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Multilevel preconditioning operators on locally modified grids*

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Abstract

Systems of grid equations that approximate elliptic boundary value problems on locally modified grids are considered. The triangulation, which approximates the boundary with second order of accuracy, is generated from an initial uniform triangulation by shifting nodes near the boundary according to special rules. This "locally modified" grid possesses several significant features: this triangulation has a regular structure, the generation of the triangulation is rather fast, this construction allows to use multilevel preconditioning (BPX-like) methods. The proposed iterative methods for solving elliptic boundary value problems approximately are based on two approaches: The fictitious space method, i.e. the reduction of the original problem to a problem in an auxiliary (fictitious) space, and the multilevel decomposition method, i.e. the construction of preconditioners by decomposing functions on hierarchical grids. The convergence rate of the corresponding iterative process with the preconditioner obtained is independent of the mesh size. The construction of the grid and the preconditioning operator for the three dimensional problem can be done in the same way.

Key words: elliptic boundary value problems, mesh generation, finite element method, multilevel methods

AMS subject classification. 65F30, 65N50, 65N55.

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1 Introduction

The most efficient solvers for large-scale systems of finite element equations are algorithms which make use of a sequence of discretizations of the boundary value problem considered or at least of a sequence of triangulations of the underlying domain. Examples for such solvers are the multi-grid method (see, e.g., [6, 10, 12, 13]) and BPX-like methods (see, e.g., [5, 21, 25]). For some practical problems, as e.g. boundary value problems in domains with a complicated geometry, it is impossible to construct such a sequence of triangulations with a sufficiently coarse grid. Then, the methods mentioned above lose their efficiency. To overcome this problem several approaches were proposed (see, e.g., [2, 7, 15, 18, 26]). In the present paper, we describe the construction of a preconditioner based on the fictitious space lemma [18] and a multilevel decomposition of functions on hierarchical grids. The construction of locally modified grids [3, 17, 23, 24] and the application of these preconditioners is presented. The convergence rate of the conjugate gradient method with the proposed preconditioner is independent of the discretization parameter. Other preconditioning operators on locally fitted grids were suggested in [9, 11, 14, 16].

The paper is organized as follows. In section 2 we introduce the boundary value problem of second order which we want to solve numerically by hierarchical methods. In section 3 we describe briefly the finite element discretization and the basic idea for the construction of the preconditioner. Section 4 is devoted to the construction of structured triangular grids which approximate the boundary of the domain with second order of accuracy. Two similar methods of the construction are suggested. In the first one the resulting triangular grid is constructed from an original uniform quadrilateral grid in a square containing the original domain Ω and in the second approach the resulting grid is based upon a uniform triangular grid in a triangle containing Ω. The main idea is to modify these uniform grids in the neighbourhood of the boundary of the domain Ω such that one gets triangulations of Ω which consists only of congruent triangles except near to the boundary. In section 5 we discuss some advantages of the constructed grids. We also show that the constructed grids are quasiuniform and thus can be used in multilevel preconditioning methods. Moreover the regular structure of the grid makes the application of such methods especially beneficial because there exists a natural one-to-one correspondence between the nodes of the resulting grid and some subset of nodes of the original uniform structured grid. In section 6, we describe the construction of preconditioners based on the fictitious space lemma and a multilevel decomposition of functions on hierarchical grids. In the last section, numerical results are given. These experiments confirm the theoretical results.

2 The original problem

Let $\Omega \in R^2$ be a domain with a Lipschitz continuous and twice piecewise continuously differentiable boundary, i.e. it belongs to the class $C^{0,1} \cap PC^2$ [27]. In the domain Ω we consider the boundary value problem:

$$
\begin{aligned}
-\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+a_0(x)u &= f(x) \quad \text{for all } x\in\Omega\,,\\
u(x) &= 0 \quad \text{for all } x\in\Gamma_0\,,\\
\frac{\partial u}{\partial N}+\sigma(x)u &= 0 \quad \text{for all } x\in\Gamma_1\,,
\end{aligned}
\tag{1}
$$

where

$$
\frac{\partial u}{\partial N}=\sum_{i,j=1}^{2}a_{ij}(x)\frac{\partial u}{\partial x_j}\cos(n,x_i)
$$

is the conormal derivative, n denotes the outward normal to $\Gamma=\partial\Omega$ and Γ_0 is a union of a finite number of curvilinear segments, $\Gamma=\Gamma_0\cup\Gamma_1$, $\Gamma_0=\bar{\Gamma}_0$. Here $\bar{\Gamma}_0$ denotes the closure of Γ_0.

We consider the weak formulation of problem (1):

$$
\text{Find } u\in H^1(\Omega,\Gamma_0)\ :\ a(u,v)=\ell(v)\ \ \forall v\in H^1(\Omega,\Gamma_0) \tag{2}
$$

with the bilinear form

$$
a(u,v)=\int_{\Omega}\left(\sum_{i,j=1}^{2}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}+a_0(x)uv\right)dx+\int_{\Gamma_1}\sigma(x)uv\,ds
$$

and the linear functional

$$
\ell(v)=\int_{\Omega}f(x)v\,dx\,.
$$

The space $H^1(\Omega,\Gamma_0)$ is a subspace of the Sobolev space $H^1(\Omega)$ defined by

$$
H^1(\Omega,\Gamma_0)=\{v\in H^1(\Omega)\ :\ v(x)=0\ \text{ for } x\in\Gamma_0\}\,.
$$

Let us suppose that the coefficients a_{ij}, $i,j=1,2$, a_0, and the right-hand side f of problem (1) are given such that the bilinear form $a(u,v)$ is symmetric, elliptic, and continuous on $H^1(\Omega,\Gamma_0)\times H^1(\Omega,\Gamma_0)$, i.e.

$$
a(u,v)=a(v,u)\ \ \forall u,v\in H^1(\Omega,\Gamma_0)
$$
$$
\alpha_1\|v\|^2_{H^1(\Omega)}\le a(v,v)\le\alpha_2\|v\|^2_{H^1(\Omega)}\ \ \forall v\in H^1(\Omega,\Gamma_0)\,,
$$

and the linear functional $\ell(v)$ is continuous on $H^1(\Omega,\Gamma_0)$, i.e.

$$
|\ell(v)|\le\alpha_3\|v\|^2_{H^1(\Omega)}\,,
$$

where α_1, α_2, and α_3 are positive constants.

It is well-known that under the above assumptions on $a(u,v)$ and $\ell(v)$ there exists a unique solution of problem (2) [1].

3 Finite element approximation

Let for a fixed positive parameter h (we always suppose that h is sufficiently small)

$$\Omega^h = \bigcup_{i=1}^{M} \tau_i$$

be a triangulation of the domain Ω (Ω^h is assumed to be a closed set). We suppose that Ω^h is a quasi-uniform triangulation [8], i.e.

1. for any h less than some h_0 the length of the edges and the area of all triangles belong to intervals $[\beta_1 h, \beta_2 h]$ and $[\gamma_1 h^2, \gamma_2 h^2]$, respectively,
2. there exists a one-to-one correspondence between points on Γ and $\Gamma^h = \partial\Omega^h$ and the distance between them is less than δh^2, (3)

where the constants β_1, β_2, γ_1, γ_2, and δ are independent of h.

If $\Gamma_1 = \Gamma$, we suppose that $\Omega \subset \Omega^h$; if $\Gamma_0 = \Gamma$, we suppose that $\Omega^h \subset \Omega$. If $\Gamma_0 \neq \emptyset$ and $\Gamma_1 \neq \emptyset$, we make the following assumption: points where the boundary condition changes should be at triangulation nodes, $\Gamma_1 \subset \Omega^h$ and $\Gamma_0 \subset \overline{R^2 \setminus \Omega^h}$. The part of Γ^h approximating Γ_0 will be denoted by Γ_0^h and that for Γ_1 by Γ_1^h. For the triangulation Ω^h, we define the space $H_h(\Omega^h, \Gamma_0^h)$ of real continuous functions which are linear on each triangle of Ω^h and vanish on Γ_0^h. We extend these functions on $\Omega \setminus \Omega^h$ by zero.

The solution of the projection problem:

$$\text{Find } u^h \in H_h(\Omega^h, \Gamma_0^h) \, : \, a(u^h, v^h) = \ell(v^h) \;\; \forall v^h \in H_h(\Omega^h, \Gamma_0^h) \tag{4}$$

is called an approximate solution of problem (2). Aspects of the approximation properties of (2) by (4) have been thoroughly studied e.g. in [8, 20]. We do not consider that here.

Each function $u^h \in H_h(\Omega^h, \Gamma_0^h)$ is put in standard correspondence with a real column vector $u \in R^N$ whose components are the values of the function u^h at the corresponding nodes of the triangulation $\Omega^h \setminus \Gamma_0^h$. Then (4) is equivalent to the system of mesh equations

$$Au = f \tag{5}$$

with

$$(Au, v) = a(u^h, v^h) \;\; \forall u^h, v^h \in H_h(\Omega^h, \Gamma_0^h)$$

and

$$(f, v) = \ell(v^h) \;\; \forall v^h \in H_h(\Omega^h, \Gamma_0^h)\,,$$

where u^h and v^h are the prolongations of the vectors u and v; $(.,.)$ is the Euclidean scalar product in R^N.

The most efficient methods for solving the system of linear algebraic equations (5) are preconditioned iterative processes with preconditioners B. The rate of convergence of these methods depends on the constants c_1 and c_2 in the inequalities

$$c_1(Bu, u) \leq (Au, u) \leq c_2(Bu, u) \;\; \forall u \in R^N\,. \tag{6}$$

To construct effective preconditioning operators the following fictitious space lemma [18] is useful.

Lemma 3.1 *Let H and $\tilde{H}$ be Hilbert spaces with the scalar products $(u,v)_H$ and $(\tilde{u},\tilde{v})_{\tilde{H}}$, respectively. Let A and $\tilde{A}$ be symmetric positive definite continuous operators in the spaces H and $\tilde{H}$:*

$$A \,:\, H \to H\,,\ \ \tilde{A} \,:\, \tilde{H} \to \tilde{H}\,.$$

Suppose that R is a linear operator such that

$$R \,:\, \tilde{H} \to H\,,\ \ (AR\tilde{v}, R\tilde{v})_H \leq c_R(\tilde{A}\tilde{v},\tilde{v})_{\tilde{H}} \ \ \forall \tilde{v} \in \tilde{H}\,,$$

and there exists an operator T such that

$$T \,:\, H \to \tilde{H}\,,\ \ RTu = u\,,\ \ c_T(\tilde{A}Tu, Tu)_{\tilde{H}} \leq (Au,u)_H \ \ \forall u \in H\,,$$

where c_R and c_T are positive constants. Then

$$c_T(A^{-1}u,u)_H \leq (R\tilde{A}^{-1}R^*u,u)_H \leq c_R(A^{-1}u,u)_H \ \ \forall u \in H\,.$$

The operator R^* is adjoint to R with respect to the scalar products $(u,v)_H$ and $(\tilde{u},\tilde{v})_{\tilde{H}}$:

$$R^* \,:\, H \to \tilde{H}\,,\ \ (R^*u,\tilde{v})_{\tilde{H}} = (u, R\tilde{v})_H\,.$$

Note that for the construction and the implementation of a preconditioner of the type $B = R\tilde{A}^{-1}R^*$ only the existence of the operator T is required. In section 6, we will discuss the construction of such a preconditioner more in detail.

In the next section, the triangulation Ω^h will be constructed in the following way. At first, we generate a uniform quadrilateral grid D^h for some domain in which the domain Ω is embedded. In a second step, this uniform mesh will be modified locally into the mesh Ω^h. In our application, the Hilbert space H in Lemma 3.1 is the finite element space $H_h(\Omega^h, \Gamma_0^h)$, and the Hilbert space $\tilde{H}$ will be defined in section 6. The operator A in Lemma 3.1 is the operator from (5) and $\tilde{A}$ will be constructed using the fictitious space lemma and a multilevel decomposition [18] (see section 6).

4 Generation of the triangular grid

Let Ω be a bounded domain in R^2 satisfying the previous conditions (see at the begin of section 2). We suppose that the boundary of the domain Ω consists of a finite number of parts C_i and each C_i is part of a twice continuously differentiable curve, i.e. part of a curve from C^2, without self-intersections. Then, the following inequalities are valid for the angle α_{ij} between C_i and C_j at the end point for any i and j:

$$0 < \alpha_0 < \alpha_{ij} < 2\pi - \alpha_0$$

with a constant α_0.

4.1 Quadrilateral source grid

The algorithm for the generation of a triangular grid satisfying the conditions (3) for domains with a boundary from C^2 without self-intersections and self-touching was suggested and investigated in [17]. The algorithm consists in the following steps:

The domain Ω is embedded into a square D, where the edges of D are parallel to the axes of a Cartesian coordinate system. Let K be the maximal module of the curvature of the parts C_i for all i. Let $R > 0$ be chosen in such a way that a circle with the radius R touching C_i in some point does not have another common point with C_i. Then, we introduce the quantities (see [19])

$$\sigma = \min\left\{\frac{R}{2}, \frac{1}{2K}\right\} \quad \text{and} \quad h_0 = \frac{\sigma}{2\sqrt{2}}\,. \tag{7}$$

Furthermore, we define two sets ω_σ and Ω_σ as follows. A point $(\bar{x}, \bar{y})$ belongs to ω_σ if $(\bar{x}, \bar{y}) \in \Omega$ and the distance between this point and the boundary $\partial\Omega$ is less than σ. Analogously, a point $(\bar{x}, \bar{y})$ belongs to Ω_σ if $(\bar{x}, \bar{y}) \notin \Omega$ and the distance between the point $(\bar{x}, \bar{y})$ and $\partial\Omega$ is less than σ. For any point $(\bar{x}, \bar{y}) \in \omega_\sigma$ there exists only one normal vector from $(\bar{x}, \bar{y})$ to the boundary $\partial\Omega$ such that the segment from $(\bar{x}, \bar{y})$ to the corresponding point on $\partial\Omega$ lies entirely within ω_σ. The same is true for any point $(\bar{x}, \bar{y}) \in \Omega_\sigma$ (see [19], p. 20).

Let $D^h_\square$ be a uniform quadrilateral grid in D with the distance between the nodes equals to $h = s/L < h_0$, where s is the length of the edges of D and L is a positive integer (see Figure 1).

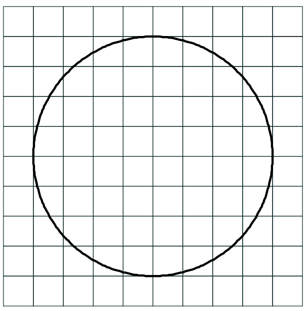

Figure 1: Domain Ω embedded into a square D and the quadrilateral grid $D^h_\square$

We denote the nodes of the grid $D^h_\square$ by $Z_{i,j}$,

$$Z_{i,j} = (x_i, y_j)\,, \quad i,j = 0,1,\ldots,L\,,$$

and the cells of $D^h_\square$ by $D_{i,j}$, where

$$D_{i,j} = \{(x,y)\ :\ x_i \le x < x_{i+1}\,,\ y_i \le y < y_{i+1}\}\,.$$

Therefore,

$$D = \bigcup_{i,j=0}^{L-1} \bar{D}_{i,j} \,.$$

Starting from the grid $D_\Box^h$ we construct a locally modified grid $\tilde{D}_\Box^h$. For each node $Z_{i,j} = (x_i, y_j) \in D_\Box^h$ we find a corresponding node $\tilde{Z}_{i,j} = (\tilde{x}_i, \tilde{y}_j) \in \tilde{D}_\Box^h$ according to the following rule. We consider rays which start from the node $Z_{i,j}$ and are parallel to the coordinate axes. The points of intersection of these rays with the boundary Γ which are the nearest to the node $Z_{i,j}$ in the corresponding direction are denoted by P_l, P_r, P_a, and P_b, see Figure 2. If some intersection point does not exist then we consider it as infinitely far away.

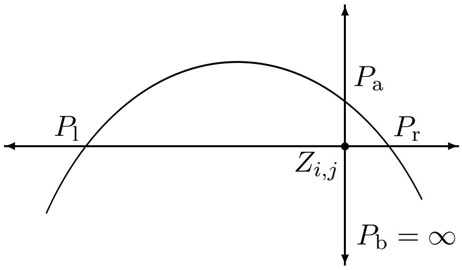

Figure 2: Definition of the intersection points P_l, P_r, P_a, and P_b

The distances from P_l, P_r, P_a, and P_b to the node $Z_{i,j}$ are denoted by d_l, d_r, d_a, and d_b, respectively. Let d be the minimum of these distances. If d is greater than $h/2$, then we set $\tilde{Z}_{i,j} = Z_{i,j}$. If d is equal to $h/2$ and the corresponding intersection point lies right-side or above the node $Z_{i,j}$ then this intersection point is $\tilde{Z}_{i,j}$, otherwise we set $\tilde{Z}_{i,j} = Z_{i,j}$. If d is less than $h/2$ then we put $\tilde{Z}_{i,j}$ equal to the corresponding intersection point. If the selection of $\tilde{Z}_{i,j}$ is not unique then we accept any of the valid points. Such a situation arises for example in the case when $d_\mathrm{a} = d_\mathrm{r} = h/2$ and $d_\mathrm{l} > h/2$. Using this approach we get from the grid $D_\Box^h$ in Figure 1 the locally modified grid $\tilde{D}_\Box^h$ in Figure 3

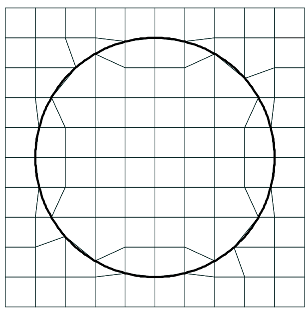

Figure 3: The locally modified quadrilateral grid $\tilde{D}_\Box^h$

We denote the correspondence between the nodes $Z_{i,j} = (x_i, y_j)$ of the grid $D_\Box^h$ and the nodes $\tilde{Z}_{i,j} = (\tilde{x}_i, \tilde{y}_j)$ of the locally modified grid $\tilde{D}^h_\square$ by the mapping

$$f^{-1} \,:\, (\tilde{x}_i, \tilde{y}_j) \rightarrow (x_i, y_j) \text{ for all nodes } (\tilde{x}_i, \tilde{y}_j) \in \tilde{D}^h_\square \,. \tag{8}$$

Lemma 4.1 *Let $Z_{i,j} = (x_i, y_j) \in D^h_\square$, $Z_{i+1,j} = (x_i + h, y_j) \in D^h_\square$, $f(Z_{i,j}) \neq Z_{i,j}$, and $f(Z_{i+1,j}) \neq Z_{i+1,j}$. If $h \leq h_0$, then one of the following conditions must be fulfilled:*

$$(f(Z_{i,j}) - Z_{i,j}, Z_{i+1,j} - Z_{i,j}) = 0 \quad \textit{and} \quad (f(Z_{i+1,j}) - Z_{i+1,j}, Z_{i+1,j} - Z_{i,j}) = 0 \tag{9}$$

or

$$(f(Z_{i,j}) - Z_{i,j}, f(Z_{i+1,j}) - Z_{i+1,j}) = 0\,, \tag{10}$$

where $(.,.)$ denotes the Euclidean scalar product in R^2.

Proof: Let us assume that neither (9) nor (10) is true. This is only possible if the points $Z_{i,j}$, $Z_{i+1,j}$, $\tilde{Z}_{i,j} = f(Z_{i,j})$, and $\tilde{Z}_{i+1,j} = f(Z_{i+1,j})$ lie on a straight line. Three possible essentially different cases of the location of these nodes are shown in Figure 4.

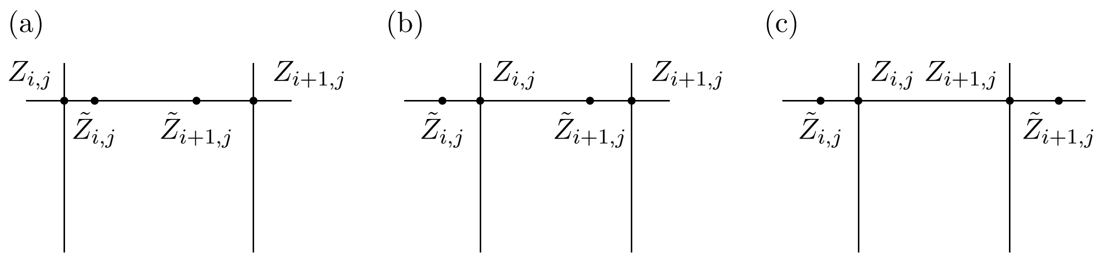

Figure 4: Different cases for the location of the nodes Z_{ij}, $Z_{i+1,j}$, $\tilde{Z}_{ij}$, $\tilde{Z}_{i+1,j}$

Let us consider the case (a) (see, Figure 4) in detail. We suppose that the boundary of the domain Ω intersects the left edge of the grid cell (see Fig. 5(a)). Let d_1 denote the distance between $Z_{i,j}$ and $\tilde{Z}_{i,j}$, d_2 the distance between $Z_{i,j}$ and $\tilde{Z}_{i+1,j}$, and d_3 the distance between $Z_{i,j}$ and the intersection point $\hat{S}$ of the boundary and the left edge of the grid cell. It is obvious that $0 < d_1 < h/2$, $h/2 < d_2 < h$, and $d_1 < d_3$. According to the mean value theorem there exists a point $\hat{P} = (\hat{x}, \hat{y})$, $x_i \leq \hat{x} \leq x_i + d_1$, $y_j - d_3 \leq \hat{y} \leq y_j$, such that the tangent to the boundary in this point is parallel to the line through the points $\tilde{Z}_{i,j} = (x_i + d_1, y_j)$ and $\hat{S} = (x_i, y_j - d_3)$. Analogously, there exists a point $\breve{P} = (\breve{x}, \breve{y})$, $x_i + d_1 \leq \breve{x} \leq x_i + d_2$, such that the tangent to the boundary in this point is parallel to the line through (x_i, y_j) and (x_{i+1}, y_j). Let us draw the normal vectors $\vec{\hat{n}} = (d_3, -d_1)^T$ and $\vec{\breve{n}} = (0, -1)^T$ to the boundary at the points $\hat{P}$ and $\breve{P}$ as it is shown in Figure 5 Let $\hat{g}$ be a straight line in direction $\vec{\hat{n}}$ through the point $\hat{S} = (x_i - d_3, y_j)$ and $\breve{g}$ a line in direction $\vec{\breve{n}}$ through $\breve{S} = \tilde{Z}_{i+1,j} = (x_i + d_2, y_j)$. Note that the point S, the intersection point of the lines with the directions $\vec{\hat{n}}$ and $\vec{\breve{n}}$ through the points $\hat{P}$ and $\breve{P}$, respectively, lies above the line $\hat{g}$ and left to the line $\breve{g}$. The straight lines $\bar{g}$ and $\breve{g}$ have the reprensentation

$$\hat{g} \,:\, \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_i \\ y_j - d_3 \end{bmatrix} + \hat{\lambda} \begin{bmatrix} d_3 \\ -d_1 \end{bmatrix}\,, \quad \breve{g} \,:\, \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_i + d_2 \\ y_j \end{bmatrix} + \breve{\lambda} \begin{bmatrix} 0 \\ -1 \end{bmatrix}\,, \quad \hat{\lambda}, \breve{\lambda} \in R,$$

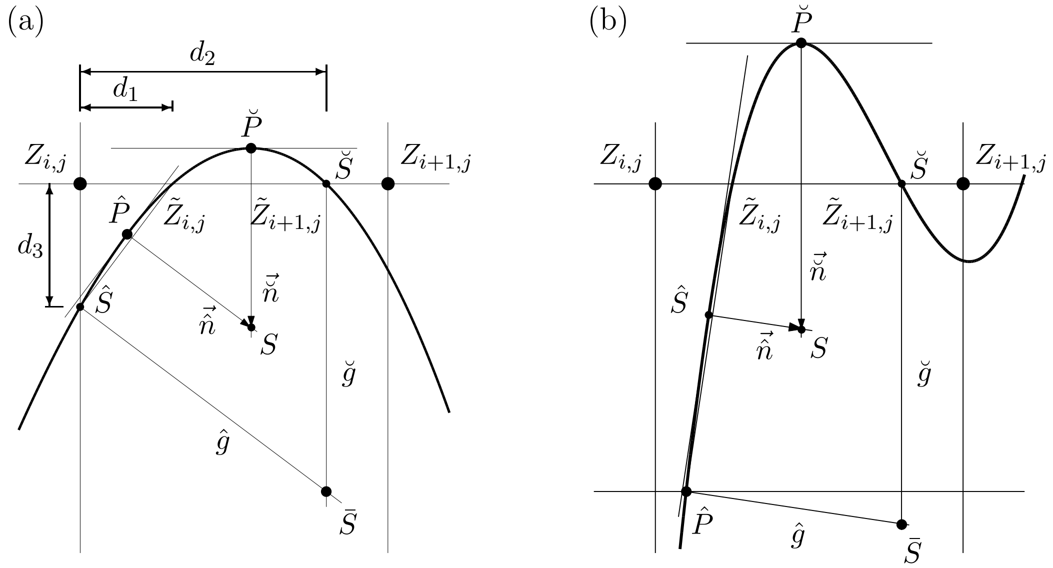

Figure 5: The boundary intersects the left and upper edge (a) or the upper and lower edge (b) of the grid cell.

and therefore, we get the intersection point $\bar{S} = (x_i + d_2, y_j - d_3 - d_1 d_2/d_3)$. The distances $\varrho(\bar{S}, \hat{S})$ and $\varrho(\bar{S}, \breve{S})$ from $\bar{S}$ to the boundary are:

$$\begin{aligned}
\varrho(\bar{S}, \hat{S}) &= d_2 \sqrt{1 + \left(\frac{d_1}{d_3}\right)^2} \leq \sqrt{2} d_2 \leq \sqrt{2} h \leq \sqrt{2} h_0 \leq \frac{\sigma}{2} < \sigma \,, \\
\varrho(\bar{S}, \breve{S}) &= d_3 + \frac{d_1 d_2}{d_3} \leq d_3 + d_2 \leq \frac{3}{2} h \leq \frac{3}{2} h_0 \leq \frac{3}{4} \frac{\sigma}{\sqrt{2}} < \sigma \,,
\end{aligned} \tag{11}$$

where σ is defined in (7). The distances from S to the boundary are less than the distances from $\bar{S}$ to the boundary, and owing to (11) less than σ. Therefore, we have found a point $S \in \omega_\sigma$ from which exist two different normal vectors to the boundary. This is in contratiction to the definition of ω_σ.

Let us now consider the situation where the boundary does not intersect the left edge of the grid cell (see Figure 5(b)). It is easy to see that it is similar to the previous case when $d_3 = h$.

Remark 4.1 *The statement of Lemma 4.1 is also true if we consider instead the node $Z_{i+1,j}$ the node $Z_{i,j+1} = (x_i, y_j + h) \in D^h_\square$.*

Let us now consider a grid cell $D_{i,j}$ with the vertices $Z_{i,j}$, $Z_{i+1,j}$, $Z_{i+1,j+1}$, and $Z_{i,j+1}$.

Lemma 4.2 *If three of the nodes $\tilde{Z}_{i,j}$, $\tilde{Z}_{i+1,j}$, $\tilde{Z}_{i+1,j+1}$, and $\tilde{Z}_{i,j+1}$ belong to $\partial\Omega$, then these nodes can not lie on the edges of the grid cell $D_{i,j}$. That means that situations like those shown in Figure 6 are impossivble.*

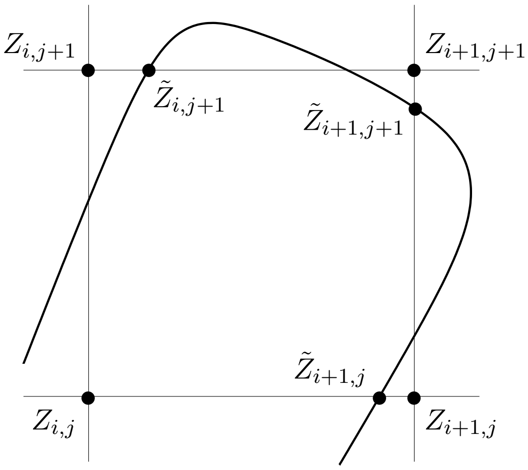

Figure 6: Situation, where the points $\tilde{Z}_{i,j+1}$, $\tilde{Z}_{i+1,j+1}$, and $\tilde{Z}_{i+1,j}$ lie on the edges of the grid cell $D_{i,j}$

Proof: The proof is analogous to the proof of Lemma 4.1. □

It is easy to prove that all quadrilaterals of the grid $\tilde{D}^h_\square$ are convex, see, e.g., [17].

Let us now divide each quadrilateral into two triangles by a diagonal. If there exist quadrilaterals having two nodes on the boundary which can be connected by a diagonal and one of the other nodes lies inside Ω the other one outside Ω, then this diagonal will be selected. For all other quadrilaterals the diagonal is selected in such a way that the minimum of the values of the sinuses of the angles in the derived triangles is as large as possible. In this way, we get the triangular grid $\tilde{D}^h_\triangle$. We denote the triangles of $\tilde{D}^h_\triangle$ by τ_i and define the triangulation Ω^h as the union of all triangles τ_i of $\tilde{D}^h_\triangle$ having at least one vertex inside Ω. The grid $\tilde{D}^h_\triangle$ obtained from the locally modified quadrilateral grid $\tilde{D}^h_\square$ in Figure 3 is shown in Figure 7.

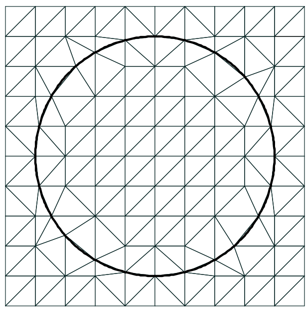

Figure 7: The locally modified triangular grid $\tilde{D}^h_\triangle$

Using Lemma 4.1 and Lemma 4.2 the following theorem can be proved (see also [17]).

Theorem 1 *The lengths of the edges and the area of the triangles obtained by the method described above belong to intervals* $[\beta_1 h, \beta_2 h]$ *and* $[\gamma_1 h^2, \gamma_2 h^2]$*, respectively, where*

$$\beta_1 = 0.5\,,\ \beta_2 = \frac{\sqrt{18}}{2}\,,\ \gamma_1 = 0.125\,,\ \textit{and}\ \gamma_2 = 1.125\,.$$

The sinus of any angle is greater than or equal to $\sqrt{10}/10$*.*

Proof: We have to consider all possible cases concerning the location of the moved nodes. In this paper, we want to present only some representative cases and the extreme case which leads to the constants β_1, β_2, γ_1, γ_2, and the smallest value of the sinuses of the angles given in the statement of the Theorem.

Let us first consider the case where two neigbour nodes of a grid cell are moved to the boundary. Owing Lemma 4.1 there exist only two possibilities concerning the location of the moved nodes (see Figure 8):

- Node $Z_{i,j+1}$ is moved along the line through the nodes $Z_{i,j}$, $Z_{i,j+1}$, and $Z_{i+1,j+1}$ is moved along the line through the nodes $Z_{i+1,j}$, $Z_{i+1,j+1}$, i.e. the nodes $Z_{i,j+1}$, $\tilde{Z}_{i,j+1}$, $Z_{i+1,j+1}$, and $\tilde{Z}_{i+1,j+1}$ have the coordinates (x_i, y_{j+1}), $(x_i, y_{j+1}+\delta_2 h)$, (x_{i+1}, y_{j+1}), and $(x_{i+1}, y_{j+1}+\delta_3 h)$, respectively (see Figure 8(a)), where $-0.5 \le \delta_2, \delta_3 \le 0.5$.
- Node $Z_{i,j+1}$ is moved along the line through the points $Z_{i,j}$, $Z_{i,j+1}$, and $Z_{i+1,j+1}$ is moved along the line through the nodes $Z_{i,j+1}$, $Z_{i+1,j+1}$, i.e. the nodes $Z_{i,j+1}$, $\tilde{Z}_{i,j+1}$, $Z_{i+1,j+1}$, and $\tilde{Z}_{i+1,j+1}$ have the coordinates (x_i, y_{j+1}), $(x_i, y_{j+1}+\delta_2 h)$, (x_{i+1}, y_{j+1}), and $(x_{i+1}+\delta_3 h, y_{j+1})$, respectively (see Figure 8(b)), where $-0.5 \le \delta_2, \delta_3 \le 0.5$.

Note that the shift of both nodes $Z_{i,j+1}$ and $Z_{i+1,j+1}$ along the line through these nodes is impossible due to Lemma 4.1.

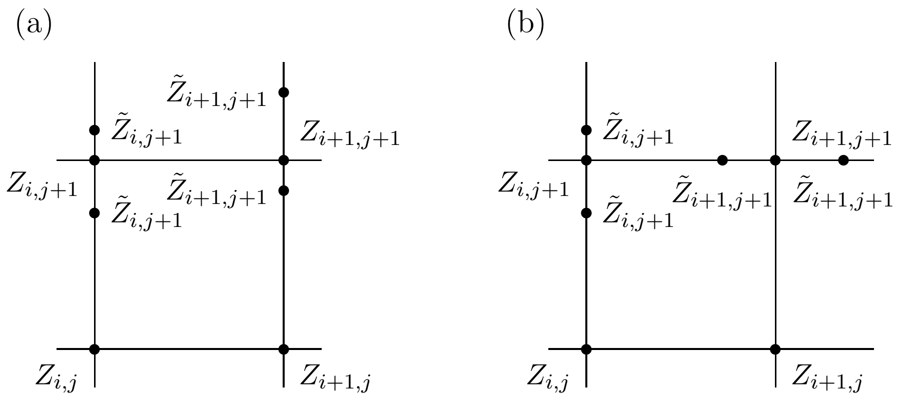

Figure 8: The possibilities for moving neighbour nodes of a cell

For the case (a), let the cell with the vertices $Z_{i,j}$, $\tilde{Z}_{i,j+1}$, $\tilde{Z}_{i+1,j+1}$, and $Z_{i+1,j}$ be divided into two triangles as shown in Figure 9. Then, the length d_i, the areas A_j of the triangles,

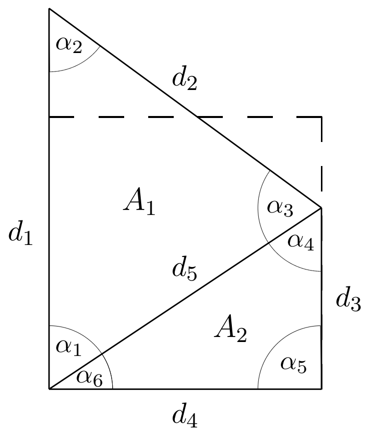

Figure 9: One possibility for dividing the quadrilateral into two triangles (case (a) of Figure 8)

and the sinuses of the angles α_k can be expressed in terms of δ_2 and δ_3 as follows:

$$\begin{aligned}
&d_1 = h(1+\delta_2), \quad d_2 = h\sqrt{1+(\delta_2-\delta_3)^2}, \quad d_3 = h(1+\delta_3), \\
&d_4 = h, \quad d_5 = h\sqrt{1+(1+\delta_3)^2}, \\
&A_1 = h^2\frac{1+\delta_2}{2}, \quad A_2 = h^2\frac{1+\delta_3}{2},
\end{aligned}$$

$$\begin{aligned}
\sin\alpha_1 &= \frac{1}{\sqrt{1+(1+\delta_3)^2}}, \quad \sin\alpha_2 = \frac{1}{\sqrt{1+(\delta_3-\delta_2)^2}}, \\
\sin\alpha_3 &= \frac{1+\delta_2}{\sqrt{1+(\delta_3-\delta_3)^2}\sqrt{1+(1+\delta_3)^2}}, \\
\sin\alpha_4 &= \frac{1}{\sqrt{1+(1+\delta_3)^2}}, \quad \sin\alpha_5 = 1, \quad \sin\alpha_6 = \frac{1+\delta_3}{\sqrt{1+(1+\delta_3)^2}}\,.
\end{aligned}$$

One gets analogous expressions if the other diagonal is selected. As described before, we choose that diagonal which leads to two triangles for which the minimal value of the sinuses of the angles is maximal. Then, a simple calculation gives

$$\frac{h}{2} \le d_i \le \frac{\sqrt{13}h}{2}\,, \ \frac{h^2}{4} \le A_j \le \frac{3h^2}{4}\,, \ \sin\alpha_k \ge \frac{\sqrt{5}}{5}\,.$$

Let us now consider case (b). Again, we distinguish the two possibilities for dividing the quadrilateral into two triangles (see Figure 10) and choose the subdividing that gives the maximum of the minima of $\sin\alpha_k$. We obtain

$$\frac{h}{2} \le d_i \le \frac{\sqrt{13}h}{2}\,, \ \frac{3h^2}{16} \le A_j \le \frac{51h^2}{50}\,, \ \sin\alpha_k \ge \frac{\sqrt{5}}{5}\,.$$

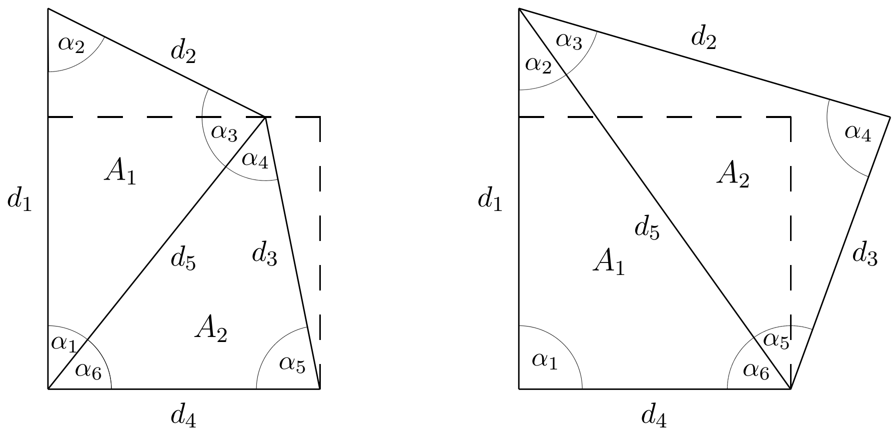

Figure 10: Two possibilities for dividing the quadrilateral into two triangles (case (ii) of Figure 8)

Finally, let us discuss the situation shown in Figure 11 which is an extreme case. We suppose that the boundary of Ω cuts the line through $Z_{i,j}$, $Z_{i,j+1}$ in the point with the coordinates $(x_i, y_j + \frac{3h}{2})$, the line through $Z_{i,j+1}$, $Z_{i+1,j+1}$ in $(x_i + \frac{h}{2}, y_j + h)$, the line through $Z_{i+1,j}$, $Z_{i+1,j+1}$ in $(x_i + h, y_j + \frac{h}{2})$, and the line through $Z_{i,j}$, $Z_{i+1,j}$ in $(x_i + \frac{3h}{2}, y_j)$. Furthermore, we suppose that one of the nodes $Z_{i,j}$, $Z_{i+1,j+1}$ lies inside Ω and the other one outside Ω. Then, we have to select the diagonal drawn in Figure 11

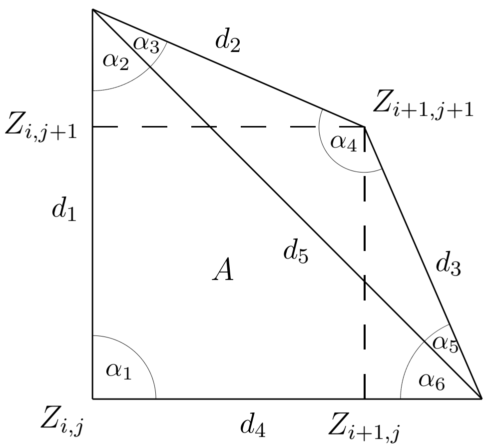

Figure 11: Extreme case

As second extreme case we consider the following: The boundary of Ω cuts the line through $Z_{i,j}$, $Z_{i+1,j}$ in the point with the coordinates $(x_i + \frac{h}{2}, y_j)$ and the line through $Z_{i,j}$, $Z_{i,j+1}$ in $(x_i, y_j + \frac{h}{2})$. Additionally, we suppose that one of the nodes $Z_{i,j}$, $Z_{i+1,j+1}$ lies inside Ω

and the other one outside Ω. Then, we get the following estimates

$$\frac{h}{2} \le d_i \le \frac{\sqrt{18}h}{2}\,,\ \frac{h^2}{8} \le S_j \le \frac{9h^2}{8}\,,\ \sin\alpha_k \ge \frac{\sqrt{10}}{10}\,.$$

All other possible cases lead to no smaller and no larger lower and upper bounds for the length of the edges, the areas of the triangles, and the sinuses of the angles, respectively. □

The algorithm described above can be easily generalized for domains with a piecewise smooth boundary. We denote by Δ_k points where smooth parts P_i and P_j of the boundary $\partial\Omega$ intersect. The construction of the correspondence between the nodes of $D^h_\square$ and $\tilde{D}^h_\square$ will be done in three steps.

First step: We consider semi-opened squares (left and bottom edges are excluded) with the center in the nodes $Z_{i,j}$, edges parallel to the axes of the coordinate system, and the length of the edges equals to h. We have to find that square which contains a given point Δ_k. Then, the midpoint $Z_{i,j}$ of this square will be moved into Δ_k and therefore corresponds to the point $\tilde{Z}_{i,j} = \Delta_k$ (see Figure 12(a)).

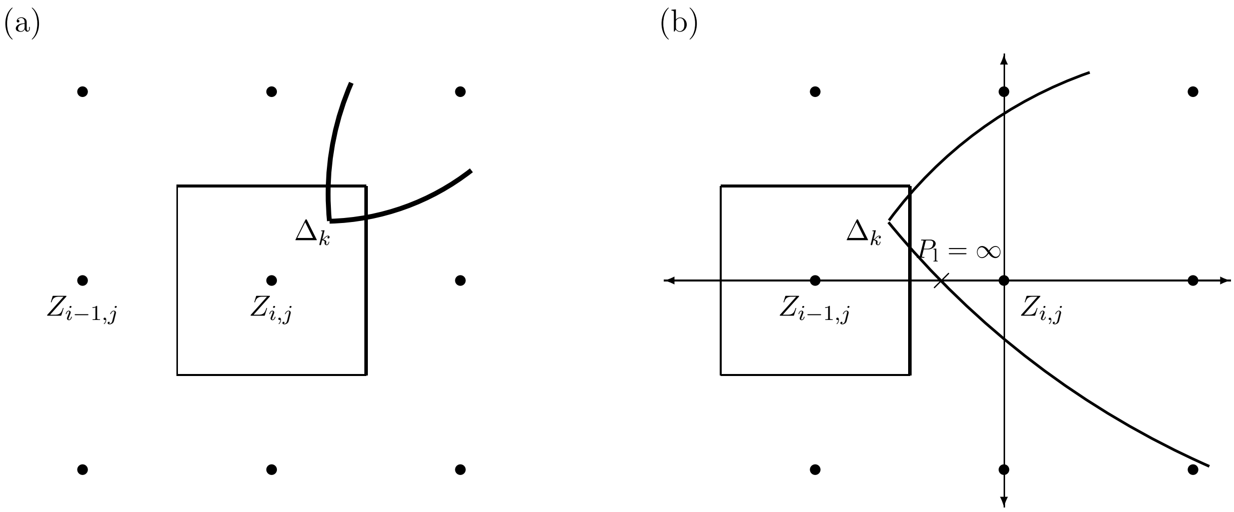

Figure 12: (a) Finding (x_i, y_j) in the first step (b) Situation, where one intersection points is considered as infinitely far away

The second step coincides almost completely with the algorithm described above for domains with a smooth boundary. The only difference is that we will treat a point of intersection of the ray starting at $Z_{i,j}$ with the boundary as infinitely far away when the nearest node in the corresponding direction has been already "shifted" to the boundary on the previous step, see Figure 12(b).

Third step: This step is necessary for handling acute angles of the boundary as shown in Figure 13. For example, in this situation the node $Z_{i+1,j+1}$ is already moved to the boundary corner Δ_k in the first step of the algorithm. Since the distance between the node $Z_{i,j+1}$ and the intersection point $\tilde{Z}_{i,j+1}$ is smaller than the distance between $Z_{i,j+1}$ and $\tilde{Z}_{i,j}$ the node $Z_{i,j+1}$ is moved into the intersection point $\tilde{Z}_{i,j+1}$ (see Figure 13). It is obvious that the node $Z_{i,j}$ is not moved in the first two steps since the distance between this node and the intersection point $\tilde{Z}_{i,j}$ is larger than $h/2$. Without moving the node $Z_{i,j}$ into the intersection point $\tilde{Z}_{i,j}$ one would get a topologically incorrect triangulation. Consequently, if the segment between the node $Z_{i,j}$ and some nearest node intersects the boundary in a point which is different from endpoints of this segment, we move this node into the intersection point regardless of the distance between the node $Z_{i,j}$ and the intersection point.

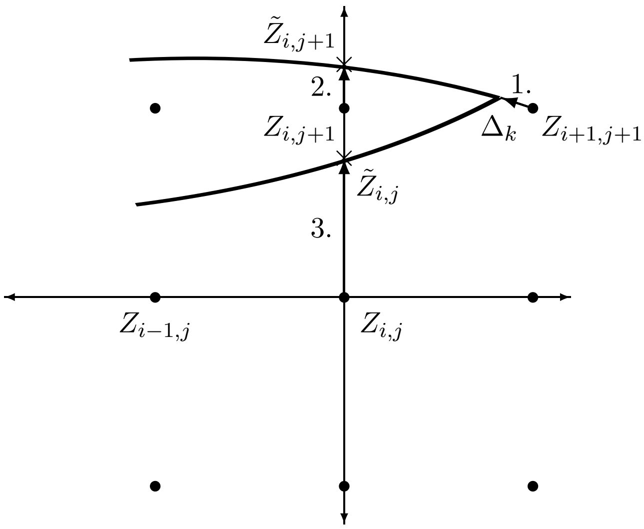

Figure 13: Handling of acute angles of the boundary

Dividing of each quadrilateral into two triangles completes the generation of the triangulation. The method of dividing was described above for domains with the smooth boundary. An example for a locally modified grid in the case of domains with a piecewise smooth boundary is shown in Figure 14.

It is easy to prove (by consideration of all possible configurations near the boundary corners) that the inequality

$$l > \frac{h \sin \alpha}{2}, \quad \alpha = \min_i \ \alpha_i \tag{12}$$

for the length of the edges of the triangles obtained by the described procedure is fulfilled. For the sinuses of the angles β of these triangles holds

$$\sin \beta > M(\alpha) > 0\,, \tag{13}$$

where $M(\alpha)$ is a positive function.

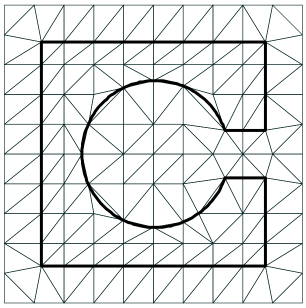

Figure 14: Locally modified grid for a domain with a piecewise smooth boundary

Remark 4.2 *If the conditions $\Omega \subset \Omega^h$ for $\Gamma_1 = \Gamma$ or $\Omega^h \subset \Omega$ for $\Gamma_0 = \Gamma$ are not fulfilled, then we can modify our mesh by shifting near boundary nodes outside or inside of Ω in a distance of the order h^2 in such a way that these conditions are satisfied. In such a case the constants β_1, β_2, γ_1, and γ_2 (see Theorem 1) will be sligthly changed but will not depend on h.*

4.2 Triangular source grid

A similar algorithm for the generation of a triangular grid can be used in the case when the original uniform grid is a triangular one. A mesh generation algorithm using a triangular source grid is also described in [22]. This algorithm can be applied to domains with a boundary that consists of a finite number of closed twice-continuously differentiable arcs which do not touch or cut each other or themselves (i.e. the domain is not necessarily simple connected). Our algorithm works also in the case of a piecewise smooth boundary.

In some sense a triangular source grid is even more preferable than a quadrilateral one, because one step of the algorithm – dividing quadrilaterals into two triangles – becomes unnecessary and the resulting grid mainly consists of optimal (equilateral) triangles. The generation of the grid starts from the embedding of the domain Ω into a big triangle D with internal angles equal to $\pi/3$. At first, we build a uniform triangulation $D^h_\triangle$ in D, see Figure 15(a).

Then, we perform exactly the same actions (except the last one) as described in subsection 4.1, i.e. at the first step we shift nearest nodes to the endpoints of parts C_i of the boundary. Then, we calculate the distances from the given node to the boundary along grid edges and shift the node to the point of intersection of the edge and the boundary if the minimal distance is less than $h/2$ and so on. Actually we can use one program for the generation of the grid in both cases. All differences are located in low level procedures like these one:

- get initial coordinates of the node,
- get the number of neighbour nodes,
- get grid coordinates of the neighbour node for the given node.

(a) (b)

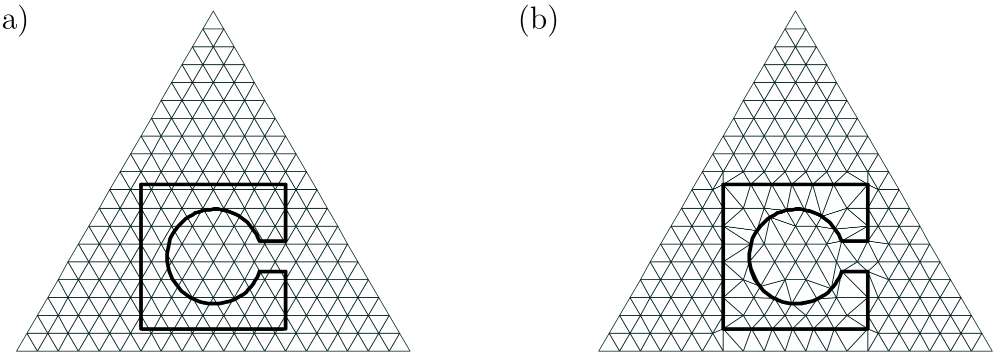

Figure 15: (a) A Domain embedded into a triangle D and the source grid $D^h_\triangle$; (b) The locally modified triangular grid $\tilde{D}^h_\triangle$

A resulting locally modified grid looks like the grid shown in Figure 15(b).

It follows from the previous considerations and may be easily proved that inequalities similar to (12) and (13) are also valid.

5 Advantages of the generated grids

Locally modified triangular grids generated by the methods described in section 4 possesses some advantages.

• **Boundary approximation**

The proposed algorithm guarantees an approximation of the boundary of the domain considered with a second order of accuracy. As it is well-known, the accurate approximation of the boundary is one of the key properties for a good approximation of the problem we want to solve numerically.

• **Regular structure**

The constructed meshes have a regular structure. This feature is difficult to overestimate because it decreases the amount of the memory required for storing the generated grid and the number of arithmetical operations required for the generation of the stiffness matrix. Indeed, we must store only coordinates of the shifted nodes because coordinates of the rest nodes can be found according to simple formula. The number of shifted nodes has the order $\mathcal{O}(l/h)$, where l is the length of the boundary. Using the well-known and widely used technique of so-called "hash tables" we may decrease the amount of memory required for holding the whole grid up to $\mathcal{O}(l/h)$. Moreover in the case of the triangular source grid we do not need to store the structure of the grid (information about links between adjacent nodes) because the structure is absolutely regular. On the other hand, due to the regular structure and congruence of all finite elements of the grid everywhere inside of the domain except narrow the band near the boundary, we have to compute only one element stiffness matrix which is the same for all these triangles in the case of constant coefficients in the equation (1). Calculations are faster even in the case of variable coefficients.

• **Local nature of the generation**

The generation of the grid has a "local" nature and requires only $\mathcal{O}(l/h)$ operations. Indeed, in the first step of the grid generation, when we shift the nearest node to the corner points of the boundary, we can find the required node with a small fixed number of operations due to the regular initial positions of the nodes. In the second step, when we shift nodes near to the boundary, we "walk" along the boundary and check only nearest nodes. It gives us $\mathcal{O}(l/h)$ operations in the second step. In the third step we have to check only nodes that fall into some neighbourhood of the corners on the boundary. At last, in the fourth step for quadrilateral source grids we must check only those quadrilaterals which have shifted nodes.

• **Applicability of multilevel preconditioners**

The generated grids are extremely suitable for using multilevel preconditioning operators (BPX-like). This will be discussed in the next section.

6 The construction of the preconditioner $\boldsymbol{B}$

Let us assume that $h = 2^{-J}s$, where s is the length of the sides of D and J is a positive integer.

Let Q^h denote the minimal figure that consists of cells $\bar{D}_{i,j}$ and contains Ω^h, i.e. $\Omega^h \subset Q^h$; let S^h be the set of boundary nodes of Q^h. We subdivide the set S^h into two subsets S_0^h and S_1^h as follows: If

$$\bar{D}_{i,j} \cap \Gamma_0 \neq \emptyset$$

all nodes of $\bar{D}_{i,j} \cap S^h$ are in S_0^h, and

$$S_1^h = S^h \setminus S_0^h \,.$$

Additionally, we consider in D a sequence of grids

$$D_0^h\,,\ D_1^h\,,\ \ldots\ ,\ D_J^h \equiv D^h$$

with the step sizes

$$h_0 = s\,,\ \ h_1 = 2^{-1}s\,,\ \ldots\ ,\ h_J \equiv h = 2^{-J}s\,.$$

We triangulate these grids hierarchically. The restriction of the triangulation D_J^h on Q^h will be denoted by $Q_\triangle^h$ and the triangles of $Q_\triangle^h$ by $\mathcal{T}_i$. Corresponding to the triangulations $D_0^h\,,\ D_1^h\,,\ \ldots\ ,\ D_J^h$ we define the finite element spaces

$$H_h(D_0^h, \partial D_0^h) \subset H_h(D_1^h, \partial D_1^h) \subset \cdots \subset H_h(D_J^h, \partial D_J^h) \equiv H_h(D^h, \partial D^h)$$

and denote by $\{\Phi_i^{(\ell)}\}_{i=1}^{N_\ell}$ the usual nodal basis of the space $H_h(D_\ell^h, \partial D_l^h)$, $\ell = 0, 1, \ldots, J$. By $\tilde{\Phi}_i^{(\ell)}$ we denote the restriction of the basis function $\Phi_i^{(\ell)}$ onto $Q_\triangle^h$.

In the following, we will use the space $H_h(Q_\triangle^h, S_0^h)$ as the fictitious space $\tilde{H}$ in Lemma 3.1. We now define the projection operator R,

$$R \,:\, H_h(Q_\triangle^h, S_0^h) \to H_h(\Omega^h, \Gamma_0^h)\,,$$

the extension operator T,

$$T \,:\, H_h(\Omega^h, \Gamma_0^h) \to H_h(Q_\triangle^h, S_0^h)\,,$$

and an easily invertible operator in the space $H_h(Q_\triangle^h, S_0^h)$. As described in section 4, there exists a one-to-one correspondence between nodes $(\tilde{x}_i, \tilde{y}_j)$ of the triangulation Ω^h and some subset of nodes (x_i, y_j) of Q^h. We want to use this correspondence in the definition of the operators R and T. Let us begin with the operator R. For a given mesh function $U^h \in H_h(Q_\triangle^h, S_0^h)$ we define a function $u^h \in H_h(\Omega^h, \Gamma_0^h)$ as follows (see also Figure 16). We put

$$u^h(\tilde{x}_i, \tilde{y}_j) = (RU^h)(\tilde{x}_i, \tilde{y}_j) = U^h(x_i, y_j) \quad \text{for all nodes } (\tilde{x}_i, \tilde{y}_j) \text{ from } \Omega^h$$

and the function u^h is equal to zero at nodes on Γ_0^h.

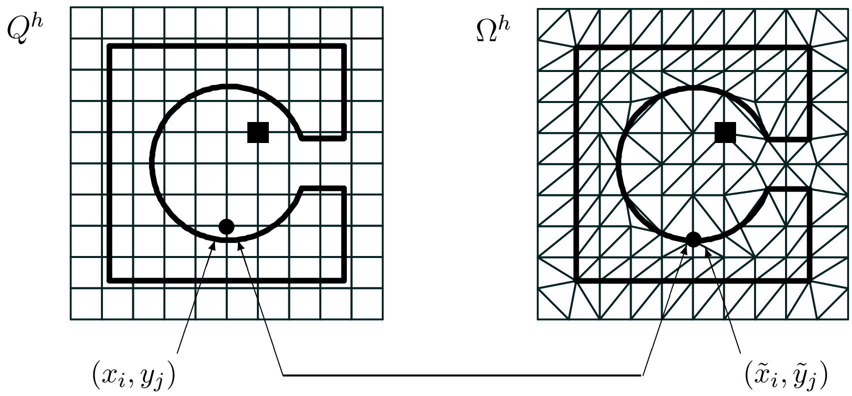

Figure 16: Correspondence between the nodes $(x_i y_j)$ and $(\tilde{x}_i, \tilde{y}_j)$

Now, we define the operator T. For a given function $u^h \in H_h(\Omega^h, \Gamma_0^h)$ we have to define a function $U^h \in H_h(Q_\triangle^h, S_0^h)$. The function U^h is equal to zero at nodes on S_0^h. At the other nodes, U^h is defined as follows. If the vertex (x_i, y_j) of the triangulation Q^h corresponds to some vertex $(\tilde{x}_i, \tilde{y}_j)$ of the triangulation Ω^h, then we put

$$U^h(x_i, y_j) = (Tu^h)(x_i, y_j) = u^h(\tilde{x}_i, \tilde{y}_j)\,.$$

If the vertex (x_i, y_j) does not belong to $f^{-1}(\Omega^h)$ (for the definition of f see (8)), and let D_{kl} be a cell which has (x_i, y_j) (e.g. the node marked by ■ in Figure 16) as a node and

$D_{kl} \cap f^{-1}(\Omega^h) \neq \emptyset$, then we consider any node (x_n, y_m) of D_{kl} which belongs to $f^{-1}(\Omega^h)$ and put

$$U^h(x_i, y_j) = U^h(x_n, y_m)\,.$$

In the following, we show that these operators R and T fulfil the conditions required in the fictitious space lemma (Lemma 3.1).

Lemma 6.1 *There exist constants c_R and c_T, which are independent of h, such that*

$$\|RU^h\|_{H^1(\Omega^h)} \leq c_R \|U^h\|_{H^1(Q^h)}\,, \tag{14}$$

$$c_T \|Tu^h\|_{H^1(Q^h)} \leq \|u^h\|_{H^1(\Omega^h)}\,, \tag{15}$$

and

$$RTu^h = u^h \ \ \forall u^h \in H_h(\Omega^h, \Gamma_0^h)\,. \tag{16}$$

Proof: Since the values of the functions u^h at the vertices of Ω^h and U^h at the vertices of $f^{-1}(\Omega^h)$ are the same, the identity (16) is obvious.

Let us introduce the following discrete norms

$$\begin{aligned}\|u^h\|_{H_h^1(\Omega^h)} &= \sum_{\tau_i \in \Omega^h} \Big\{h^2\Big((u^h(z_{i_1}))^2 + (u^h(z_{i_2}))^2 + (u^h(z_{i_3}))^2\Big) \\ &\qquad + (u^h(z_{i_1}) - u^h(z_{i_2}))^2 + (u^h(z_{i_2}) - u^h(z_{i_3}))^2 \\ &\qquad + (u^h(z_{i_3}) - u^h(z_{i_1}))^2\Big\}\end{aligned}$$

and

$$\begin{aligned}\|U^h\|_{H_h^1(Q_\triangle^h)} &= \sum_{\mathcal{T}_i \in Q_\triangle^h} \Big\{h^2\Big((U^h(Z_{i_1}))^2 + (U^h(Z_{i_2}))^2 + (U^h(Z_{i_3}))^2\Big) \\ &\qquad + (U^h(Z_{i_1}) - U^h(Z_{i_2}))^2 + (U^h(Z_{i_2}) - U^h(Z_{i_3}))^2 \\ &\qquad + (U^h(Z_{i_3}) - U^h(Z_{i_1}))^2\Big\}\,,\end{aligned}$$

where z_{i_1}, z_{i_2}, z_{i_3} are the vertices of the triangle $\tau_i \in \Omega^h$ and Z_{i_1}, Z_{i_2}, Z_{i_3} are the vertices of the triangle $\mathcal{T}_i \in Q_\triangle^h$, respectively.

It is well-known that these discrete norms and the corresponding Sobolev norms are equivalent with constants independent of h, supposed that the properties (3) are fulfilled for the triangulations (see, e.g., [20]). It is easy to prove that there exist constants $\tilde{c}_R$ and $\tilde{c}_T$, independent of h, such that

$$\|RU^h\|_{H_h^1(\Omega^h)} \leq \tilde{c}_R \|U^h\|_{H_h^1(Q_\triangle^h)}$$

and

$$\tilde{c}_T \|Tu^h\|_{H_h^1(Q_\triangle^h)} \leq \|u^h\|_{H_h^1(\Omega^h)}\,.$$

From these inequalities and from the equivalence of the discrete norms and the Sobolev norms we get the statement of the lemma. □

Remark 6.1 *If D^h is a triangular grid, then $\Omega^h = f(Q^h)$ and the matrix representation of the restriction operator R and of the extension operator T are the identity operators.*

Next, we construct the preconditioner B. We define an operator $\tilde{A}$ in the following way:

$$\tilde{A}^{-1}U^h = \sum_{\ell=0}^{J} \sum_{\substack{\mathrm{supp}\Phi_i^{(\ell)} \cap Q^h \neq \emptyset \\ \mathrm{supp}\Phi_i^{(\ell)} \cap S_0^h = \emptyset}} (U^h, \Phi_i^{(\ell)})_{L_2(Q^h)} \tilde{\Phi}_i^{(\ell)} \quad \forall U^h \in H_h(Q_\triangle^h, S_0^h)$$

and the preconditioner B by

$$B^{-1} = R\tilde{A}^{-1}R^* \,. \tag{17}$$

From [4, 18, 21, 25] we have constants c_3 and c_4, independent of h, such that

$$c_3\|U^h\|^2_{H^1(Q^h)} \leq (\tilde{A}u, u) \leq c_4\|U^h\|^2_{H^1(Q^h)} \tag{18}$$

for arbitrary $U^h \in H_h(Q_\triangle^h, S_0^h)$. Here $\tilde{A}$ is the matrix representation of the operator $\tilde{A}$. Since the matrix A generates a H^1-equivalent norm in the space $H_h(\Omega^h, \Gamma_0^h)$, then from Lemma 6.1, Lemma 3.1, and (18) we get the following theorem.

Theorem 2 *There exist positive constants c_5 and c_6, independent of h, such that*

$$c_5(A^{-1}u, u) \leq (B^{-1}u, u) \leq c_6(A^{-1}u, u) \;\; \forall u \in R^N \,.$$

7 The results of the numerical experiments

In this subsection, we apply our algorithms to the boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \,, \end{aligned} \tag{19}$$

where Ω is the unit square (domain 1), the circle, which is embedded into the unit square (domain 2), see Figure 1, or a square with the extracted circle and the wide cut (domain 3), see Figure 14. We study the convergence behaviour of the algorithm presented in previous section.

Problem (19) is discretized as explained in section 3. For the construction of the locally modified grids both the quadrilateral and the triangular source grids are used. We solve the systems of algebraic finite element equation (5) by means of the preconditioned conjugate gradient method with the preconditioner defined by (18) in section 6.

To be able to measure the error of the iterates in the A-energetic norm we choose in (19) the right-hand side $f(x) = 0$ (i.e. the exact solution is $u = 0$). As initial guess for the iteration process, a vector is used whose components correspond to the values of the function $1 - 2\sqrt{(x-0.5)^2 + (y-0.5)^2}$ in the nodes of the finite element meshes. In the Tables 1 and 2, we present the numbers of iterations which we need to solve the systems of finite element equations with a relative error of 10^{-5} measured in the A-energetic norm.

In the Tables 1 and 2, J is the characteristic of the initial regular grid, such that the number of nodes in each directions is equal to 2^{-J}.

Table 1: Number of iterations (#it) for quadrilateral source grid

	J	5	6	7	8	9	10
#it	domain 1	9	10	10	14	16	16
	domain 2	10	14	17	19	22	23
	domain 3	7	11	13	15	17	19

Table 2: Number of iterations (#it) for triangular source grid

	J	6	7	8	9	10
#it	domain 1	9	11	11	12	12
	domain 2	15	16	19	20	21
	domain 3	7	9	11	13	14

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