Coupling of finite and boundary elements via fictitious domains

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Abstract

For the solution of inhomogeneous boundary value problems in complex three–dimensional domains we propose a coupled finite–boundary element method. The problem is reformulated as boundary integral equation to avoid a volume triangulation of the complex structure. To compute the Newton potential a finite element method in a simpler fictitious domain is applied. Corresponding error estimates and a numerical example are given.

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1 Introduction

Boundary element methods are well established when solving homogeneous partial differential equations. This is mainly due to the fact, that only a boundary triangulation is needed. When dealing with inhomogeneous partial differential equations, a volume potential requires an integration over the whole domain and the advantage of the boundary element method is lost. On the other hand, finite element methods include volume forces in a natural way, but a triangulation of complex structures and efficient solution methods (when no mesh hierarchy is given) may still be a difficult task.

In this paper we propose a new approach to couple both methods to keep the advantages of each. If a particular solution of the inhomogeneous partial differential equation is known, the volume potential (Newton potential) is given implicitly by boundary potentials only. In the engineering literature, this principle is known as Dual Reciprocity Method, c.f. [1]. While numerical results are convincing, there is no numerical analysis available yet.

To compute a particular solution, we will use a finite element method, which is applied to an appropriately defined boundary value problem in some fictitious domain. This can be done in such a way that both a hierarchy of triangulations and an efficient iterative solution method by means of multilevel techniques are easy to implement. While the indirect computation of Newton potentials by finite elements was already considered for the 2D case in [8], we use this

approach for the complete numerical solution of boundary value problems in 3D.
The paper is organized as follows: In Section 2 we present the boundary value problems considered and their reformulation as boundary integral equations. We describe a fictitious domain approach for computing a particular solution of the inhomogeneous boundary value problem. Section 3 is devoted to the discussion of the finite element discretization of the corresponding boundary value problem in the fictitious domain and to the boundary element discretization of the integral equations. Error estimates for the approximate Newton potential and the solutions obtained by using the described boundary element discretizations are given. Furthermore, the whole coupled finite element/boundary element algorithm is explained. The numerical example presented in Section 4 confirms the theoretical results.

2 A coupled finite–boundary element formulation

Let $\Omega \subset I\!R^3$ be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. For $x \in \Omega$ we consider the second order partial differential equation

$$Lu(x) = -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}\left(a_{ji}\frac{\partial}{\partial x_i}u(x)\right) + c\,u(x) \;=\; f(x) \tag{2.1}$$

where the associated bilinear form is given by

$$a(u,v) = \int\limits_{\Omega} \left[\sum_{i,j=1}^{3} a_{ji}\frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}v(x) + c\,u(x)v(x)\right] dx.$$

We assume that L is an elliptic partial differential operator with constant coefficients. In case of piecewise constant coefficients we may consider a corresponding domain decomposition and use the proposed approach within the subdomains. For simplicity in the presentation we consider the case of scalar equations only, a generalization to formally positive elliptic systems is straight forward. According to (2.1) the conormal derivative operator is given by

$$T_x u(x) \;=\; \sum_{i,j=1}^{3} n_j(x) a_{ji}\frac{\partial}{\partial x_i}u(x) \quad \text{for } x \in \Gamma \tag{2.2}$$

almost everywhere using the outer normal vector $n(x)$. Let $t(x) := T_x u(x)$ for $x \in \Gamma$. In addition to the partial differential equation (2.1) we consider either Dirichlet boundary conditions

$$u(x) \;=\; g(x) \quad \text{for } x \in \Gamma \tag{2.3}$$

or Robin type boundary conditions described by

$$t(x) + \kappa(x)u(x) \;=\; g(x) \qquad \text{for } x \in \Gamma \tag{2.4}$$

where $\kappa(\cdot) \in W^1_\infty(\Gamma)$ and $\kappa(x) \geq \kappa_0 > 0$ for $x \in \Gamma$. Instead of (2.3) or (2.4) we may also consider boundary conditions of mixed type including Neumann boundary conditions, as well as nonlinear boundary conditions as it was done in [4] for homogeneous partial differential equations. However, all of these more general situations can be handled as in the case of Robin boundary conditions (2.4) and will not be considered here.

2.1 Boundary integral equations

We assume that there is given a fundamental solution $U^*(x,y)$ of the partial differential operator L in (2.1). Define $T^*(x,y) := T_y U^*(x,y)$. The solution of the boundary value problem (2.1) and (2.3) or (2.4) can be described by the representation formula for $\tilde{x} \in \Omega$,

$$u(\tilde{x}) \;=\; \int\limits_\Gamma U^*(\tilde{x},y)t(y)ds_y - \int\limits_\Gamma T^*(\tilde{x},y)u(y)ds_y + \int\limits_\Omega U^*(\tilde{x},y)f(y)dy. \qquad (2.5)$$

To use (2.5) we need to know all the Cauchy data $[u(x), t(x)]$ for $x \in \Gamma$. Taking in (2.5) the limes $\Omega \ni \tilde{x} \to x \in \Gamma$ this results in the well known boundary integral equation

$$(Vt)(x) \;=\; (\sigma I + K)u(x) - (Nf)(x) \qquad \text{for } x \in \Gamma. \qquad (2.6)$$

Here we used the standard boundary integral operators for $x \in \Gamma$, in particular, the single and double layer potentials

$$(Vt)(x) \;=\; \int\limits_\Gamma U^*(x,y)t(y)ds_y, \qquad (Ku)(x) \;=\; \int\limits_\Gamma T^*(x,y)u(y)ds_y\,,$$

and the Newton potential

$$(Nf)(x) \;=\; \int\limits_\Omega U^*(x,y)f(y)dy\,. \qquad (2.7)$$

In (2.6),

$$\sigma(x) \;=\; \lim_{\varepsilon\to 0}\frac{\text{meas}B_\varepsilon(x)\cap\Omega}{\text{meas}B_\varepsilon(x)} \qquad \text{for } x \in \overline{\Omega}. \qquad (2.8)$$

For the Dirichlet boundary value problem (2.1) and (2.3) we have to find the yet unknown Neumann datum $t(x)$. For this we use the boundary integral equation to find $t \in H^{-1/2}(\Gamma)$ such that

$$(Vt)(x) \;=\; (\sigma I + K)g(x) - (Nf)(x) \qquad \text{for } x \in \Gamma. \qquad (2.9)$$

It is well known that the single layer potential $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is bounded and $H^{-1/2}(\Gamma)$–elliptic, see for example [6], i.e.,

$$\langle Vw, \tau\rangle_{L_2(\Gamma)} \;\leq\; c_2^V \cdot ||w||_{H^{-1/2}(\Gamma)}||\tau||_{H^{-1/2}(\Gamma)} \qquad (2.10)$$

$$\langle Vw, w\rangle_{L_2(\Gamma)} \;\geq\; c_1^V \cdot ||w||^2_{H^{-1/2}(\Gamma)} \qquad (2.11)$$

hold for all $w, \tau \in H^{-1/2}(\Gamma)$. Therefore, we conclude that the boundary integral equation (2.9) admits a unique solution $t \in H^{-1/2}(\Gamma)$.
For the Robin boundary value problem (2.1) and (2.4) both Cauchy data $[u(x), t(x)]$ for $x \in \Gamma$ are unknown. To formulate an appropriate boundary integral equation to find $u \in H^{1/2}(\Gamma)$ we may proceed as follows: Since the single layer potential V is invertible, from (2.6) we get

$$t(x) \;=\; V^{-1}(\sigma I + K)u(x) - V^{-1}(Nf)(x) \quad \text{for } x \in \Gamma.$$

Defining the Steklov–Poincaré operator

$$(Su)(x) \;=\; V^{-1}(\sigma I + K)u(x) \quad \text{for } x \in \Gamma, \tag{2.12}$$

we get a Dirichlet–Neumann map as

$$t(x) \;=\; (Su)(x) - V^{-1}(Nf)(x) \quad \text{for } x \in \Gamma. \tag{2.13}$$

Inserting (2.13) into the Robin type boundary condition (2.4) this gives a boundary integral equation to find $u \in H^{1/2}(\Gamma)$ such that

$$(Su)(x) + \kappa(x)u(x) \;=\; g(x) + V^{-1}(Nf)(x) \quad \text{for } x \in \Gamma. \tag{2.14}$$

It follows from the properties of all boundary integral operators involved (see for example [3]) that the boundary integral equation (2.14) has a unique solution $u \in H^{1/2}(\Gamma)$.
Note that in both the representation formula (2.5) (for $x = \widetilde{x} \in \Omega$) and the boundary integral equation (2.9) (for $x \in \Gamma$) we need to compute the Newton potential

$$\eta(x) \;=\; \int\limits_{\Omega} U^*(x,y) f(y)dy\,. \tag{2.15}$$

Moreover, in the boundary integral equation (2.14) related to the Robin boundary value problem we also need to compute

$$\omega \;=\; V^{-1}\eta(x) \quad \text{for } x \in \Gamma$$

which is equivalent to find the solution of the boundary integral equation

$$(V\omega)(x) \;=\; \eta(x) \quad \text{for } x \in \Gamma.$$

A direct evaluation of (2.15) for arbitrary $x \in \overline{\Omega}$ seems to be inefficient. Moreover, such an approach requires some volume triangulation of Ω which contradicts the advantage of using boundary elements. Hence we will compute (2.15) by using an indirect approach: Since the representation formula (2.5) and therefore the boundary integral equation (2.6) hold for all pairs of Cauchy data associated to a solution of the partial differential equation (2.1), the Newton potential can be computed by

$$\eta(x) \;=\; \sigma(x)u_p(x) + (Ku_{p|\Gamma})(x) - (Vt_{p|\Gamma})(x) \quad \text{for } x \in \overline{\Omega} \tag{2.16}$$

where u_p is an arbitrary particular solution of (2.1) and $t_p(x) := T_x u_p(x)$ for $x \in \Gamma$. In engineering applications this principle is known as *Dual Reciprocity Method* [1]. In the DRM the given function f is approximated by using radial basis functions $\psi_k = \psi(|x - x_k|)$ where an explicit solution of $L\vartheta_k = \psi_k$ can be computed. The resulting particular solution is then used in (2.16). Here we will follow the concept outlined in [8] to compute a particular solution by means of an efficient finite element approach for an arbitrary boundary value problem in a fictitious domain allowing a simple hierarchical triangulation.

2.2 A fictitious domain approach

To compute a particular solution of the inhomogeneous partial differential equation (2.1) we consider an artificial Dirichlet or Robin boundary value problem. Let $\widetilde{\Omega} \subset I\!R^3$ be a convex domain containing Ω. Let $\widetilde{f} \in L_2(\widetilde{\Omega})$ be an arbitrary but fixed extension of the given data $f \in L_2(\Omega)$. Note that the partial differential operator L is defined within $\widetilde{\Omega}$ due to the assumption of constant coefficients. Then, we may consider the Dirichlet boundary value problem

$$L\widetilde{u}(x) = \widetilde{f}(x) \quad \text{for } x \in \widetilde{\Omega}, \qquad \widetilde{u}(x) = 0 \quad \text{for } x \in \partial\widetilde{\Omega}$$

or in variational form to find $\widetilde{u} \in H_0^1(\widetilde{\Omega})$ satisfying

$$\widetilde{a}(\widetilde{u}, \widetilde{v}) \;=\; \langle F, \widetilde{v}\rangle \quad \text{for all } \widetilde{v} \in H_0^1(\widetilde{\Omega}) \tag{2.17}$$

where

$$\widetilde{a}(\widetilde{u}, \widetilde{v}) \;=\; \int\limits_{\widetilde{\Omega}} \left[\sum_{i,j=1}^{3} a_{ji} \frac{\partial}{\partial x_i} \widetilde{u}(x) \frac{\partial}{\partial x_j} \widetilde{v}(x) + c\widetilde{u}(x)\widetilde{v}(x) \right] dx \tag{2.18}$$

and

$$\langle F, \widetilde{v}\rangle \;=\; \int\limits_{\widetilde{\Omega}} \widetilde{f}(x)\widetilde{v}(x)dx\,.$$

Alternatively, we can also consider an artificial Robin boundary value problem,

$$L\widetilde{u}(x) = \widetilde{f}(x) \quad \text{for } x \in \widetilde{\Omega}, \quad (T_x\widetilde{u})(x) + \alpha(x)\widetilde{u}(x) = 0 \quad \text{for } x \in \partial\widetilde{\Omega}$$

with $\alpha(x) \in W^1_\infty(\partial\widetilde{\Omega})$ and $\alpha(x) > 0$ for all $x \in \partial\widetilde{\Omega}$. The variational formulation of them is given by: Find $\widetilde{u} \in H^1(\widetilde{\Omega})$ such that

$$\widetilde{a}(\widetilde{u}, \widetilde{v}) \;=\; \langle F, \widetilde{v}\rangle \quad \text{for all } \widetilde{v} \in H^1(\widetilde{\Omega}) \tag{2.19}$$

holds, where

$$\widetilde{a}(\widetilde{u}, \widetilde{v}) \;=\; \int\limits_{\widetilde{\Omega}} \left[\sum_{i,j=1}^{3} a_{ji} \frac{\partial}{\partial x_i} \widetilde{u}(x) \frac{\partial}{\partial x_j} \widetilde{v}(x) + c\widetilde{u}(x)\widetilde{v}(x) \right] dx + \int\limits_{\partial\widetilde{\Omega}} \alpha(x)\widetilde{u}(x)\widetilde{v}(x)ds \tag{2.20}$$

and

$$\langle F, \widetilde{v}\rangle \;=\; \int\limits_{\widetilde{\Omega}} \widetilde{f}(x)\widetilde{v}(x)dx.$$

Due to the assumptions made above the bilinear forms (2.18) and (2.20) are W–bounded and W–elliptic, i.e.,

$$
\begin{array}{rcll}
|\tilde{a}(\tilde{u},\tilde{v})| & \leq & c_2^W \cdot ||\tilde{u}||_{H^1(\widetilde{\Omega})} ||\tilde{v}||_{H^1(\widetilde{\Omega})} & \text{for all } \tilde{u},\tilde{v} \in W \\
\tilde{a}(\tilde{v},\tilde{v}) & \geq & c_1^W \cdot ||\tilde{v}||^2_{H^1(\widetilde{\Omega})} & \text{for all } \tilde{v} \in W
\end{array}
$$

with $W = H_0^1(\widetilde{\Omega})$ and $W = H^1(\widetilde{\Omega})$, respectively. Therefore, the variational problems (2.17) and (2.19) admit a unique solution $\tilde{u} \in W$.

Lemma 2.1 *Let $\tilde{f} \in L_2(\widetilde{\Omega})$ be a fixed extension of $f \in L_2(\Omega)$. Let $\tilde{u}$ be the unique solution of the variational problem (2.17) or (2.19). Then, $\tilde{u}_{|\Omega} \in H^1(\Omega)$ is a weak solution of the partial differential equation (2.1).*

Proof. Let $\tilde{u}$ be the unique solution of the variational problem (2.17) or (2.19). Then, we consider the Dirichlet boundary value problem

$$Lu(x) = f(x) \quad \text{for } x \in \Omega, \qquad u(x) = \tilde{u}(x) \quad \text{for } x \in \Gamma$$

which is equivalent to find $u \in H^1(\Omega)$, $u(x) = \tilde{u}(x)$ for $x \in \Gamma$ such that

$$a(u,v) = \int_\Omega f(x)v(x)dx \quad \text{for all } v \in H_0^1(\Omega).$$

Since $H_0^1(\Omega) = \{v \in H^1(\widetilde{\Omega}) \, : \, v = 0 \text{ on } \overline{\widetilde{\Omega}} \setminus \Omega\} \subset H_0^1(\widetilde{\Omega}) \subset H^1(\widetilde{\Omega})$ we get

$$a(u,v) = \int_\Omega f(x)v(x)dx = \int_{\widetilde{\Omega}} \tilde{f}(x)v(x)dx = \tilde{a}(\tilde{u},v) = a(\tilde{u},v)$$

for all $v \in H_0^1(\Omega)$. Hence, $w = u - \tilde{u}$ is a solution of the variational problem

$$a(w,v) = 0 \quad \text{for all } v \in H_0^1(\Omega), \qquad w(x) = 0 \quad \text{for } x \in \Gamma,$$

from which $w = 0$ follows. Hence, $u(x) = \tilde{u}_{|\Omega}$ is a weak solution of the inhomogeneous partial differential equation (2.1). ■

To apply (2.16) to compute the Newton potential we need to have also the conormal derivative on Γ of the particular solution $\tilde{u}$. Using (2.2) we have to find $\underline{p} = \nabla\tilde{u} \in [L_2(\Gamma)]^3$ satisfying

$$\int_{\widetilde{\Omega}} \underline{p}(x) \cdot \underline{q}(x)dx = \int_{\widetilde{\Omega}} \nabla\tilde{u}(x) \cdot \underline{q}(x)dx \qquad \text{for all } \underline{q} \in [L_2(\widetilde{\Omega})]^3. \tag{2.21}$$

Then we define

$$\tilde{t}(x) = \sum_{i,j=1}^3 n_j(x)a_{ji}p_i(x) \qquad \text{for } x \in \Gamma \text{ almost everywhere,}$$

and the Newton potential (2.15) can be computed by

$$\eta(x) = \sigma(x)\tilde{u}(x) + (K\tilde{u})(x) - (V\tilde{t})(x) \quad \text{for } x \in \overline{\Omega}.$$

3 The discrete method

In this section, finite element discretizations of the problems (2.17), (2.19), and (2.21) are described. Using the finite element solutions of these problems we compute an approximation of the Newton potential (2.15) via (2.16). We present corresponding error estimates for the approximate Newton potential which were proved in [8]. Then, we discuss the numerical solution of the boundary integral equations (2.9) and (2.14). We replace the exact Newton potential in these boundary integral equations by the approximate Newton potential and apply then boundary element discretizations. Error estimates for the approximate solutions of (2.9) and (2.14) obtained are proved.

3.1 Finite element discretization

Let $W_{\tilde h}(\widetilde\Omega) = \mathrm{span}\{\psi_k\}_{k=1}^{M_{\widetilde\Omega}}$ be a finite element trial space with $W_{\tilde h}(\widetilde\Omega) \subset H^1_0(\widetilde\Omega)$ for problem (2.17) and $W_{\tilde h}(\widetilde\Omega) \subset H^1(\widetilde\Omega)$ for problem (2.19), respectively. Let $\tilde h$ be the mesh size of the underlying triangulation. We assume that there holds an approximation property in $W_{\tilde h}(\widetilde\Omega)$, in particular, when using piecewise linear continuous finite elements we assume

$$\inf_{\widetilde v_{\tilde h}\in W_{\tilde h}(\widetilde\Omega)} ||\widetilde v - \widetilde v_{\tilde h}||_{H^1(\widetilde\Omega)} \;\le\; c_A \cdot \tilde h \cdot |\widetilde v|_{H^2(\widetilde\Omega)} \quad \text{for all } \widetilde v \in H^2(\widetilde\Omega). \tag{3.1}$$

The Galerkin formulation of (2.17) and (2.19) is to find $\widetilde u_{\tilde h} \in W_{\tilde h}(\widetilde\Omega)$ such that

$$\widetilde a(\widetilde u_{\tilde h}, \widetilde v_{\tilde h}) \;=\; \langle F, \widetilde v_{\tilde h}\rangle \qquad \text{for all } \widetilde v_{\tilde h} \in W_{\tilde h}(\widetilde\Omega). \tag{3.2}$$

Using standard arguments we get from (3.1) the error estimate

$$||\widetilde u - \widetilde u_{\tilde h}||_{H^1(\widetilde\Omega)} \;\le\; c\cdot \tilde h \cdot |\widetilde u|_{H^2(\widetilde\Omega)} \tag{3.3}$$

as well as the pointwise estimate [2]

$$||\widetilde u - \widetilde u_{\tilde h}||_{L_\infty(\widetilde\Omega)} \;\le\; c\cdot \tilde h^2 \cdot \ln|\tilde h|^{3/2} \cdot |\widetilde u|_{W_2^\infty(\widetilde\Omega)}.$$

Let $\Pi_{\tilde h}(\widetilde\Omega) \subset [L_2(\widetilde\Omega)]^3$ be another finite element trial space to approximate $\underline{p} = \nabla\widetilde u$. The Galerkin formulation of (2.21) is to find $\underline{p}_{\tilde h} \in \Pi_{\tilde h}(\widetilde\Omega)$ satisfying

$$\int_{\widetilde\Omega} \underline{p}_{\tilde h}(x)\cdot \underline{q}_{\tilde h}(x)dx \;=\; \int_{\widetilde\Omega} \nabla\widetilde u(x)\cdot \underline{q}_{\tilde h}(x)dx \quad \text{for all } \underline{q}_{\tilde h} \in \Pi_{\tilde h}(\widetilde\Omega).$$

Since $\widetilde u$ is not known we replace it by the finite element approximation $\widetilde u_{\tilde h}$ and consider the problem: Find $\widetilde{\underline{p}}_{\tilde h} \in \Pi_{\tilde h}$ such that

$$\int_{\widetilde\Omega} \widetilde{\underline{p}}_{\tilde h}(x)\cdot \underline{q}_{\tilde h}(x)dx \;=\; \int_{\widetilde\Omega} \nabla\widetilde u_{\tilde h}(x)\cdot \underline{q}_{\tilde h}(x)dx \quad \text{for all } \underline{q}_{\tilde h} \in \Pi_{\tilde h}(\widetilde\Omega) \tag{3.4}$$

holds. As it was shown in [8, Section 3] there holds the error estimate

$$||\underline{p} - \widetilde{\underline{p}}_{\tilde h}||_{[L_2(\widetilde\Omega)]^3} \;\le\; c\cdot \tilde h \cdot |\widetilde u|_{H^2(\widetilde\Omega)}. \tag{3.5}$$

Now we are able to compute an approximate Newton potential as

$$\eta_{\tilde h}(x) \;=\; \sigma(x)\widetilde u_{\tilde h}(x) + (K\widetilde u_{\tilde h})(x) - (V\widetilde t_{\tilde h})(x) \quad \text{for } x \in \overline{\Omega} \tag{3.6}$$

where

$$\widetilde t_{\tilde h}(x) \;:=\; \sum_{i,j=1}^{3} n_j(x) a_{ji} \widetilde p_{\tilde h,i}(x) \quad \text{for } x \in \Gamma \text{ almost everywhere.}$$

Since we compute the approximation $\widetilde t_{\tilde h}$ pointwise, the choice of continuous basis functions to define $\Pi_{\tilde h}(\widetilde\Omega)$ is favorable, hence we will use piecewise linear ones, too.

Lemma 3.1 *Let $\Omega \subset \widetilde\Omega$, $\widetilde\Omega$ convex and $f \in L_2(\Omega) \cap L_\infty(\Omega)$. Let $\eta_{\tilde h}$ be the approximation (3.6) of the Newton potential (2.15), (2.16). Then there hold the error estimates*

$$||\eta - \eta_{\tilde h}||_{H^{1/2}(\Gamma)} \;\leq\; c \cdot \tilde h \cdot |\widetilde u|_{H^2(\widetilde\Omega)} \tag{3.7}$$

and

$$||\eta - \eta_{\tilde h}||_{L_\infty(\Omega)} \;\leq\; c \cdot \tilde h^2 \cdot \Big\{ |\widetilde u|_{H^2(\widetilde\Omega)} + |\ln \tilde h\,|^{3/2} \cdot ||\widetilde u||_{W_2^\infty(\widetilde\Omega)} \cdot \Big\}$$

For a proof, see [8, Theorem 3.1 and 3.2]. Note, that using duality arguments, one can show the error estimate

$$||\eta - \eta_{\tilde h}||_{L_2(\Gamma)} \;\leq\; c \cdot \tilde h^{3/2} \cdot |\widetilde u|_{H^2(\widetilde\Omega)}. \tag{3.8}$$

3.2 Boundary element discretization for the Dirichlet problem

In this section we describe a discrete scheme for the numerical solution of the boundary integral equation (2.9) which is related to the Dirichlet boundary value problem (2.1) and (2.3). The variational problem of (2.9) is to find $t \in H^{-1/2}(\Gamma)$ such that

$$\langle Vt, \tau\rangle_{L_2(\Gamma)} \;=\; \langle (\sigma I + K)g, \tau\rangle_{L_2(\Gamma)} - \langle \eta, \tau\rangle_{L_2(\Gamma)} \quad \text{for all } \tau \in H^{-1/2}(\Gamma). \tag{3.9}$$

Let $Z_h(\Gamma) = \operatorname{span}\{\varphi_k^0\}_{k=1}^{N_\Gamma} \subset H^{-1/2}(\Gamma)$ be a boundary element trial space of piecewise constant basis functions defined with respect to a boundary element mesh of mesh size h. We assume an approximation property in $Z_h(\Gamma)$,

$$\inf_{w_h \in Z_h(\Gamma)} ||w - w_h||_{H^{-1/2}(\Gamma)} \;\leq\; c \cdot h^{s+1/2} \cdot ||w||_{H^s(\Gamma)} \quad \text{for all } w \in H^s(\Gamma)$$

with $s \in [-\frac{1}{2}, 1]$.
Replacing η by its approximation $\eta_{\tilde h}$, the Galerkin variational formulation of (3.9) is to find $\hat t_h \in Z_h(\Gamma)$ satisfying

$$\langle V\hat t_h, \tau_h\rangle_{L_2(\Gamma)} \;=\; \langle (\sigma I + K)g, \tau_h\rangle_{L_2(\Gamma)} - \langle \eta_{\tilde h}, \tau_h\rangle_{L_2(\Gamma)} \quad \text{for all } \tau_h \in Z_h(\Gamma). \tag{3.10}$$

Theorem 3.1 *Let $t \in H^\varrho(\Gamma)$ for some $\varrho > -\frac{1}{2}$ be the solution of the boundary integral equation (2.9). Let $\hat{t}_h \in Z_h$ be the Galerkin solution of (3.10). Then,*

$$\begin{aligned} ||t - \hat{t}_h||_{H^{-1/2}(\Gamma)} &\leq c \cdot \left\{ \inf_{\tau_h \in Z_h} ||t - \tau_h||_{H^{-1/2}(\Gamma)} + \tilde{h} \cdot |\tilde{u}|_{H^2(\tilde{\Omega})} \right\} \\ &\leq c \cdot \left\{ h^{s+1/2} \cdot ||t||_{H^s(\Gamma)} + \tilde{h} \cdot |\tilde{u}|_{H^2(\tilde{\Omega})} \right\} \end{aligned} \tag{3.11}$$

for all $-\frac{1}{2} \leq s \leq \min\{\varrho, 1\}$.

Proof. In addition to (3.10) we consider the Galerkin variational problem to find $t_h \in Z_h(\Gamma)$ such that

$$\langle V t_h, \tau_h \rangle_{L_2(\Gamma)} = \langle (\sigma I + K) g, \tau_h \rangle_{L_2(\Gamma)} - \langle \eta, \tau_h \rangle_{L_2(\Gamma)} \quad \text{for all } \tau_h \in Z_h(\Gamma).$$

Using the $H^{-1/2}(\Gamma)$–ellipticity of the single layer potential V we get

$$\begin{aligned} c_1^V \cdot ||t_h - \hat{t}_h||^2_{H^{-1/2}(\Gamma)} &\leq \langle V(t_h - \hat{t}_h), t_h - \hat{t}_h \rangle_{L_2(\Gamma)} \\ &= \langle \eta_{\tilde{h}} - \eta, t_h - \hat{t}_h \rangle_{L_2(\Gamma)} \\ &\leq ||\eta - \eta_{\tilde{h}}||_{H^{1/2}(\Gamma)} ||t_h - \hat{t}_h||_{H^{-1/2}(\Gamma)} \end{aligned}$$

and therefore

$$||t_h - \hat{t}_h||_{H^{-1/2}(\Gamma)} \leq \frac{1}{c_1^V} \cdot ||\eta - \eta_{\tilde{h}}||_{H^{1/2}(\Gamma)}.$$

Hence, (3.11) follows from the triangle inequality, (3.7) and standard arguments. ■

As a conclusion of Theorem 3.1 we state the following result:

Corollary 3.1 *From (3.11) we conclude that for a given boundary element discretization parameter h the finite element discretization should be defined with*

$$\tilde{h} = h^{\min\{\varrho,1\}+1/2}.$$

If the weak solution u of the boundary value problem (2.1), (2.3) belongs only to the space $H^2(\Omega)$, then $t \in H^{1/2}(\Gamma)$ and therefore, $\tilde{h} = h$.

Corollary 3.2 *Let the assumptions as in Theorem 3.1 be satisfied. Using the Aubin–Nitsche trick [7] and (3.8) one can prove*

$$||t - \hat{t}_h||_{H^{-1}(\Gamma)} \leq c \cdot \left\{ h^{s+1} \cdot ||t||_{H^s(\Gamma)} + \tilde{h}^{3/2} \cdot |\tilde{u}|_{H^2(\tilde{\Omega})} \right\}.$$

Note that due to (3.8) no error estimates in $H^\tau(\Gamma)$ for $\tau < -1$ are valid. Assuming an inverse inequality in $Z_h(\Gamma)$ one can show an error estimate in $L_2(\Gamma)$,

$$||t - \hat{t}_h||_{L_2(\Gamma)} \leq c \cdot \left\{ h^s \cdot ||t||_{H^s(\Gamma)} + h^{-1/2} \cdot \tilde{h} \cdot |\tilde{u}|_{H^2(\tilde{\Omega})} \right\}.$$

3.3 Boundary element discretization for the Robin problem

The variational problem of the boundary integral equation (2.14) according to the Robin boundary value problem (2.1) and (2.4) is to find $u \in H^{1/2}(\Gamma)$ such that

$$\langle Su + \kappa u, v\rangle_{L_2(\Gamma)} \;=\; \langle g + \omega, v\rangle_{L_2(\Gamma)} \quad \text{for all } v \in H^{1/2}(\Gamma) \tag{3.12}$$

where $\omega = V^{-1}\eta$. Let $W_h(\Gamma) = \text{span}\{\vartheta_k\}_{k=1}^{M_\Gamma} \subset H^{1/2}(\Gamma)$ a boundary element trial space with the mesh size parameter h, for example of piecewise linear continuous trial functions. For $s \in [\frac{1}{2}, 2]$ we assume the approximation property

$$\inf_{v_h \in W_h(\Gamma)} ||v - v_h||_{H^{1/2}(\Gamma)} \;\leq\; c \cdot h^{s-1/2} \cdot ||v||_{H^s(\Gamma)} \quad \text{for all } v \in H^s(\Gamma).$$

The approximate Galerkin scheme of (3.12) is to find $\hat{u}_h \in W_h(\Gamma)$ satisfying

$$\langle S\hat{u}_h + \kappa\hat{u}_h, v_h\rangle_{L_2(\Gamma)} \;=\; \langle g + \hat{\omega}_h, v_h\rangle_{L_2(\Gamma)} \quad \text{for all } v_h \in W_h(\Gamma) \tag{3.13}$$

where $\hat{\omega}_h \in Z_h(\Gamma)$ is the unique solution of

$$\langle V\hat{\omega}_h, \tau_h\rangle_{L_2(\Gamma)} \;=\; \langle \eta_{\tilde{h}}, \tau_h\rangle_{L_2(\Gamma)} \quad \text{for all } \tau_h \in Z_h(\Gamma). \tag{3.14}$$

Theorem 3.2 *Let $u \in H^\varrho(\Gamma)$ for some $\varrho > \frac{1}{2}$ be the solution of the boundary integral equation (2.14). Let $\hat{u}_h \in W_h(\Gamma)$ be the Galerkin solution of (3.13). Then,*

$$\begin{aligned}
||u - \hat{u}_h||_{H^{1/2}(\Gamma)} \;&\leq\; c \cdot \left\{ \inf_{v_h \in W_h(\Gamma)} ||u - v_h||_{H^{1/2}(\Gamma)} \right. \\
&\qquad \left. + \inf_{\tau_h \in Z_h(\Gamma)} ||\omega - \tau_h||_{H^{-1/2}(\Gamma)} + \tilde{h} \cdot |\tilde{u}|_{H^2(\tilde{\Omega})} \right\}. \\
&\leq\; c \cdot \left\{ h^{s-1/2} \cdot ||u||_{H^s(\Gamma)} + h \cdot ||\tilde{u}||_{H^2(\tilde{\Omega})} + \tilde{h} \cdot |\tilde{u}|_{H^2(\Omega)} \right\}
\end{aligned}$$

with $\frac{1}{2} \leq s \leq \min\{2, \varrho\}$.

Proof. Let $u_h \in W_h(\Gamma)$ be the unique solution of the variational problem

$$\langle Su_h + \kappa u_h, v_h\rangle_{L_2(\Gamma)} \;=\; \langle g + \omega, v_h\rangle_{L_2(\Gamma)} \quad \text{for all } v_h \in W_h(\Gamma).$$

Applying standard arguments we get Cea's lemma,

$$||u - u_h||_{H^{1/2}(\Gamma)} \;\leq\; c \inf_{v_h \in W_h(\Gamma)} ||u - v_h||_{H^{1/2}(\Gamma)}.$$

On the other hand,

$$||u_h - \hat{u}_h||_{H^{1/2}(\Gamma)} \;\leq\; c \cdot ||\omega - \hat{\omega}_h||_{H^{-1/2}(\Gamma)}.$$

Hence we have

$$||u - \hat{u}_h||_{H^{1/2}(\Gamma)} \;\leq\; c \cdot \left\{ \inf_{v_h \in W_h(\Gamma)} ||u - v_h||_{H^{1/2}(\Gamma)} + ||\omega - \hat{\omega}_h||_{H^{-1/2}(\Gamma)} \right\}$$

and we need to estimate the second term. For this we consider the variational problem to find $\omega_h \in Z_h(\Gamma)$ such that

$$\langle V\omega_h, \tau_h\rangle_{L_2(\Gamma)} = \langle \eta, \tau_h\rangle_{L_2(\Gamma)} \quad \text{for all } \tau_h \in Z_h(\Gamma).$$

Applying Cea's lemma we conclude the error estimate

$$||\omega - \omega_h||_{H^{-1/2}(\Gamma)} \leq c \cdot \inf_{\tau_h \in Z_h} ||\omega - \tau_h||_{H^{-1/2}(\Gamma)}.$$

Using

$$||\omega_h - \hat{\omega}_h||_{H^{-1/2}(\Gamma)} \leq c \cdot ||\eta - \eta_{\tilde{h}}||_{H^{1/2}(\Gamma)}$$

the first estimate follows from the triangle inequality and (3.7). Using the approximation properties in $W_h(\Gamma)$ and $Z_h(\Gamma)$ gives the second inequality where we used

$$||\omega||_{H^{1/2}(\Gamma)} \leq c \cdot ||\widetilde{u}||_{H^2(\widetilde{\Omega})}.$$

■

3.4 The coupled FE/BE algorithm

In the following, we describe the algorithm for solving the inhomogeneous boundary value problems (2.1) – (2.3) and (2.1) – (2.4) approximately. Since we want to use a boundary element method for solving these problems we have to compute the Newton potential (2.7) in certain points $x \in \overline{\Omega}$, see Sect. 2.1. As mentioned before, a direct computation of the Newton potential is in general inefficient due to the complexity. But if a particular solution of the differential equation (2.1) is known we can compute the Newton potential by using formula (2.16). Such a particular solution will be obtained by solving an appropriate boundary value problem in a fictitious domain $\widetilde{\Omega}$ containing the original domain Ω (see Section 2.2). For solving these auxiliary problem in $\widetilde{\Omega}$ we want to apply a multilevel finite element method. Therefore, we have to perform the following steps in our coupled finite element/boundary element algorithm:

1. Define a simple fictitious domain $\widetilde{\Omega}$ with $\Omega \subset \widetilde{\Omega}$. Construct a coarse triangulation $\mathcal{T}_1$ of $\widetilde{\Omega}$ by using tetrahedral elements.

2. Consider a corresponding Dirichlet or Robin boundary value problem in the fictitious domain $\widetilde{\Omega}$ for finding an approximation of a particular solution $\widetilde{u}$. Apply a full multilevel strategy based on a sequence of nested triangulations $\mathcal{T}_q$, $q = 1, 2, \ldots \ell$, and finite element subspaces $W_{\tilde{h}_1}(\widetilde{\Omega}) \subset W_{\tilde{h}_2}(\widetilde{\Omega}) \subset \cdots \subset W_{\tilde{h}_\ell}(\widetilde{\Omega})$ spanned, e.g., by piecewise linear functions. We solve the systems of finite element equations on each level by means of a multigrid algorithm. This approach gives us an approximate solution $\widetilde{\widetilde{u}}_{\tilde{h}_\ell}$ for which the error estimate

$$||\widetilde{u} - \widetilde{\widetilde{u}}_{\tilde{h}_\ell}||_{H^1(\widetilde{\Omega})} \leq c \cdot \tilde{h}_\ell \cdot |\widetilde{u}|_{H^2(\widetilde{\Omega})} \tag{3.15}$$

holds (see, e.g., [5]). The cost of arithmetical work for getting the approximation $\widetilde{\widetilde{u}}_{\tilde{h}_\ell}$ is proportional to the number of nodes in the finest triangulation $\mathcal{T}_\ell$, i.e., we apply an asymptotically optimal algorithm for solving the problem in the fictitious domain $\widetilde{\Omega}$. The estimate (3.15) is of the same order as the estimate (3.3) used in the error analysis in Sections 3.2 and 3.3, and consequently, all our estimates given in these sections remain true if we replace $\widetilde{u}_{\tilde{h}}$ by $\widetilde{\widetilde{u}}_{\tilde{h}_\ell}$.

3. Compute an approximate gradient $\underline{\widetilde{\widetilde{p}}}_{\tilde{h}_\ell}$ of $\nabla\tilde{u}$ by a $L_2(\widetilde{\Omega})$ projection, i.e., solve problem (3.4). Here, we use again a sequence of finite element discretizations based on tetrahedral elements with piecewise linear ansatz functions. The corresponding systems of algebraic finite element equations are solved by means of a full multilevel strategy. We use on each level the preconditioned conjugate gradient method with the diagonal part of the mass matrix as preconditioner. Since the mass matrix and its diagonal part are spectrally equivalent the convergence of the preconditioned conjugate gradient method is independent of the discretization parameter. In this way, we get an approximation $\underline{\widetilde{\widetilde{p}}}_{\tilde{h}_\ell}$ for which the error estimate
$$||\underline{p} - \underline{\widetilde{\widetilde{p}}}_{\tilde{h}_\ell}||_{[L_2(\widetilde{\Omega})]^3} \le c \cdot \tilde{h}_\ell \cdot |\tilde{u}|_{H^2(\widetilde{\Omega})}$$
holds. This estimate is of the same order as the estimate (3.5).

4. Construct boundary element discretizations for the variational problems (3.9) or (3.12) by using piecewise constant and piecewise linear basis functions, respectively. For computing the right–hand side in the boundary element schemes (3.10) and (3.14) we need the Newton potential in points $x \in \Gamma$. Applying formula (3.6) we have to compute $\widetilde{\widetilde{u}}_{\tilde{h}_\ell}(x)$ and $\widetilde{\widetilde{t}}_{\tilde{h}_\ell}(x)$ in points $x \in \Gamma$. To determine the value of the finite element solution $\widetilde{\widetilde{u}}_{\tilde{h}_\ell}$ and the conormal derivative $\widetilde{\widetilde{t}}_{\tilde{h}_\ell}$ in such points we use also the hierarchy of the triangulations $\mathcal{T}_q$. More precisely, at first we have to find that tetrahedral element of the coarsest mesh $\mathcal{T}_1$ which contains a given point $x \in \Gamma$. Then, we consider the eight subtetrahedra in $\mathcal{T}_2$ and determine that subtetrahedron which contains the point x. The same steps are performed until we have found the tetrahedron of the finest mesh $\mathcal{T}_\ell$ containing x. In this way, we get a very efficient search algorithm. The number of search steps is bounded by the number of elements in the coarsest triangulation $\mathcal{T}_1$ times the number of levels ℓ. The corresponding values of $\widetilde{\widetilde{u}}_{\tilde{h}_\ell}(x)$ and $\widetilde{\widetilde{t}}_{\tilde{h}_\ell}(x)$ are obtained by linear interpolation.

5. Solve the discrete boundary element equations by some preconditioned conjugate gradient scheme. As preconditioner one can use the approach as described in [11] yielding a constant bound for the spectral condition number. Hence, the arithmetical costs to solve the BEM equations are due to the matrix vector products to be performed, in a standard approach this is of order N^2; using fast algorithms such as panel clustering or multipol expansions this can be reduced significantly.

6. Compute the solution of the boundary value problem using an adaptive post processing algorithm [9] via the representation formula (2.5) and the Newton potential (2.15).

We want to remark that we have to perform the steps 1 – 3, i.e. the finite element discretization, and the computation of the approximate Newton potential in step 4 only once also in the case of nonlinear boundary conditions.

4 Numerical results

As a simple model problem we consider the Dirichlet boundary value problem

$$-\Delta u(x) \;=\; f(x) \quad \text{for } x \in \Omega \subset I\!R^3, \qquad u(x) \;=\; g(x) \quad \text{for } x \in \Gamma$$

where Ω is given by (see Figure 1)

$$\begin{aligned}
\overline{\Omega} &= \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3 \\
\Omega_1 &= \Big((x,y,z) \in I\!R^3 \;:\; x^2 + y^2 + (z+0.6)^2 \;<\; 0.16\Big\} \\
\Omega_2 &= \Big((x,y,z) \in I\!R^3 \;:\; x^2 + y^2 + (z-0.6)^2 \;<\; 0.16\Big\} \\
\Omega_3 &= \Big\{(x,y,z) \in I\!R^3 \;:\; x^2 + y^2 < 0.0375,\; -0.25 < z < 0.25\Big\}\,.
\end{aligned}$$

The boundary integral equation (2.9) is discretized by a Galerkin method using piecewise constant test and trial functions. For the approximation of the Newton potential we applied the algorithm as described in Section 3.4. As fictitious domain we choose

$$\widetilde{\Omega} \;=\; (-0.5, 0.5) \times (-0.5, 0.5) \times (-1.1, 1.1)\,.$$

The functions $f(x)$ and $g(x)$ are chosen such that

$$u(x) = -x^3$$

is the exact solution
We have tested our approach for both solving the artificial boundary value problem in $\widetilde{\Omega}$ with either Dirichlet (Table 1,2) or Robin (Table 3,4) boundary conditions. Due to Theorem 3.1 the rate of convergence in the energy norm is given by $\max\{h^{3/2}, \widetilde{h}\}$ (Table 2,4) while in $L_2(\Gamma)$ the rate of convergence is given by $\max\{h, h^{-1/2}\widetilde{h}\}$ (Table 1,3). Hence, for coarse $\widetilde{h}$ the convergence will fail while for small $\widetilde{h}$ the convergence due to the boundary element solution dominates. This is well documented for both cases. From the numerical results we conclude that there is no essential difference in solving a Dirichlet of Robin boundary value problem in the fictitious domain. The norm computations in $H^{-1/2}(\Gamma)$ were done by using an equivalent multilevel norm as described in [10]. Note that N_{BEM} denotes the number of boundary elements while M_{FEM} is the number of vertices of the finite element mesh.

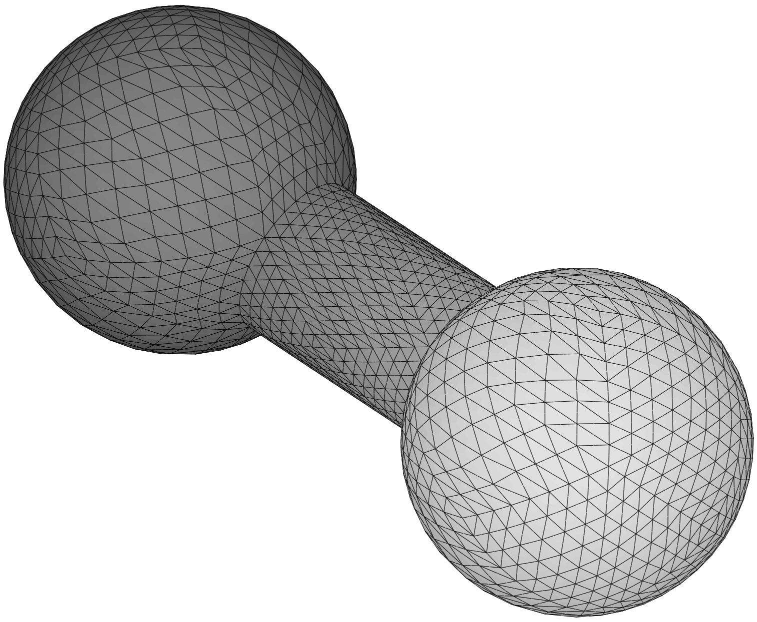

Figure 1: $N_\Gamma = 3584, M_\Gamma = 1794$

N_{BEM}	$M_{\mathrm{FEM}} = 225$	$M_{\mathrm{FEM}} = 1377$	$M_{\mathrm{FEM}} = 9537$	$M_{\mathrm{FEM}} = 70785$
224	8.184 –2	5.107 –2	4.859 –2	4.836 –2
896	1.021 –1	3.681 –2	2.797 –2	2.733 –2
3584	1.167 –1	3.768 –2	1.690 –2	1.415 –2

Table 1: Error $||t - \hat{t}_h||_{L_2(\Gamma)}$. Dirichlet boundary value problem (FEM).

N_{BEM}	$M_{\mathrm{FEM}} = 225$	$M_{\mathrm{FEM}} = 1377$	$M_{\mathrm{FEM}} = 9537$	$M_{\mathrm{FEM}} = 70785$
224	2.899 –2	1.531 –2	1.425 –2	1.412 –2
896	2.811 –2	8.719 –3	5.754 –3	5.500 –3
3584	2.317 –2	7.145 –3	2.706 –3	2.014 –3

Table 2: Error $||t - \hat{t}_h||_{H^{-1/2}(\Gamma)}$. Dirichlet boundary value problem (FEM).

N_{BEM}	$M_{\mathrm{FEM}} = 225$	$M_{\mathrm{FEM}} = 1377$	$M_{\mathrm{FEM}} = 9537$	$M_{\mathrm{FEM}} = 70785$
224	7.740 –2	4.948 –2	4.765 –2	4.746 –2
896	9.715 –2	3.611 –2	2.764 –2	2.710 –2
3584	1.117 –1	3.670 –2	1.670 –2	1.415 –2

Table 3: Error $||t - \hat{t}_h||_{L_2(\Gamma)}$. Robin boundary value problem (FEM).

N_{BEM}	$M_{\mathrm{FEM}} = 225$	$M_{\mathrm{FEM}} = 1377$	$M_{\mathrm{FEM}} = 9537$	$M_{\mathrm{FEM}} = 70785$
224	2.706 –2	1.450 –2	1.373 –2	1.362 –2
896	2.665 –2	8.508 –3	5.623 –3	5.408 –3
3584	2.221 –2	6.967 –3	2.665 –3	2.014 –3

Table 4: Error $||t - \hat{t}_h||_{H^{-1/2}(\Gamma)}$. Robin boundary value problem (FEM).

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