b-continuity

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Abstract

A *b-coloring* of a graph G by k colors is a proper coloring of the vertices of G such that in each color class exists a *color dominating vertex* that is a vertex having neighbors in all other $k-1$ color classes. The *b-chromatic number* $b(G)$ of a graph G is the maximum k for which G has a b-coloring by k colors. It is easy to see that every G has a b-coloring by $\chi(G)$ colors.

A graph G is called b-continuous if there are b-colorings of G by k colors for all $\chi(G) \leq k \leq b(G)$. It is known that there are graphs which are not b-continuous. In this short note we show that chordal graphs and some planar graphs are b-continuous.

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1 Introduction

A *b-coloring* of a graph G by k colors is a proper coloring of the vertices of G such that in each color class exists a vertex having neighbors in all other $k-1$ color classes. Such a vertex is called a *color dominating vertex*. The *b-chromatic number* $b(G)$ of a graph G is the maximum k for which G has a b-coloring by k colors.

Obviously, a coloring of a graph G with $\chi(G)$ colors where $\chi(G)$ is the chromatic number of G is a b-coloring. Thus $b(G) \geq \chi(G)$ for all G. To give an upper bound on $b(G)$ we assume that the vertices $v_1, \ldots, v_n$ of G are ordered such that $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n)$. Let $t(G) := \max\{i \ ; \ d(v_i) \geq i-1\}$

be the maximum number i such that G contains at least i vertices of degree $\geq i-1$. Obviously, $t(G) \leq \Delta(G)+1$ where $\Delta(G)$ denotes the maximum degree of G. It is observed in [3] and [4] that $t(G)$ is an upper bound for $b(G)$. Furthermore this upper bound is tight. On the other hand $b(G)$ may be arbitrarily far from $t(G)$ shown by the complete bipartite graph $K_{n,n}$ [3]. Moreover also the differences $t(G)-\chi(G)$ and $\Delta(G)-t(G)$ can be arbitrarily large as seen by $K_{n,n}$ and the star $K_{1,n}$ respectively.

Considering b-colorings we observe further interesting properties. There are graphs for which exist b-colorings by k_1 colors and k_2 colors but no b-colorings by k colors where $k_1 < k < k_2$. Thus it is reasonable to define the *b-spectrum* $S_b(G)$ of a graph G as the set of integers k for which there is a b-coloring of G by k colors. We call a graph G *b-continuous* if $S_b(G)$ is an interval that is there are b-colorings of G by k colors for all $\chi(G) \leq k \leq b(G)$. Barth, Cohen and Faik [1] proved that for every set of integers I there exist a graph G with $S_b(G) = I$. Moreover they have shown that, even if b-colorings of cardinality $\chi(G)$ and $b(G)$ are given, the problem of knowing if G is b-continuous is NP-complete.

It is an interesting open problem to find classes of graphs which are b-continuous. In [1] it is proved that interval graphs are b-continuous. In this note we will give a short proof that even all chordal graphs are b-continuous. Moreover we prove that also some planar graphs have this property.

2 Chordal graphs

Theorem 1 *Chordal graphs are b-continuous.*

Proof: We would like to prove: if a chordal graph G has a b-coloring with $k > \chi(G)$ colors then it has also a b-coloring with $k-1$ colors.

Assume G is a counterexample. Consider a given b-coloring by k colors. Let v be a simplicial vertex. Note that v cannot be a color dominating vertex since the neighborhood builds a clique of at most $\chi(G)-1 < k-1$ vertices. Remove v from the graph. If there is still a color dominating vertex for every of the k colors continue to remove simplicial vertices until not all colors have a color dominating vertex. Assume the last removed vertex v has color 1 in the b-coloring and denote the new graph by G'.

Then there are some vertices $w_1, \ldots, w_s$ $(s \geq 1)$ in the neighborhood of v such that each of these vertices is the only color dominating vertex for its color and

v is the only vertex in its neighborhood having color 1. Assume w_1 has color k. Color w_1 by 1 and all other vertices of color k by a suitable color (they are not dominating for color k therefore it is possible). Since $w_1, \ldots, w_s$ build a clique $w_2, \ldots, w_s$ are now dominating vertices for the $k-1$-coloring of G' which is in fact a b-coloring. Coloring the removed vertices by suitable colors we obtain a b-coloring of G with $k-1$ colors contradicting the assumption. $\square$

3 Planar graphs

For a given connected graph G denote by $dist(v, w)$ the number of edges of a shortest path connecting v and w.

Theorem 2 *Let G be a connected planar graph of girth at least 5. If G contains t vertices $v_1, \ldots, v_t$ such that $d(v_i) \geq t-1$ for all $i = 1, \ldots, t$ and $dist(v_i, v_j) \geq 5$ for all $i \neq j$ then $b(G) \geq t$ and G has b colorings by s colors for all $\chi(G) \leq s \leq t$.*

Proof. We try to find a b coloring by $s \leq t$ colors precoloring vertices $v_1, \ldots, v_s$ and the neighborhoods such that v_i is colored by i and all other colors occur in its neighborhood.
Assume first $s \geq 4$. The precoloring is possible in a proper way because we have $s \geq 4$ available colors, G is planar and $dist(v_i, v_j) \geq 5$.
Now we claim that every non-colored vertex has at most one colored neighbor. This is true because of the girth- and the distance constraint. Thus for every non-colored vertex there are at least $s-1 \geq 3$ available colors. Thomassen [6] proved that every planar graph of girth at least 5 is 3-choosable. Thus it is possible to extend the precoloring to the whole graph.
Note that $\chi(G) \leq 3$ by the girth condition. If $\chi(G) = 3$ we are done since there is a proper coloring of G by 3 colors which is in fact a b-coloring. If $\chi(G) = 2$ it remains to show that G has a b-coloring by 3-colors. We claim that G has such a coloring if it is not a star or a path of at most 4 vertices. Note that for the star we have $t = 2$ and are done and a short path does not fulfill the assumption of the Theorem for $t = 3$. If G is a tree beside these exceptions we can find the required b-coloring using for example Lemma 4.4 of [5]. Otherwise consider an order $v_i, \ldots, v_n$ of the vertices of G such that $d_{G_i}(v_i) \leq 2$ where $G_i = G[v_1, \ldots, v_i]$. Such an order exists since

$\sum_{v \in V(G)} d(v) \leq 3|V(G)| - 12$ because of Eulers formula, the girth condition and $\chi(G) = 2$. Let j be the smallest integer such that G_j contains a cycle. Note that G_j has exactly one cycle since $d_{G_j}(v_j) \leq 2$ and no G_s with $s < j$ has a cycle. It is easy to find a b-coloring of G_j by 3-colors coloring 5 vertices of the cycle by $1, 2, 3, 1, 2$ and extending it to a 3-coloring of G_j. Finally extend the 3-coloring to G coloring the vertices in the order $v_{j+1}, \ldots, v_n$. □

If in the assumption of the above Theorem $t = t(G)$ then we can conclude that these graphs are b-continuous.

Corollary 3 *Let G be a connected planar graph of girth at least 5 and $t = t(G)$. If G contains t vertices $v_1, \ldots, v_t$ such that $d(v_i) \geq t - 1$ for all $i = 1, \ldots, t$ and $dist(v_i, v_j) \geq 5$ for all $i \neq j$ then $b(G) = t(G)$ and G is b-continuous.*

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