neq 0$) to

exchange $y_2 \leftrightarrow x_2$

Thus we get (if exchange $y_2 \leftrightarrow x_2$ possible)

T2=ET y_1 y_2 1
————————————————————————————
  x_1 | ■   ■   □
  x_2 | ■   ■   □   unique solution
————————————————————————————

If $a_{22}+a_{12}\cdot\dfrac{a_{21}}{-a_{11}}=0$ than T1=ET (end table with followig

decisions):

T1=ET y_1 x_2=t   1
————————————————————————————
  x_1 | ■   □   □
  y_2 | ■   0   [0]   non−unique solution
                                                   (with parameter t)
————————————————————————————

T1=ET y_1 x_2=t   1
————————————————————————————
  x_1 | ■   □   □
  y_2 | ■   0   [$\neq 0$]   (no solution,
                   contradiction in the y−row $y\neq 0$)
————————————————————————————
stop

Now using the new created functions LinEqSys and AVRank in Main-menu

(To use the LinEqSys and AVRank program in an eActivity the program must be stored in the Library-folder!)

ST	x_1	x_2	x_3	1
y_1	a_{11}	a_{12}	a_{13}	$-\beta_1$
y_2	a_{21}	a_{22}	a_{23}	$-\beta_2$
y_3	a_{31}	a_{32}	a_{33}	$-\beta_3$
y_4	a_{41}	a_{42}	a_{43}	$-\beta_4$

pivot $a_{11}\neq 0$ to
exchange $y_1 \leftrightarrow x_1$

$$\begin{bmatrix} 3 & 2 & t & 0 \\ 0 & 1 & -4 & 1 \\ 1 & 3 & 0 & -1 \\ -1 & 0 & 2 & -1 \end{bmatrix} \Rightarrow matST$$

$$\begin{bmatrix} 3 & 2 & t & 0 \\ 0 & 1 & -4 & 1 \\ 1 & 3 & 0 & -1 \\ -1 & 0 & 2 & -1 \end{bmatrix}$$

LinEqSys(matST,1,1)

done

syntax: **LinEqSys**(matrix, row index i of a_{ik}, column index k of a_{ik}), if a_{ik} pivot

matnew⇒matT1

$$\begin{bmatrix} -\dfrac{2}{3} & \dfrac{-t}{3} & 0 \\ 1 & -4 & 1 \\ \dfrac{7}{3} & \dfrac{-t}{3} & -1 \\ \dfrac{2}{3} & \dfrac{t}{3}+2 & -1 \end{bmatrix}$$

T1	y_1	x_2	x_3	1
x_1 $\mid$	■	□	□	□
y_2 $\mid$	■	□	□	□
y_3 $\mid$	■	□	□	□
y_4 $\mid$	■	□	□	□

pivot $a_{22} \neq 0$ to

exchange $y_2 \leftrightarrow x_2$

LinEqSys(matT1, 2, 1)

done

matnew⇒matT2

$$\begin{bmatrix} \dfrac{-(t+8)}{3} & \dfrac{2}{3} \\ 4 & -1 \\ \dfrac{-(t-28)}{3} & -\dfrac{10}{3} \\ \dfrac{t+14}{3} & -\dfrac{5}{3} \end{bmatrix}$$

T2	y_1	y_2	x_3	1
x_1 $\mid$	■	■	□	□
x_2 $\mid$	■	■	□	□
y_3 $\mid$	■	■	□	□
y_4 $\mid$	■	■	□	□

pivot $a_{33} \neq 0$ to

exchange $y_3 \leftrightarrow x_3$ or

pivot $a_{43} \neq 0$ to

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exchange $y_4 \leftrightarrow x_3$

If $t \neq 28$ than we have a third exchange-step $x_3 \leftrightarrow y_3$

LinEqSys(matT2,3,1)

done

matnew➔matET

$$\begin{bmatrix} \dfrac{4 \cdot (t+2)}{t-28} \\ \dfrac{-40}{t-28}-1 \\ \dfrac{-10}{t-28} \\ \dfrac{-5 \cdot t}{t-28} \end{bmatrix}$$

T3=ET	y_1	y_2	y_3	1
x_1	■	■	■	$\dfrac{4 \cdot (t+2)}{t-28}$
x_2	■	■	■	$\dfrac{-40}{t-28}-1$
x_3	■	■	■	$\dfrac{-10}{t-28}$
y_4	■	■	■	$\dfrac{-5 \cdot t}{t-28}$ $y_4=0$, if $t=0$

matET is the solution, if the last element $\dfrac{-5 \cdot t}{t-28}$ equals 0, i.e. $t=0$

matET|t=0

$$\begin{bmatrix} -\dfrac{2}{7} \\[8pt] \dfrac{3}{7} \\[8pt] \dfrac{5}{14} \\[8pt] 0 \end{bmatrix}$$

Thus x=−2/7, y=3/7, z=5/14

If t≠14 than we have a third exchange−step $x_3 \leftrightarrow y_4$
We start again with matST:

LinEqSys(matST,1,1)

done

LinEqSys(matnew,2,1)

done

LinEqSys(matnew,4,1)

done

matnew➔matET

$$\begin{bmatrix} \dfrac{-(t+4)}{t+14} \\[8pt] \dfrac{20}{t+14}-1 \\[8pt] \dfrac{-5 \cdot t}{t+14} \\[8pt] \dfrac{5}{t+14} \end{bmatrix}$$

T3=ET y_1 y_2 y_4 1
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$$x_1 \quad | \quad \blacksquare \qquad \blacksquare \qquad \blacksquare \qquad \frac{-(t+4)}{t+14}$$

$$x_2 \quad | \quad \blacksquare \qquad \blacksquare \qquad \blacksquare \qquad \frac{20}{t+14}-1$$

$$y_3 \quad | \quad \blacksquare \qquad \blacksquare \qquad \blacksquare \qquad \frac{-5 \cdot t}{t+14} \qquad y_3=0, \text{ if } t=0$$

$$x_3 \quad | \quad \blacksquare \qquad \blacksquare \qquad \blacksquare \qquad \frac{5}{t+14}$$

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matET is the solution, if the 3rd element y_3 equals 0,
i.e. t=0

matET|t=0

$$\begin{bmatrix} -\dfrac{2}{7} \\[6pt] \dfrac{3}{7} \\[6pt] 0 \\[6pt] \dfrac{5}{14} \end{bmatrix}$$

Thus x=−2/7, y=3/7, z=5/14

stop

Remark: rank of matST is 3 for t=0 and 4 otherwise

AVRank(matST,1,1)

done

syntax:

AVRank(matrix, row index i of a_{ik}, column index k of a_{ik}), if a_{ik} pivot

The rank of a matrix is the number of possible exchange steps

matnew➜matT1

$$\begin{bmatrix} 1 & -4 & 1 \\ \dfrac{7}{3} & \dfrac{-t}{3} & -1 \\ \dfrac{2}{3} & \dfrac{t}{3}+2 & -1 \end{bmatrix}$$

T1 y_1 x_2 x_3 1

————————————————————

x_1 | ■   ■   ■   ■   delete the old pivot

                                      row and pivot column

y_2 | ■   □   □   □   pivot $a_{11} \neq 0$ to

                                      exchange $x_2 \leftrightarrow y_2$

y_3 | ■   □   □   □

y_4 | ■   □   □   □

————————————————————

AVRank(matT1,1,1)

                                                done

matnew➜matT2

$$\begin{bmatrix} \dfrac{-(t-28)}{3} & -\dfrac{10}{3} \\ \dfrac{t+14}{3} & -\dfrac{5}{3} \end{bmatrix}$$

T2 y_1 y_2 x_3 1

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x_1 | ■   ■   ■   ■

x_2 | ■   ■   ■   ■   delete the old pivot

y₃ | ■ ■ □ □   pivot **a₁₁≠0** to

exchange x₂ ↔ y₂

y₄ | ■ ■ □ □

————————————————————

If t≠28 we compute:

AVRank(matT2, 1, 1)

done

matnew➔matT3

$$\left[\frac{-5 \cdot t}{t-28}\right]$$

T3	y₁	y₂	x₃	1
x₁	■	■	■	■
x₂	■	■	■	■
y₃	■	■	■	■
y₄	■	■	■	$\frac{-5 \cdot t}{t-28}$

delete the old
pivot row and pivot column again

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If t≠−14 we compute:

AVRank(matT2, 2, 1)

done

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matnew➔matT3

$$\left[\frac{-5 \cdot t}{t+14}\right]$$

T3	y_1	y_2	x_3	1
x_1 |	■	■	■	■
x_2 |	■	■	■	■
y_3 |	■	■	■	$\dfrac{-5 \cdot t}{t+14}$
y_4 |	■	■	■	■

delete the old pivot row and pivot column again

Thus we have 3 steps, i.e. rank equals 3, and for $t \neq 0$ we get rank equals 4

$$\mathrm{rank}\left(\begin{bmatrix} 3 & 2 & t & 0 \\ 0 & 1 & -4 & 1 \\ 1 & 3 & 0 & -1 \\ -1 & 0 & 2 & -1 \end{bmatrix}\right)$$

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The rank(matrix) function can not differ between 3 or 4 in dependence of parameter t.