On Uniquely 4-Colorable Planar Graphs

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April 15, 1998

Abstract

A k-chromatic graph G is called *uniquely k-colorable* if every k-coloring of the vertex set V of G induces the same partition of V into k color classes. There is an infinite class $\mathcal{C}$ of uniquely 4-colorable planar graphs obtained from the K_4 by repeatedly inserting new vertices of degree 3 in triangular faces.

In this paper we are concerned with the well-known conjecture (see [6]) that every uniquely 4-colorable planar graph belongs to $\mathcal{C}$. We shall show that a minimal counterexample to this conjecture is 5-connected.

1 Introduction

Let $G = (V, E)$ be a graph without loops and multiple edges where V is the vertex set and E is the edge set of G. A (proper) *k–coloring* of G is a mapping c of V into the (color-) set $\{1, 2, \ldots, k\}$ such that $c(x) \neq c(y)$ whenever $xy \in E$. If G admits a k–coloring, then G is said to be *k–colorable*. The graph G is called *k–chromatic*, if G is k–colorable but not $(k-1)$–colorable. Every k–coloring c of G induces a partition of its vertex set V into k color classes $V_1, \ldots, V_k$ where a vertex x belongs to V_i if $c(x) = i$. In this case we briefly write $c = (V_1, \ldots, V_k)$. Note that a color class V_i of c may be empty.

A k–chromatic graph G is called *uniquely k–colorable* if every k–coloring of G induces the same partition of its vertex set.

Uniquely colorable graphs were introduced and studied at the end of the sixties by Cartwright and Harary [1] and Harary, Hedetniemi and Robinson [5]. Chartrand and Geller [2] started to investigate this question for planar graphs. Jensen and Toft [6] raised the problem to give a structural characterization of uniquely 3- and 4-colorable planar graphs. It is easy to construct an infinite family of uniquely 4-colorable planar graphs. Denote by $\mathcal{C}$ the class of all planar graphs which can be obtained from the K_4 by repeatedly inserting new vertices of degree 3 in triangular faces. Obviously, the graphs belonging to $\mathcal{C}$ are planar and uniquely 4-colorable. There is no planar uniquely 4-colorable graph known that is not in $\mathcal{C}$. This leads to the following conjecture.

Conjecture 1 ([6]) *Every uniquely* 4*-colorable planar graph belongs to* $\mathcal{C}$*.*

Further equivalent or stronger conjectures in terms of edge-colorings and hamiltonian cycles can be found in several papers, such as Fiorini and Wilson [3], Greenwell and Kronk [4] and Tutte [7].

The main result of this paper given in Theorem 2 deals with the connectivity of uniquely 4-colorable planar graphs. A graph G is *m–connected* for some integer $m \geq 1$ if the removal of any ℓ vertices with $0 \leq \ell \leq m-1$ neither disconnects G nor reduces it to the trivial graph consisting of a single vertex.

In the sequel, we denote the class of all planar uniquely 4-colorable graphs by $\mathcal{U}$. The complete graph K_4 is obviously the graph with the smallest number of vertices in $\mathcal{U}$. On the one hand, by a result of Chartrand and Geller (Lemma 4), every graph belonging to $\mathcal{U}$ is 3-connected. On the other hand, it is known that planar graphs are at most 5-connected.

Our main result given in the next theorem is proved in section 3.

Theorem 2 *A minimal counterexample to Conjecture* 1 *is* 5*-connected.* □

2 Structure of uniquely colorable graphs

In this section we present some interesting and mostly known results needed for later investigations.

Chartrand and Geller investigated already the connectivity of uniquely colorable graphs.

Lemma 3 ([2]) *For any n-coloring of a uniquely n-colorable graph G, the subgraph induced by the union of any two color classes is connected.* □

Lemma 4 ([2]) *Every uniquely n-colorable graph G is $(n\text{-}1)$-connected.* □

Now, consider a uniquely 4-colorable graph $G = (V, E)$ and let $c = (V_1, V_2, V_3, V_4)$ be a 4-coloring of G. For $1 \leq i < j \leq 4$, let G_{ij} be the subgraph of G induced by the union of the color classes V_i and V_j. Lemma 3 implies $|E_{ij}| \geq |V_{ij}| - 1$ where E_{ij} is the edge set and V_{ij} is the vertex set of G_{ij}. Therefore, we obtain

$$\begin{aligned} |E| &= |E_{12}| + |E_{13}| + |E_{14}| + |E_{23}| + |E_{24}| + |E_{34}| \\ &\geq |V_{12}| + |V_{13}| + |V_{14}| + |V_{23}| + |V_{24}| + |V_{34}| - 6 \\ &= 3(|V_1| + |V_2| + |V_3| + |V_4|) - 6 = 3|V| - 6. \end{aligned}$$

By well known facts about planar graphs, this implies the following result.

Lemma 5 ([2]) *Every uniquely 4-colorable planar graph is a triangulation.* □

Furthermore, if the uniquely 4-colorable graph G is planar, then every subgraph G_{ij} has exactly $|V_{ij}| - 1$ edges. By Lemma 3, this implies the following result.

Lemma 6 *For any 4-coloring of a uniquely 4-colorable planar graph G, the subgraph induced by the union of any two color classes is a tree.* □

Clearly, if a graph G is not $(t+1)$-connected and G is not the complete graph K_{t+1}, then there exists a set S of at most t vertices in G such that G disintigrates into at least two parts after removing these vertices. The set S is usually called a *cut set* of G.

3 Proof of Theorem 3

Let $G = (V, E)$ be a minimal counterexample to Conjecture 1, that is

(a) $G \in \mathcal{U} - \mathcal{C}$, and

(b) $|V|$ is minimum subject to (a).

We consider an embedding of G into the plane (also denoted by G). Obviously, G is not a K_5. In order to prove that G is 5-connected, we investigate the set $\mathcal{K}$ of all cut sets of G. We want to show that $|S| \geq 5$ for all $S \in \mathcal{K}$. For $v \in V$, let $d(v)$ denote the degree of the vertex v in G. Since $G \in \mathcal{U}$, the following result is a consequence of Lemma 4 and Lemma 5.

(3.1) G is a 3-connected triangulation.

(3.2) $d(v) \geq 4$ for all $v \in V$.

Proof. Suppose that (3.2) is not true. From (3.1) it then follows that $d(v) = 3$ for some vertex $v \in V$. Then, clearly, the graph $G' = G - v$ obtained from G by deleting the vertex v is uniquely 4-colorable and planar, i.e. $G' \in \mathcal{U}$. Therefore, by (b), $G' \in \mathcal{C}$ implying $G \in \mathcal{C}$, a contradiction. □

(3.3) $|S| \geq 4$ for all $S \in \mathcal{K}$.

Proof. Suppose that (3.2) is not true. Then, because of (3.1), $|S| = 3$ for some $S \in \mathcal{K}$, say $S = \{x, y, z\}$. By (3.1), G is a triangulation and, therefore, S is a 3-cycle in G. Denote the set of vertices inside this 3-cycle by V_I and the set of vertices outside this 3-cycle by V_O. Let $G_I = G[V_I \cup \{x, y, z\}]$ ($G_O = G[V_O \cup \{x, y, z\}]$) be the subgraph of G induced by the union of V_I (respectively, V_O) and the cut set S. Both subgraphs G_I and G_O are planar and uniquely 4-colorable since otherwise G would not belong to $\mathcal{U}$. Since G_I and G_O have fewer vertices than G, both graphs are elements of $\mathcal{C}$. It is easy to prove that every graph in $\mathcal{C}$ is either K_4 or has at least two vertices of degree 3 which are non-adjacent. Since $G_I, G_O \in \mathcal{C}$, this implies that G contains a vertex of degree 3, a contradiction to (3.2) □

Before we are able to prove that $|S| \geq 5$ for all $S \in \mathcal{K}$ we need the following result.

(3.4) $d(x) \geq 5$ for all $x \in V$.

Proof. Suppose that (3.4) is not true. Then, by (3.2), there is a vertex $v \in V$ such that $d(v) = 4$. Denote its neighbors by w, x, y and z. Since G is a triangulation, these four vertices form a 4-cycle in G, say with edges wx, xy, yz, and zw.

Now, consider a 4-coloring $c = (V_1, V_2, V_3, V_4)$ of G. For $1 \leq i < j \leq 4$, let G_{ij} denote the subgraph of G induced by $V_i \cup V_j$. A path in G_{ij} is usually called an (i, j)-Kempe chain (with respect to c). By Lemma 6, G_{ij} is a tree for $1 \leq i < j \leq 4$.

Therefore, the vertices in $A = \{w, x, y, z\}$ cannot be colored with two colors. If the vertices in A are colored with four colors, then vertex v cannot be colored properly. Thus the vertices in A receive three colors, say $c(w) = c(y) = 1$, $c(x) = 2$ and $c(z) = 4$ implying that $c(v) = 3$.

Next, we construct a graph $\widetilde{G}$ by deleting the vertex v from G and adding the edge wy. Clearly, $\widetilde{G}$ is planar and we claim that $\widetilde{G}$ is uniquely 4-colorable.

First, we show that $\widetilde{G}$ is 4-colorable. If we remove the vertex v from G, then the vertices w and y belong to different components of $G_{13} - v$. Thus, we may change the colors 1 and 3 in the component of the forest $G_{13} - v$ containing w. This results in a 4-coloring $\widetilde{c}$ of $\widetilde{G}$.

In order to prove that $\widetilde{G}$ is uniquely 4-colorable, consider an arbitrary 4-coloring $c' = (V_1', V_2', V_3', V_4')$ of $\widetilde{G}$. Then the vertices in A are colored by four colors with respect to c', say $c'(w) = 1, c'(x) = 2, c'(y) = 3$ and $c'(z) = 4$. Otherwise, c' uses three colors for the vertices in A and, therefore, c' can be extended to a 4-coloring c^* of G where $c^*(w) \neq c^*(y)$. But this implies that G is not uniquely 4-colorable, a contradiction.

Furthermore, x and z are joined by a (2,4)-Kempe chain with respect to c', since otherwise we would obtain a 4-coloring of $\widetilde{G}$ that uses only three colours for the vertices in A, a contradiction. Consequently, in the planar graph G' obtained from $\widetilde{G}$ by deleting the edge wy, the vertices w and y are not joined by a (1,3)-Kempe chain with respect to c'.

Now, change the colors 1 and 3 in the component of the subgraph of G' induced by $V_1' \cup V_3'$ that contains the vertex y and, moreover, color v with 3. This results in a 4-colouring c_1 of G. Since G is uniquely 4-colorable, we infer that $c_1 = c$ and, therefore, $c' = \widetilde{c}$. This proves that $\widetilde{G}$ is uniquely 4-colorable.

Therefore, $\widetilde{G}$ belongs to $\mathcal{C}$ and, therefore, $\widetilde{G}$ has a vertex of degree three.

First, assume that x is a vertex of degree three in $\widetilde{G}$ and denote the neighbor of x different from w and y by x'. Then $x' \neq z$, since otherwise $\{x, y, z\}$ or $\{x, w, z\}$ would be a cut set in G, a contradiction to (3.3). Hence x' is adjacent to w and y in both graphs $\widetilde{G}$ and G because these graphs are triangulations. But this implies that x' cannot be colored properly with respect to c. Clearly, $c(x')$ is neither 1 nor 2 because x' has neighbors of this colors and moreover, $c(x')$ cannot be 3 or 4 since otherwise there would be a cycle either in G_{13} or in G_{14}, a contradiction. By analogous arguments we obtain that z cannot be a vertex of degree 3 in $\widetilde{G}$. For all other vertices of $\widetilde{G}$ the degree in $\widetilde{G}$ equals the degree in G.

Consequently, G has a vertex of degree 3, a contradiction to (3.2). This proves (3.4) □

(3.5) $|S| \geq 5$ for all $S \in \mathcal{K}$.

Proof. Suppose that (3.5) is not true. Then, by (3.3), there is a cut set S of G with $|S| = 4$, say $S = \{w, x, y, z\}$. Since G is a triangulation, these four vertices form a 4-cycle C in G, say with edges wx, xy, yz, and zw.

Let $c = (V_1, V_2, V_3, V_4)$ be a 4-coloring of G. For $1 \leq i < j \leq 4$, let G_{ij} denote the subgraph of G induced by $V_i \cup V_j$. By Lemma 6, G_{ij} is a tree for $1 \leq i < j \leq 4$. Therefore, the vertices in S receive at least three colors with respect to c implying that $c(w) \neq c(y)$ or $c(x) \neq c(z)$, say $c(x) = 2$ and $c(z) = 3$. Then w and y are colored with 1 or 4. Furthermore, x and z are joined by a (2,3)-Kempe chain P with respect to c. We may assume that P lies inside the 4-cycle C. Denote the set of vertices inside this 4-cycle by V_I and the set of vertices outside this 4-cycle by V_O. Furthermore, denote the subgraph of G induced by the union of V_I (respectively, V_O) and the cut set S by G_I (respectively, G_O). We assume that S is chosen among all cut sets $S' \in \mathcal{K}$ with $|S'| = 4$ such that $|V_I|$ (the part containing the Kempe chain P) is maximal.

Next, we construct a graph $\widetilde{G}$ from $G_O = G - V_I$ by adding the edge xz. Obviously $\widetilde{G}$ is planar and we claim that this graph is uniquely 4-colourable.

Clearly, the restriction $\tilde{c}$ of c onto $\widetilde{G}$ is a 4-coloring of $\widetilde{G}$. Now, consider an arbitrary 4-colouring c' of $\widetilde{G}$. Then $c'(x) \neq c'(z)$, say $c'(x) = c(x) = 2$

and $c'(z) = c(z) = 3$. We have to show that both c' and $\tilde{c}$ induce the same paprtition of the vertex set of $\widetilde{G}$.

First, consider the case that $c(w) \neq c(y)$. Then, w.l.o.g. $c(w) = 1$ and $c(y) = 4$. If $c'(w) \neq c'(y)$, say $c'(w) = 1$ and $c'(y) = 4$, then $c' = \tilde{c}$, since otherwise we would obtain a 4-coloring of G that induces a different partition than c, a contradiction. If $c'(w) = c'(y) = c$, say with $c = 1$, then we argue as follows. Since the (2,3)-Kempe chain P belongs to G_I, there is no (1,4)-Kempe chain with respect to c in G_I. Let H be the subgraph of $G_{1,4}$ induced by the vertices of G_I. Then y and w belong to different components of H. If we change the colours 1 and 4 in the componet of H containing y, we obtain from c a coloring c^* of G_I such that $c'(v) = c^*(v)$ for $v \in S$. Then $c' \cup c^*$ is a 4-coloring of G that induces a different partition than c, a contradiction.

Now, consider the case that $c(w) = c(y) = c$. Then, w.l.o.g. $c = 1$. If $c'(w) = c'(y)$, say $c'(w) = 1$ and $c'(y) = 1$, then $c' = \tilde{c}$, since otherwise we would obtain a 4-coloring of G that induces a different partition than c, a contradiction. If $c'(w) \neq c'(y)$, say $c'(w) = 1$ and $c'(y) = 4$, then based on a similar argument as in the fomer case, we obtain a 4-coloring of G that induces a different partition than c, a contradiction.

Thus $\widetilde{G}$ is uniquely 4-colorable planar graph with fewer vertices than G. Consequently, $\widetilde{G} \in \mathcal{C}$ and, therefore, $\widetilde{G}$ has a vertex v of degree three. Obviously, $v \in S$, since otherwise G has a vertex of degree 3, a contradiction to (3.2).

If $v = x$ or $v = z$ there is an edge wy in $\widetilde{G}$ as well as in G because both graphs are triangulations. Thus there is a cut set of three elements in G, a contradiction to (3.3).

Therefore, $v \in S - \{x, z\}$, say $v = w$. Denote the neighbor of w in $\widetilde{G}$ different from x and z by w'. Then, w' is adjacent to x and z because $\widetilde{G}$ is a triangulation. Hence the vertices of $S' = \{w', x, y, z\}$ form a 4-cycle C' in G. Because of (3.4). the degree of w' in G and, therefore, also in $\widetilde{G}$ is at least five. This implies that w' has at least one neighbor in G outside the rectangle C'. But then S' is a cut set of four vertices in G, contradicting the choice of the cut set S.

Thus (3.5) and, therefore, Theorem 2 is proved. $\square$

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